

THE AGGREGATE CLAIMS DISTRIBUTION  
AND STOP-LOSS REINSURANCE

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ABSTRACT

A recursive definition of the aggregate claims distribution is developed. The computation of the aggregate claims distribution does not require the usual computation of the convolutions of the distribution of claim amounts. It is independent of the number of lives in the group, and hence is particularly useful for large groups. Related formulas are developed for moments of stop-loss claims and retained claims. A numerical illustration shows that the computation is so efficient that one can easily calculate the aggregate claims distribution and related values manually.

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I. INTRODUCTION

**T**HE problem of calculating the aggregate claims distribution has concerned actuaries for many years. The customary procedure, as given in the *Part 5 Study Notes* on risk theory [5], requires a very large number of calculations to generate the various convolutions of the distribution of claim amounts given that a certain number of claims occur. In this paper, we show that the computation of the aggregate claims distribution and related values can be carried out without conditioning upon the number of claims. In fact, reference to numbers of claims will be made only in the theory sections of the paper. The only things needed to calculate the exact distribution are the sum of the forces of mortality over the entire group, and the expected amount of claims for each amount of insurance in the group.

In the case of life insurance, claim amounts are of fixed size, whereas in the case of medical, health, and general insurance, claim amounts are variable for risks within any risk category. The necessary theory, as well as an example, are given for the fixed-claim situation first, and then for the variable claim situation, which requires some additional theory.

In Section VII, formulas are developed for moments of both stop-loss claims and retained claims, using only that portion of the aggregate claims distribution to the left of the stop-loss level. These formulas

eliminate the need to calculate values of the aggregate claims distribution beyond the stop-loss level.

## II. THEORY: FIXED CLAIM SIZE

Consider a group life contract under which each life is insured for a fixed amount of death benefit. In accordance with the collective risk model, it is assumed that lives dying during the year are replaced immediately by lives of identical risk. Hence, the number of claims arising from each certificate is a Poisson-distributed random variable.

The amounts of insurance will be measured by some convenient unit  $U$ , such as \$1,000. Computations are minimized when the unit chosen is the greatest common divisor of the amounts of insurance of the certificates in the group. In a paper by Mereu [4], the amounts of insurance in the numerical example are \$4,000, \$6,000, \$8,000, \$10,000, \$12,000, \$14,000, \$16,000, \$20,000, and \$25,000. The appropriate unit for this example is \$1,000, the greatest common divisor of these amounts. The choice of this unit minimizes computations, since the aggregate claims distribution is calculated at integral multiples of the unit, and because we require amounts of insurance to be integral multiples of the unit.

Let  $Y_i$  denote the random variable representing the number of claims of amount  $iU$ . Since the sum of independent Poisson random variables is itself a Poisson random variable,  $Y_i$  is a Poisson random variable.

Let  $\theta_i$  denote the sum of the forces of mortality for all lives with amount of insurance equal to  $iU$ . Then  $Y_i$  is Poisson-distributed with parameter  $\theta_i$ .

Let  $nU$  represent the largest amount of insurance in the group. Then there are at most  $n$  independent Poisson random variables  $Y_i$ ,  $i = 1, 2, \dots, n$ , with parameters  $\theta_i$ ,  $i = 1, 2, \dots, n$ . If there are no certificates of amount  $jU$ , then  $\theta_j$  is set equal to zero.

Table 1 summarizes the data in Table 1 of [4] by summing the forces of mortality ( $t_i$  in Mereu's notation) over each amount. The value  $E_j$  is the expected units of claims of size  $jU$ . Table 1 contains all the information needed to generate the aggregate claims distribution; no intermediate calculations are necessary.

The probability that exactly  $k$  claims of amount  $iU$  occur is given by

$$\Pr \{Y_i = k\} = \frac{e^{-\theta_i} \theta_i^k}{k!}, \quad i = 1, 2, \dots, n, \quad (1)$$

which equals zero if  $\theta_i = 0$ .

Let  $X_i$  denote the random variable representing the aggregate claims of amount  $iU$ . Then

$$\Pr \{X_i = iUk\} = \frac{e^{-\theta_i} \theta_i^k}{k!}, \quad i = 1, 2, \dots, n. \quad (2)$$

Let  $X = X_1 + X_2 + \dots + X_n$ . Then  $X$  represents the aggregate claims over all amount categories.

Let  $P_i$  represent the probability that the aggregate claims will be exactly  $iU$ ; that is,

$$P_i = \Pr \{X = iU\}.$$

We now apply a result first proved by Adelson in a paper dealing with an inventory control problem in operations research.<sup>1</sup> He shows that the probability generating function of  $X$  is given by

$$\exp\left(-\sum_{i=1}^n \theta_i\right) \exp\left(\sum_{i=1}^n \theta_i i^i U\right). \quad (3)$$

Using this probability generating function, he proves that there is a recurrence relation

$$iP_i = \sum_{j=1}^i j\theta_j P_{i-j}, \quad i = 1, 2, 3, \dots, \quad (4)$$

TABLE 1  
AMOUNTS OF INSURANCE, TOTAL FORCE OF MORTALITY,  
AND EXPECTED CLAIM AMOUNTS

$j$	$jU$	$\theta_j$	$j\theta_j = E_j$
4.....	\$ 4,000	0.034606	0.138424
6.....	6,000	0.017823	0.106938
8.....	8,000	0.025323	0.202584
10.....	10,000	0.023590	0.235900
12.....	12,000	0.021329	0.255948
14.....	14,000	0.024705	0.345870
16.....	16,000	0.021995	0.351920
20.....	20,000	0.040867	0.817340
25.....	25,000	0.015878	0.396950
Total.....		0.226116	2.851874

<sup>1</sup> Reported in N. L. Johnson and S. Kotz, *Distributions in Statistics: Discrete Distributions* (New York: Wiley, 1969), p. 112.

where

$$P_0 = \Pr \{X = 0\} = \exp \left( - \sum_{j=1}^n \theta_j \right).$$

The proof of this recurrence relation is not given in this paper, the purpose of which is to adapt Adelson's result to a problem in actuarial science. Interested readers can find the proof in Adelson's paper [1].

