THE COST OF MISMATCH IN STOCHASTIC INTEREST RATE MODELS

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1. Introduction

Evaluation of assets and liabilities of insurance companies is heavily dependent on interest rates. Actuaries should take that dependence into account. One commonly used measure is the "cost of mismatch". Roughly speaking, this cost is the difference in present value at evaluation time between the net cash flows (assets minus liabilities) discounted with a plausible ("base case") interest rate scenario and the net cash flows discounted with a different ("worst case") scenario.

Traditionally, this cost is computed by using scenario testing. This is, for instance, recommended by the Cash Flow Valuation Method (CFVM) of the Valuation Technique Paper N^{o} 9 (VTP9), published by the Canadian Institute of Actuaries. A few scenarios are usually considered: typically, scenarios where interest rates pop up, pop down, increase then decrease, or decrease then increase.

In a stochastic world consideration of only a few deterministic scenarios can potentially be dangerous. The aim of the present research is to study the computation of the cost of mismatch using stochastic interest rate models instead of scenario testing. As far as we know, there has not been any justification (apart from common sense) that scenario testing gives a sound answer. Comparing stochastic results with scenario testing results in some simple cases would either give more weight to or destroy the relevance of the scenario testing approach.

In this paper, two commonly used stochastic interest rate models are considered: the one introduced by Vasicek (1977) and the one proposed by Cox, Ingersoll, and Ross (1985). Insurance cash flows are assumed to be independent of the level of interest rates, but affected by a random mortality. The cost of mismatch is redefined as the difference between the present value under the "base case" scenario and some percentile of the present value distribution. This approach has much in common with the concept of "value at risk", as pointed out by Brender (1999).

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The study of solvency testing under stochastic interest and mortality has been drawing attention for some time. Recent work on the subject has been contributed, among others, by Beekman and Fuelling (1993, 1991, 1990), Borowiak (1999), Bruno, Camerini and Tomassetti (2000), Bühlmann (1992), Frees (1990), Levikson and Yosef (2001), Norberg and Møller (1996), and Parker (1997, 1994a, 1994b, 1994c, 1994d, 1993).

The first half of this paper (Sections 2, 3, and 4) summarizes the CFVM, introduces a stochastic approach, and compares the two methods. The second half of the paper (Sections 5 and 6) examines the stochastic approach in more detail. Approximations (which become exact when the same interest rate is used for both lending and borrowing) are obtained for the first three moments of the future and present values of the portfolio of life insurance contracts. An approximation for the distributions of the future and present values is also proposed.

2. General Framework

Consider a portfolio of life insurance contracts. For $i = 0, \ldots, n$, let X_i be the **cash flow** occurring at time t_i generated by this portfolio, where $0 = t_0 < t_1 < \cdots < t_n$. Let us also introduce the three continuous-time processes $R^0 = \{R^0(t); 0 \leq t \leq t_n\}$, $R^1 = \{R^1(t); 0 \leq t \leq t_n\}$, and $R = \{R(t); 0 \leq t \leq t_n\}$, to be interpreted as follows.

- $R^0(t)$ represents the instantaneous **rate of lending** of the life-insurance company on its investments at time t.
- $R^1(t)$ stands for the instantaneous **rate of borrowing** at time t. In this paper, it is simply assumed that $R^1(t) = R^0(t) + \delta$ for some non-negative constant δ.
- $R(t)$ represents the rate actually used in the accumulation process, depending on whether the company is in a lending or in a borrowing situation at time t. A more precise definition of the process R will be given shortly.

To simplify notation, let us introduce for $0 \leq i \leq j \leq n$ the **accumulation factors**

$$A_{i,j} = \exp\left[\int_{t_i}^{t_j} R(t)\, dt\right]$$

and the **discount factors**

$$V_{i,j} = \exp\left[-\int_{t_i}^{t_j} R(t)\, dt\right]$$

associated with the process R. We will also use the notation $A^0_{i,j}$ and $V^0_{i,j}$ for the accumulation and discount factors associated in the obvious way with the process R^0.

Let $S_i(X, R)$ be the **surplus** at time t_i just **after** the occurrence of the cash flow X_i. The initial surplus is simply

$$S_0(X, R) = X_0.$$

From then, positive surplus are accumulated onto the next period using the lending rate R^0, while negative surplus are accumulated using the borrowing rate R^1. That is, for $i = 1, \ldots, n$

we let

$$R(t) = \begin{cases} R^0(t) & \text{if } S_{i-1}(X,R) \geq 0, \\ R^1(t) & \text{if } S_{i-1}(X,R) < 0, \end{cases} \quad \text{over } t_{i-1} \leq t < t_i \tag{2.1}$$

and

$$S_i(X,R) = S_{i-1}(X,R) A_{i-1,i} + X_i. \tag{2.2}$$

Note that it is also possible to express $S_i(X,R)$ as

$$S_i(X,R) = \sum_{k=0}^{i} X_k A_{k,i}. \tag{2.3}$$

The very last surplus $S_n(X,R)$ represents the **future value** of the cash flows X under the interest rate R; we will note this quantity by $F(X,R)$. The **present value** of the same cash flows, noted by $P(X,R)$, will be defined later. The two processes X and R^0 could either be deterministic (as in the scenario testing approach of the CFVM), or random (as examined later in this paper).

The CFVM is essentially a present value computation and the cost of mismatch, a measure of sensitivity to interest rates, different from assumptions. Comparing what happens when we move from the deterministic framework of the CFVM to a stochastic framework should give an idea of the relevance of scenario testing.

We are not going to look at examples with interest sensitive cash flows in this paper since we want to examine the relevance of scenario testing even in situations where interest rate fluctuations of the cash flows is a priori not the prime cause of problems.

Clearly, if cash flows are entirely deterministic, perfect hedging is theoretically possible (for instance, with bonds of appropriate maturities) and the computation of the cost of mismatch is useless. We propose to look at examples where mortality is the sole other source of randomness. Even in this simple situation, we will show that scenario testing might give a false sense of security.

VTP9 recommends that " ... *positive cash flows might be assumed to be reinvested for a period up to the next negative net cash flow. On the other hand, negative net cash flows may require borrowing or the sale of assets. The reinvestment/disinvestment practice should be consistent with the company's expected investment practice.*"

Instead of fixing a well-defined and elaborate investment strategy involving assets of different maturities, we take in (2.2) a somewhat macroscopic view and model the global rate of return of the company by the short term rate R. This short term rate is then used as suggested in VTP9.

The rationale behind this approach is that the cost of mismatch is a macroscopic measure *per se*. At a microscopic (i.e. product specific) level, interest rate risk should be taken into account at pricing. Pricing in turn should incorporate evaluation of the assets supporting the liabilities, including interest rate fluctuations. The whole idea of the cost of mismatch is somehow to perform the microscopic analysis with simplified assumptions (a base case scenario) and to take care of interest rate risk afterwards, i.e. at a macroscopic level. So we assume for convenience that the global rate of return of the company is described by a short rate model. This rate of return is used as the investment rate R^0 for positive surplus and, together with the borrowing premium δ, as the borrowing rate R^1 for negative surplus.

One of the advantages of this relatively simple framework is its tractability. Several analytic results will be derived in Sections 5 and 6. This would be much more difficult if cash flows were interest rate dependent, or if an investment strategy involving assets of different maturities was considered (requiring a term structure model of interest rates instead of a short term model).

3. The Cash Flow Valuation Method

We summarize in this section the main points of the CFVM. The interested reader is invited to consult the reference paper VTP9 for a more complete exposition.

3.1. **Interest rate scenarios.** In scenario testing we change the interest rate scenarios for R^0 in order to study the impact on present value. The technical paper VTP9 suggests to consider 4 types of scenarios, in addition to the base case:

- POP-UP: interest rates grow higher than in the base case in a relatively short period of time and stay higher than in the base case for the rest of the study period.
- POP-DOWN: interest rates fall lower than in the base case in a relatively short period of time and stay lower than in the base case for the rest of the study period.
- UP-DOWN: interest rates grow higher than in the base case in a relatively short period of time, stay higher than in the base case for some time, then fall down lower than in the base case and stay lower than in the base case for the rest of the study period.
- DOWN-UP: interest rates fall lower than in the base case in a relatively short period of time, stay lower than in the base case for some time, then grow higher than in the base case and stay higher than in the base case for the rest of the study period.

The paper also recommends to take fluctuations of at least 3%.

3.2. **Cash flows.** The CFVM suggests to construct a deterministic cash flow process X for each scenario, since insurance cash flows are potentially dependent on the level of interest rates. As explained in the previous section, the aim of the present paper is only to compare the CFVM against a stochastic approach under interest rate independent cash flows. We will hence use the same cash flow process X for every interest rate scenario.

3.3. **Present value.** Let us introduce the modified cash flow process $X^{\oplus x} = \left\{X_i^{\oplus x}; i = 0, \ldots, n\right\}$, defined by

$$X_i^{\oplus x} = \begin{cases} X_0 + x & \text{for } i = 0; \\ X_i & \text{for } i = 1, \ldots, n. \end{cases}$$

In words, the modified cash flows $X^{\oplus x}$ are the same as the original cash flows X except for an additional amount of x units at time 0. The present value, as defined by the CFVM, is

$$P^{\text{CFVM}}(X,R) = \frac{F(X,R)}{F(X^{\oplus 1}, R^{\oplus 1}) - F(X,R)}, \tag{3.1}$$

where R is the interest rate process used to accumulate the cash flows X and $R^{\oplus 1}$, the one used to accumulate the modified cash flows $X^{\oplus 1}$ (because of the initial additional unit, borrowing may occur over different periods).

The idea behind this way of discounting is as follows. The difference $F(X^{\oplus 1}, R^{\oplus 1}) - F(X,R)$ represents the future value at time t_n of the initial additional unit. The inverse of this difference is then used as a "discount factor" to find the present value at time 0 of the future value $F(X,R)$.

By successively letting R^0 be each one of the interest rate scenario, the corresponding present value is computed using (2.1), (2.2) and (3.1). We will note by P_k^{CFVM} the present value computed under scenario k.

