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EXTENSIONS OF LIDSTONE'S THEOREM

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ABSTRACT

This paper deals with the problem of determining the effect on reserves of variations in the assumptions. A general approach to this problem is presented and then used to extend Lidstone's theorem, mainly to cases involving a nonmonotone critical function. The setting is a general type of insurance policy that allows benefits, interest rates, and premiums to vary freely by duration. In addition, the paper discusses an auxiliary problem that arises in applications of Lidstone's theorem, namely, the determination of monotonicity properties of reserves and costs of insurance.

I. INTRODUCTION

WHAT is the effect on reserves when changes are made in the underlying assumptions? The major work dealing with this question is the 1905 paper of Lidstone [5]. A partial account of this material is given in the book by Spurgeon [7]. Some other references dealing with the subject are the papers of Baillie [1], Gershenson [2], and Milgrom [6].

In this paper we will summarize this work, and supplement some of the intuitive arguments used in previous accounts with rigorous mathematical proofs. We then give various new extensions and applications.

The main conclusions of Lidstone's work are well known through the name of Lidstone's theorem as described, for example, in [3]. However, the complete details of his ideas do not seem to appear in North American actuarial literature. We will elaborate on this after first introducing some notation.

We will consider throughout a policy of duration n. Let

B_t = Death benefit payable at time $t + 1$, should death occur in policy year $t + 1$;
P_t = Premium payable at time t;
q_t = Rate of mortality for policy year $t + 1$;
i_t = Rate of interest for policy year $t + 1$ (note that the subscript here differs from those of [5] and [6];

E = Endowment payment payable at time n, should the insured be then alive (this can, of course, be zero, as in the case of term insurance);
${}_tV$ = Terminal reserve at the end of year t (note that ${}_nV = E$).

In all cases the subscript t takes on the values 0, 1, . . . , $n - 1$. When any of the foregoing symbols are primed, they will refer to an alternate basis. Let

$$\Delta_t = {}_tV' - {}_tV .$$

Note that we allow for all quantities to vary with time. The varying interest necessitates some additional notation. Let

$$v_t = (1 + i_t)^{-1} ,$$

$$v(t) = v_0 v_1 \ldots v_{t-1} \text{ for } t > 0, \qquad v(0) = 1 .$$

Annuities can be defined as usual, with $v(t)$ replacing v^t. For example, we have

$$\ddot{a}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v(t)\, {}_tp_x .$$

We now define the following (for $t = 0, 1, \ldots, n - 1$):

$$c_t = ({}_tV + P_t)(i'_t - i_t) + q_t(B_t - {}_{t+1}V) - q'_t(B'_t - {}_{t+1}V) ,$$

$$L_t = -\frac{c_t}{(1 + i'_t)} ,$$

$$J_t = L_t - (P'_t - P_t) .$$

Note that c_t above corresponds to the so-called critical function in [3] (where it also is called c_t), while $(1 + i'_t)\, J_t$ corresponds to $-S_t$ in the notation of [3] or $-R_t$ in the notation of [1].

It is clear from the discussion in [5] and [6] that c_t represents the value at time $t + 1$ of the gain that occurs in policy year $t + 1$ when original assumptions are replaced by alternate ones. L_t is then the value at time t of the loss in policy year $t + 1$. (We discount with interest only, since this loss is occasioned by each policyholder who is alive at time t.) J_t equals L_t adjusted by the extra premium and therefore represents a kind of net loss at time t.

We need one final definition to cover the possible loss occurring at maturity if the new basis provides a different endowment amount. We define

$$L_n = J_n = E' - E .$$

At this point we observe that we easily could have incorporated additional generality (as was done in [6]) and considered such factors as withdrawal rates, cash values, expenses, benefit payments in midyear, and so on. The definition of c_t would require appropriate adjustment, but the definitions of L_t and J_t, as well as all that follows concerning these quantities, would remain unchanged.

One of Lidstone's key observations was to note that the present value of the extra premiums paid must be equal to the present value of the losses, where present values are computed using the alternate assumptions. The demonstration of this first appears in [4], an earlier paper of Lidstone. Similar results can be found in the basic theorem of [6] and in formula (6) of [1]. This observation suggests the following point of view.

We consider the policy with the alternate assumptions as being decomposed into two separate contracts. The first of these is just the original policy. The second is a life annuity that provides a payment of L_t at time t and carries premiums of $(P_t' - P_t)$ at time t. Its operation is governed by the new basis for interest and mortality. We will refer to this second contract as the *auxiliary annuity*. Lidstone referred to it as the *variation fund*. (It is quite possible, of course, for the auxiliary annuity premiums and/or payments to be negative, but this does not affect the theory.)

Let W_t be the auxiliary annuity reserve at time t calculated *before* premium and benefit payments are made. It is easy to see that

$$\Delta_t = W_t . \tag{1.1}$$

This shows that the question of whether reserves on the new basis are greater than or less than standard reserves can be replaced by the question of whether auxiliary annuity reserves are positive or negative.

We now can give a verbal explanation of the classical Lidstone theorem. Consider first the case when the original and alternate assumptions provide for level premiums of P and P', respectively. Suppose that L_t is increasing. The auxiliary annuity then consists of an increasing sequence of payments that must be paid for by the level premium $P' - P$. This naturally causes positive reserves. Similarly, a decreasing sequence of benefits will result in negative auxiliary annuity reserves.

Now consider the general case of varying premiums. We simply observe that the reserves on any annuity contract are obviously unchanged if at any duration we adjust both the premium and benefit by the same constant amount. (Indeed, the extra amount of premium is paid out immediately as a benefit.) Accordingly, for the purpose of calculating auxiliary annuity reserves, we can add $(P_t - P'_t)$ to both premiums and benefits for each duration t. The result is a contract with benefit payments of J_t but now with *level* premiums of zero amount. We thus conclude that for increasing J_t, auxiliary annuity reserves will be positive and new reserves will exceed standard ones. The reverse is true for decreasing J_t.

All previous writers on Lidstone's theorem have indicated that there is a difference between the level premium and the varying premium case, but the true nature of this difference has not been fully explained. The above analysis should help to clarify the situation. As shown above, the basic theoretical considerations are the same regardless of the premium pattern. In all cases, reserve changes are determined by the behavior of J_t. We are faced, however, with the problem of trying to determine this behavior. In many instances, it is easy to determine the effect on L_t of given interest or mortality changes. In the level premium situation, J_t differs from L_t by a constant and will therefore have the same monotonicity properties. This is the great advantage of this case. For the general varying premium policy, on the other hand, we may know a great deal about L_t but still find it difficult to obtain information about J_t.

To illustrate the difficulty, we consider a limited payment policy. Assume that there is an index $k < n - 1$ such that P_t and P'_t are both positive constants for $t \leq k$ and are both equal to zero for $t > k$. Suppose we have a change in assumptions that results in a decreasing L_t. If L_t is negative, then the premiums will decrease. Therefore, $(P'_t - P_t)$ is increasing, since it starts out negative and eventually becomes zero. Hence, J_t is also decreasing, and we know that reserves will be reduced. However, if L_t is positive, then the premium increases and we can no longer infer that J_t is decreasing. It will, of course, be decreasing over each of the intervals $0 \leq t \leq k$ and $k + 1 \leq t \leq n - 1$, but if the constant extra premium is greater than $L_k - L_{k+1}$, then J_{k+1} will be greater than J_k. Special cases of this situation were considered by Baillie ([1], Sec. V, cases *a* and *b*). He considered a policy with reserves increasing with duration. An increase in interest rates and a constant increase in mortality rates both result in a decreasing L_t, but premiums decrease in the former case and increase in the latter. We will return to this example later in Appendix A.

One of the main limitations of Lidstone's theorem is that it applies only when J_t is either decreasing or increasing. It is natural to inquire what happens when J_t is no longer monotone. We will use the general framework that we have established to investigate this problem in certain cases. Of course, the situation is much more complicated than the monotone case and there are no longer any easily stated conclusions.

A case of particular interest is the one that, in some sense, is closest to the monotone case, namely, when J_t changes direction exactly once over its entire range. This is of practical significance, because such behavior of J_t arises in many circumstances (examples are given in Sec. III). In this case, we are able to give a reasonably complete description of the resulting reserve changes, which can be summarized as follows: Just as a monotone sequence $\{J_t\}$ produces a sequence $\{\Delta_t\}$ that is of constant sign, the case where $\{J_t\}$ changes direction exactly once produces a sequence $\{\Delta_t\}$ that changes sign at most once. Complete details are given in conclusion E of Section II. This result is not new. It follows intuitively from the observations of Articles 18 and 19 of [5], but Lidstone did not pursue this idea. In addition to the applications, we extend this case by investigating the effect on the point of sign change when we vary the alternate assumptions.

The remainder of the paper is organized as follows. The main conclusions are listed in Section II, for the most part without derivations. In Section III, we give a few miscellaneous applications of the Section II results. Section IV provides some insight into the method of deriving these results. In particular, we present a formula for annuity reserves that shows that the problem can be reduced to some mathematical questions involving weighted averages of finite sequences of real numbers. For the interested reader this mathematical theory is developed in Appendix A, where we have proceeded in a somewhat more general manner than is needed for present purposes. However, since weighted averages play a role in many actuarial considerations, it is possible that the results in Appendix A will have more universal applicability.