Since in our applications in group life insurance there may be no certificates at many of the amount levels  $iU$ ,  $i = 1, 2, 3, \dots, n$ , the calculation can be simplified by summing only over those values of  $j$  for which  $\theta_j$  is not zero. Furthermore, the number of terms in the sum is limited by the fact that  $j$  must not be greater than  $i$ , lest the subscript of  $P_{i-j}$  be negative. Thus, the largest  $j$  that can be used in the sum is the minimum of  $i$  and  $n$ , written as  $\min(i, n)$ . Hence, equation (4) can be rewritten as

$$P_i = \frac{1}{i} \sum_{\substack{j=1 \\ E_j \neq 0}}^{\min(i, n)} E_j P_{i-j}, \quad (5)$$

where

$$P_0 = \exp \left( - \sum_{\substack{j=1 \\ E_j \neq 0}}^n \theta_j \right).$$

Let  $m$  represent the number of positive values of  $E_j$ ,  $j = 1, 2, \dots, n$ . Then the maximum number of terms in the recursion formula (5) is  $\min(i, m)$ . There is an upper bound on the number of terms required to compute values of  $P_j$  from  $P_1$  to  $P_k$ . If  $k \geq m$ , the maximum number of terms is given by

$$1 + 2 + 3 + \dots + m + m + \dots + m = m(m+1)/2 + m(k-m).$$

If  $k < m$ , the maximum number of terms is  $k(k+1)/2$ . This maximum number is reached only if the amounts of insurance are  $U, 2U, \dots, mU$ . In practical situations the actual total number of terms is much less than the maximum. However, it is useful to note that the computation of  $P_i$  from  $P_1$  to  $P_k$  ( $k \geq m$ ) requires no more than  $m(m+1)/2 + m(k-m)$  multiplications,  $k$  fewer additions than multiplications, and exactly  $k$  divisions. In addition, very little computer capacity is required, since only the values of  $P_i$  and  $E_i$  for  $i = 1, 2, \dots, k$  need be stored.

### III. EXAMPLE: FIXED CLAIM SIZE

We shall continue to examine the data of [4], which are summarized in Table 1 of this paper. For these data we see that  $n = 25$ ,  $m = 9$ , and

$U = \$1,000$ . We first compute  $P_0$ , the probability of no claims, which is given by

$$P_0 = \exp\left(-\sum_{i=1}^n \theta_i\right) = \exp(-0.226116) = 0.79762557.$$

Using formula (5), we compute recursively

$$P_4 = \frac{1}{4}E_4P_0 = 0.02760263,$$

$$P_6 = \frac{1}{6}E_6P_0 = 0.01421608,$$

$$P_8 = \frac{1}{8}(E_4P_4 + E_8P_0) = 0.02067588,$$

$$P_{10} = \frac{1}{10}(E_4P_6 + E_6P_4 + E_{10}P_0) = 0.01930795,$$

$$P_{12} = \frac{1}{12}(E_4P_8 + E_6P_6 + E_8P_4 + E_{12}P_0) = 0.01784373,$$

$$P_{14} = \frac{1}{14}(E_4P_{10} + E_6P_8 + E_8P_6 + E_{10}P_4 + E_{14}P_0) = 0.02072499,$$

$$P_{16} = \frac{1}{16}(E_4P_{12} + E_6P_{10} + E_8P_8 + E_{10}P_6 + E_{12}P_4 + E_{16}P_0) = 0.01874013,$$

$$P_{18} = \frac{1}{18}(E_4P_{14} + E_6P_{12} + E_8P_{10} + E_{10}P_8 + E_{12}P_6 + E_{14}P_4) = 0.00148619.$$

These values agree with those developed by Mereu [4] and those who discussed his paper.

Table 2 gives  $P_i$ ,  $F_i = \sum_{j=0}^i P_j$ ,  $G_i = \sum_{j=0}^i F_j$ , and  $H_i = \sum_{j=0}^i G_j$  for  $i = 0, 1, \dots, 26$ . The value  $F_i$  is the cumulative probability that aggregate claims will not exceed  $iU$ , that is,

$$F_i = \Pr \{X \leq iU\}.$$

The functions  $G_i$  and  $H_i$  will be used in later sections.

TABLE 2  
VALUES OF  $P_i$ ,  $F_i$ ,  $G_i$ , AND  $H_i$

$i$	$iU$	$P_i$	$F_i$	$G_i$	$H_i$
0.....	\$ 0	0.79762557	0.79762557	0.79762557	0.79762557
4.....	4,000	0.02760263	0.82522820	4.01573049	11.99198621
6.....	6,000	0.01421608	0.83944428	5.68040297	22.51334787
8.....	8,000	0.02067588	0.86012016	7.37996742	36.41316255
10....	10,000	0.01930795	0.87942811	9.11951570	53.77276583
12....	12,000	0.01784373	0.89727185	10.89621565	74.66792529
14....	14,000	0.02072499	0.91799684	12.71148434	99.17289713
16....	16,000	0.01874013	0.93673697	14.56621815	127.36859646
18....	18,000	0.00148619	0.93822316	16.44117828	159.31272986
20....	20,000	0.03424170	0.97246487	18.35186631	195.04399762
22....	22,000	0.00125971	0.97372457	20.29805575	234.66638455
24....	24,000	0.00227777	0.97600234	22.24778266	278.18594754
25....	25,000	0.01266470	0.98866704	23.23644970	301.42239724
26....	26,000	0.00147878	0.99014582	24.22659552	325.64899276