3.4. **Cost of mismatch.** Let us denote the CFVM present value computed according to the base case scenario by $P_{\text{BASE}}^{\text{CFVM}}$. This quantity is of particular importance: it represents the targeted profit of the insurance company. The difference $C_k^{\text{CFVM}} = P_{\text{BASE}}^{\text{CFVM}} - P_k^{\text{CFVM}}$ is thus the amount in current dollars missing to attain the expected situation if scenario k gets realized instead of the base case. This missing amount C_k^{CFVM} is taken as the measure of the risk associated to scenario k. The cost of mismatch is defined as the missing amount in the worst case: $C^{\text{CFVM}} = \max\left\{C_k^{\text{CFVM}}\right\}$.

3.5. **Numerical examples.** For all numerical examples in this paper, we will assume a borrowing premium of $\delta = 0.02$ and base the mortality on the life table N° 305 available from the Society of Actuaries web site (`www.soa.org/tablemgr/tablemgr.asp`).

As a first example, consider a 10 year term life insurance contract, with a monthly (gross) premium of \$15 and a benefit of \$100 000 payable at the end of the month of death. Let us call X_{RAN1} the (random) cash flow process generated by a portfolio of 1 000 such contracts, on 1 000 independent lives of age 30. Since the CFVM deals with deterministic cash flows, let us replace each random cash flow by its expectation, and call X_{DET1} the resulting (deterministic) cash flow process. Equivalently, we could assume that mortality risk pooling eliminates fluctuations in the mean.

For the BASE scenario, we take a constant rate of 6% over the 10 year period. For the four other scenarios, we take fluctuations of 3%, as suggested by VTP9, and assume that the increase or decrease happen in 6 months. For the UP-DOWN and DOWN-UP scenarios, we divide the whole period equally between the up and down parts. Figure 3.1 illustrates all those scenarios and Table 3.1 presents the future and present values computed under each scenario. (The reader should not pay attention for now to the last row in Table 3.1.) The cost of mismatch turns out to be \$11 400, about 11% of the expected profit, coming up from the difference between the present value under the BASE scenario and the present value under the POP-DOWN scenario.

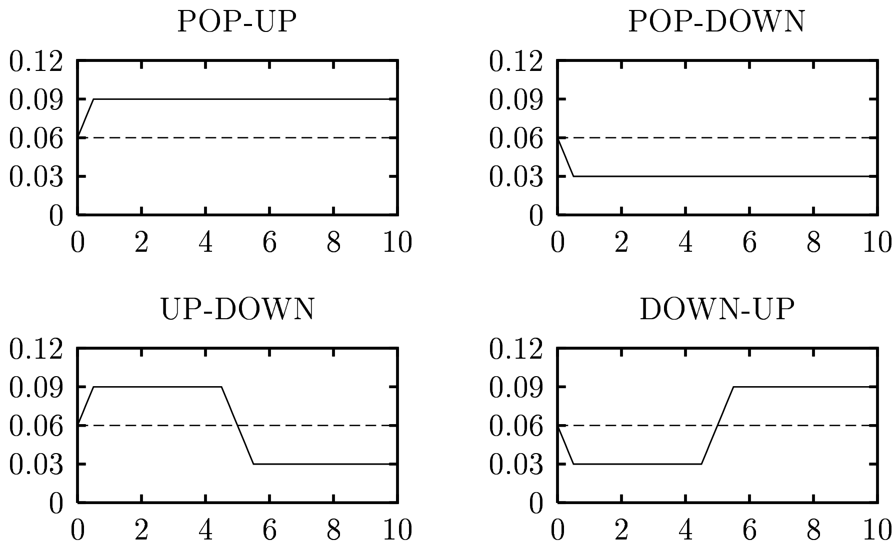

FIGURE 3.1. The four alternate scenarios (solid line) along with the base scenario (dotted line).

	BASE	POP-UP	POP-DOWN	UP-DOWN	DOWN-UP
$F(X_{\text{DET1}}, R)$	183 790	264 439	121 676	170 871	198 033
$P^{\text{CFVM}}(X_{\text{DET1}}, R)$	100 866	108 322	89 466	94 482	107 871
$P(X_{\text{DET1}}, R)$	99 192	106 293	88 231	93 035	105 931

TABLE 3.1. CFVM analysis of the cash flows X_{DET1}.

As a second example, consider a 30 year temporary life annuity contract providing monthly payments of \$1 000 for an initial (gross) cost of \$150 000. Let us examine a portfolio of 1 000 such annuities, on 1 000 independent lives of age 60. We will call X_{RAN2} the resulting (random) cash flow process, and X_{DET2} the deterministic process obtained by replacing each cash flow by its expectation.

The five CFVM interest scenarios are constructed in the same way as in the first example, but over a period of 30 years instead of 10 years. Table 3.2 displays the results of the CFVM analysis. This time, the cost of mismatch turns out to be \$30 423 990, about 82% of the expected profit.

4. Stochastic Approach

Let us point out a few shortcomings of the CFVM.

- While VTP9 does offer some guidelines, the number of scenarios, and their specific construction, is left at the discretion of the valuation actuary. This does not provide an objective measure of risk.

	BASE	POP-UP	POP-DOWN	UP-DOWN	DOWN-UP
$F(X_{\mathrm{DET2}}, R)$	2.239×10^8	8.553×10^8	1.631×10^7	3.320×10^8	7.872×10^7
$P^{\mathrm{CFVM}}(X_{\mathrm{DET2}}, R)$	3.700×10^7	5.792×10^7	6.579×10^6	5.530×10^7	1.292×10^7
$P(X_{\mathrm{DET2}}, R)$	3.700×10^7	5.792×10^7	6.579×10^6	5.530×10^7	1.292×10^7

TABLE 3.2. CFVM analysis of the cash flows X_{DET2}.

- One of the aims of the CFVM is to measure the interest rate risk, in order to establish a provision for adverse interest rate fluctuations. In life insurance, another provision is generally made for adverse mortality. However, stochastic interest rate and mortality could interact in complex ways, which cannot be captured by two separate provisions. It would be preferable to take into account the random nature of mortality and interest rate at the same time, and establish a single provision which encompasses both.
- The present value is defined in a peculiar way. If we use, say, \$1 000 as our monetary unit instead of \$1, we may very well obtain a different present value, since borrowing could occur over different periods in definition (3.1). This lack of numeraire invariance is not very satisfactory.

The purpose of the stochastic approach, presented in this section, is to examine the objectivity of the scenario approach. Also, we propose an alternate (numeraire invariant) definition of the present value.

4.1. **Interest rate process.** Two well-known stochastic models for interest rates are the one introduced by Vasicek (1977),

$$dR^0(t) = \kappa\left[\theta - R^0(t)\right]dt + \sigma dW(t) \tag{4.1}$$

and the one introduced by Cox, Ingersoll, and Ross (1985),

$$dR^0(t) = \kappa\left[\theta - R^0(t)\right]dt + \sigma\sqrt{R^0(t)}\,dW(t)\,. \tag{4.2}$$

These two stochastic differential equations are generally used to model the short term rate in the market, and their parameters estimated accordingly. However, in this case, the parameters should probably not be estimated from historical values of the short term rate. Doing so would suppose the company is constantly “rolling over” all its assets in short term instruments (such as 30-day T-bills), a rather naive investment strategy. Instead, the parameters should be consistent with the internal rate of return of the insurance company. Also, the parameters should describe the model under the real “physical” probability measure, not under a martingale “risk-neutral” probability measure, since we are concerned with solvency testing, not pricing.

More sophisticated interest rate models could be used, for instance term-structure models such as Heath, Jarrow, and Morton (1992), but this lies beyond the scope of the present paper.

4.2. **Cash flow process.** Let $X^{(1)} = \left\{X_i^{(1)}; i = 0, \ldots, n\right\}$ be the cash flow process generated by a **single** life-insurance contract. We will assume that each random variable $X_i^{(1)}$ could be expressed as

$$X_i^{(1)} = x_i + y_i I_{\{t_i < T\}} + z_i I_{\{t_{i-1} < T \leq t_i\}}, \tag{4.3}$$

where T is the future lifetime of the insured and $I_{\{E\}}$ is the indicator function of the event E. The quantities x_i, y_i, and z_i are to be interpreted as follows.

- x_i: A base amount received by the company at time t_i, whether the insured is alive or not. This is useful for modeling deterministic cash-flows.
- y_i: An additional amount received by the company at time t_i, but only if the insured is still alive at that time. Positive values of y_i are used to model premiums and cotisations, while negative values are used for endowments and annuity benefits paid by the company to the insured.
- z_i: An additional amount received by the company at time t_i, but only if the insured dies between t_{i-1} and t_i. Negative values of z_i are used to model death benefits.

This model for $X^{(1)}$ is general enough to represent the most common insurance contracts: whole life, term, deferred, endowment, increasing and decreasing life insurances and annuities, and many more. Generating a sample path of the process $X^{(1)}$ is easy: we simply need to simulate a random draw of the future lifetime T.

We are now in position to consider a portfolio of individual contracts. The cash flow process X generated by such a **portfolio** of w independent contracts is

$$X_i = \sum_{v=1}^{w} X_i^{(v)}, \tag{4.4}$$

where $X_i^{(v)}$ is the cash flow occurring at time t_i from the v^{th} contract. We do not require those contracts to be similar. In other words, $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ and the distribution of the random variable T may differ from one contract to another.

To generate a sample path of the process X, we simulate a sample path of each process $X^{(1)}, \ldots, X^{(w)}$ and add them up according to (4.4). Thus, generating a sample path of X boils down to simulating w future lifetimes $T^{(1)}, \ldots, T^{(w)}$, one for each contract in the portfolio.

4.3. **Equilibrium present value.** We are going to view discounting in a way different from VTP9. The basic idea behind the definition of present value is the following. With a constant interest rate i, when we say that the present value of 1 unit to be cashed in n periods is $v^n = (1+i)^{-n}$, we mean that, if we borrow v^n today, we will be able to pay the debt in n periods when we cash the 1 unit, hence ending with a zero net situation at the end of the project.

Faced with uncertain investment opportunities, we can no longer discount in the usual way since we do not know when we will be in a borrowing or lending situation. The important point, however, is that we want to keep the zero net situation at the end of the project.

We thus define the present value P as the amount of money that should be subtracted today in order to be in equilibrium over the whole period under consideration, i.e.

$$F\left(X^{\oplus(-P)}, R^{\oplus(-P)}\right) = 0.$$

If X and R are stochastic, the amount P computed along a trajectory is the exact amount that we should subtract today so that we attain a zero situation at the end of the project provided that particular trajectory gets realized.