In order to make effective use of Lidstone's theorem and its extensions, it is often necessary to have knowledge of the patterns of reserves and costs of insurance. This point is illustrated by some of the examples in Section III, and is well known from the familiar corollaries to the classical theorem that deal with constant increases in mortality and interest rates. In both cases, the hypothesis requires that reserves increase by duration. Appendix B is devoted to an investigation of the monotonicity properties of reserves and costs of insurance.

We conclude this section with a few remarks on terminology and notation. Throughout the paper (except for Appendix B), we will be dealing with functions $f(t)$ that are defined only on the integers. Accordingly, when we speak of the behavior of $f(t)$ on the interval $a \leq t \leq b$, where a and b are fixed integers, we are referring only to the integers in this interval, that is, $t = a, a + 1, \ldots, b$.

When we say that $f(t)$ is *increasing* on the interval $a \leq t \leq b$, we mean that

$$f(t) \leq f(t + 1) \quad \text{for } a \leq t \leq b - 1 \,. \tag{1.2}$$

In other words, we do not mean *strictly* increasing. We prefer this to the alternate choice of *nondecreasing,* which some writers would choose to indicate relationship (1.2). Decreasing functions are defined analogously. Note that under this definition a constant function is both increasing and decreasing!

II. MAIN CONCLUSIONS

We consider the policy introduced in Section I, and we use the notation of that section throughout. In addition, let

$$\begin{aligned} m &= n - 1 \quad && \text{if } E' = E \\ &= n && \text{if } E' \neq E \,. \end{aligned}$$

In some sense, m is the last pertinent duration for the auxiliary annuity. We always have $\Delta_{m+1} = 0$.

The classical Lidstone theorem tells us the sign of $\{\Delta_t\}$ when the sequence $\{J_t\}$ is monotone. In general, $\{J_t\}$ is not monotone, but it is natural to investigate the behavior of reserve changes for a time period corresponding to a monotone subsequence of $\{J_t\}$. We will therefore consider the following question. Given a monotone sequence

$$\{J_s, J_{s+1}, \ldots, J_v\} \,, \tag{2.1}$$

where $0 \leq s \leq v \leq m$, what can we say about the signs of the entries in the following sequence?

$$\{\Delta_s, \Delta_{s+1}, \ldots, \Delta_{v+1}\} \,. \tag{2.2}$$

The conclusions are as follows:

A. *General case*. Without further conditions, our information is somewhat sparse, but we still can say something.

1. Suppose that sequence (2.1) is increasing. Then the entries of sequence (2.2) are limited to the following possibilities:
   a) They may be of constant sign, that is, all nonpositive or all nonnegative.
   b) They may change sign exactly once, either from positive to negative or the other way.
   c) They may change sign exactly twice, from negative to positive and back to negative.
2. Suppose that sequence (2.2) is decreasing. The conclusions are the same, except that in case *c* the change is from positive to negative and back to positive. Indeed, by applying the reasoning in part 1 to the sequence $\{-J_s, \ldots, -J_v\}$, we must reach exactly the same conclusion but with positive and negative interchanged.

B. *Case of an initial segment*. We can say considerably more in the case of an extreme segment of durations, that is, either $s = 0$ or $v = m$. Now the double sign change no longer occurs, and there are further restrictions for the other cases. Moreover, even if sequence (2.1) is not monotone, we can still infer results when its entries are of constant sign. Suppose, for example, $s = 0$ in sequence (2.1). Then we have the following:
1. Suppose that (2.1) is increasing. Then
   a) If $J_0 < 0$, either the entries in (2.2) are positive throughout (except for Δ_0, which is always equal to zero), or they change sign exactly once, from positive to negative.
   b) If $J_0 > 0$, the entries in (2.2) (other than Δ_0) are all negative.
   c) If $J_0 = 0$, the entries in (2.2) are negative except for an initial string of zeros ending one duration later than the initial string of zeros in (2.1). That is, if $J_0 = J_1 = \ldots = J_{j-1} = 0$ and $J_j \neq 0$, then $\Delta_t = 0$ for $0 \leq t \leq j$, and $\Delta_t < 0$ for $j < t \leq v$.
2. Suppose that (2.1) is decreasing. As indicated in conclusion A for the general case, we can simply write down the same result as in part 1, except that positive and negative (>0 and <0) are interchanged throughout.
3. If the entries of (2.1) are ≤ 0, those of (2.2) are ≥ 0.
4. If the entries of (2.1) are ≥ 0, those of (2.2) are ≤ 0.

C. *Case of a terminal segment*. Suppose that $v = m$ in sequence (2.1). We then have the following:
1. Suppose that (2.1) is increasing. Then
   a) If $J_m > 0$, either the entries in (2.2) are positive throughout (except for $\Delta_{m+1} = 0$), or they change sign exactly once, from negative to positive.

b) If $J_m < 0$, the entries in (2.2) (other than Δ_{m+1}) are all negative.

c) If $J_m = 0$, the entries in (2.2) are all negative except for a final string of zeros starting at the same point as the final string of zeros in (2.1). That is, if $J_m = J_{m-1} = \ldots = J_i = 0$ and $J_{i-1} \neq 0$, then $\Delta_t = 0$ for $i \leq t \leq m + 1$, and $\Delta_t < 0$ for $s \leq t < i$.

2. Suppose that (2.1) is decreasing. The conclusion is as in part 1, with positive and negative (>0 and <0) interchanged throughout.
3. If the entries of (2.1) are all ≤ 0, those of (2.2) are all ≤ 0.
4. If the entries of (2.1) are all ≥ 0, those of (2.2) are all ≥ 0.

D. *Classical Lidstone theorem.* Suppose that $\{J_t\}$ is monotone over its entire range. Since the present value of this sequence is zero under the alternate assumptions, we cannot have J_0 and J_m both positive or both negative. Comparing conclusions B and C, we obtain the following.

1. If $\{J_t\}$ is increasing for $0 \leq t \leq m$, and it is not a constant zero sequence, then

$$\Delta_t > 0 \quad \text{for } 1 \leq t \leq m \,.$$

2. If $\{J_t\}$ is decreasing for $0 \leq t \leq m$, and is not a constant zero sequence, then

$$\Delta_t < 0 \quad \text{for } 1 \leq t \leq m \,.$$

3. If $J_t = 0$ for $0 \leq t \leq m$, then

$$\Delta_t = 0 \quad \text{for } 0 \leq t \leq m + 1 \,.$$

E. *Case of one change of direction in* $\{J_t\}$. We now arrive at the key result mentioned in the Introduction:

1. Suppose that for some index q $(0 < q < m)$

$$J_0 \leq J_1 \leq \ldots \leq J_q > J_{q+1} \geq \ldots > J_m \,. \tag{2.3}$$

a) If $J_0 > 0$, then $\Delta_t < 0$, $1 \leq t \leq m$.

b) If $J_m > 0$, then $\Delta_t > 0$, $1 \leq t \leq m$.

c) If J_0 and J_m are both <0, there is an index p, $1 < p < m$, such that

$$\Delta_t > 0, \quad 1 \leq t < p \,;$$

$$< 0 \,, \quad p < t \leq m \,.$$

d) If $J_0 = J_1 = \ldots = J_{j-1} = 0$, and $J_j \neq 0$, then

$$\begin{aligned} \Delta_t &= 0, \quad 1 \leqslant t \leqslant j\,; \\ &< 0, \quad j < t \leqslant m\,. \end{aligned}$$

e) If $J_m = J_{m-1} = \ldots = J_{i+1} = 0$, and $J_i \neq 0$, then

$$\begin{aligned} \Delta_t &> 0, \quad 1 \leqslant t \leqslant i\,; \\ &= 0, \quad i < t \leqslant m\,. \end{aligned}$$

2. Suppose that for some index q ($0 < q < m$)

$$J_0 \geqslant J_1 \geqslant \ldots \geqslant J_q < J_{q+1} \leqslant \ldots \leqslant J_m\,. \tag{2.4}$$

We can write down the same conclusion as in part 1, with positive and negative ($<$ and $>$) interchanged throughout in all statements about J and Δ.

Suppose now that we have a second set of alternate assumptions, and we wish to compare the resulting reserve changes with those produced by the first set. Let $\hat{J}$ and $\hat{\Delta}$ denote the quantities calculated with respect to the new alternate basis.