## IV. THEORY: VARIABLE CLAIM SIZE

Consider a portfolio of  $r$  independent Poisson risks with expected values  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r$ . Let  $N_j$  be a random variable denoting the number of claims resulting from the  $j$ th risk. Then the probability that  $m$  claims arise from the  $j$ th risk is

$$\Pr \{N_j = m\} = \frac{e^{-\lambda_j} \lambda_j^m}{m!}, \quad j = 1, 2, \dots, r. \quad (6)$$

The probability generating function for the Poisson-distributed random variable  $N_j$  is given by the formula

$$G_{N_j}(t) = \exp[-\lambda_j(1-t)]. \quad (7)$$

We now introduce the distribution of the amount of claims given that a single claim has occurred. As before, we assume that the amount of a claim is an integral multiple of some convenient unit  $U$ . Let  $X_{kj}$  denote the random variable representing the amount of the  $k$ th claim arising from the  $j$ th risk.

Let

$$s_j(i) = \Pr \{X_{kj} = iU\} \quad (8)$$

denote the probability that a claim arising from the  $j$ th risk will be of amount  $iU$ . Then the probability generating function of  $X_{kj}$  is

$$G_{X_{kj}}(t) = \sum_{i=1}^n s_j(i) t^{iU}, \quad (9)$$

where  $nU$  is the maximum possible claim over all risks.

Let

$$X_j = \sum_{k=0}^{N_j} X_{kj}$$

denote the aggregate claims resulting from the  $j$ th risk. Then, following Halmstad [2], we can compute the probability generating function of  $X_j$  by compounding the probability generating functions, as

$$\begin{aligned} G_{X_j}(t) &= G_{N_j}[G_{X_{kj}}(t)] \\ &= \exp \left[ -\lambda_j \left\{ 1 - \left[ \sum_{i=1}^n s_j(i) t^{iU} \right] \right\} \right]. \end{aligned} \quad (10)$$

Let  $X = X_1 + X_2 + \dots + X_r$  denote the distribution of aggregate claims over the entire portfolio. Then, because of the independence of the risks,  $S$  has the probability generating function

$$\begin{aligned}
 G_X(t) &= \prod_{j=1}^r G_{X_j}(t) \\
 &= \exp \left[ - \sum_{j=1}^r \lambda_j \left\{ 1 - \left[ \sum_{i=1}^n s_j(i) t^{iU} \right] \right\} \right],
 \end{aligned}
 \tag{11}$$

which can be written as

$$G_X(t) = \exp \left( - \sum_{j=1}^r \lambda_j \right) \exp \left\{ \sum_{i=1}^n \left[ \sum_{j=1}^r \lambda_j s_j(i) t^{iU} \right] \right\}.
 \tag{12}$$

Now, because  $\sum_{i=1}^n s_j(i) = 1$ , the first exponent in equation (12) can be rewritten as

$$- \sum_{j=1}^r \lambda_j \sum_{i=1}^n s_j(i) = - \sum_{i=1}^n \left[ \sum_{j=1}^r \lambda_j s_j(i) \right].
 \tag{13}$$

Substituting

$$\theta_i = \sum_{j=1}^r \lambda_j s_j(i)
 \tag{14}$$

in equation (12), and using equation (13), we see that the probability generating function of the aggregate claims over the entire portfolios can be written as

$$G_X(t) = \exp \left( - \sum_{i=1}^n \theta_i \right) \exp \left( \sum_{i=1}^n \theta_i t^{iU} \right).
 \tag{15}$$

The right-hand side of this equation is identically equal to expression (3). Hence, if we let  $P_i = \Pr \{X = iU\}$ , the recurrence relation given by equation (5) holds; that is, the aggregate claims distribution can be written as

$$P_i = \frac{1}{i} \sum_{\substack{j=1 \\ E_j \neq 0}}^{\min(i,n)} E_j P_{i-j},
 \tag{16}$$

where

$$P_0 = \exp \left( - \sum_{\substack{j=1 \\ E_j \neq 0}}^n \theta_j \right)$$

and  $E_j = j\theta_j$ .

#### V. INTERPRETATION OF THE RESULT

The quantity  $\theta_i$  in equation (14) is the weighted sum of the Poisson parameters over all risks, where the weights are the associated probabilities of having a claim of amount  $iU$ . It thus can be interpreted as the expected number of claims of size  $iU$  in the portfolio. The quantity  $E_j$

is then the expected aggregate amount of claims of size  $jU$  (as measured in units of  $U$ ).

Equation (14) indicates that, for each of the risks, one should prorate the Poisson parameters to each amount class in proportion to the probability for that amount class, and then sum over each of the amount classes. The calculation of the aggregate claims distribution for the portfolio depends on the number of amount classes and not on the number of risks in the portfolio.

One sees from equation (15) that computation of the aggregate claims distribution for a portfolio of risk with variable claim size is equivalent to the computation for a portfolio of fixed claim size, when the data are appropriately summarized.

#### VI. EXAMPLE: VARIABLE CLAIM SIZE

Since all risks with the same amount distribution can be treated as a single "risk" with a Poisson parameter, that is, the sum of the Poisson parameters over the risks comprising the single "risk," one need list only those new "risks" having different amount distributions. Consider a group medical expense insurance contract whose risks are classified according to marital status and employment status. Summary information related to this contract is given in Table 3.

Table 4 shows the aggregate claims distribution, calculated by using equation (16), and the stop-loss premiums, calculated by using equation (21), which will be developed in Section VII.

