On a Formula of Nesbitt

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In [B-H-N] C. J. Nesbitt gave the formula

\[ \int_0^t \frac{1}{\bar{a}_{n-h}} \, dh = \ln \left( \frac{\bar{s}_n}{\bar{s}_{n-t}} \right), \quad 0 \leq t < n. \]  

(1)

Variants of this formula can be found in Exercise 20.21 of [B-G-H-J-N]. By considering \( \delta \) as \(-\delta\), we obtain from (1) the formula

\[ \int_0^t \frac{1}{\bar{s}_{n-h}} \, dh = \ln \left( \frac{\bar{a}_n}{\bar{a}_{n-t}} \right), \quad 0 \leq t < n, \]

or

\[ \exp(-\int_0^t \bar{p}_{n-h} \, dh) = \left( \frac{\bar{a}_{n-t}}{\bar{a}_n} \right), \quad 0 \leq t < n. \]  

(2)

Formula (2) can be generalized as

\[ \exp(-\int_0^t [\bar{p}(\bar{A}_{x+h:n-h}) - \mu_{x+h}] \, dh) = \left( \frac{\bar{a}_{x+t:n-t}}{\bar{a}_{x:n}} \right), \quad 0 \leq t < n, \]  

(3)

which had been given by Vasmoen [Va]. (See also [Be].) The right-hand side of (3) is, of course, \( 1 - \bar{V}(\bar{A}_{x:n}) \), the net amount at risk at duration \( t \) of a continuous \( n \)-year endowment insurance with unit face amount issued to \( x \).
I received a letter from Nesbitt, dated January 29, 1985, in which he pointed out that (3) can be derived by means of the formula

$$\int_0^t \frac{1}{a_{x+h}} \, dh = \ln \left( \frac{N_x}{N_{x+t}} \right). \quad (4)$$

It is interesting to note that (4) is a key tool in a recent paper by Sundt [Su].

Let

$$tW(\overline{A}_{x:n}) = \frac{tV(\overline{A}_{x:n})}{\overline{A}_{x+t:n-t}}. \quad (5)$$

This is the amount of paid-up insurance that can be provided on an n-year continuous endowment insurance at duration t by the full net level premium reserve. If we rewrite (1) as

$$\exp(-\int_0^t \frac{1}{\overline{a}_{n-h}} \, dh) = \frac{1/P_{n-t}}{1/P_n}, \quad (6)$$

we see that the following is a generalization,

$$\exp(-\int_0^t \left( \frac{1}{\overline{a}_{x+h:n-h}} - \frac{\mu_{x+h}}{\overline{A}_{x+h:n-h}} \right) \, dh) = \frac{1/P(\overline{A}_{x+t:n-t})}{1/P(\overline{A}_{x:n})} \quad (7)$$

$$= 1 - tW(\overline{A}_{x:n}). \quad (8)$$

Formula (8) can be derived by means of Exercise 16.5 in [B-G-H-J-N].

The following two formulas hold for all positive numbers s and t, s + t < n,

$$[1 - sV(\overline{A}_{x:n})][1 - tV(\overline{A}_{x+s:n-s})] = [1 - stV(\overline{A}_{x:n})] \quad (9)$$

and

$$[1 - sW(\overline{A}_{x:n})][1 - tW(\overline{A}_{x+s:n-s})] = [1 - stW(\overline{A}_{x:n})]. \quad (10)$$

Two other analogous formulas in Life Contingencies are
\[ s_{\text{tp} x} = s_{\text{tp} x} \]  
\[ (11) \]

and
\[ s_{\text{tEx} x} = s_{\text{tEx} x} \]  
\[ (12) \]

These four equations are examples of the functional equation
\[ f(\alpha, \beta) f(\beta, \gamma) = f(\alpha, \gamma), \]  
\[ (13) \]

which had been studied by the Russian mathematician D. M. Sinzow about 100 years ago. (In [Ac], Sinzow is spelled Sincov.) The general solution of (13) is of the form
\[ f(\alpha, \beta) = \frac{g(\beta)}{g(\alpha)}. \]  
\[ (14) \]

If the function \( g \) is differentiable, then (14) becomes
\[ f(\alpha, \beta) = \exp\left( \int_{\alpha}^{\beta} \frac{g'(h)}{g(h)} \, dh \right). \]  
\[ (15) \]

Thus, (3) and (7) follow from
\[ \frac{d}{dh} \ln(\overline{a_{x+h-n-h}}) = \mu_{x+h} - \overline{\bar{\mu}(\overline{A_{x+h-n-h}})} \]  
\[ (16) \]

and
\[ \frac{d}{dh} \ln[1/\overline{\bar{\mu}(\overline{A_{x+h-n-h}})}] = \frac{\mu_{x+h}}{\overline{A_{x+h-n-h}}} - \frac{1}{\overline{a_{x+h-n-h}}}, \]  
\[ (17) \]

respectively. Formula (16) can be found in [Va].

Finally, we note the formulas
\[ s_{\text{p} x} = \ell_{x+1}^{\text{t} x} = \exp(-\int_{0}^{t} \mu_{x+h} \, dh) \]  
\[ (18) \]

and
\[ s_{\text{e} x} = \frac{D_{x+1}^{t} x}{D_{x}} = \exp(-\int_{0}^{t} (\mu_{x+h} + \delta) \, dh). \]  
\[ (19) \]
References