We will continue to discuss the case where $\{J_t\}$ has one change of direction. Now in many familiar applications there is a simple linear relationship between J and $\hat{J}$—for example, in the case of constant multiples of mortality or constant additions to mortality or interest on level premium policies. (These are discussed more fully in Sec. III.) In such cases, $\{\hat{J}_t\}$ will also have one change of direction and, by conclusion E, we can expect both $\{\Delta_t\}$ and $\{\hat{\Delta}_t\}$ to exhibit one change of sign. A natural procedure is to compare the point at which this change occurs. The relationship between Δ_t and $\hat{\Delta}_t$ is more complicated than that between J_t and $\hat{J}_t$, since the auxiliary annuity reserves are calculated according to the alternate mortality and interest assumptions, which differ in each case. We need to consider the ratios

$$\lambda_t = \frac{v''(t)\ {}_tp''_x}{v'(t)\ {}_tp'_x}\,, \tag{2.5}$$

where double-primed symbols refer to the second alternate basis. When $\{\lambda_t\}$ is monotone, we obtain the following conclusions:

F. *Point of sign changes for alternate assumptions*. Suppose that there are constants $r >_0$ and h, such that

$$\hat{J}_t = rJ_t + h , \quad 0 \leqslant t \leqslant m ,$$

and suppose that J_t and hence $\hat{J}_t$ satisfy (2.3) or (2.4).

1. If $\{\lambda_t\}$ is increasing, then the point of sign change in $\{\hat{\Delta}_t\}$ occurs later (or at the same time) for the second set of alternate assumptions, that is,
   a) If (2.3) holds, then $\Delta_t \geqslant$ (resp. $>$)[1] 0 implies that $\hat{\Delta}_t \geqslant$ (resp. $>$) 0.
   b) If (2.4) holds, then $\Delta_t \leqslant$ (resp. $<$) 0 implies that $\hat{\Delta}_t \leqslant$ (resp. $<$) 0.
   By interchanging the two sets of alternate assumptions, we see that
2. If $\{\lambda_t\}$ is decreasing, then the point of sign change in $\{\Delta_t\}$ occurs earlier (or at the same time) for the second set of alternate assumptions, that is,
   a) If (2.3) holds, then $\Delta_t \leqslant$ (resp. $<$) 0 implies that $\hat{\Delta}_t \leqslant$ (resp. $<$) 0.
   b) If (2.4) holds, then $\Delta_t \geqslant$ (resp. $>$) 0 implies that $\hat{\Delta}_t \geqslant$ (resp. $>$) 0.

III. EXAMPLES AND APPLICATIONS

A. *Limited Payment Policies*

We return to the situation introduced in Section I where we have a change in assumptions producing values of L_t that are decreasing and positive. We want to determine the effect on reserves for a limited payment policy for which premiums are level during the premium-paying period. We can use the results of Section II to conclude in general, as Baillie does in a particular case ([1], Sec. V, case *b*), that new reserves may begin lower than standard ones but will eventually become higher and remain so. Indeed, this pattern is given by conclusion B, part 2, in Section II, during the premium-paying period. After the premium-paying period, when $J_t = L_t > 0$, new reserves are higher, from conclusion C, part 4.

B. *Mortality Increases on Endowment Policies*

Suppose that interest rates remain unchanged but mortality rates increase. Under what conditions will the new reserves be higher than standard at *all*

[1] The notation (resp.) means that the preceding symbol should be replaced throughout the expression by the symbol in parentheses, thus producing an alternate expression.

durations prior to maturity on a level premium endowment policy? The simple but somewhat surprising answer is that this is never true. This was brought out in the discussions of [1] and [2]. In fact, simply using the assumptions that

$$B_{n-1} = B'_{n-1} = E' = E \tag{3.1}$$

and

$$P'_{n-1} > P_{n-1}\,, \tag{3.2}$$

it follows that

$${}_{n-1}V' = \left(\frac{E}{1 + i_{n-1}} - P'_{n-1}\right) < \left(\frac{E}{1 + i_{n-1}} - P_{n-1}\right) = {}_{n-1}V\,.$$

We see that there cannot be a higher reserve in the last duration prior to maturity.

This is readily verified by general reasoning. Assumption (3.1) means that the extra mortality has no effect in the final policy year, while assumption (3.2) means that there is still a payment due for the extra mortality at the beginning of this year. If we consider the auxiliary annuity at the end of $n - 1$ years, we see that the insured has received all the benefits from this annuity but has not yet paid for them. This naturally means a negative reserve.

One can ask, however, for conditions that ensure that new reserves will be higher than standard at all durations up to some fixed point. This was done by Greville in the discussion of [2]. A special case of this result was also derived by Baillie in his author's review of the discussion of [1]. We will state Greville's result and show how it may be derived from the results in Section II.

Consider a level premium n-year endowment policy issued at age x. (We will be considering here the normal conditions, that is, i is constant and $E = B_t$ for all t.) For ease in notation, let

$$\ddot{a}_t = \ddot{a}_{x+t:\overline{n-t}|}\,.$$

Suppose that mortality rates change and i remains the same. Let θ_t and z_t be defined by

$$q'_t = q_t + \frac{\theta_t}{v\ddot{a}_{t+1}}$$

and

$$\ddot{a}_t = (1 + z_t)\ddot{a}'_t \, .$$

THEOREM (Greville). *Suppose that r is an index $<n$ such that*

$$\theta_t \textit{ is increasing (resp. decreasing)}, \quad \textit{for } 0 \leqslant t \leqslant r - 1 , \tag{3.3}$$

and

$$z_r \geqslant \textit{(resp. } \leqslant\textit{)} \ \theta_{r-1} \, ; \tag{3.4}$$

then ${}_tV' \geqslant$ (resp. $\leqslant$) ${}_tV$ for $0 \leqslant t \leqslant r$.

Proof (for the case where θ_t is increasing). Assume that $E = 1$. Using standard formulas for endowment reserves and premiums ([3], formulas [4.7] and [5.6]), we have

$$L_t = v(q'_t - q_t)(1 - {}_{t+1}V) = \frac{\theta_t}{\ddot{a}_0}$$

and

$$J_t = \frac{\theta_t}{\ddot{a}_0} - \left(\frac{1}{\ddot{a}_0} - \frac{1}{\ddot{a}'_0}\right) = \left(\frac{\theta_t - z_0}{\ddot{a}_0}\right) , \tag{3.5}$$

which shows that J_t is increasing. We apply conclusion B of Section II to the sequence $\{J_0, J_1, \ldots, J_{r-1}\}$.

Suppose that $J_{r-1} \leqslant 0$. Then $J_t \leqslant 0$ for $0 \leqslant t \leqslant r - 1$, and the conclusion follows from conclusion B, part 3.

Suppose, on the other hand, that $J_{r-1} > 0$. Then (3.4) and (3.5) show that

$$z_r > z_0 \, ,$$

implying that

$$\Delta_r = \left(\frac{\ddot{a}_r}{\ddot{a}_0} - \frac{\ddot{a}'_r}{\ddot{a}'_0}\right) = \frac{\ddot{a}'_r}{\ddot{a}'_0}\left(\frac{z_r - z_0}{1 + z_0}\right) > 0 \, .$$

This last inequality precludes the second possibility in conclusion B, part 1, *a*, and the conclusion of the theorem follows.

C. *Multiples of q_x*

We consider the common type of mortality change where

$$q'_t = (1 + k)q_t$$

for some constant k. If all other assumptions are unchanged, we have

$$L_t = vkq_t(B_t - {}_{t+1}V) .$$

We will assume that $k > 0$ throughout the discussion. For $k < 0$, we simply interchange the original and new bases.

We see that in the level premium case the monotonicity properties of J_t depend only on the corresponding properties of the sequence

$$\{K_t\} = q_t(B_t - {}_{t+1}V) .$$

The behavior of $\{K_t\}$ depends, of course, on many factors, such as the type of policy, the mortality table, the range of ages, and even the interest rate. Observations show, however, that in many of the familiar cases we tend to obtain a sequence with one change of direction. Some mathematical evidence of this statement is provided in Appendix B.

Consider, for example, the normal type of endowment policy with $B_t = E = 1$. It is usual for $\{K_t\}$ to increase at the early durations because of the effect of the increasing values of q_t. As reserves begin to build up, the decreasing nature of the $(1 - {}_{t+1}V)$ term often dominates, and $\{K_t\}$ will decrease until a value of zero is reached for $t = n - 1$. We can apply conclusion E of Section II to conclude that, typically, new reserves will begin higher than standard and then eventually become lower than standard.

A natural problem that now arises is to determine the effect of changing k. Suppose that $0 < k < \hat{k}$, and let us compare the results of changing mortality rates to $(1 + k)q_t$ and $(1 + \hat{k})q_t$, respectively. Let $\hat{J}_t$ and P'' (the level premium) refer to quantities calculated with respect to $\hat{k}$. It is not hard to see that J and $\hat{J}$ are linearly related. Indeed,

$$\hat{J}_t = \frac{\hat{k}}{k} J_t + h , \quad 0 \leq t \leq n - 1 ,$$

where

$$h = \left(1 - \frac{\hat{k}}{k}\right)P + \frac{\hat{k}}{k}P' - P'' .$$

The ratio in expression (2.5) is given by

$$\lambda_{t+1} = \left[\frac{1 - (1 + k)q_0}{1 - (1 + \hat{k})q_0}\right] \cdots \left[\frac{1 - (1 + k)q_t}{1 - (1 + \hat{k})q_t}\right] . \tag{3.6}$$

It is clear that each factor in expression (3.6) is less than 1, and, hence, $\{\lambda_t\}$ is decreasing. We can apply conclusion F of Section II to obtain the result that, as we increase the multiple of q_x, the substandard reserves remain higher than standard for longer periods.