This equilibrium present value P is computed by initializing $F = 0$ and by solving recursively for the previous surplus. To simplify the discussion, let us introduce the two processes S' and R'. One should think of $-S'_i(X, R')$ as the surplus of the modified cash flows $X^{\oplus(-P)}$ just **before** the occurrence of the cash flow at time t_i, and of R' as the rate used to accumulate the modified cash flows $X^{\oplus(-P)}$. Thus,

$$S'_n(X, R') = X_n.$$

Note that, if $S'_i > 0$, the modified surplus has been negative between t_{i-1} and t_i. From then, positive values of S' are discounted onto the previous period using the borrowing rate R^1, while negative values of S' are discounted using the lending rate R^0. That is, working backward for $i = n-1, \ldots, 0$, we let

$$R'(t) = \begin{cases} R^1(t) & \text{if } S'_{i+1}(X, R') \geq 0, \\ R^0(t) & \text{if } S'_{i+1}(X, R') < 0, \end{cases} \quad \text{over } t_i \leq t < t_{i+1} \tag{4.5}$$

and

$$S'_i(X, R') = S'_{i+1}(X, R')\, V'_{i,i+1} + X_i, \tag{4.6}$$

where

$$V'_{i,j} = \exp\left[-\int_{t_i}^{t_j} R'(t)\, dt\right]$$

is the discount factor associated with R' .

Note that it is also possible to express $S'_i(X, R')$ as

$$S'_i(X, R') = \sum_{k=i}^{n} X_k V'_{i,k}. \tag{4.7}$$

The equilibrium present value $P(X, R)$ is then given by $S'_0(X, R')$.

When $\delta = 0$, that is, when lending and borrowing rates are the same, the two definitions (P^{CFVM} and P) are equivalent. However, this is no longer the case when $\delta \neq 0$. We think the equilibrium approach is a more natural definition of the present value which should be used when $\delta > 0$, instead of the CFVM approach. Moreover, the symmetry between the way the future value was defined by (2.1), (2.2) and (2.3), and the way the equilibrium present value is defined by (4.5), (4.6) and (4.7) will be exploited later.

4.4. **Cost of mismatch.** In this stochastic framework, we could simulate a whole bunch of randomly generated trajectories for X and R^0, instead of a few deterministically chosen trajectories. In scenario testing, we got a few values for C_k^{CFVM} and the cost of mismatch C^{CFVM} was chosen as the missing amount in the worst case. In stochastic models, we get a sample of values C_k and the cost of mismatch C can be defined probabilistically. We suggest to define the cost of mismatch as some (say 99^{th}) percentile of the sample distribution of the missing amount:

$$\Pr\left[P_{\mathrm{BASE}}(X,R) - P(X,R) < C\right] = 0.99.$$

Since this is the same as

$$\Pr\left[P(X,R) \leq P_{\mathrm{BASE}}(X,R) - C\right] = 0.01,$$

the cost of mismatch is $C = P_{\mathrm{BASE}}(X,R) - \alpha$, where α is the 1^{st} percentile of $P(X,R)$.

4.5. **Numerical examples.** For the remaining examples in this paper, we will take $\theta = 0.06$, $\kappa = 0.3$, and $\sigma = 0.02$ in the Vasicek model and $\theta = 0.06$, $\kappa = 0.3$, and $\sigma = 0.08$ in the CIR model. We will also set the initial value of the lending interest rate at $R^0(0) = 0.06$. With those parameters, both models give $E\left[R^0(t)\,|R^0(0)\right] = 0.06$ for every t, which is exactly the BASE scenario used in the CFVM. This allows for a relatively fair comparison between the CFVM and the stochastic approach.

Let us reconsider the random cash flow process X_{RAN1} introduced in Section 3.5. Figure 4.1 displays the distribution of the present value $P\left(X_{\mathrm{RAN1}}, R\right)$, estimated from 10 000 simulations (i.e., 10 000 trajectories of X_{RAN1}, each one based on 1 000 random death times, and 10 000 trajectories of R^0). The 1^{st} percentile of that distribution is -677 045 in the Vasicek model and -643 660 in the CIR model. Assuming the Vasicek model, the interpretation of that percentile is the following. If the company wishes to end up with a non-negative future value with a probability of 0.99, then \$677 045 needs to magically appear in its bank account today. In other words, \$677 045 is the amount missing today to be 99% sure of not going bankrupt. The same interpretation holds for the CIR model.

The reader could now look at the last row of Table 3.1, which gives the equilibrium present value $P\left(X_{\mathrm{DET1}}, R\right)$ under each CFVM scenario. For the BASE case, this present value is \$99 192. Thus, the insurance company could distribute \$99 192 today to its shareholders and still end up with a zero net situation if the realization of R^0 happens to be the BASE scenario and the realization of X_{RAN1} happens to be X_{DET1}. By analogy with the CFVM, we take this amount of \$99 192 as the targeted profit.

Again, let us concentrate on the Vasicek model for a moment. Since the 1^{st} percentile of $P\left(X_{\mathrm{RAN1}}, R\right)$ is -677 045, the cost of mismatch (with respect to the targeted profit of \$99 192) turns out to be \$776 237. Its interpretation is the following. If the company distributes \$99 192 today to its shareholders and still wishes to end up with a non-negative future value with a probability of 0.99, then \$776 237 needs to magically appear in its bank account today. In other words, \$776 237 is the amount missing today to be 99% sure of attaining the targeted profit.

Now, let us compare the scenario testing approach with the stochastic approach. The equilibrium present values of the CFVM scenarios from Table 3.1 are plotted on Figure 4.1. Obviously, they do not give a good idea of the spread of the distribution, but this is because most of the randomness in $P(X_{\mathrm{RAN1}}, R)$ comes from the mortality, not from the interest rate. This is not a defect of the CFVM: its purpose is only to measure the interest rate risk.

In order to do a reasonable comparison, let us replace X_{RAN1} by X_{DET1} in the stochastic approach. Figure 4.2 displays the distribution of the present value $P(X_{\mathrm{DET1}}, R)$, estimated from 10 000 simulations under the Vasicek and CIR models. Now that random mortality has been replaced by deterministic mortality, the distribution is much more concentrated, confirming that mortality was the main source of risk in $P(X_{\mathrm{RAN1}}, R)$. This time, the CFVM scenarios seem to give a better idea of the spread of the present value distribution: they cover the percentiles 0.072 to 0.879 in the Vasicek model, and the percentiles 0.036 to 0.876 in the CIR model.

Even with deterministic cash flows (in principle, this situation can be fully hedged) we see that scenario testing does not provide a safe answer. The scenarios give a rather good idea of the spread of the distribution, but we are still far from a 99% probability that the extra provision, represented by the cost of mismatch, is sufficient. The best we get with scenario testing is a 93% probability in the Vasicek model and a 96% probability in the CIR model. The cost of mismatch computed from scenario testing in this example thus gives a false sense of security.

Finally, let us look at the two cash flow processes X_{RAN2} and X_{DET2}. Figure 4.3 displays the distribution of the present value $P(X_{\mathrm{RAN2}}, R)$, again based on 10 000 simulations. The 1$^{\mathrm{st}}$ percentile of this distribution is 5 126 116 in the Vasicek model and 13 067 926 in the CIR model. Taking $P(X_{\mathrm{DET2}}, R)$ under the BASE scenario (see Table 3.2) as our reference, the cost of mismatch is thus \$31 877 284 in the Vasicek case, and \$23 935 474 in the CIR case.

Figure 4.4 presents the distribution of the present value $P(X_{\mathrm{DET2}}, R)$. Note that there is almost no difference between Figures 4.3 and 4.4. Thus, mortality is not an important source of risk in $P(X_{\mathrm{RAN2}}, R)$. Again, the CFVM scenarios furnish a good idea of the spread of the distribution of $P(X_{\mathrm{DET2}}, R)$: they cover the percentiles 0.011 to 0.989 in the Vasicek model, and the percentiles 0.001 to 0.978 in the CIR model.

However, this is a significant difference from the percentiles of the previous example. Now the spread of the distribution is very well represented by the scenarios and we get a safer answer: 99% in the Vasicek model and essentially 100% in the CIR case.

We conclude that the scenario testing approach is conservative and might be appropriate in some cases, but that it does not offer a measure of risk which is totally consistent with stochastic interest rate models across different situations. It is not even reliable with simple deterministic cash flows. How could it be reliable with fully stochastic cash flows … with interest sensitive cash flows?

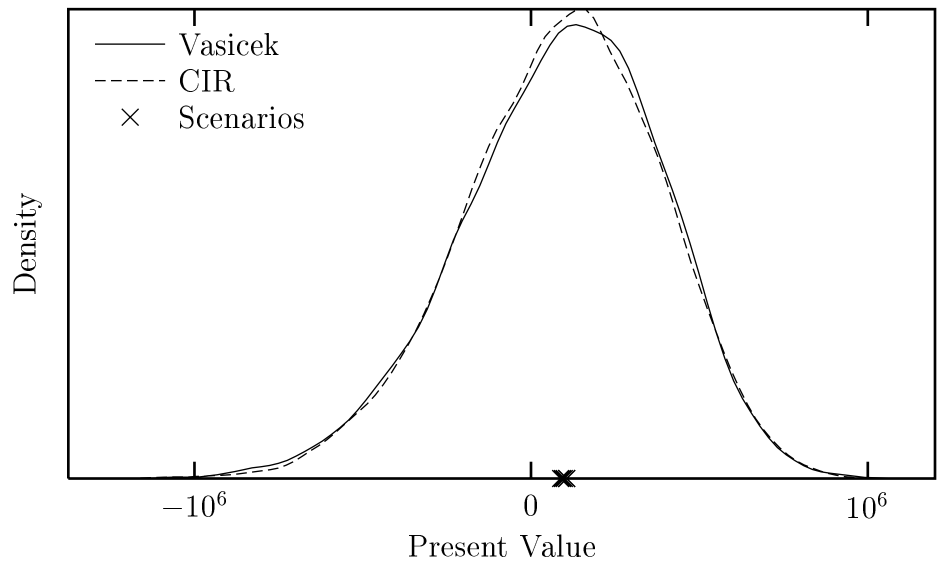

FIGURE 4.1. Distribution of $P\left(X_{\mathrm{RAN1}}, R\right)$.