#### VII. STOP-LOSS REINSURANCE

Mereu [4] computes the mean and variance of both aggregate claims and stop-loss claims. The mean and variance of aggregate claims are easy to compute; let them be denoted by  $U\mu$  and  $U^2\sigma^2$ , respectively. Then

$$\mu = \sum_{j=1}^n j\theta_j, \quad \text{and} \quad \sigma^2 = \sum_{j=1}^n j^2\theta_j, \quad (17)$$

which can be computed from Table 1. Alternatively,

$$\mu = \sum_{i=1}^{\infty} iP_i, \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{\infty} (i - \mu)^2 P_i, \quad (18)$$

neither of which can be calculated conveniently. However, we shall need equation (18) for later development.

We now develop formulas for the mean and variance of both retained claims and stop-loss claims; these values can be computed from a single line of Table 2. The chief advantage of these formulas is that the moments



TABLE 3  
VALUES OF  $\lambda_j$  AND  $s_j(i)$  FOR SAMPLE GROUP CONTRACT

STATUS	$j$	$\lambda_j$	$s_j(i)$							
			$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
Active, single . . . . .	1	40.2	0.20	0.15	0.15	0.10	0.10	0.10	0.10	0.10
Active, married . . . . .	2	100.1	0.05	0.15	0.15	0.20	0.15	0.10	0.10	0.10
Retired, single . . . . .	3	5.3	0.20	0.15	0.10	0.05	0.05	0.10	0.15	0.20
Retired, married . . . . .	4	8.6	0.05	0.15	0.10	0.10	0.10	0.15	0.20	0.15
$\theta_i = \sum_{j=1}^4 \lambda_j s_j(i)$ . . . . .			14.535	23.13	22.435	25.165	20.16	15.85	16.545	16.38

$$\sum_{j=1}^4 \theta_i = 154.20, \quad \sum_{j=1}^4 \lambda_j = 154.20$$

of both types of claims can be computed from the left-hand end of the aggregate claims distributions, necessitating calculation of the aggregate claims distribution only up to the stop-loss level.

Let  $UW(s)$  and  $UR(s)$  denote random variables representing stop-loss claims and retained claims, for stop-loss level  $U$ s. We shall use the operators  $E$  and  $V$  to represent mean and variance.

The first two raw moments of retained claims are

$$UE[R(s)] \quad \text{and} \quad U^2E[R(s)^2],$$

where

$$E[R(s)] = \sum_{i=0}^s iP_i + \sum_{i=s+1}^{\infty} sP_i = \sum_{i=0}^s iP_i + s(1 - F_s) \quad (19)$$

TABLE 4  
AGGREGATE CLAIMS DISTRIBUTION

Aggregate Claim	Probability of Claim	Cumulative Probability	Stop-Loss Premium
0 .....	0.00000000	0.00000000	\$671.51
1 .....	0.00000000	0.00000000	670.51
...			
500 .....	0.00008770	0.00149819	171.54
...			
600 .....	0.00338668	0.11837528	74.77
...			
670 .....	0.00660896	0.50006997	24.84
...			
700 .....	0.00578013	0.68897060	12.65
...			
800 .....	0.00072096	0.98127073	0.45
...			
900 .....	0.00000948	0.99983773	0.00
...			
1,000 .....	0.00000002	0.99999977	0.00

Expected claims: \$671.51; variance of claims: 3,645.24;  
standard deviation: 60.38

and

$$E[R(s)^2] = \sum_{i=0}^s i^2 P_i + s^2(1 - F_s). \tag{20}$$

Similarly, for stop-loss claims,

$$\begin{aligned} E[W(s)] &= \sum_{i=s+1}^{\infty} (i - s)P_i = \sum_{i=0}^{\infty} (i - s)P_i - \sum_{i=0}^s (i - s)P_i \\ &= \mu - s - \sum_{i=0}^s (i - s)P_i, \end{aligned} \tag{21}$$

and

$$\begin{aligned} E[W(s)^2] &= \sum_{i=s+1}^{\infty} (i - s)^2 P_i = \sum_{i=0}^{\infty} (i - s)^2 P_i - \sum_{i=0}^s (i - s)^2 P_i \\ &= \sigma^2 + (s - \mu)^2 - \sum_{i=0}^s (i - s)^2 P_i. \end{aligned} \tag{22}$$

The four sums in equations (19)–(22) are easily simplified by using a generalized summation-by-parts formula as given by Kellison [3, p. 131], that is,

$$\sum u_x v_x = u_x \bar{v}_x - \Delta u_x \sum^2 v_{x+1} + \Delta^2 u_x \sum^3 v_{x+2} - \dots \tag{23}$$

The results are given by

$$\sum_{i=0}^s i P_i = (s + 1)F_s - G_s; \tag{24}$$

$$\sum_{i=0}^s i^2 P_i = (s + 1)^2 F_s - (2s + 3)G_s + 2H_s; \tag{25}$$

$$\sum_{i=0}^s (i - s) P_i = F_s - G_s; \tag{26}$$

and

$$\sum_{i=0}^s (i - s)^2 P_i = F_s - 3G_s + 2H_s. \tag{27}$$

Substituting equations (24)–(27) in equations (19)–(22) yields

$$E[R(s)] = s + F_s - G_s; \tag{28}$$

$$E[R(s)^2] = s^2 + (2s + 1)F_s - (2s + 3)G_s + 2H_s; \tag{29}$$

$$E[W(s)] = \mu - s - F_s + G_s; \tag{30}$$

and

$$E[W(s)^2] = \sigma^2 + (s - \mu)^2 - F_s + 3G_s - 2H_s. \tag{31}$$

Hence,

$$V[R(s)] = E[R(s)^2] - E[R(s)]^2, \quad (32)$$

which reduces to

$$V[R(s)] = F_s - 3G_s + 2H_s - (G_s - F_s)^2. \quad (33)$$

Similarly,

$$V[W(s)] = \sigma^2 - F_s + 3G_s - 2H_s - (G_s - F_s)^2 + 2(s - \mu)(G_s - F_s). \quad (34)$$

For the example under consideration, with  $s = 18$ , substituting values from Table 2 in equations (28), (30), (33), and (34), with  $\mu = 2.851874$  and  $\sigma^2 = 44.989822$ , yields

$$E[R(18)] = 2.49704488, \quad E[W(18)] = 0.35482912;$$

$$V[R(18)] = 29.8985304, \quad V[W(18)] = 4.08949160.$$

Higher moments of stop-loss claims and retained claims can be computed in similar fashion.