For a particular example of how one may use such results, consider a twenty-year level premium endowment policy for which $\{K_t\}$ exhibits the typical pattern described above. Suppose we know that at 150 percent mortality, the fifteenth terminal reserve is higher than the standard one. We then know that, for all multiples higher than 150 percent, the tth substandard terminal reserve is higher than standard for all $t \leq 15$.

It is, of course, possible that K_t will be decreasing over the entire duration of the policy. In that case, it is of interest to note that no matter what k is, *all* substandard reserves will be *less* than standard. In Appendix B, one can find sufficient conditions for a decreasing K_t in the continuous case. (See Theorem B.2 and inequality [B.18].)

D. *Term Insurance*

The familiar corollaries to the classical version of Lidstone's theorem do not apply to term insurance, since the reserves do not increase by duration. In the typical case, the reserves will increase up to a point and then decrease to a final value of zero. Our extensions of Lidstone's theorem can be exploited to yield information in this case. In this subsection we will, for simplicity, consider a level premium policy with constant interest rate, $B_t = 1$ for all t, and $E = 0$.

Suppose first that $q'_t = q_t + c$ for some positive constant c, while all other assumptions remain unchanged. Then

$$L_t = vc(1 - {}_{t+1}V) .$$

In the ordinary life or endowment case, L_t decreases and new reserves are lower. In the term insurance case, L_t (and hence J_t) exhibits the pattern of (2.4). From conclusion E of Section II we obtain the result that new reserves typically will begin lower than, but eventually become higher than, the original. Using an expression similar to (3.6), with $(1 + k)q$ replaced by $q + c$, and applying conclusion F of Section II, it is clear that as c increases the period during which the new reserves are lower is increased. We can also verify this last observation intuitively. If c increases to $1 - q_{n-1}$, then $q'_{n-1} = 1$ and we are essentially in the ordinary life case.

Next we consider the case (discussed in the previous section for endowment policies) where $q'_t = (1 + k)q_t$, for some positive constant k, and all other assumptions remain unchanged. It is clear now that since reserves eventually are decreasing, we can expect K_t eventually to be increasing. In fact, observations show that K_t tends to be increasing over the entire duration of the policy. Again, mathematical evidence of this can be found in Appendix B. (See Corollary B.4, part *a*.) The classical Lidstone theory will apply, and we conclude that new reserves are higher than original.

We now consider interest changes for the term policy. Suppose that $i' = i + c$, for some positive constant c, and all other assumptions remain unchanged. Then

$$L_t = -c({}_tV + P_t)$$

will follow the pattern of (2.4). We conclude that for early durations the new reserves may be lower, as they would be for an endowment policy, but we expect them eventually to be higher. Let us also compare two constant interest changes, c and $\hat{c}$, where $0 < c < \hat{c}$. The relevant ratio of (2.5) is

$$\lambda_t = \left(\frac{1 + i + \hat{c}}{1 + i + c}\right)^{-t},$$

which decreases. Therefore, as c increases the new reserves remain lower for longer periods. Again, this seems intuitively correct. As interest rates become larger, the present value of an endowment payment diminishes, and the term and endowment situations tend to coalesce.

IV. A FORMULA FOR ANNUITY RESERVES

Consider a life annuity that provides for a *net* payment of b_t at age $x + t$, $t = 0, 1, \ldots, n - 1$. We mean by this that b_t equals the annuity payment less the premium. As indicated in Section I, we can think of the benefit payment as being equal to b_t and the premium as being equal to zero, thus viewing the annuity as a level premium contract.

As in Section I, we let W_t denote the reserve at age $x + t$ computed *before* the payment of b_t is made. One method of deriving a formula for W_t is to view our annuity as n separate contracts and value each separately. The rth such contract, $r = 0, 1, \ldots, n - 1$, provides for a single payment of b_r at age $x + r$ and carries level annual premiums of

$$Q_r = \frac{b_r v(r)\ {}_rp_x}{\ddot{a}_{x:\overline{n}|}}, \tag{4.1}$$

payable for n years beginning at age x.

To simplify notation, we introduce the function

$$F_t = \frac{\ddot{a}_{x:\overline{t|}}\ddot{a}_{x+t:\overline{n-t|}}}{\ddot{a}_{x:\overline{n|}}}, \quad t = 1, \ldots, n. \tag{4.2}$$

Let $W_{t,r}$ be the reserve at time t applicable to the rth contract. For $r \geq t$, we take a retrospective viewpoint, since there are no annuity payments prior to time t. For $r < t$, it is easier to view the situation prospectively. The result is

$$\begin{aligned} W_{t,r} &= Q_r \ddot{s}_{x:\overline{t|}}, & r \geq t \\ &= -Q_r \ddot{a}_{x+t:\overline{n-t|}}, & r < t, \end{aligned}$$

and, using (4.1) and (4.2),

$$\begin{aligned} W_{t,r} &= F(t)\left[\frac{b_r v(r)\ {}_r p_x}{{}_{t|n-t}\ddot{a}_x}\right], & r \geq t \\ &= -F(t)\left[\frac{b_r v(r)\ {}_r p_x}{\ddot{a}_{x:\overline{t|}}}\right], & r < t. \end{aligned} \tag{4.3}$$

Let

$$w_t = v(t)\ {}_t p_x, \quad t = 0, 1, \ldots, n-1; \tag{4.4}$$

then, using (4.3), we have, for $t = 1, \ldots, n-1$,

$$\begin{aligned} W_t &= \sum_{r=0}^{n-1} W_{t,r} \\ &= F_t\left[\frac{\sum_{r=t}^{n-1} b_r w_r}{\sum_{r=t}^{n-1} w_r} - \frac{\sum_{r=0}^{t-1} b_r w_r}{\sum_{r=0}^{t-1} w_r}\right]. \end{aligned}$$

Let the two terms in the bracketed expression above be denoted by E_t and I_t, respectively; then

$$W_t = F_t[E_t - I_t].$$

Since F_t is >0 for all t, W_t is $\geqslant 0$ or $\leqslant 0$ according as $E_t \geqslant I_t$ or $\leqslant I_t$. These latter quantities are weighted averages associated with the sequence $\{b_0, b_1, \ldots, b_{n-1}\}$ and the sequence of weights $\{w_0, w_1, \ldots, w_{n-1}\}$. E_t is the weighted average of a final segment, and I_t the weighted average of an initial segment. Now, since the weighted average of the entire sequence is zero, it is intuitively clear that E_t and I_t are of opposite signs or are both zero. Applying this to our auxiliary annuity where $b_t = J_t$, and using (1.1), we see that

$$\Delta_t > 0 \quad \text{if and only if} \quad I_t < 0 \, ,$$

or, equivalently,

$$\Delta_t > 0 \quad \text{if and only if} \quad E_t > 0 \, .$$

Most of the results in Section II can be derived intuitively from these observations. Consider, for example, conclusion B, part 1, *a*. In this case, I_t begins negatively and Δ_t will be positive at the beginning. If I_t becomes positive, it must remain so due to the increasing nature of J_t, and Δ_t will remain negative.

As mentioned earlier, precise derivations of all the Section II statements can be found in Appendix A, as follows:

Section II	*Appendix A*
A	Theorem A.4, part *a* (see also Theorem A.4, part *b*, for further refinements concerning the possible pattern of zero entries)
B and C	Theorem A.4, parts *c* and *d*; Theorem A.1
D	Corollary A.5
E	Theorem A.6
F	Theorem A.7 and the remark following that theorem

APPENDIX A

SOME FACTS ABOUT WEIGHTED AVERAGES

Let $\{b_0, b_1, b_2, \ldots, b_m\}$ be a finite sequence of arbitrary real numbers and let $\{w_0, w_1, \ldots, w_m\}$ be a sequence with $w_i > 0$ for each i. We will think of the latter as a sequence of weights. We define

$$I_t = \frac{\sum_{i=0}^{t-1} b_i w_i}{\sum_{i=0}^{t-1} w_i} \, , \qquad 1 \leqslant t \leqslant m + 1 \, ;$$

$$E_t = \frac{\sum_{i=t}^{m} b_i w_i}{\sum_{i=t}^{m} w_i}, \qquad 0 \leq t \leq m ;$$

$$Y_t = E_t - I_t , \qquad 1 \leq t \leq m .$$

Note that I_t is the weighted average of an initial segment of the sequence $\{b_i\}$, while E_t is the weighted average of a final segment.

Our main goal in this appendix is to establish conditions for determining the signs of the entries in the sequence $\{Y_i\}$.

It will be convenient to extend the definitions of the above quantities. Let

$$\mu = I_{m+1} = E_0 ,$$

the weighted average of the entire sequence.