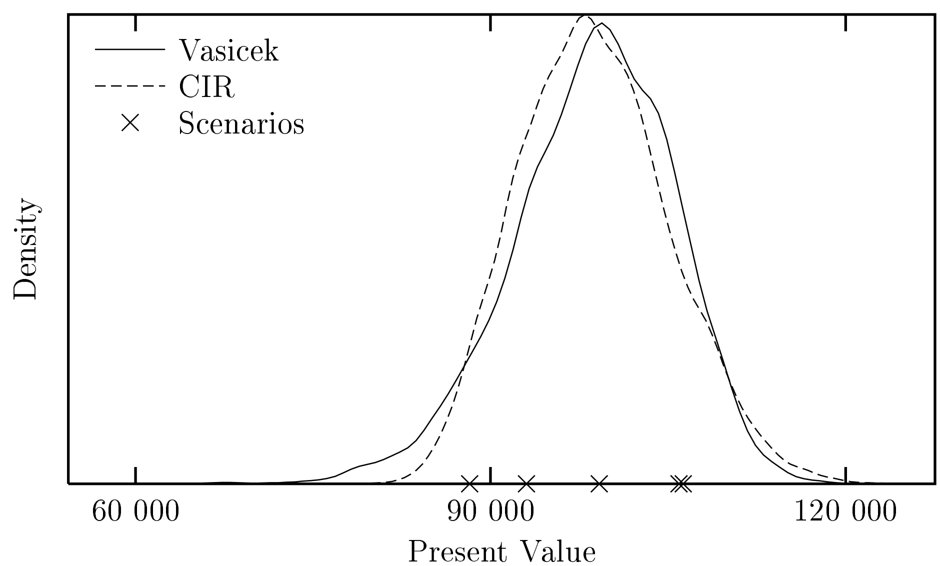

FIGURE 4.2. Distribution of $P\left(X_{\mathrm{DET1}}, R\right)$.

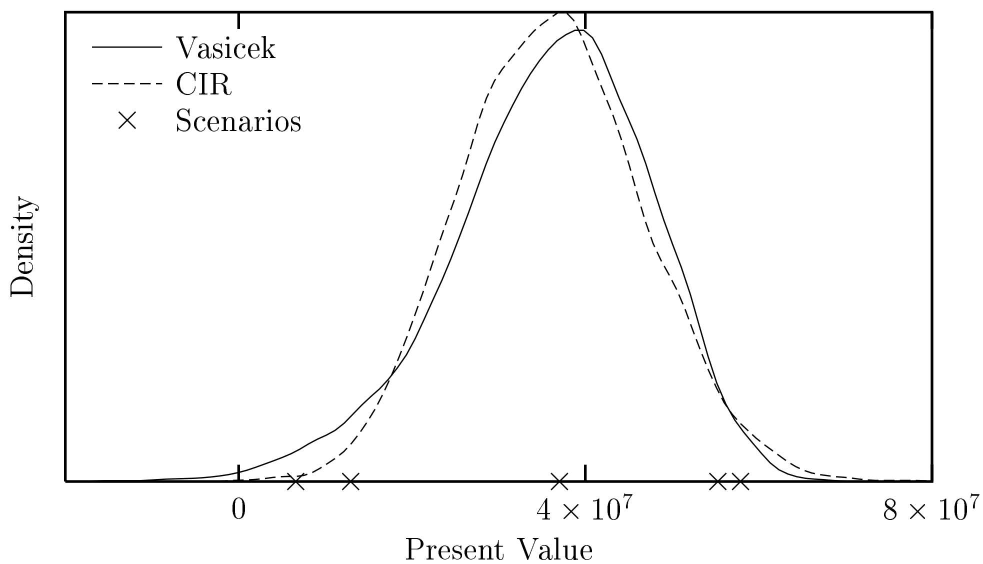

FIGURE 4.3. Distribution of $P(X_{\mathrm{RAN2}}, R)$.

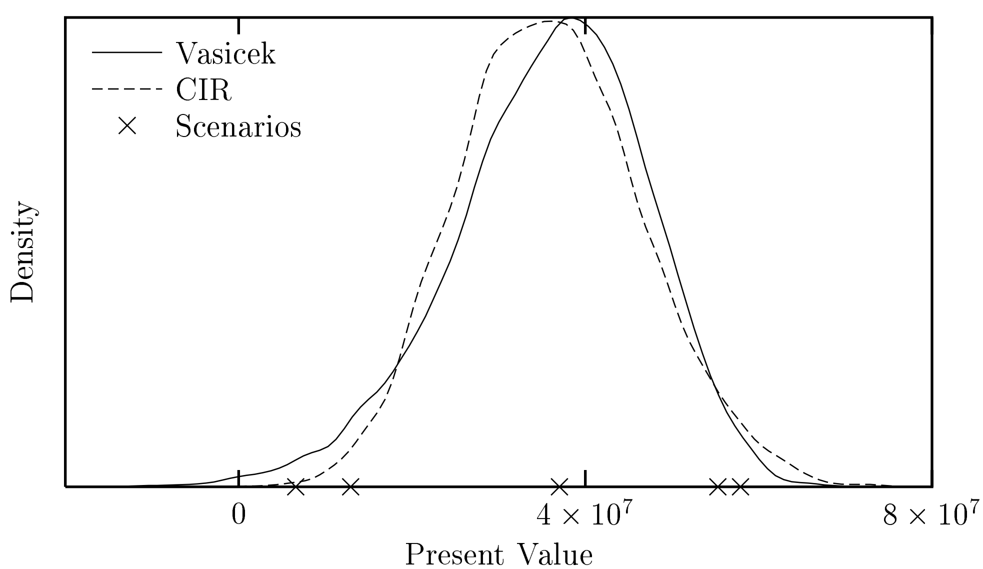

FIGURE 4.4. Distribution of $P(X_{\mathrm{DET2}}, R)$.

5. Moment Approximations

In this section, approximations are derived for the first three moments of the two random variables $F(X,R)$ and $P(X,R)$. Those approximations will be used in the next section to construct an approximation of the distribution of $F(X,R)$ and $P(X,R)$.

5.1. Moments of the interest rate process.

Let us start by presenting a useful Lemma. We prefer to omit the proof (see for instance Lamberton and Lapeyre (1996) for an idea of the techniques involved).

Lemma 5.1. *Let the process R^0 be given by the Vasicek model (4.1) or by the CIR model (4.2). Then, for $\zeta, \xi, s, u, v \in \mathbb{R}$ such that $s \le u \le v$,*

$$E\left[\exp\left(-\zeta R^0(v) - \xi \int_u^v R^0(t)\,dt\right) | R^0(s)\right] = \exp\left[\phi(s,u,v,\zeta,\xi) R^0(s) + \psi(s,u,v,\zeta,\xi)\right],$$

where

$$\begin{aligned}
\phi(s,u,v,\zeta,\xi) &= \left(\frac{\xi}{\kappa} - \zeta\right) e^{-\kappa(v-s)} - \frac{\xi}{\kappa} e^{-\kappa(u-s)}, \\
\psi(s,u,v,\zeta,\xi) &= \frac{\sigma^2}{4\kappa^3} \begin{pmatrix} 2\xi(\xi - \zeta\kappa)\left[e^{-\kappa(v+u-2s)} + e^{-\kappa(v-u)}\right] \\ -(\xi-\zeta\kappa)^2 e^{-2\kappa(v-s)} + (\zeta\kappa+\xi)^2 \\ -\xi^2\left[e^{2\kappa(s-u)} - 2\kappa(v-u) + 3\right] \end{pmatrix} \\
&\quad + \theta\left[e^{-\kappa(v-s)}\left(\zeta - \frac{\xi}{\kappa}\right) + \frac{\xi}{\kappa} e^{-\kappa(u-s)} - \zeta - \xi(v-u)\right]
\end{aligned}$$

in the Vasicek model and

$$\begin{aligned}
\phi(s,u,v,\zeta,\xi) &= -\frac{2\kappa\lambda e^{-\kappa(u-s)}}{\lambda\sigma^2\left[1 - e^{-\kappa(u-s)}\right] + 2\kappa}, \\
\psi(s,u,v,\zeta,\xi) &= -\eta\theta\kappa - \frac{2\theta\kappa}{\sigma^2} \ln\left[\frac{\lambda\sigma^2\left[1 - e^{-\kappa(u-s)}\right] + 2\kappa}{2\kappa}\right], \\
\eta &= -\frac{2}{\sigma^2} \ln\left(\frac{2\gamma e^{\frac{\gamma+\kappa}{2}(v-u)}}{\zeta\sigma^2\left[e^{\zeta(v-u)} - 1\right] + (\gamma+\kappa) e^{\gamma(v-u)} + \gamma - \kappa}\right), \\
\lambda &= \frac{\zeta\left[(\gamma+\kappa) + (\gamma-\kappa) e^{\gamma(v-u)}\right] + 2\xi\left[e^{\gamma(v-u)} - 1\right]}{\zeta\sigma^2\left[e^{\zeta(v-u)} - 1\right] + (\gamma+\kappa) e^{\gamma(v-u)} + \gamma - \kappa}, \\
\gamma &= \sqrt{\kappa^2 + 2\sigma^2\xi}
\end{aligned}$$

in the CIR model.

Now, for $0 \leq h \leq i \leq j \leq k \leq m \leq n$, let us define the joint moments of the accumulation factors $A_{i,j}^0$,

$$\begin{aligned}
\alpha_{i,m} &= E\left[A_{i,m}^0 | R^0(t_0)\right], \\
\alpha_{i,j,m} &= E\left[A_{i,m}^0 A_{j,m}^0 | R^0(t_0)\right], \\
\alpha_{i,j,k,m} &= E\left[A_{i,m}^0 A_{j,m}^0 A_{k,m}^0 | R^0(t_0)\right],
\end{aligned}$$

and the joint moments of the discount factors $V_{i,j}^0$,

$$\begin{aligned}
\nu_{h,i} &= E\left[V_{h,i}^0 | R^0(t_0)\right], \\
\nu_{h,i,j} &= E\left[V_{h,i}^0 V_{h,j}^0 | R^0(t_0)\right], \\
\nu_{h,i,j,k} &= E\left[V_{h,i}^0 V_{h,j}^0 V_{h,k}^0 | R^0(t_0)\right].
\end{aligned}$$

Those joint moments are easily computed using the previous Lemma, as the next theorem shows.