#### VIII. CONCLUSIONS

Formula (5) can be used to generate efficiently the exact aggregate claims distribution. If the number of different amounts is large, as would be the case if a company were computing the distribution of claims as measured by the net amount at risk for a block of its business, one could group the amounts into convenient units such as \$1,000, \$5,000, or \$10,000, to reduce the number of amount cells. Once the data are summarized as in Table 1 or Table 2, the computation depends upon the number of cells and not the number of lives or policies.

Formulas (28), (30), (33), and (34) can be used to determine the mean and variance of both stop-loss claims and retained claims. The formulas could be simplified, but the simplification would involve values with subscripts not equal to  $s$ .

The results of this paper are unique to the Poisson distribution, that is, the collective risk model. The unique properties of this distribution allow the development of the recursive formula (5).

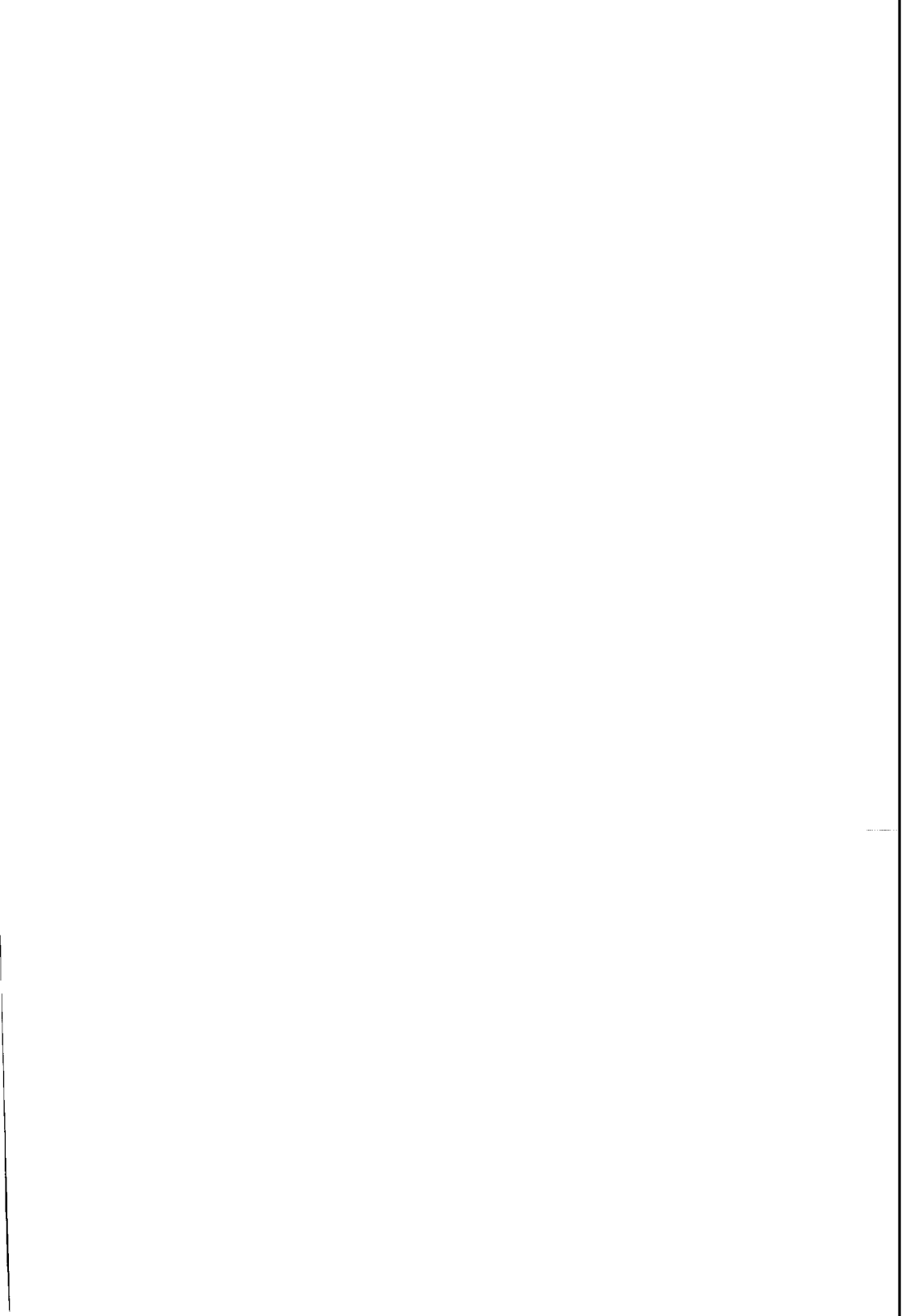
#### IX. ACKNOWLEDGMENT

After this paper had been accepted for publication, but before it appeared in print, formula (5) appeared (in somewhat different notation) in a paper by R. E. Williams, A.S.A., entitled "Computing the Probability Density Function for Aggregate Claims" (to appear in the *Pro-*

ceedings of the Canadian Institute of Actuaries). Mr. Williams independently derived the result of Adelson [1] and presented a number of applications in group insurance.

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## DISCUSSION OF PRECEDING PAPER

HANS BÜHLMANN\* AND HANS U. GERBER:

The most important aspect of this paper is the new method of calculating the compound Poisson distribution. Now there are three methods.