We define

$$b_{m+1} = b_{-1} = \mu \quad \text{and} \quad E_{m+1} = I_0 = \mu ,$$

from which it follows that

$$Y_{m+1} = Y_0 = 0 .$$

We now define

$$\alpha_t = \sum_{i=0}^{t-1} w_i , \qquad \beta_t = \sum_{i=t}^{m} w_i ,$$

$$\gamma_t = \frac{w_t}{\alpha_{t+1}} , \qquad \delta_t = \frac{w_t}{\beta_t} .$$

The three formulas below follow immediately from the definitions.

$$I_{t+1} = \gamma_t b_t + (1 - \gamma_t) I_t , \qquad 0 \leq i \leq m ; \tag{A.1}$$

$$E_t = \delta_t b_t + (1 - \delta_t) E_{t+1} , \qquad 0 \leq t \leq m ; \tag{A.2}$$

$$\mu = \frac{\alpha_t I_t + \beta_t E_t}{\alpha_t + \beta_t} , \qquad 1 \leq t \leq m . \tag{A.3}$$

From (A.3) we can derive two expressions for Y_t. First solve for E_t in terms of I_t and then subtract I_t. Alternatively, solve for $-I_t$ in terms of E_t and then add E_t. The resulting formulas are (for $t \neq 0$)

$$Y_t = (\mu - I_t)\frac{(\alpha_t + \beta_t)}{\beta_t} \tag{A.4}$$

and

$$Y_t = (E_t - \mu)\frac{(\alpha_t + \beta_t)}{\alpha_t}. \tag{A.5}$$

From (A.4) and (A.5) it is clear that, for $1 \leqslant t \leqslant m$,

$$Y_t > \text{(resp. } <) \ 0 \text{ if and only if } I_t < \text{(resp. } >) \ \mu, \tag{A.6}$$

$$Y_t > \text{(resp. } <) \ 0 \text{ if and only if } E_t > \text{(resp. } <) \ \mu. \tag{A.7}$$

We can now state our first main result concerning the signs of $\{Y_i\}$.

THEOREM A.1. *Let r be a fixed index ($1 \leqslant r \leqslant m$); then*

a) If $b_t \leqslant$ (resp. $\geqslant$) μ for $0 \leqslant t \leqslant r - 1$, then $Y_t \geqslant$ (resp. $\leqslant$) 0 for $1 \leqslant t \leqslant r$;

b) If $b_t \geqslant$ (resp. $\leqslant$) μ for $r \leqslant t \leqslant m$, then $Y_t \geqslant$ (resp. $\leqslant$) 0 for $r \leqslant t \leqslant m$.

Proof. Part *a* follows immediately from (A.6) after it is noted that the condition on $\{b_i\}$ implies that $I_t \leqslant$ (resp. $\geqslant$) μ for $1 \leqslant t \leqslant r$. Similarly, part *b* follows from (A.7).

This theorem gives information strictly from quantitative data concerning the sequence $\{b_i\}$. All subsequent results will depend on the monotonicity properties of this sequence. Consider first the behavior over an increasing segment. Suppose that, for some indices s and v ($0 \leqslant s \leqslant v \leqslant m$),

$$b_s \leqslant b_{s+1} \leqslant \ldots \leqslant b_{v-1} \leqslant b_v. \tag{A.8}$$

We wish to investigate the signs of entries in the associated sequence

$$Y_s, Y_{s+1}, \ldots, Y_v, Y_{v+1}. \tag{A.9}$$

LEMMA A.2. *Suppose that (A.8) holds. Let r be any index in the interval $s \leqslant r < v$, satisfying*

$$\mu \leqslant I_r \leqslant b_r; \tag{A.10}$$

then $Y_t \leq 0$ *for* $r < t \leq v + 1$. *Moreover, if either inequality in (A.10) is strict, then* $Y_t < 0$ *for* $r < t \leq v + 1$.

Proof. From (A.1) and the fact that $0 < \gamma_r \leq 1$, we see that I_{r+1} is between I_r and b_r, and, in fact, it is strictly between, unless $I_r = b_r$. Hence

$$\mu \leq I_{r+1} \leq b_r \leq b_{r+1} ,$$

where we use (A.8) for the third inequality. We see that (A.10) holds for index $r + 1$ as well. By induction, (A.10) will hold for all t with $r < t \leq v$, and the first inequality will hold for $t = v + 1$. The desired result now follows from (A.6). The statements regarding strict inequality are clear from the proof.

COROLLARY A.3. *Suppose that (A.8) holds. Given any indices i and j with* $s \leq i < j \leq v$,

a) $Y_i > 0$ *and* $Y_j \leq 0$ *implies* $Y_t < 0$ *for* $j < t \leq v + 1$;
b) $Y_i = 0$ *and* $Y_j < 0$ *implies* $Y_t < 0$ *for* $j < t \leq v + 1$;
c) $Y_i = 0$ *and* $Y_j = 0$ *implies* $Y_t \leq 0$ *for* $j < t \leq v + 1$.

Proof. In part a, let r be the smallest index such that $i < r$ and $Y_r \leq 0$. From (A.6) we see that $I_{r-1} < \mu \leq I_r$, and from (A.1) and (A.8) we see that $I_r < b_r$. Since $r \leq j$, the conclusion follows directly from Lemma A.2. Parts b and c follow in a similar manner. In part b, for example, we take r to be the smallest index such that $i < r$ and $Y_r < 0$.

We can now state our main result, namely,

THEOREM A.4. *Suppose that (A.8) holds. We then have the following:*

a) Either

(i) $Y_t < 0$ *for* $s \leq t \leq v + 1$, *or*

(ii) *There are indices k and l, with* $s \leq k \leq l \leq v + 1$, *such that*

$$Y_t \begin{cases} < 0 , & s \leq t < k \\ \geq 0 , & k \leq t \leq l \\ < 0 , & l < t \leq v + 1 . \end{cases}$$

b) If part a(ii) *holds, then either*

(i) $Y_t > 0$ *for* $k < t < l$, *or*

(ii) $Y_t = 0$ *for* $k \leq t \leq l$. *If this occurs, then* $b_t = \mu$ *for* $k \leq t \leq l - 1$. *If* $k > s$, *then* $b_{k-1} < \mu$, *and if* $l < v + 1$, *then* $b_l > \mu$.

c) *Suppose that* $s = 0$. *Then part* a(ii) *holds with* $k = 0$. *Condition* b(ii) *will hold if and only if* $b_0 \geq \mu$, *and in this case* l *is the lowest index for which* $b_l > \mu$ *(or* $l = v$ *if* $b_v = \mu$).

d) *Suppose that* $v = m$. *Then part* a(ii) *holds with* $l = m + 1$. *Condition* b(ii) *will hold if and only if* $b_m \leq \mu$, *and in this case* k *is the highest index for which* $b_k < \mu$ *(or* $k = s$ *if* $b_s = \mu$).

Proof. If part a(i) does not hold, let k and l be the smallest and largest indices, respectively, between k and v for which $Y_t \geq 0$. From parts a and b of Corollary A.3, we see that $Y_t \geq 0$ for $k \leq t \leq l$, establishing part a of the theorem.

Suppose that $Y_t = 0$ for $k \leq t \leq l$. The rest of part b(ii) follows from the fact that any two of the three statements

$$Y_t = 0 \,, \quad Y_{t+1} = 0, \quad b_t = \mu$$

imply the third. Use (A.1) and (A.6).

Suppose, on the other hand, that $Y_r > 0$ for some r between k and l. Part a of Corollary A.3 implies that $Y_t > 0$ for $r \leq t < l$, while part c of this corollary implies that $Y_t > 0$ for $k < t \leq r$. Hence, part b(i) holds.

If $s = 0$, the fact that $Y_0 = 0$ shows that part a(ii) holds with $k = 0$. If $b_0 < \mu$, then $l_1 < \mu$ and $Y_1 > 0$, so part b(i) holds. If $b_0 \geq \mu$, $Y_1 \leq 0$ and the statement in part b(ii) completes the proof of part c.

If $v = m$, reasoning analogous to the above establishes part d.

Suppose we apply Theorem A.4 to the case where $s = 0$ and $v = m$, that is, the entire sequence is increasing. Both of parts c and d apply, and we can conclude the following.

COROLLARY A.5. *Suppose that (A.8) holds for* $s = 0$ *and* $v = m$. *Then either* $Y_t > 0$ *for* $1 \leq t \leq m$ *or* $Y_t = 0$ *for all* t. *The latter condition holds if and only if* $\{b_i\}$ *is a constant sequence.*

Of course, it is not difficult to deduce this corollary directly in this simple case.

At this point, it is natural to consider the case of a decreasing segment. Suppose that there are indices $s \leq v$ such that

$$b_s \geq b_{s+1} \geq \ldots \geq b_{v-1} \geq b_v \,. \tag{A.8*}$$

We obtain the same conclusions except with all inequalities reversed.

In fact, for any statement involving the quantities b, I, E, Y, we can obtain a new statement by interchanging $\leq$ and $\geq$, and $<$ and $>$. We will call this new statement the *dual* of the original. For any valid theorem concerning these quantities, the dual theorem is also valid. We can see this by considering the sequence $\{-b_r\}$ in place of $\{b_r\}$. The signs of I and E change, and all relevant inequalities are reversed. Sometimes we write a statement together with its dual by using parentheses, as in Theorem A.1. This is not always notationally convenient, however. Instead, for a statement or theorem referenced by number n, we will reference the dual by number n^*, as in condition (A.8*). The dual statement itself will not always be written out explicitly.