Theorem 5.2. *(Joint moments of the accumulation and discount factors.) We have,*

$$\begin{aligned}
\alpha_{i,m} &= \exp\left[\phi_1 R^0(t_0) + \psi_1\right], \\
\alpha_{i,j,m} &= \exp\left[\phi_3 R^0(t_0) + \psi_2 + \psi_3\right], \\
\alpha_{i,j,k,m} &= \exp\left[\phi_6 R^0(t_0) + \psi_4 + \psi_5 + \psi_6\right],
\end{aligned}$$

with

$$\begin{aligned}
&\phi_1 = \phi(t_0, t_i, t_m, 0, -1), && \psi_1 = \psi(t_0, t_i, t_m, 0, -1), \\
&\phi_2 = \phi(t_j, t_j, t_m, 0, -2), && \psi_2 = \psi(t_j, t_j, t_m, 0, -2), \\
&\phi_3 = \phi(t_0, t_i, t_j, \phi_2, -1), && \psi_3 = \psi(t_0, t_i, t_j, \phi_2, -1), \\
&\phi_4 = \phi(t_j, t_j, t_k, 0, -3), && \psi_4 = \psi(t_j, t_j, t_k, 0, -3), \\
&\phi_5 = \phi(t_i, t_i, t_j, \phi_4, -2), && \psi_5 = \psi(t_i, t_i, t_j, \phi_4, -2), \\
&\phi_6 = \phi(t_0, t_i, t_j, \phi_5, -1), && \psi_6 = \psi(t_0, t_i, t_j, \phi_5, -1).
\end{aligned}$$

Also,

$$\begin{aligned}
\nu_{h,i} &= \exp\left[\phi_1' R^0(t_0) + \psi_1'\right], \\
\nu_{h,i,j} &= \exp\left[\phi_3' R^0(t_0) + \psi_2' + \psi_3'\right], \\
\nu_{h,i,j,k} &= \exp\left[\phi_6' R^0(t_0) + \psi_4' + \psi_5' + \psi_6'\right],
\end{aligned}$$

with

$$\begin{aligned}
&\phi_1' = \phi(t_0, t_h, t_i, 0, 1), && \psi_1' = \psi(t_0, t_h, t_i, 0, 1), \\
&\phi_2' = \phi(t_i, t_i, t_j, 0, 1), && \psi_2' = \psi(t_i, t_i, t_j, 0, 1), \\
&\phi_3' = \phi(t_0, t_h, t_i, -\phi_2', 2), && \psi_3' = \psi(t_0, t_h, t_i, -\phi_2', 2), \\
&\phi_4' = \phi(t_j, t_j, t_k, 0, 1), && \psi_4' = \psi(t_j, t_j, t_k, 0, 1), \\
&\phi_5' = \phi(t_i, t_i, t_j, -\phi_4', 2), && \psi_5' = \psi(t_i, t_i, t_j, -\phi_4', 2), \\
&\phi_6' = \phi(t_0, t_h, t_i, -\phi_5', 3), && \psi_6 = \psi(t_0, t_h, t_i, -\phi_5', 3).
\end{aligned}$$

The functions ϕ and ψ are given by Lemma 5.1, depending on the model (Vasicek or CIR).

Proof. We prove only the result for $\alpha_{i,j,m}$; the other proofs are similar. Since $t_0 \leq t_i \leq t_j \leq t_m$, we could rewrite $\alpha_{i,j,m}$ as

$$\alpha_{i,j,m} = E\left[A^0_{i,j}\left(A^0_{j,m}\right)^2 |R^0(t_0)\right].$$

Then, by the "tower property" of conditional expectation,

$$\alpha_{i,j,m} = E\left[A^0_{i,j}E\left[\left(A^0_{j,m}\right)^2 |R^0(t_j)\right]|R^0(t_0)\right].$$

Finally, using Lemma 5.1 twice, we get

$$\begin{aligned}\alpha_{i,j,m} &= E\left[A^0_{i,j}\exp\left(\phi_2 R^0(t_j)+\psi_2\right)|R^0(t_0)\right]\\ &= \exp(\psi_2)\,E\left[A^0_{i,j}\exp\left(\phi_2 R^0(t_j)\right)|R^0(t_0)\right]\\ &= \exp\left[\phi_3 R^0(t_0)+\psi_2+\psi_3\right].\end{aligned}$$

□

Note that the first three moments, in the form given by the Theorem, follow a straightforward pattern. Higher moments follow the same pattern and could thus be found with very little effort, but are not needed in this paper.

5.2. **Moments of the cash flow process.** Let us start by computing moments of the cash flows resulting from a single contract. To simplify the next formulas, we will depart from standard actuarial notation and define

$$\begin{aligned}p_i &= \Pr[t_i < T] \quad \text{for } i=0,\ldots,n;\\ q_i &= \begin{cases}0 & \text{for } i=0,\\ \Pr[t_{i-1} < T \leq t_i] & \text{for } i=1,\ldots,n.\end{cases}\end{aligned}$$

Theorem 5.3. *(Joint moments of a single contract.) Let the process $X^{(1)}$ be defined by (4.3). Then,*

$$E\left[X^{(1)}_i\right] = x_i + p_i y_i + q_i z_i;$$

also

$$E\left[X^{(1)}_i X^{(1)}_j\right] = \begin{cases} q_j y_i z_j + c_{ij} & \text{for } i<j,\\ q_i z_i^2 + c_{ij} & \text{for } i=j,\end{cases}$$

$$c_{ij} = p_i y_i x_j + p_j y_j (y_i + x_i) + q_i z_i x_j + q_j x_i z_j + x_i x_j;$$

and

$$E\left[X^{(1)}_i X^{(1)}_j X^{(1)}_k\right] = \begin{cases} q_j y_i z_j x_k + q_k z_k (y_i y_j + y_i x_j + x_i y_j) + c'_{ijk} & \text{for } i<j<k,\\ q_j z_j (y_i z_j + 2y_i x_j + x_i z_j) + c'_{ijk} & \text{for } i<j=k,\\ q_i z_i^2 x_k + q_k y_i z_k (y_i + 2x_i) + c'_{ijk} & \text{for } i=j<k,\\ q_i z_i^2 (z_i + 3x_i) + c'_{ijk} & \text{for } i=j=k,\end{cases}$$

$$
\begin{aligned}
c'_{ijk} &= p_i y_i x_j x_k + p_j y_j x_k (y_i + x_i) + p_k y_k (y_i + x_i)(y_j + x_j) \\
&\quad + q_i z_i x_j x_k + q_j x_i z_j x_k + q_k x_i x_j z_k + x_i x_j x_k .
\end{aligned}
$$

We prefer to omit the proof, since it is straightforward but rather long. Performing standard manipulations on sums, we could now obtain the moments of the cash flows for our portfolio of individual contracts.

Theorem 5.4. *(Joint moments of a portfolio.) Let the process X be defined by (4.4). Then,*

$$
\begin{aligned}
E[X_i] &= \sum_{v=1}^{w} E\left[X_i^{(v)}\right], \\
E[X_i X_j] &= \sum_{v=1}^{w} \left(E\left[X_i^{(v)} X_j^{(v)}\right] - E\left[X_i^{(v)}\right] E\left[X_j^{(v)}\right] \right) + E[X_i] E[X_j], \\
E[X_i X_j X_k] &= \sum_{v=1}^{w} \left(\begin{array}{c} E\left[X_i^{(v)} X_j^{(v)} X_k^{(v)}\right] - E\left[X_i^{(v)} X_j^{(v)}\right] E\left[X_k^{(v)}\right] \\ -E\left[X_i^{(v)} X_k^{(v)}\right] E\left[X_j^{(v)}\right] - E\left[X_j^{(v)} X_k^{(v)}\right] E\left[X_i^{(v)}\right] \\ +2E\left[X_i^{(v)}\right] E\left[X_j^{(v)}\right] E\left[X_k^{(v)}\right] \end{array} \right) \\
&\quad + E[X_i] \, Cov[X_j, X_k] + E[X_j] \, Cov[X_i, X_k] \\
&\quad + E[X_k] \, Cov[X_i, X_j] + E[X_i] E[X_j] E[X_k] .
\end{aligned}
$$

Moments of $X^{(v)}$ are given, for each v, by Theorem 5.3.

Let us say that two contracts are **similar** if $\{x_i\}$, $\{y_i\}$, $\{z_i\}$, $\{p_i\}$ and $\{q_i\}$ are the same for the two contracts. In this situation, Theorem 5.4 reduces to the following Corollary.

Corollary 5.5. *(Joint moments of a portfolio of similar contracts.) Let $X^{(S)}$ be a portfolio of w similar contracts. Then,*

$$
\begin{aligned}
E\left[X_i^{(S)}\right] &= wE\left[X_i^{(1)}\right], \\
E\left[X_i^{(S)} X_j^{(S)}\right] &= wE\left[X_i^{(1)} X_j^{(1)}\right] + w(w-1) E\left[X_i^{(1)}\right] E\left[X_j^{(1)}\right], \\
E\left[X_i^{(S)} X_j^{(S)} X_k^{(S)}\right] &= wE\left[X_i^{(1)} X_j^{(1)} X_k^{(1)}\right] + w(w-1) E\left[X_i^{(1)} X_j^{(1)}\right] E\left[X_k^{(1)}\right] \\
&\quad + w(w-1) E\left[X_i^{(1)} X_k^{(1)}\right] E\left[X_j^{(1)}\right] + w(w-1) E\left[X_j^{(1)} X_k^{(1)}\right] E\left[X_i^{(1)}\right] \\
&\quad + w(w-1)(w-2) E\left[X_i^{(1)}\right] E\left[X_j^{(1)}\right] E\left[X_k^{(1)}\right] .
\end{aligned}
$$

Moments of a portfolio consisting of several sub-portfolios of similar contracts (e.g. a sub-portfolio of 10 similar contracts of one type and a sub-portfolio of 20 similar contracts of another type) could be found efficiently by combining Theorem 5.4 and Corollary 5.5. We simply need to replace $X^{(v)}$ in Theorem 5.4 by $X^{(S_v)}$, the cash flow process of the v^{th} sub-portfolio. Moments of $X^{(S_v)}$ are given, for each v, by Corollary 5.5.