1. The first is the customary convolution method (as illustrated in the study notes on risk theory).

2. In a second approach, let  $X$  have a compound Poisson distribution with Poisson parameter  $\theta$ , and let  $p(i)$  denote the probability that a given claim is  $i$  ( $i = 1, \dots, n$ ). Then

$$X = Y_1 + 2Y_2 + \dots + nY_n,$$

where  $Y_i$  is the number of claims of size  $i$ . This method is based on the facts that (a) the distribution of  $Y_i$  is Poisson with parameter  $\theta_i = \theta p(i)$ , and (b)  $Y_1, \dots, Y_n$  are mutually independent. Thus it is easy to calculate the distribution of  $iY_i$ , and then  $n - 1$  convolutions have to be performed.

3. The new method starts with  $\Pr(X = 0) = e^{-\theta}$  and computes  $\Pr(X = i)$  recursively from the formula

$$\Pr(X = i) = \frac{1}{i} \sum_{j=1}^{i \wedge n} j \theta_j \Pr(X = i - j). \quad (1)$$

This method is certainly the easiest to program (on a computer or a calculator).

As the author indicated, the recursion (1) can be derived from the generating function. We shall give an alternative derivation that provides more insight.

Let  $Z_1, Z_2, \dots$  be independent, identically distributed random variables such that the possible values of  $Z_i$  are  $0, 1, 2, \dots$ . Let  $S_m = Z_1 + \dots + Z_m$ . Then

$$\begin{aligned} \frac{i}{m} &= E(Z_m | S_m = i) \\ &= \frac{\sum_j j \Pr(Z_m = j) \Pr(S_{m-1} = i - j)}{\Pr(S_m = i)}. \end{aligned}$$

Thus

$$\Pr(S_m = i) = \frac{m}{i} \sum_j j \Pr(Z_m = j) \Pr(S_{m-1} = i - j). \quad (2)$$

Now let  $Z_k$  be compound Poisson with Poisson parameter  $\theta/m$  and claim amount distribution  $p(\cdot)$ . Note that for any  $m$ , the distribution

\* Mr. Bühlmann, not a member of the Society, is president of the Swiss Association of Actuaries.

of  $S_m$  is that of  $X$ . In the limit  $m \rightarrow \infty$  formula (2) produces the recursive formula (1).

JAMES A. TILLEY:

In the first part of my discussion, I will comment briefly on the usefulness of the new method for calculating the aggregate claims distribution. Next, I will give an alternate derivation of the principal result of Dr. Panjer's paper. Finally, I will develop new recursion formulas applicable to compound binomial and negative binomial processes. The recursion formula derived by Dr. Panjer is a limiting case of each of these new formulas. The practical importance of the negative binomial process is described briefly.

#### *Usefulness of the New Method*

As described by Dr. Panjer, the conventional approach to calculating the probability density function for aggregate claims in a collective risk model has been to break the problem into two pieces: (1) the determination of the probability density for the number of claims in a specified period and (2) the computation of  $n$ -fold convolutions of the individual claim amount distribution. It has not been usual to expend much effort on item 1—generally one uses the Poisson process with the appropriate expected number of claims. The item 2 computations, however, consume considerable computer time because high-order convolutions often are needed to determine accurately the right-hand tail of the aggregate claims distribution.

In the new method developed independently by Dr. Panjer and Mr. Williams (reference in Sec. IX of Dr. Panjer's paper), the aggregate claims distribution is computed from a recursion formula that does not require the calculation of convolutions of the individual claim distribution. Each author presented a different derivation of the recursion formula. In my opinion, what is most significant about their new method is how remarkably practical it is.

1. To achieve a high level of accuracy in determining the aggregate claims distribution, especially in the right-hand tail, the recursion method generally consumes much less computer time than the convolution method, potentially by orders of magnitude, depending on the expected number of claims for the group and the shape of the individual claim amount distribution.
2. Programs that capitalize on the power of particular computer languages such as APL to make the convolution method more efficient with respect to computer execution time typically use considerably more computer storage than the recursion method.



3. In the convolution method, as a higher-order convolution is included in the calculation, all the values of the prior approximation to the aggregate claims distribution (except for the "leftmost" values) are altered. By contrast, in the recursion method, one determines the aggregate claims distribution sequentially from its leftmost value (zero) toward its right-hand tail, and, once computed, each value is exact (to the accuracy of the computing device) and is not altered as the computation is carried further.
4. The recursion formula calculations are so efficient on today's computers that one can afford to calculate by brute force the entire aggregate claims distribution (for all practical purposes) and then determine stop-loss premiums directly from the density function without resort to any of the special techniques described previously in the actuarial literature.

The third point may be of theoretical interest only, since both the recursion formula and the convolution method produce accurate results if they are terminated when the cumulative distribution function is sufficiently close to unity, for example, to within  $10^{-6}$ .

The only practical limitation of the applicability of the Panjer-Williams recursion formula is that it is valid only if the number of claims for the collective is a Poisson random variate. In the next section, I will present another derivation of the recursion formula sufficiently general that it can be extended to the compound binomial and negative binomial processes, and perhaps, with sufficient ingenuity (and good fortune), to some other processes.

#### *Derivation of the Panjer-Williams Recursion Formula*

In the following derivation, no distinction is drawn between the fixed and variable claim size situations because the probability density function of individual claim amounts appears *explicitly* in the equations. This approach contrasts with that of Panjer and Williams, who did not refer to the individual claim amount distribution *explicitly* but classified the data by amount cells and identified appropriate characteristics of the risks in those cells. There is no practical difference between the two approaches, however, since one would have to classify the data by cells in order to determine the individual claim amount distribution.