Corollary A.5 and its dual say that a monotone sequence $\{b_i\}$ gives rise to a sequence $\{Y_i\}$ of constant sign. Theorem A.4 can be used to extend this idea. We can show in general that a small number of changes of direction in $\{b_i\}$ will mean a small number of changes of sign in $\{Y_i\}$. We will analyze in detail the simplest case, next to the monotone one. This is where the entire sequence $\{b_i\}$ changes direction exactly once. The conclusion is that $\{Y_i\}$ changes sign at most once.

We consider the case where there is an index q $(0 < q < m)$ such that

$$b_0 \leq b_1 \leq \ldots \leq b_q > b_{q+1} \geq \ldots \geq b_{m-1} \geq b_m \,. \tag{A.11}$$

THEOREM A.6. *Suppose that (A.11) holds. We then have the following:*

a) If $b_m \geq \mu$, *then*

$$\begin{aligned} Y_t &> 0\,, \quad 1 < t \leq i \\ &= 0\,, \quad i < t \leq m, \end{aligned}$$

where i is the largest index for which $b_i > \mu$;

b) If $b_0 \geq \mu$, *then*

$$\begin{aligned} Y_t &= 0\,, \quad 0 \leq t \leq j \\ &< 0\,, \quad j < t < m\,, \end{aligned}$$

where j is the smallest index for which $b_j > \mu$;

c) If b_m *and* b_0 *are both* $<\mu$, *there is an index p, with* $1 < p < m$, *such that*

$$\begin{aligned} Y_t &> 0\,, \quad 1 < t < p \\ &< 0\,, \quad p < t < m\,. \end{aligned}$$

Proof. We apply Theorem A.4, part c, to the segment $(b_0, b_1, \ldots, b_q)$ and Theorem A.4*, part d, to the segment $(b_q, b_{q+1}, \ldots, b_m)$.

a) If $b_m \geqslant \mu$, it is clear from (A.11) that $b_q \geqslant \mu$. Now Theorem A.4*, part d, gives us the indicated pattern for $q \leqslant t < m$ with a value of $i > q$. The pattern of (A.11) and the fact that μ is the weighted average of the entire sequence show that $b_0 < \mu$. Using this together with the fact that $Y_q > 0$, we conclude from Theorem A.4 that $Y_t > 0$, $1 < t \leqslant q$.

b) Our argument is similar to that in part a. If $b_0 \geqslant \mu$, it is clear from (A.11) that $b_q > \mu$ and $b_m < \mu$. Theorem A.4, part c, gives the indicated pattern for $1 \leqslant t \leqslant q$, and Theorem A.4*, part d, shows that $Y_t < 0$ for $q \leqslant t \leqslant m$.

c) Let l and k^* be the indices obtained from Theorem A.4, part c, and Theorem A.4*, part d, respectively. If b_0 and b_m are both $<\mu$, we know that $0 < l$ and $k^* < m$, as well as the fact that part b(i) of each theorem holds. If $1 < l \leqslant q$, then $Y_t > 0$ for $1 < t < l$ and $Y_t < 0$ for $l < t \leqslant q$. We see as in part b above that $Y_t < 0$ for $q + 1 < t \leqslant m$, which shows that part c holds with $p = l$. If $l = 1$, we can take $p = 2$, since $I_1 = b_0 < 0$. If $l = q + 1$, we can show in a similar fashion that part c holds with $p = k^*$.

We next consider the problem of determining what happens to the various quantities when we change the sequence of weights. Let $\{\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_m\}$ be a new weight sequence. We will use symbols with carets above them to denote quantities computed with this new sequence.

Let

$$\lambda_i = \frac{\hat{w}_i}{w_i}, \qquad i = 0, 1, \ldots, m .$$

Writing $\hat{w}_i$ as $\lambda_i w_i$, we see that, for all applicable values of t,

$$I_t - \hat{I}_t = \sum_{i,j=0}^{t-1} \frac{w_i w_j (b_i - b_j)(\lambda_j - \lambda_i)}{\alpha_t \hat{\alpha}_t} . \tag{A.12}$$

A similar expression holds for $(E_t - \hat{E}_t)$. We must use β instead of α in the denominator and let t and m replace zero and $t - 1$, respectively, as the limits of summation.

From (A.1) and (A.2) we obtain, with a minor amount of algebraic manipulation, the recursion formulas

$$(I_{t+1} - \hat{I}_{t+1}) = (\gamma_t - \hat{\gamma}_t)(b_t - I_t) + (I - \hat{\gamma}_t)(I_t - \hat{I}_t) , \tag{A.13}$$

$$(\hat{E}_t - E_t) = (\hat{\delta}_t - \delta_t)(b_t - \hat{E}_{t+1}) + (1 - \delta_t)(\hat{E}_{t+1} - E_{t+1}) . \quad \text{(A.14)}$$

It is tempting to try to use the above formulas to make statements concerning the relative magnitude of the original and capped quantities. This is difficult or impossible without further assumptions. If we impose some monotonicity requirements on $\{\lambda_i\}$ and $\{b_i\}$ we can reach some conclusions.

Suppose that $\{b_i\}$ satisfies (A.11) and that $\{\lambda_i\}$ is decreasing. For $0 \leq i \leq j \leq q$ (q is as in [A.11]), we have $(b_i - b_j) \leq 0$ and $(\lambda_j - \lambda_i) \leq 0$. From (A.12) we conclude that

$$\hat{I}_t \leq I_t , \quad 1 \leq t \leq q + 1 . \quad \text{(A.15)}$$

Similar reasoning shows that

$$E_t \leq \hat{E}_t , \quad q \leq t \leq m \quad \text{(A.16)}$$

as well as

$$\hat{\gamma}_t \leq \gamma_t , \quad 0 \leq t \leq m \quad \text{(A.17)}$$

and

$$\delta_t \leq \hat{\delta}_t , \quad 0 \leq t \leq m . \quad \text{(A.18)}$$

In fact, for $0 < t < m$, (A.17) and (A.18) follow directly from (A.15) and (A.16), respectively. For fixed t, we apply these latter formulas to the sequence which has an entry of 1 in the position indexed by t and zero entries elsewhere. For such a sequence, which obviously satisfies (A.11) with $q = t$, we have $\gamma_t = I_{t+1}$ and $\delta_t = E_t$.

We can now formulate and prove our main result concerning change of weights. If $\{b_i\}$ satisfies (A.11), we know from Theorem A.6 that $\{Y_i\}$ will, in general, change from positive to negative. A natural problem is to determine how the weight sequence affects the point of change. It turns out that if the ratios of new weights to old are decreasing, then the point of change is deferred and $\{\hat{Y}_i\}$ will, in general, have more (or as many) positive entries. The precise statement is as follows.

THEOREM A.7. *Suppose that* $\{b_i\}$ *satisfies (A.11). Let* (w_i) *and* $(\hat{w}_i)$ *be two weight sequences such that* $\hat{w}_i w_i^{-1}$ *is decreasing. Then* $Y_r \geq$ *(resp.* $>$*)* 0 *implies that* $\hat{Y}_r \geq$ *(resp.* $>$*)* 0.

Proof. We prove the statement for $\geq$. The case of strict inequality requires nothing more than modifying the second inequality sign in (A.23) and (A.24).

Suppose first that $r \geqslant q$ and assume that $Y_r \geqslant 0$. We then have

$$I_t \leqslant E_t \leqslant b_t , \quad q \leqslant t \leqslant r , \tag{A.19}$$

where the first inequality follows from Theorem A.6 and the second follows from the definition of q. Now using formulas (A.13), (A.17), and (A.19), we obtain by induction

$$\hat{I}_r \leqslant I_r , \tag{A.20}$$

where we use (A.15) to start the induction at index q. From (A.15),

$$E_r \leqslant \hat{E}_r ,$$

which, combined with (A.20), yields

$$\hat{Y}_r \geqslant Y_r \geqslant 0 . \tag{A.21}$$

Next, suppose that $r < q$. We will derive the contrapositive of the desired result, namely, that $\hat{Y}_r < 0$ implies $Y_r < 0$. If $\hat{Y}_r < 0$, we have

$$\hat{E}_t < \hat{I}_t \leqslant b_t , \quad r \leqslant t \leqslant q , \tag{A.22}$$

where the first inequality follows from Theorem A.6 and the second follows from the definition of q. Now, using formulas (A.16), (A.18), and (A.22), we obtain by induction

$$E_r \leqslant \hat{E}_r , \tag{A.23}$$

where we use (A.16) to start the induction at index q and work *backward* to index r. From (A.15),

$$\hat{I}_r \leqslant I_r ,$$

which, combined with (A.23), gives

$$Y_r \leqslant \hat{Y}_r < 0 , \tag{A.24}$$

completing the proof.

Suppose we have a sequence satisfying (A.11*); that is, it consists of an initial decreasing segment followed by a final increasing one. What happens to $\{Y_i\}$ now if we change to weights with a decreasing sequence of ratios? Theorem A.6* tells us that $\{Y_i\}$ will, in general, change from negative to positive, and Theorem A.7* says that $Y_r \leq 0$ implies that $\hat{Y}_r \leq 0$. Exactly as in the case of (A.11), the point of change is deferred.