5.3. **Moments of the future value.** The process R, defined by (2.1), depends on the process X: for fixed paths of R^0 and R^1, different paths of X may result in different paths of R. For this reason any quantity involving R is not very tractable. For instance, moments of $A_{i,j}$ would be difficult to obtain (unlike moments of $A_{i,j}^0$), and the task would be even more challenging for moments of $F(X,R)$. Thus, although it is in theory possible to compute moments of $F(X,R)$ exactly (after a formidable amount of work), the result would probably be complex to the point of being useless, and we are going to consider approximations instead.

5.3.1. *First moment.* Let $\mathcal{B} = \{0,1\}^n$ be the set of all binary n-tuples. That is, each element b of $\mathcal{B}$ could be written as $b = (b_1, \ldots, b_n)$ with $b_i = 0$ or 1 for $i = 1, \ldots, n$. Now, for every $b \in \mathcal{B}$, let us introduce the process $R^b = \left\{R^b(t); 0 \leq t \leq t_n\right\}$ defined by

$$R^b(t) = R^{b_i}(t) \quad \text{over } \ t_{i-1} \leq t < t_i,$$

for $i = 1, \ldots, n$.

For every $b \in \mathcal{B}$, let us also define

$$\begin{aligned} A_{i,j}^b &= \exp\left[\int_{t_i}^{t_j} R^b(t)\,dt\right] \\ a_{i,j}^b &= \exp\left[\delta \sum_{k=i+1}^{j} (t_k - t_{k-1})\, b_k\right], \end{aligned}$$

with $0 \leq i \leq j \leq n$.

To clarify these definitions, fix the paths of X and R^0, and consider the binary n-tuple $B \in \mathcal{B}$ such that $R^B = R$. Because of its dependence on X and R^0, this binary n-tuple B is in fact a stochastic process, that indicates over which periods the insurance company is in a borrowing situation: when $B_i = 0$ the company is lending over the period $[t_{i-1}, t_i)$, while when $B_i = 1$, the company is borrowing.

For an arbitrary $b \in \mathcal{B}$, $A_{i,j}^b$ is the accumulation factor associated to R^b. In particular, $A_{i,j}^B = A_{i,j}$ for every $0 \leq i \leq j \leq n$, since $R^B = R$.

As for $a_{i,j}^b$, it represents the appropriate adjustment that must be applied to the lending accumulation factor $A_{i,j}^0$ in order to get the accumulation factor $A_{i,j}^b$ associated to R^b. That is, $a_{i,j}^b$ is such that $A_{i,j}^0 a_{i,j}^b = A_{i,j}^b$ for every $0 \leq i \leq j \leq n$. In particular, $A_{i,j}^0 a_{i,j}^B = A_{i,j}$ for every $0 \leq i \leq j \leq n$.

Theorem 5.6. *For every path of the processes X and R^0,*

$$F(X,R) = \min_{b \in \mathcal{B}} \left\{F\left(X, R^b\right)\right\}.$$

The proof is done by induction, after fixing the paths of X and R^0 and replacing $F(X,R)$ by $S_n(X,R)$.

Corollary 5.7. *Let k be an odd integer. Then,*

$$E\left[F^k(X,R)\right] \leq \min_{b \in \mathcal{B}} \left\{E\left[F^k\left(X, R^b\right)\right]\right\}.$$

Proof. By taking the expectation of Theorem 5.6 over all paths of X and R^0,

$$\begin{aligned} E\left[F^k(X,R)\right] &= E\left[\left(\min_{b\in\mathcal{B}}\left\{F\left(X,R^b\right)\right\}\right)^k\right] \\ &= E\left[\min_{b\in\mathcal{B}}\left\{F^k\left(X,R^b\right)\right\}\right]. \end{aligned}$$

Then, we use the fact that the expectation of the minimum of several random variables is always less than or equal to the expectation of each random variable. □

Unlike moments of $F(X,R)$, moments of $F\left(X,R^b\right)$ are easily computed thanks to the independence of X and R^b. For instance, the first moment is given by

$$\begin{aligned} E\left[F\left(X,R^b\right)\right] &= E\left[\sum_{i=0}^n X_i A_{i,n}^b\right] \\ &= \sum_{i=0}^n E[X_i]\, E\left[A_{i,n}^b\right] \\ &= \sum_{i=0}^n E[X_i]\, E\left[A_{i,n}^0\right] a_{i,n}^b \\ &= \sum_{i=0}^n E[X_i]\, \alpha_{i,n} a_{i,n}^b. \end{aligned}$$

Unfortunately, in general $E[F(X,R)]$ and $E\left[F\left(X,R^b\right)\right]$ are not the same. By Corollary 5.7, the latter overestimates the former:

$$E[F(X,R)] \le E\left[F\left(X,R^b\right)\right], \quad \text{for every } \; b\in\mathcal{B}. \tag{5.1}$$

The idea is then to find the binary n-tuple b which minimizes $E\left[F\left(X,R^b\right)\right]$, and use it to approximate $E[F(X,R)]$. Let us call $\tilde{b}$ this particular n-tuple; i.e.,

$$E\left[F\left(X,R^{\tilde{b}}\right)\right] = \min_{b\in\mathcal{B}}\left\{E\left[F\left(X,R^b\right)\right]\right\}.$$

An iterative construction of $\tilde{b}$ is given by the next theorem.

Theorem 5.8. *Let $s_0 = E[X_0]\,\alpha_{0,n}$. Then, for $i=1,\ldots,n$,*

$$\begin{aligned} \tilde{b}_i &= \begin{cases} 1 & \text{if } s_{i-1} < 0, \\ 0 & \text{otherwise,} \end{cases} \\ s_i &= s_{i-1} a_{i-1,i}^{\tilde{b}} + E[X_i]\,\alpha_{i,n}. \end{aligned}$$

The proof is done by induction on i, noting that $s_n = E\left[F\left(X,R^{\tilde{b}}\right)\right]$.

Empirical evidence suggests that the upper bound given by (5.1) using $\tilde{b}$ is rather close to the correct value of $E\left[F\left(X,R\right)\right]$. This leads us to propose the approximation

$$E\left[F\left(X,R\right)\right]\simeq E\left[F\left(X,R^{\tilde{b}}\right)\right]. \tag{5.2}$$

5.3.2. *Second and third moments.* As mentioned before, B indicates over which time periods the company is in a borrowing situation. Approximation (5.2) replaces this stochastic process B by the deterministic n-tuple $\tilde{b}$. In other words, it assumes that borrowing always occurs over the same time periods[1], regardless the paths of X and R^0. Using the same idea to approximate the second and third moment, we get

$$\begin{aligned}
E\left[F\left(X,R\right)^2\right] &\simeq \sum_{i=0}^{n} E\left[X_i^2\right]\alpha_{i,i,n}\left(a_{i,n}^{\tilde{b}}\right)^2 + 2\sum_{0\le i<j\le n} E\left[X_iX_j\right]\alpha_{i,j,n}a_{i,n}^{\tilde{b}}a_{j,n}^{\tilde{b}}.\\
E\left[F\left(X,R\right)^3\right] &\simeq \sum_{i=0}^{n} E\left[X_i^3\right]\alpha_{i,i,i,n}\left(a_{i,n}^{\tilde{b}}\right)^3 + 3\sum_{0\le i<j\le n} E\left[X_i^2X_j\right]\alpha_{i,i,j,n}\left(a_{i,n}^{\tilde{b}}\right)^2 a_{j,n}^{\tilde{b}}\\
&\quad +3\sum_{0\le i<j\le n} E\left[X_iX_j^2\right]\alpha_{i,j,j,n}a_{i,n}^{\tilde{b}}\left(a_{j,n}^{\tilde{b}}\right)^2\\
&\quad +6\sum_{0\le i<j<k\le n} E\left[X_iX_jX_k\right]\alpha_{i,j,k,n}a_{i,n}^{\tilde{b}}a_{j,n}^{\tilde{b}}a_{k,n}^{\tilde{b}}.
\end{aligned}$$

Note that when $\delta=0$, that is, when borrowing and lending rates are the same, we have

$$R^1=R^0=R=R^b \quad \text{for every } b\in\mathcal{B};$$

thus,

$$F\left(X,R\right)=F\left(X,R^b\right) \quad \text{for every } b\in\mathcal{B}.$$

In that situation, the preceding approximations for the first three moments of $F\left(X,R\right)$ become exact.

5.4. **Moments of the present value.** Recall that the future value and the equilibrium present value were defined in very similar ways. We could thus apply to the present value the ideas we just used for the future value, and derive the same kind of moment approximations. Let us simply state the result without giving any intermediate details.

[1] Interestingly, the borrowing periods indicated by $\tilde{b}$ are not necessarily those periods where the company expects to borrow money. To see this, note that whether the company expects to be in a borrowing or lending situation over $[t_{i-1},t_i)$ depends only on what happens at time t_i or before. On the other hand, the construction of $\tilde{b}_i$ by Theorem 5.8 relies on $\alpha_{i-1,n}$, which depends on $[t_{i-1},t_n]$.

