On a Formula of Nesbitt

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In [B-H-N] C. J. Nesbitt gave the formula

$$\int_0^t \frac{1}{\bar{a}_{\overline{n-h|}}}\, dh = \ln\left(\frac{\bar{s}_{\overline{n|}}}{\bar{s}_{\overline{n-t|}}}\right), \qquad 0 \le t < n. \qquad (1)$$

Variants of this formula can be found in Exercise 20.21 of [B-G-H-J-N]. By considering δ as –δ, we obtain from (1) the formula

$$\int_0^t \frac{1}{\bar{s}_{\overline{n-h|}}}\, dh = \ln\left(\frac{\bar{a}_{\overline{n|}}}{\bar{a}_{\overline{n-t|}}}\right), \qquad 0 \le t < n,$$

or

$$\exp\left(-\int_0^t \bar{P}_{\overline{n-h|}}\, dh\right) = \frac{\bar{a}_{\overline{n-t|}}}{\bar{a}_{\overline{n|}}}, \qquad 0 \le t < n. \qquad (2)$$

Formula (2) can be generalized as

$$\exp\left(-\int_0^t [\bar{P}(\bar{A}_{x+h:\overline{n-h|}}) - \mu_{x+h}]\, dh\right) = \frac{\bar{a}_{x+t:\overline{n-t|}}}{\bar{a}_{x:\overline{n|}}}, \qquad 0 \le t < n, \qquad (3)$$

which had been given by Vasmoen [Va]. (See also [Be].) The right-hand side of (3) is, of course, $1 - {}_t\bar{V}(\bar{A}_{x:\overline{n|}})$, the *net amount at risk* at duration t of a continuous n-year endowment insurance with unit face amount issued to (x).

I received a letter from Nesbitt, dated January 29, 1985, in which he pointed out that (3) can be derived by means of the formula

$$\int_0^t \frac{1}{\bar{a}_{x+h}}\, dh \;=\; \ln\!\left(\frac{\bar{N}_x}{\bar{N}_{x+t}}\right). \tag{4}$$

It is interesting to note that (4) is a key tool in a recent paper by Sundt [Su].

Let

$${}_t\bar{W}(\bar{A}_{x:\overline{n}|}) \;=\; \frac{{}_t\bar{V}(\bar{A}_{x:\overline{n}|})}{\bar{A}_{x+t:\overline{n-t}|}}. \tag{5}$$

This is the amount of *paid-up insurance* that can be provided on an n-year continuous endowment insurance at duration t by the full net level premium reserve. If we rewrite (1) as

$$\exp\Bigl(-\int_0^t \frac{1}{\bar{a}_{\overline{n-h}|}}\, dh\Bigr) \;=\; \frac{1/\bar{P}_{\overline{n-t}|}}{1/\bar{P}_{\overline{n}|}}, \tag{6}$$

we see that the following is a generalization,

$$\exp\Bigl(-\int_0^t \left(\frac{1}{\bar{a}_{x+h:\overline{n-h}|}} - \frac{\mu_{x+h}}{\bar{A}_{x+h:\overline{n-h}|}}\right) dh\Bigr) \;=\; \frac{1/\bar{P}(\bar{A}_{x+t:\overline{n-t}|})}{1/\bar{P}(\bar{A}_{x:\overline{n}|})} \tag{7}$$

$$=\; 1 \;-\; {}_t\bar{W}(\bar{A}_{x:\overline{n}|}). \tag{8}$$

Formula (8) can be derived by means of Exercise 16.5 in [B-G-H-J-N].