Note that if the ratios in the sequence $\{\lambda_i\}$ are increasing rather than decreasing, we simply interchange the two weight sequences to conclude that, with the new weights, the point of change in sign of $\{Y_i\}$ occurs earlier, in the case of either condition (A.11) or condition (A.11*).

Remark. In certain cases we can apply Theorem A.7 to situations where changes occur in the sequence $\{b_i\}$ as well as the weight. Suppose we have a new sequence

$$b_i = kb_i + h$$

for some constants $k > 0$ and h, and let $\hat{Y}$ be calculated with respect to $\{b_i\}$ as well as $\{W_i\}$. It is clear that $\hat{Y} = kY$, and hence the conclusions of Theorem A.7 will still hold.

APPENDIX B

MONOTONICITY PROPERTIES OF ${}_tV$ AND K_t

In this appendix, we show that, under certain general and natural conditions, both reserves and costs of insurance either are monotone or change direction exactly once. The examples in Section III provide the motivation for such results and their relationship with the previous sections of the paper.

For mathematical convenience, we replace the discrete model that we have used up to now with a continuous one. This permits the use of calculus and facilitates the derivations. Similar conclusions could, no doubt, be obtained in the discrete case, but it appears that some rather complicated algebraic manipulations would be involved.

We will adopt the notation and terminology of Section I, with suitable modifications. The variable t now will assume *all* values in the interval $0 \leq t \leq n$. The benefit B_t is payable at the moment of death. Premiums are paid continuously and P_t now denotes the annual rate of payment at time t. We now have ${}_tV$ denoting the reserve at time t, rather than at the end of year t. In place of q_t and i_t, respectively, we will consider

$$\mu_t = \text{Force of mortality at age } x + t$$

and

$$\delta_t = \text{Force of interest at time } t \ .$$

The cost of insurance K is now defined by

$$K_t = \mu_t(B_t - {}_tV) \ . \tag{B.1}$$

We will assume that, for all values of t, B_t, P_t, $\delta_t \geqslant 0$ and $\mu_t > 0$, and that $E \geqslant 0$.

We also assume that all functions under discussion are twice differentiable on the closed interval $[0, n]$ (the appropriate one-sided derivatives existing at the endpoints).

Note carefully that, in this section, primes will denote differentiation with respect to t (a change from their use in previous sections).

To simplify the notation, we will often omit the subscript t.

We remind the reader that we will use the same conventions as in Section I regarding the use of the words "increasing" and "decreasing."

The starting point in our development is to observe that the monotonicity properties of a function often can be obtained by looking at differential equations that the function satisfies. We now state a general lemma in this regard, which will be applied later to both reserves and costs of insurance.

LEMMA B.1. *Let f be a function on $[0, n]$ satisfying*

$$f(0) \geqslant 0 \tag{B.2}$$

and the differential equation

$$f' = af - b \ , \tag{B.3}$$

where a and b are functions such that

(i) $a(t) > 0$ *and* $a'(t) \geqslant 0$ *for all t, and*
(ii) *For all t such that* $b(t) \geqslant 0$, (B.4)

$$b'(t) \geqslant \frac{a'(t)}{a(t)} b(t) \ . \tag{B.5}$$

Then f either is monotone or changes direction exactly once, from increasing to decreasing.

Proof. Differentiating (B.3) and substituting for f in terms of f',

$$\begin{aligned} f'' &= af' + a'\left(\frac{f' + b}{a}\right) - b' \\ &= \left(a + \frac{a'}{a}\right)f' - \left(b' - \frac{a'}{a}b\right). \end{aligned} \tag{B.6}$$

We now define a point r as follows. Take r to be the smallest zero of b, if such exists. If not, take $r = n$ if $b < 0$ or $r = 0$ if $b > 0$. Assumptions (B.4) and (B.5) show that if $b(t) \geq 0$, then $b'(t)$ is also ≥ 0. Hence, once b becomes nonnegative, it is increasing and remains nonnegative. In any event, we see that

$$\begin{aligned} b(t) &< 0\,, \quad 0 < t < r \\ &\geq 0\,, \quad r < t \leq n\,. \end{aligned} \tag{B.7}$$

Using (B.7) together with (B.4), we can infer from equation (B.3) that, for t on the interval $0 \leq t < r$, if $f(t) \geq 0$, then $f'(t)$ is also ≥ 0; hence, f is increasing and remains ≥ 0 in this interval. Now, invoking (B.2), it is not difficult to see that f, and therefore f', are nonnegative in the interval $[0, r)$. By continuity, this holds in the closed interval $[0, r]$.

Suppose that $f'(t) < 0$. Then, as we saw above, we must have $t > r$. From (B.6), using (B.4), (B.5), and (B.7), we see that $f''(t)$ is also <0. Hence, once f' becomes negative, it is decreasing and remains negative. This shows that once f begins to decrease it continues to decrease, and the conclusion of the lemma is evident.

Remarks. Assumption (B.5) can be put into more enlightening forms. Since

$$\left(\frac{b}{a}\right)' = \frac{1}{a}\left(b' - \frac{a'}{a}b\right),$$

the inequality in (B.5), together with the fact that $a > 0$, says that (b/a) is increasing. Also, for $b > 0$, (B.5) is equivalent to

$$\frac{b'}{b} \geq \frac{a'}{a}.$$

In other words, (B.5) really says that after b becomes positive, its *relative* rate of growth is greater than that of a.

We next apply the lemma to the case where $f = {}_tV$. We have the familiar differential equation

$$\begin{aligned} V' &= P + \delta V - K \\ &= V(\mu + \delta) - (B\mu - P)\,, \end{aligned} \tag{B.8}$$

which is in the form of (B.3) with

$$a = (\mu + \delta)\,, \quad B = (B\mu - P)\,. \tag{B.9}$$

We define a new function

$$\rho = \frac{\mu'}{\mu} = (\ln \mu)'\,,$$

which will play an important role in the remainder of the section.

THEOREM B.2. *Suppose that*

(i) μ *is increasing;*
(ii) B *is increasing;*
(iii) $P' \leqslant \rho P$;
(iv) $-\rho\mu \leqslant \delta' \leqslant \rho\delta$.

Then V is either (a) increasing, or (b) increasing then decreasing. Moreover, if $E \geqslant B_n$, *then (a) holds; and if* $E = 0$, *then (b) holds (unless* ${}_tV = 0$ *for all t).*

Proof. The main conclusion follows by simply verifying the conditions of Lemma B.1 for the functions a and b given in (B.9). Since $V = 0$ for $t = 0$, (B.2) holds. The first statement of (B.4) is clear, and the second follows from the first inequality in part (iv). We now observe that

$$\begin{aligned} b' &= B'\mu + B\mu' - P' \\ &\geqslant B\mu' - P'\,, && \text{using part (ii)} \\ &\geqslant \rho(B\mu - P) = \rho b\,, && \text{using part (iii)}\,, \end{aligned} \tag{B.10}$$

while, using the second inequality in part (iv), we have

$$a' = \mu' + \delta' \leqslant \rho(\mu + \delta) = \rho a \tag{B.11}$$

and (B.5) is immediate from (B.10) and (B.11).

The statement regarding $E = 0$ is now evident from the fact that ${}_nV = E$, while the remaining statement follows from the fact that

$$V_n' = P_n + \delta E - \mu_n(B_n - E)$$

is ≥ 0 when $E \geq B_n$.

Remarks on the assumptions of Theorem B.2. Note that assumption (i) is natural, while the others all hold in the familiar case where B, P, and δ are constant. Assumption (iii), in fact, allows for any decreasing P (as would be the case in a continuous version of a limited payment policy). It also allows for increasing P, provided that the relative rate of increase is less than that of μ. It is precisely this upper bound on P' that rules out the possibility of negative reserves (which, it is easy to see, are precluded by the conclusion of the theorem). We could not have a negative reserve produced by premiums being less than the benefit cost at early durations, since there would be no chance for premiums to increase sufficiently to make up this deficit. In fact, with the assumptions of the theorem, it is clear that $V' \geq 0$ at $t = 0$ and hence, from (B.8), that

$$P_0 \geq B_0\mu_0 . \tag{B.12}$$

The concluding statements of the theorem reflect the expected patterns of reserves in the case of endowment and term insurances.

We now wish to apply Lemma B.1 to K. From formula (B.1) we solve for V in terms of K and substitute in (B.8), to obtain

$$V' = P + \delta B - K\left(\frac{\delta}{\mu} + 1\right) . \tag{B.13}$$

Differentiating (B.1), we have

$$K' = \mu'(B - V) + \mu B' - \mu V' . \tag{B.14}$$

Now, noting that $\mu'(B - V) = \rho K$, and substituting for V' from (B.13), we obtain, from (B.14),

$$K' = (\mu + \delta + \rho)K - \mu(P + \delta B - B') , \tag{B.15}$$

which is in the form of (B.1) with

$$a = (\mu + \delta + \rho) , \quad b = \mu(P + \delta B - B') . \tag{B.16}$$

We now give some sufficient conditions to enable Lemma B.1 to be applied to K. One of these conditions (inequality [B.17] below) may appear somewhat unusual at first. We will attempt to clarify it later and show that it does indeed hold in many cases.