Construct $\{s'_i; i = 0, \ldots, n\}$ and $\tilde{b}'_i \in \mathcal{B}$ as follows. First, let $s'_n = E[X_n]\, \nu_{0,n}$. Then, for $i = n, \ldots, 1$, let

$$\begin{aligned} \tilde{b}'_i &= \begin{cases} 0 & \text{if } s'_i < 0, \\ 1 & \text{otherwise,} \end{cases} \\ s'_{i-1} &= s'_i v^{\tilde{b}'}_{i-1,i} + E[X_{i-1}]\, \nu_{0,i-1}, \end{aligned}$$

where

$$v^b_{i,j} = \exp\left[-\delta \sum_{k=i+1}^{j} (t_k - t_{k-1})\, b_k\right].$$

The first moments of $P(X,R)$ are then approximated by

$$\begin{aligned} E[P(X,R)] &\simeq \sum_{i=0}^{n} E[X_i]\, \nu_{0,i} v^{\tilde{b}'}_{0,i}, \\ E\left[P(X,R)^2\right] &\simeq \sum_{i=0}^{n} E\left[X_i^2\right] \nu_{0,i,i} \left(v^{\tilde{b}'}_{0,i}\right)^2 + 2 \sum_{0 \le i < j \le n} E[X_i X_j]\, \nu_{0,i,j} v^{\tilde{b}'}_{0,i} v^{\tilde{b}'}_{0,j}, \\ E\left[P(X,R)^3\right] &\simeq \sum_{i=0}^{n} E\left[X_i^3\right] \nu_{0,i,i,i} \left(v^{\tilde{b}'}_{0,i}\right)^3 + 3 \sum_{0 \le i < j \le n} E\left[X_i^2 X_j\right] \nu_{0,i,i,j} \left(v^{\tilde{b}'}_{0,i}\right)^2 v^{\tilde{b}'}_{0,j} \\ &\quad + 3 \sum_{0 \le i < j \le n} E\left[X_i X_j^2\right] \nu_{0,i,j,j} v^{\tilde{b}'}_{0,i} \left(v^{\tilde{b}'}_{0,j}\right)^2 \\ &\quad + 6 \sum_{0 \le i < j < k \le n} E[X_i X_j X_k]\, \nu_{0,i,j,k} v^{\tilde{b}'}_{0,i} v^{\tilde{b}'}_{0,j} v^{\tilde{b}'}_{0,k}. \end{aligned}$$

Once again, these approximations become exact when $\delta = 0$. Otherwise, as with the future value, the first and third moments are overestimated.

5.5. **Numerical examples.** Tables 5.1 to 5.4 compare our approximations for the first three moments against 95% confidence intervals of the same moments (based on 10 000 simulations), for the cash flow processes X_{RAN1} and X_{RAN2}. Those two examples are interesting since the main source of randomness is mortality in the first case, and interest rate in the second case. Moreover, borrowing occurs quite often in the first case, in a random and unpredictable manner. On the other hand, borrowing almost never happens in the second case. This is probably why the approximations seem better for the process X_{RAN2} than for the process X_{RAN1}.

Moment	Model	Approximation	95% bilateral confidence interval
$E[F(X_{\text{RAN1}}, R)]$	Vasicek	186 786	(161 583, 184 816)
	CIR	186 768	(161 016, 183 747)
$E\left[F(X_{\text{RAN1}}, R)^2\right]$	Vasicek	3.549×10^{11}	$(3.709 \times 10^{11},\ 3.916 \times 10^{11})$
	CIR	3.552×10^{11}	$(3.560 \times 10^{11},\ 3.760 \times 10^{11})$
$E\left[F(X_{\text{RAN1}}, R)^3\right]$	Vasicek	1.480×10^{17}	$(9.937 \times 10^{16},\ 1.350 \times 10^{17})$
	CIR	1.490×10^{17}	$(1.076 \times 10^{17},\ 1.405 \times 10^{17})$

TABLE 5.1. Moments of $F(X_{\text{RAN1}}, R)$.

Moment	Model	Approximation	95% bilateral confidence interval
$E[P(X_{\text{RAN1}}, R)]$	Vasicek	98 655	(83 354, 95 059)
	CIR	98 695	(83 421, 94 904)
$E\left[P(X_{\text{RAN1}}, R)^2\right]$	Vasicek	9.822×10^{10}	$(9.462 \times 10^{10},\ 9.960 \times 10^{10})$
	CIR	9.811×10^{10}	$(9.134 \times 10^{10},\ 9.617 \times 10^{10})$
$E\left[P(X_{\text{RAN1}}, R)^3\right]$	Vasicek	2.041×10^{16}	$(1.336 \times 10^{16},\ 1.738 \times 10^{16})$
	CIR	2.043×10^{16}	$(1.452 \times 10^{16},\ 1.833 \times 10^{16})$

TABLE 5.2. Moments of $P(X_{\text{RAN1}}, R)$.

Moment	Model	Approximation	95% bilateral confidence interval
$E[F(X_{\text{RAN2}}, R)]$	Vasicek	2.464×10^{8}	$(2.437 \times 10^{8},\ 2.495 \times 10^{8})$
	CIR	2.470×10^{8}	$(2.445 \times 10^{8},\ 2.506 \times 10^{8})$
$E\left[F(X_{\text{RAN2}}, R)^2\right]$	Vasicek	8.170×10^{16}	$(8.057 \times 10^{16},\ 8.491 \times 10^{16})$
	CIR	8.594×10^{16}	$(8.303 \times 10^{16},\ 8.930 \times 10^{16})$
$E\left[F(X_{\text{RAN2}}, R)^3\right]$	Vasicek	3.426×10^{25}	$(3.360 \times 10^{25},\ 3.743 \times 10^{25})$
	CIR	4.296×10^{25}	$(3.771 \times 10^{25},\ 4.848 \times 10^{25})$

TABLE 5.3. Moments of $F(X_{\text{RAN2}}, R)$.

Moment	Model	Approximation	95% bilateral confidence interval
$E[P(X_{\text{RAN2}}, R)]$	Vasicek	3.595×10^{7}	$(3.567 \times 10^{7},\ 3.612 \times 10^{7})$
	CIR	3.603×10^{7}	$(3.590 \times 10^{7},\ 3.632 \times 10^{7})$
$E\left[P(X_{\text{RAN2}}, R)^2\right]$	Vasicek	1.419×10^{15}	$(1.404 \times 10^{15},\ 1.434 \times 10^{15})$
	CIR	1.408×10^{15}	$(1.399 \times 10^{15},\ 1.430 \times 10^{15})$
$E\left[P(X_{\text{RAN2}}, R)^3\right]$	Vasicek	5.944×10^{22}	$(5.866 \times 10^{22},\ 6.045 \times 10^{22})$
	CIR	5.881×10^{22}	$(5.821 \times 10^{22},\ 6.014 \times 10^{22})$

TABLE 5.4. Moments of $P(X_{\text{RAN2}}, R)$.

6. Distribution Approximation

Assume $\delta = 0$ and consider the following two cases.

- Let R^0 be given by the Vasicek model and let the cash flow process X be as simple as possible: a single deterministic cash flow. Then, $F(X, R)$ and $P(X, R)$ both follow a log-normal distribution.
- Let X be a portfolio of many similar contracts and let the interest process R^0 be as simple as possible: a deterministic scenario. Then, $F(X, R)$ and $P(X, R)$ both follow a normal distribution.

These two examples represent two very simple but somewhat opposite situations. Hence, if we want to approximate the distribution of $F(X, R)$ and $P(X, R)$ by means of an analytic distribution, it seems desirable for this analytic distribution to fit well both the normal and the log-normal distributions. The translated log-normal distribution, defined below, possesses this property.

6.1. **Translated log-normal distribution.** Let $Z \sim N(\mu, \sigma^2)$ be a normal random variable with mean μ and variance σ^2, and let τ be a real constant. If

$$L = \tau + e^Z, \tag{6.1}$$

or if

$$L = \tau - e^Z, \tag{6.2}$$

then L is called a **translated log-normal** random variable. The distribution function of L is

$$F_L(x) = \begin{cases} 0 & \text{for } x \leq \tau, \\ F_Z[\ln(x - \tau)] & \text{for } x > \tau, \end{cases}$$

in the case (6.1), or

$$F_L(x) = \begin{cases} 1 - F_Z[\ln(\tau - x)] & \text{for } x \leq \tau, \\ 1 & \text{for } x > \tau, \end{cases}$$

in the case (6.2), where F_Z is the distribution function of the normal random variable Z.

The central moments of the translated log-normal distribution follow from those of the log-normal distribution:

$$\begin{aligned} m_1 &= E[L] = \tau \pm e^{\mu + \frac{1}{2}\sigma^2}, \\ m_2 &= E\left[(L - m_1)^2\right] = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right), \\ m_3 &= E\left[(L - m_1)^3\right] = \pm e^{3\mu + \frac{3}{2}\sigma^2}\left(e^{3\sigma^2} - 3e^{\sigma^2} + 2\right). \end{aligned}$$

If $\mu \to \infty$, $\sigma^2 \to 0$, and $\tau \to \mp\infty$ such that m_1 and m_2 stay the same, the translated log-normal distribution converges towards a normal distribution of mean m_1 and variance m_2. Moreover, when $\tau = 0$, the translated log-normal distribution reduces to a log-normal distribution. Hence, the normal distribution and the log-normal distribution are just two special cases of the translated log-normal distribution.

When the number of contracts is sufficiently large, a translated log-normal distribution seems to be a good fit, in most cases, for the distributions of $F(X,R)$ and $P(X,R)$. Just as the normal distribution is traditionally used to approximate the present and future values of a portfolio of many insurance contracts under deterministic interest rate, we propose to use the translated log-normal distribution in the case of stochastic interest rate.

Solving the three equations above for the parameters, we obtain the moment estimator of the translated log-normal distribution:

$$\begin{aligned}
\widehat{\sigma}^2 &= \ln\left[\frac{\left(\sqrt{\lambda^2+4\lambda}+\lambda+2\right)^{2/3}+2^{2/3}}{\left(2\sqrt{\lambda^2+4\lambda}+2\lambda+4\right)^{1/3}}-1\right], \\
\widehat{\mu} &= \frac{1}{2}\ln\left(\frac{m_2}{e^{2\widehat{\sigma}^2}-e^{\widehat{\sigma}^2}}\right), \\
\widehat{\tau} &= \begin{cases} m_1+e^{\widehat{\mu}+\frac{1}{2}\widehat{\sigma}^2} & \text{if } m_3<0, \\ m_1-e^{\widehat{\mu}+\frac{1}{2}\widehat{\sigma}^2} & \text{if } m_3\geq 0; \end{cases}
\end{aligned}$$

where $\lambda = (m_3)^2/(m_2)^3$. We could then estimate the parameters of the translated log-normal distribution based on the moment approximations of Section 5.

6.2. **Numerical examples.** Figures 6.1 to 6.4 compare the distributions obtained by simulation against the translated log-normal distributions whose parameters are obtained from the moment approximations of the previous section. From the pictures, we see that the translated log-normal distribution is able to fit a variety of shapes: skewed to the left, skewed to the right, or centered.

We could also use the distribution function of the translated log-normal distribution to approximate the percentiles of the present and future value distributions, without relying on Monte-Carlo simulations. Tables 7.1 and 7.2 display the results for $P(X_{\mathrm{RAN1}},R)$ and $P(X_{\mathrm{RAN2}},R)$. Those percentile approximations could then be used to approximate the cost of mismatch, as explained in section 4.4.