The following two formulas hold for all positive numbers s and t, s + t < n,

$$[1 \;-\; {}_s\bar{V}(\bar{A}_{x:\overline{n}|})][1 \;-\; {}_t\bar{V}(\bar{A}_{x+s:\overline{n-s}|})] \;=\; [1 \;-\; {}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|})] \tag{9}$$

and

$$[1 \;-\; {}_s\bar{W}(\bar{A}_{x:\overline{n}|})][1 \;-\; {}_t\bar{W}(\bar{A}_{x+s:\overline{n-s}|})] \;=\; [1 \;-\; {}_{s+t}\bar{W}(\bar{A}_{x:\overline{n}|})]. \tag{10}$$

Two other analogous formulas in Life Contingencies are

$$ {}_{s}p_{x}\,{}_{t}p_{x+s} = {}_{s+t}p_{x} \tag{11} $$

and

$$ {}_{s}E_{x}\,{}_{t}E_{x+s} = {}_{s+t}E_{x}. \tag{12} $$

These four equations are examples of the functional equation

$$ f(\alpha, \beta)\, f(\beta, \gamma) = f(\alpha, \gamma), \tag{13} $$

which had been studied by the Russian mathematician D. M. Sinzow about 100 years ago. (In [Ac], Sinzow is spelled Sincov.) The general solution of (13) is of the form

$$ f(\alpha, \beta) = \frac{g(\beta)}{g(\alpha)}. \tag{14} $$

If the function g is differentiable, then (14) becomes

$$ f(\alpha, \beta) = \exp\Bigl(\int_{\alpha}^{\beta} \frac{g'(h)}{g(h)} dh\Bigr). \tag{15} $$

Thus, (3) and (7) follow from

$$ \frac{d}{dh}\ln(\bar{a}_{x+h:\overline{n-h|}}) = \mu_{x+h} - \bar{P}(\bar{A}_{x+h:\overline{n-h|}}) \tag{16} $$

and

$$ \frac{d}{dh}\ln[1/\bar{P}(\bar{A}_{x+h:\overline{n-h|}})] = \frac{\mu_{x+h}}{\bar{A}_{x+h:\overline{n-h|}}} - \frac{1}{\bar{a}_{x+h:\overline{n-h|}}}, \tag{17} $$

respectively. Formula (16) can be found in [Va].

Finally, we note the formulas

$$ {}_{t}p_{x} = \frac{\ell_{x+t}}{\ell_{x}} = \exp\Bigl(-\int_{0}^{t} \mu_{x+h}\, dh\Bigr) \tag{18} $$

and

$$ {}_{t}E_{x} = \frac{D_{x+t}}{D_{x}} = \exp\Bigl(-\int_{0}^{t} (\mu_{x+h} + \delta)\, dh\Bigr). \tag{19} $$

References

[Ac] J. Aczél, *Lectures on Functional Equations and Their Applications*. New York: Academic Press, 1966.

[Be] A. Berger, "Über den Einfluss einer Änderung der Sterblichkeit auf die Prämienreserve," *Skandinavisk Aktuarietidskrift*, 19 (1936), 52-54.

[B-G-H-J-N] N.L. Bowers, Jr., H.U. Gerber, J.C. Hickman, D.A. Jones and C.J. Nesbitt, *Actuarial Mathematics*, 2nd edition, Schaumburg, IL: Society of Actuaries, 1997.

[B-H-N] N.L. Bowers, Jr., J.C. Hickman and C.J. Nesbitt, “Notes on the Dynamics of Pension Funding,” *Actuarial Research Clearing House,* 1981.1, 21-38. Reprinted in *Insurance: Mathematics and Economics*, 1 (1982), 261-270.

[Sh] E.S.W. Shiu, “Some Functional Equations in Actuarial Mathematics,” *Actuarial Research Clearing House,* 1988.2, 191-208.

[Su] B. Sundt, “Pension and Semi-Retirement,” *Bulletin of the Swiss Association of Actuaries,* (2001), 165-169.

[Va] P. Vasmoen, “Über den Einfluss einer Änderung der Sterblichkeit auf die Prämienreserve und andere damit verwandten Fragen,” *Skandinavisk Aktuarietidskrift*, 18 (1935), 1-34.