THEOREM B.3. *Suppose that*

(i) μ *and* μ' *are increasing;*
(ii) B *is increasing and* B' *is decreasing;*
(iii) $0 \leqslant \delta'$;
(iv) $0 \leqslant P'$;
(v) $\rho' \leqslant \rho^2 + \rho\delta - \delta'$. (B.17)

Then K *is either (a) decreasing, (b) increasing, or (c) increasing then decreasing. Moreover, (a) holds if and only if*

$$P_0 \geqslant B_0\mu_0 + B_0\rho_0 + B_0' \, . \tag{B.18}$$

Proof. We verify the conditions of Lemma B.1 for a and b as given in (B.16). Since $K_0 = \mu_0 B_0 \geqslant 0$, we see that (B.2) holds. From parts (i) and (iii), we easily obtain (B.4).

Let $c = P + \delta B - B'$. Using parts (ii), (iii), and (iv),

$$c' = P' + \delta' B + \delta B' - B'' \geqslant 0 \, ,$$

so

$$b' = \mu' c + \mu c' \geqslant \mu' c = \rho\mu c = \rho b \, . \tag{B.19}$$

From (B.17),

$$a' = \mu' + \delta' + \rho' \leqslant \rho\mu + \rho^2 + \rho\delta = \rho a \, , \tag{B.20}$$

and (B.5) is immediate from (B.19) and (B.20).

The final statement of the theorem follows from the fact that

$$K_0' \mp \mu_0[B_0(\mu_0 + \rho_0) + B_0' - P_0] \, ,$$

which we obtain by setting $t = 0$ in (B.15).

COROLLARY B.4. *Assume the conditions of Theorem B.3.*

a) If reserves are eventually decreasing (i.e., decreasing on the interval $[s, n]$ *for some* $s < n$*), then* K *is increasing.*

b) If

$$E \geqslant B_n - \frac{\mu_0}{\mu_n} B_0 \, , \tag{B.21}$$

then K *either is decreasing or is increasing then decreasing.*

Proof. Part *a*: In this case, K is eventually increasing, and, by Theorem B.3, it must be increasing in the entire interval $[0, n]$. Part *b:* In this case, $K_n \leqslant K_0$, so the conclusion is obvious.

In the following corollary, the assumptions are those of Theorems B.2 and B.3 combined, so the conclusions of both theorems will hold.

COROLLARY B.5. *Suppose that (B.17) holds and that*

(i) μ *and* μ' *are increasing;*
(ii) B *is increasing and* B' *is decreasing;*
(iii) $0 \leqslant \delta' \leqslant \rho\delta$, *and*
(iv) $0 \leqslant P' \leqslant \rho P$.

Then V is increasing under either of the following conditions:

a) P_0 *satisfies (B.18);*
b) E *satisfies (B.21).*

Proof. In both cases, K is not increasing. Now apply Corollary B.4, parts *a* and *b*, respectively.

We can view Corollary B.5 as follows. Basically, a sufficiently large endowment payment is the feature that makes reserves on a policy increase. This is reflected directly in part *b*. In Theorem B.2 we saw that $E \geqslant B_n$ was sufficient to cause increasing reserves. We now see that, with the added conditions of Theorem B.3, we can reach this conclusion with the smaller value of E given by (B.21). Of course, without direct knowledge of E, we can still infer that it must be large if premiums are sufficiently large. This is the idea of part *a*. It is of interest to compare this part with (B.12), which, as shown earlier, is always true under the given assumptions.

Under the assumptions of Theorem B.3, if K does not decrease, the inequality in (B.18) is reversed. This gives an upper bound for the initial premium rate. We can apply this to term policies, and therefore also to ordinary life policies, since they can be viewed as a limiting case of a term policy. For a simple example, consider an ordinary life policy with a constant death benefit of 1, constant interest, and level premiums. Assume that μ, together with its derivative, is increasing and satisfies (B.17). All assumptions are satisfied, and we may invoke both (B.12) and the negation of (B.18), to obtain the premium estimate

$$\mu_0 \leqslant P \leqslant \mu_0 + \frac{\mu_0'}{\mu_0}. \tag{B.22}$$

The lower bound in (B.22) is obvious in this case, but the upper bound is decidedly not so. In fact, it is somewhat surprising at first, since it involves the values of μ and μ' only at issue age. Note, however, that assumption (B.17) does establish a connection between the values of μ at various points.

Remarks on the assumptions of Theorem B.3. Assumption (i) is reasonable and reflects the expected behavior of μ over most ages. It is clear from the proof that assumptions (ii), (iii), and (iv) can be weakened. We need only conditions on B, P, and δ that will ensure that c' and a' are non-negative. The chosen conditions are easy to state, and they hold when all quantities are constant.

Assumption (v) is of a different nature, and it is not easy to grasp its meaning. It may become more transparent if we write it in a somewhat different manner. For simplicity, we will restrict our discussion to the case where δ is constant (so $\delta' = 0$). For $\rho > 0$, (B.17) can then be written as

$$\frac{\rho'}{\rho} \leqslant \frac{\mu'}{\mu} + \delta \,, \tag{B.23}$$

which involves a comparison of the relative growth rates of ρ and μ. When $\delta = 0$, further simplification is possible. In this case, (B.23) is equivalent to the statement that

$$\frac{\mu'}{\mu^2} \text{ is decreasing} \,. \tag{B.24}$$

We go from (B.24) to (B.23) by writing μ'/μ^2 as ρ/μ, applying ln, and differentiating. The steps can be reversed to show that (B.23) with $\delta = 0$ implies (B.24).

We see then that condition (B.17) restricts the growth rate of μ'. The underlying principle is for μ' to grow at a relative rate that is less than the relative growth rate of μ^2. If this happens, then the condition will hold regardless of δ. However, some relaxation of this growth restriction is allowed as δ increases. In fact, no matter what the mortality basis is, the condition can always be made to hold by taking δ sufficiently large.

Perhaps the best test of the reasonableness of (B.17) is to see whether it holds in cases where we have a known analytic expression for μ. In the case of Gompertz's law, ρ is a constant, and the condition holds automatically for all constant δ (in fact, for all δ satisfying assumption [iv] of Theorem B.2). In the more general case of Makeham's law, this universality no longer applies, but (B.17) does hold in a wide variety of circumstances. To see this, suppose that

$$\mu_t = A + Bc^{x+t} \,,$$

where A, B, and c are positive constants. (Recall that x is the issue age.) Suppose also that δ is constant and satisfies

$$\frac{\delta}{\ln c} \geqslant \left(2\frac{A}{\mu_0} - 1\right). \tag{B.25}$$

Observe that

$$\mu' = (\mu - A)\ln c ,$$

so that

$$\rho = \ln c - \frac{A \ln c}{\mu}. \tag{B.26}$$

Since μ is increasing, (B.25) holds when μ_0 is replaced by any μ_t, and we can write

$$\begin{aligned} \rho + \delta &\geqslant \ln c\left[\left(1 - \frac{A}{\mu}\right) + \left(2\frac{A}{\mu} - 1\right)\right] \\ &= \ln c\frac{A}{\mu}. \end{aligned} \tag{B.27}$$

Differentiating (B.26) gives

$$\frac{\rho'}{\rho} = \rho'\frac{\mu}{\mu'} = \left(\ln c\frac{A\mu'}{\mu^2}\right)\frac{\mu}{\mu'} = \ln c\frac{A}{\mu},$$

and comparison with (B.27) yields (B.23). We conclude that condition (B.17) holds under Makeham's law, provided that (B.25) also holds. Since the right-hand side of this latter inequality is <1, condition (B.17) will always hold for $\delta \geqslant \ln c$. Moreover, as soon as the issue age is high enough so that $\mu_0 \geqslant 2A$, the right-hand side of (B.25) is nonpositive and condition (B.17) will hold for all δ.

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DISCUSSION OF PRECEDING PAPER

THOMAS KABELE:

It is interesting to see a paper on the famous Lidstone theorem concerning net premium reserves and the remainder function. Mr. Promislow has given new proofs of Lidstone's results.

Lidstone's remainder function is essentially our contribution dividend formulas, and, in fact, his 1905 paper on reserve charges was a sequel to his earlier 1895 paper on the contribution dividend formula. (See my discussion of Don Cody's paper, which appears elsewhere in this volume.)

Lidstone's treatment was remarkably general. He considered lapse rates and varying interest rates, and dealt with a nonmonotone remainder function. Note that Lidstone's "equation of equilibrium" is *not* the same as Sheppard Homans's "equation of equilibrium." Lidstone's equation states the equality between the "normal" and "special" reserves. Homans's equation relates the ending reserve to the beginning reserve, plus premium and interest, less claims and expenses. Homans's equation is also called the "retrospective reserve formula" or the "Fackler accumulation formula" or the "Elizur Wright accumulation formula." Lidstone called Homans's equation the "fundamental principle."

(AUTHOR'S REVIEW OF DISCUSSION)

S. DAVID PROMISLOW:

Mr. Kabele has added some interesting historical remarks to my paper. I would like to thank him for these, and also for some helpful discussions that I had with him previously concerning the work of Lidstone.