7. Conclusion

In this paper, we defined the equilibrium present value of future cash flows as the amount of money that should be subtracted today in order to arrive at a zero accumulated value at the end of the time horizon. The cost of mismatch is then some high percentile of the difference between the present value under a base scenario and the present value under stochastic interest rates. We took a macroscopic point of view where a short rate models the global rate of return of the company. We considered two different short rate generators (Vasicek and CIR).

Based on the examples of this paper, it appears that a scenario testing approach gives a rough measure of the interest rate risk, but this measure is not entirely consistent with a stochastic approach. In particular, the method does not provide an idea of the spread of the distributions that is consistent across different types of portfolios. This statement is true even with deterministic cash flows, or cash flows that are stochastic only because of mortality. The lesson to draw is that scenario testing should be used with extreme caution: the extra provision for interest rate risk (the cost of mismatch) may be sufficient with very different probabilities, depending on the type of the portfolio.

We considered a relatively simple model, for which it was possible to derive analytic (approximate) expressions, allowing us to compute the cost of mismatch without resorting to simulation. Eventually, the stochastic approach used in this paper should be extended to term structure interest rate models, more complex and realistic investment strategies (hedging, risk minimizing, etc), and interest or equity dependent cash flows. Of course we then cannot expect to obtain analytic expressions as in the present paper.

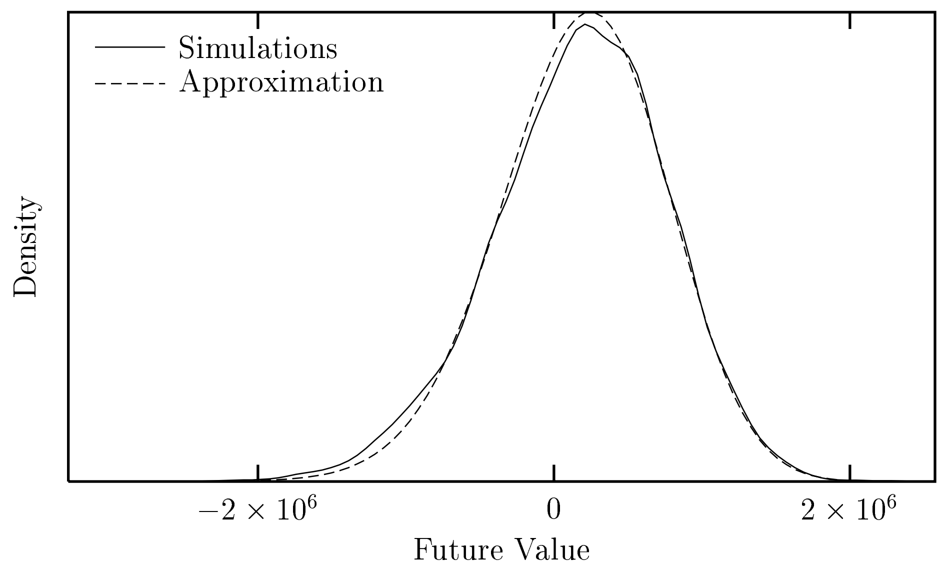

FIGURE 7.1. Distribution of $F(X_{\text{RAN1}}, R)$ in the Vasicek model.

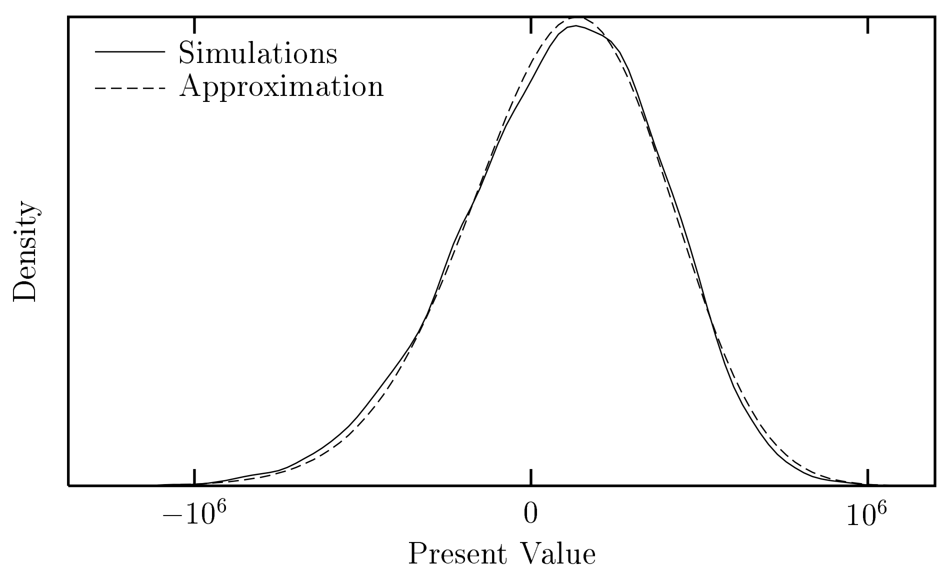

FIGURE 7.2. Distribution of $P(X_{\text{RAN1}}, R)$ in the Vasicek model.

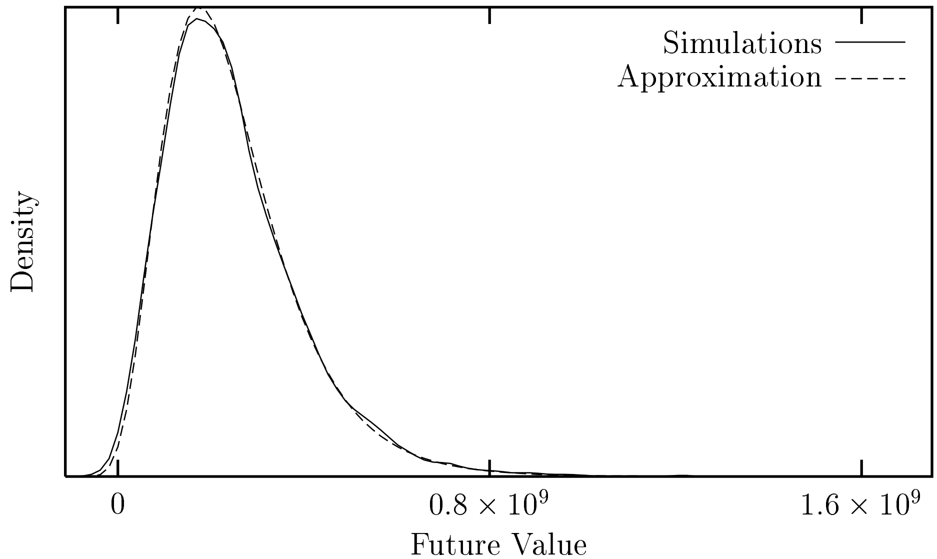

FIGURE 7.3. Distribution of $F(X_{\mathrm{RAN2}}, R)$ in the Vasicek model.

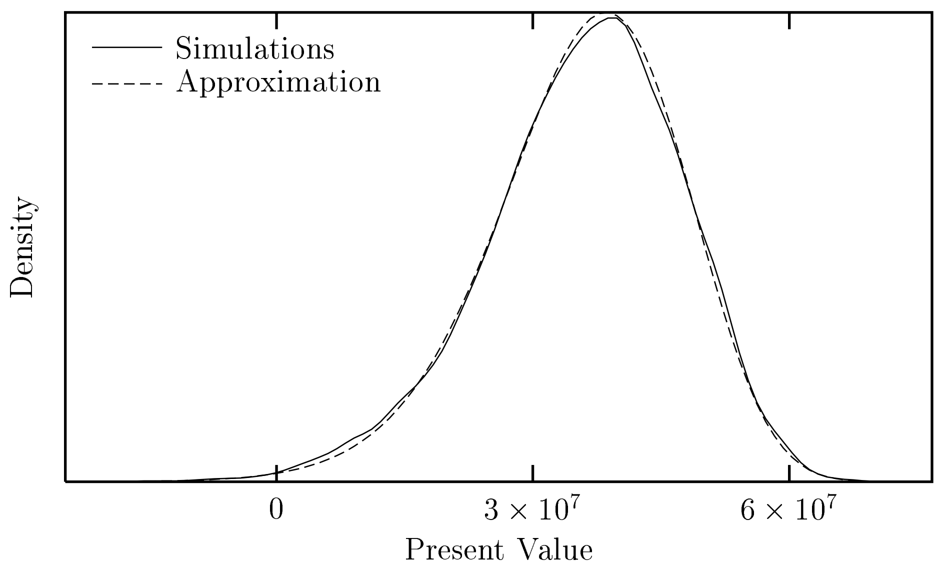

FIGURE 7.4. Distribution of $P(X_{\mathrm{RAN2}}, R)$ in the Vasicek model.

Percentiles	Vasicek		CIR	
	Approximation	Simulations	Approximation	Simulations
.01	-649 787	-677 045	-648 966	-643 660
.05	-411 049	-434 541	-410 582	-416 656
.10	-289 320	-304 651	-289 004	-293 852
.25	-94 125	-102 907	-94 009	-101 443
.50	111 257	109 412	111 222	104 410
.75	305 190	299 870	305 071	295 328
.90	470 477	458 510	470 334	454 994
.95	565 388	540 270	565 253	545 832
.99	735 709	700 152	735 628	706 128

TABLE 7.1. Percentiles of $P\left(X_{\mathrm{RAN1}}, R\right)$.

Percentiles	Vasicek		CIR	
	Approximation	Simulations	Approximation	Simulations
.01	5 770 449	5 126 116	12 732 876	13 067 926
.05	16 077 044	15 202 589	19 230 731	19 309 393
.10	21 134 671	20 949 178	22 779 878	22 748 398
.25	28 964 398	28 929 571	28 845 945	28 809 219
.50	36 827 358	36 832 909	35 789 910	36 033 601
.75	43 895 935	43 932 449	42 955 362	43 134 732
.90	49 645 151	49 917 436	49 599 709	49 802 545
.95	52 831 291	52 772 491	53 666 828	53 650 239
.99	58 336 643	58 520 239	61 484 106	61 326 416

TABLE 7.2. Percentiles of $P\left(X_{\mathrm{RAN2}}, R\right)$.

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