Let  $g(n, t)$  represent the probability density function of the number of claims in a period  $t$ . Assuming a Poisson process,

$$g(n, t) = \frac{e^{-\theta t}(\theta t)^n}{n!}, \quad (1)$$

where  $\theta$  is the expected number of claims in a unit time period. Let  $p(x)$  represent the probability density of individual claim amounts, and let

$f(x)$  be the probability density of aggregate claim amounts. Finally, let  $p^{n*}(x)$  denote the  $n$ -fold convolution of  $p(x)$ . It will be assumed in this discussion that all probability density functions are well defined. The theory can be developed for discontinuous cumulative distribution functions by treating the densities as measures rather than functions, and by using the concepts of Laurent-Schwarz distribution theory.

The aggregate claims density of the compound Poisson process is

$$f(x) = \sum_{n=0}^{\infty} \frac{e^{-\theta t} (\theta t)^n}{n!} p^{n*}(x). \quad (2)$$

Denote the Laplace transform of a function by a bar over the function. For example,

$$\bar{f}(k) = \int_0^{\infty} f(x) e^{-kx} dx. \quad (3)$$

By taking Laplace transforms of both sides of equation (2) and using the convolution theorem for Laplace transforms, we obtain

$$\begin{aligned} \bar{f}(k) &= \sum_{n=0}^{\infty} \frac{e^{-\theta t} (\theta t)^n}{n!} [\bar{p}(k)]^n \\ &= e^{\theta t \bar{p}(k) - \theta t}. \end{aligned} \quad (4)$$

Next, by differentiating equation (4) with respect to  $k$ , we derive

$$\frac{d\bar{f}}{dk} = \theta t \bar{f}(k) \frac{d\bar{p}}{dk}. \quad (5)$$

Using the relation

$$\frac{d\bar{f}}{dk} = - \int_0^{\infty} x f(x) e^{-kx} dx, \quad (6)$$

taking the inverse Laplace transform of equation (5), and using the convolution theorem, we finally obtain

$$x f(x) = \theta t \int_0^x y p(y) f(x-y) dy. \quad (7)$$

Equation (7) is the continuous form of the Panjer-Williams recursion formula for the aggregate claims density  $f(x)$ . In the continuous form it is an integral equation for  $f(x)$  and is not very useful. By replacing  $p(y)$  with a discrete density function, however, we recover the Panjer-Williams equation.

Let  $U$  be the largest positive number such that all individual claim

amounts are expressible as positive integral multiples of  $U$ . Let  $p_j$  denote the probability that an individual claim amount is  $jU$ . Similarly, let  $f_i$  denote the probability that the aggregate claims amount is  $iU$ . Then equation (7) becomes

$$if_i = \theta t \sum_{j=0}^i j p_j f_{i-j}, \quad i \geq 1. \quad (8)$$

The initial condition is  $f_0 = e^{-\theta t}$ . Identifying  $\theta_j = (\theta t)p_j$  for  $j \geq 1$ , and noting that the  $j = 0$  term vanishes in the summation, we see that equation (8) above is equivalent to equation (4) in Dr. Panjer's paper.

#### *Compound Negative Binomial Process*

The actuarial literature is rife with examples of the applicability of the negative binomial distribution in collective risk theory. Some of the reasons for its importance are the following:

1. The Poisson distribution can be obtained as a limiting case of the negative binomial distribution.
2. In many situations, the negative binomial distribution accords with actual data better than the Poisson distribution.
3. The negative binomial distribution has been used for processes exhibiting contagion—those in which the probability of a further claim increases whenever a claim has already occurred.

The negative binomial distribution has larger variance and longer tails than the Poisson distribution and consequently results in larger stop-loss premiums than does the Poisson process with the same expected number of claims. Thus it is important to examine whether or not the negative binomial distribution is more appropriate for the collective than the Poisson distribution.

As an example of how the negative binomial distribution can arise, consider a group life insurance case. On the basis of the size of the group, the age, sex, and income distribution of its members, and any other relevant available data, the group underwriter establishes the *standard* expected number of claims  $\theta$  per unit time period. Let  $r$  denote the "mortality level" of the group. We assume that aggregate claims in a period of length  $t$  would follow a compound Poisson process with Poisson parameter  $r\theta t$  if  $r$  were known with certainty. Then, in either of the following two situations, the distribution of aggregate claims would be given by a compound negative binomial process.

1. The mortality level of the group is not constant over time but fluctuates randomly and independently from period to period according to a gamma distribution.

2. The mortality level of the group is constant over time, but its exact value is unknown. On the basis of prior experience, we know that the mortality levels of similar groups can be represented by a gamma distribution.

A Bayesian approach is useful in the second situation. Let  $h_0(r)$  denote our "prior" distribution of the mortality level based on any previous experience with similar groups, or, if there is none, on our judgment.

$$h_0(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}. \quad (9)$$

Suppose that  $k_1$  claims occur in a period of duration  $t$ . The "posterior" distribution of the mortality level based on  $h_0(r)$  and the observation of  $k_1$  claims is

$$h_1(r | k_1) = \frac{(\beta + \theta t)^{\alpha+k_1}}{\Gamma(\alpha + k_1)} r^{\alpha+k_1-1} e^{-(\beta+\theta t)r}. \quad (10)$$

If  $k_2, \dots, k_m$  claims are observed in subsequent successive periods of length  $t$ , and the posterior from one period is used as the prior for the next, we obtain, for  $0 \leq j \leq m$ ,

$$h_j(r | k_1, \dots, k_j) = \frac{(\beta + j\theta t)^{\alpha+K_j}}{\Gamma(\alpha + K_j)} r^{\alpha+K_j-1} e^{-(\beta+j\theta t)r}, \quad (11)$$

where  $K_j = \sum_{i=1}^j k_i$ .

The distribution  $h_j(r | k_1, \dots, k_j)$  is a gamma density function. The same distribution is obtained whether the initial distribution is "updated" sequentially from the initial period to the most recent completed period or whether it is updated only once on the basis of the combined number of claims over all the periods. When sufficiently many periods have elapsed that  $m\theta t \gg \beta$  and  $K_m \gg \alpha$ , our revised estimate of the group's level of mortality depends mostly on its own experience and very little on our initial estimate. This is consistent with the usual notion of credibility.

From a practical viewpoint, it may be impossible to distinguish between the time-heterogeneous process (situation 1 above) and the time-homogeneous process (situation 2 above). Since both result in a compound negative binomial process, however, it may be possible to determine an aggregate claims distribution acceptable for group underwriting purposes without resolving which of the two situations is applicable.

Consider the following compound process.

$$f(x | r) = \sum_{n=0}^{\infty} \frac{e^{-r\theta t} (r\theta t)^n}{n!} p^{n*}(x); \quad (12)$$

$$h(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}, \quad \alpha, \beta > 0. \quad (13)$$

where  $h(r)$  is the gamma distribution with shape parameter  $\alpha$  and scale parameter  $\beta^{-1}$ .

The density of aggregate claims is

$$f(x) = \int_0^\infty f(x|r)h(r)dr \quad (14)$$

$$= \sum_{n=0}^\infty \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} (1-q)^\alpha q^n p^{n*}(x),$$

where  $q = \theta t(\beta + \theta t)^{-1}$  and  $0 < q < 1$ . Thus the probability density of the number of claims in a period  $t$  is

$$g(n, t) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} (1-q)^\alpha q^n, \quad n \geq 0. \quad (15)$$

This is the negative binomial distribution with mean  $\alpha q(1-q)^{-1}$  and variance  $\alpha q(1-q)^{-2}$ .

Proceeding as in the derivation of the Panjer-Williams recursion formula, we derive the following expressions.

$$\bar{f}(k) = \sum_{n=0}^\infty \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} (1-q)^\alpha q^n [\bar{p}(k)]^n, \quad (16)$$

$$\frac{d\bar{f}}{dk} = q\bar{p}(k) \frac{d\bar{f}}{dk} + \alpha q \bar{f}(k) \frac{d\bar{p}}{dk}, \quad (17)$$

$$xf(x) = q \int_0^x (x-y)p(y)f(x-y)dy + \alpha q \int_0^x yp(y)f(x-y)dy. \quad (18)$$

Replacing  $p(y)$  and  $f(x)$  with discrete distributions as before, and assuming that individual claim amounts are positive, that is,  $p_0 = 0$ , we derive the compound negative binomial recursion formula

$$if_i = q \sum_{j=1}^i [i + (\alpha - 1)j] p_j f_{i-j}, \quad i \geq 1. \quad (19)$$

The initial condition is  $f_0 = (1-q)^\alpha$ .

In the limit  $q \rightarrow 0$  and  $\alpha \rightarrow \infty$  with  $\alpha q = \theta t$ , a constant, the negative binomial distribution "approaches" the Poisson distribution with mean  $\theta t$ , equation (19) "approaches" equation (8), and the initial condition "approaches"  $f_0 = e^{-\theta t}$ .

*Compound Binomial Process*

As a final example of the technique described in this discussion, consider a compound binomial process.

$$f(x) = \sum_{n=0}^m \binom{m}{n} q^n (1-q)^{m-n} p^{n*}(x). \quad (20)$$

In equation (20),  $m$  is a positive integer and  $q$  is a real number satisfying  $0 < q < 1$ .

The following equations are analogous to those derived previously.

$$\frac{d\bar{f}}{dk} = \left( \frac{q}{1-q} \right) \left[ m\bar{f}(k) \frac{d\bar{p}}{dk} - \bar{p}(k) \frac{d\bar{f}}{dk} \right], \quad (21)$$

$$xf(x) = \left( \frac{q}{1-q} \right) \int_0^x [(m+1)y - x] p(y) f(x-y) dy, \quad (22)$$

$$if_i = \left( \frac{q}{1-q} \right) \sum_{j=1}^i [(m+1)j - i] p_j f_{i-j}, \quad i \geq 1. \quad (23)$$

The initial condition is  $f_0 = (1-q)^m$ . The results for the compound Poisson process are recovered in the limit  $q \rightarrow 0$ ,  $m \rightarrow \infty$ , with  $mq = \theta t$ , a constant.

*Acknowledgment*

I would like to thank Mr. Walter B. Lowrie for reading an advance copy of this discussion and directing me to Mr. Ethan Stroh's paper "Actuarial Note: The Distribution Functions of Collective Risk Theory as Linear Compounds," published in *ARCH*, 1978.1. Mr. Stroh used discrete Laplace transforms directly to derive the recursion formulas for both the Poisson and the negative binomial distributions. In light of the independent derivations of the Poisson formula by Messrs. Stroh and Williams and Dr. Panjer, one can only wonder how many other actuaries have also discovered it.

I decided to offer my discussion of Dr. Panjer's paper, not because my derivations are original but because Mr. Stroh's valuable work has apparently missed the attention of many actuaries. It is unfortunate that *ARCH*, which was intended originally to be a vehicle for the rapid dissemination of current actuarial research, has not fulfilled its purpose recently and is not as widely read as it should be, even by research-minded actuaries.

## (AUTHOR'S REVIEW OF DISCUSSION)

HARRY H. PANJER:

Both discussions give alternate derivations of the recursive formula for the distribution of aggregate claims for the compound Poisson model. Drs. Bühlmann and Gerber provide a very elegant constructive proof to obtain the recursion. Dr. Tilley uses Laplace transforms to obtain the result. In addition, he uses the same method to obtain recursions for the compound negative binomial and the compound binomial distributions. He also provides a very lucid exposition on the importance of the negative binomial distribution.

I would like to thank the discussants for their important additions to this paper, and Dr. Tilley for the additional reference to Mr. Stroh's paper.

