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AMORTIZATION OF GAINS AND LOSSES UNDER CERTAIN PROJECTED BENEFIT COST METHODS

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ABSTRACT

This paper demonstrates how, under the projected benefit cost methods (whereby the gains are reflected in the normal cost rate), gains determined at a given valuation date are amortized over the future working lifetime of the active employees included in the valuation as of that date.

THE purpose of this paper is to examine how gains and losses are amortized under the projected benefit cost methods, with gains spread over the future working lifetime of the population. We can assume without any loss of generality that there is no unfunded liability at time $t = 0$ of the first valuation and also, for ease of expression, that the experience always results in gains.

First we shall consider the case of a closed group of employees receiving no salary increases from date of entry. Then we shall turn to the most general case of an open group receiving salary increases. The following notation is used:

TL_t = Present value at time t of pension and fringe benefits;
A_t = Assets;
NC_t = Cash contribution rate = N_t/D_t;
N_t = $TL_t - A_t$ (unfunded liability if it exists);
D_t = Present value of future salaries;
S_t = Total salary at time t;
NG_t = Numerator gain, calculated on the basis of expected salary;
DG_t = Denominator gain, calculated on the basis of expected salary;
${}_{t-1}G_t$ = Dollar gain in tth year = $NG_t - NC_{t-1}DG_t$;
D_t^{NE} = Present value of new entrants' future salaries at time t;
N_t^{NE} = Present value at time t of pension and fringe benefits for new entrants;
$\Delta^S N_C$ = Excess of actual value over expected value of the numerator due to unexpected increases in salaries, calculated for all active employees at time t included in the valuation at $t - 1$;
$\Delta^S D_C$ = Excess of actual value over expected value of the denominator due to unexpected increases in salaries calculated as above.

I. THE CLOSED GROUP—NO SALARY INCREASES

At $t = 0$ the normal cost NC_0 (equal to N_0/D_0) is determined. Payments made at the beginning of each year i (as long as there remains an active employee) in the amount of S_iNC_0 will fully fund the liability of the plan if the experience conforms to the actuarial assumptions. This rarely happens, and accordingly gains eventually will be detected at time $t = 1$ and $t = 2$.

For illustrative purposes, consider the simple case of a group of employees all under age 65, covered by a plan providing only for a pension B_{65} at age 65. The plan is funded using the aggregate cost method, and liabilities are discounted for interest and mortality. Then

$$NC_1 = \frac{N_1}{D_1}$$

$$= \frac{TL_0(1+i) + NER - NAR - [A_0(1+i) + NC_0S_0(1+i) + IG]}{D_0(1+i) - S_0(1+i) + DER - DAR},$$

where

NER = Numerator of expected release of liability on account of death
= $\Sigma q_{x-1}B_{65}N_{65}/D_x$ (for all employees active at the beginning of the year);

NAR = Numerator of actual release of liability on account of death
= $\Sigma B_{65}N_{65}/D_x$ (for the employees who actually died during the year).

Similarly,

$$DER = \sum q_{x-1}\frac{N_x - N_{65}}{D_x}S_x\,; \qquad DAR = \sum \frac{N_x - N_{65}}{D_x}S_x\,.$$

IG is the interest gain. Since $TL_0 - A_0 = NC_0D_0$,

$$NC_1 - NC_0 = -\frac{[(NAR - NC_0DAR) - (NER - NC_0DER)] + IG}{D_1}$$

$$= -\frac{\text{Mortality gain} + \text{interest gain}}{D_1}$$

$$= -\frac{{}_0G_1}{D_1}.$$

$(NAR - NC_0DAR)$ is the sum of expressions of the form $B_{65}N_{65}/D_x - NC_0[(N_x - N_{65})/D_x]S_x$ for all employees who actually died, that is, the sum of actual release of *accrued liabilities*. The expected release of liability $(NER - NC_0DER)$ is the sum of expressions of the form $q_{x-1}\{B_{65}N_{65}/D_x - NC_0[(N_x - N_{65})/D_x]S_x\}$ for all active employees at the beginning of the year, that is, the sum of expected release of *accrued liabilities*. Therefore, the concept of gain, for the methods under consideration, is in no way different from the concept of gain under the unit credit cost method or the entry age normal method, once the term "accrued liability" has been defined.

In an actual case, ${}_{t-1}G_t$ will include other terms similar to the mortality gain derived above and computed on the other decrements for which liabilities are discounted; it will also include terms such as excess of the liability expected to be set up over the liability actually set up on account of vested terminations, excess of expected pension payments over actual pension payments, and so on.

Generally, at time $t = 1$ the normal cost rate decreases to NC_1, reflecting the gains ${}_0G_1$ experienced during the first year.

$$NC_1 = \frac{N_1}{D_1} = \frac{(N_0 - S_0NC_0)(1+i) - NG_1}{(D_0 - S_0)(1+i) - DG_1}; \tag{1}$$

$$NC_0 - NC_1 = \frac{NG_1 - DG_1NC_0}{D_1} = \frac{{}_0G_1}{D_1}. \tag{2}$$

From equation (2), ${}_0G_1 = (NC_0 - NC_1)D_1$, showing that the gain is amortized over the lifetime of the population by yearly amounts equal, in the absence of any further gains, to $S_i(NC_0 - NC_1)$. In any year i, therefore, the contribution is still S_iNC_0 but is made up of cash payments equal to S_iNC_1 and of the portion of the gain allocated to that year equal to $S_i(NC_0 - NC_1)$. If the funding of the plan is viewed in this manner, the situation at $t = 1$ is better described by writing that the normal cost rate is still NC_0, as shown in equation (3):

$$NC_0 = \frac{N_1 + {}_0G_1}{D_1}, \tag{3}$$

or, more explicitly,

$$NC_0 = \frac{TL_1 - (A_1 - {}_0G_1)}{D_1}. \tag{4}$$

Equation (4) recognizes that, because of the gain ${}_0G_1$, a portion of the assets equal to that gain is now considered as advance contributions to be released timely as described above. Equation (1) gives the cash contribution rate, while equation (3) describes the funding position, including the amortization of gains.

At time $t = 2$, first assume that ${}_1G_2 = 0$. If the cash contribution made at $t = 1$ was equal to S_1NC_1, we may summarize the funding position of the plan at $t = 2$ by equations (5) and (6), similar to equations (1) and (3), respectively except that no gain occurred during the second year. Equation (6) considers the gain amortization $S_1(NC_0 - NC_1)$ as a contribution, and NC_0' is the resulting normal cost rate.

$$NC_2 = \frac{N_2}{D_2} = \frac{(N_1 - S_1NC_1)(1+i)}{(D_1 - S_1)(1+i)} = NC_1 , \tag{5}$$

$$NC_0' = \frac{[N_1 + {}_0G_1 - S_1NC_1 - S_1(NC_0 - NC_1)](1+i)}{(D_1 - S_1)(1+i)} = NC_0 , \tag{6}$$

since, from equation (3), $N_1 + {}_0G_1 = D_1NC_0$. Therefore,

$$NC_0 = \frac{N_2 + [{}_0G_1 - S_1(NC_0 - NC_1)](1+i)}{D_2} ,$$

where $[{}_0G_1 - S_1(NC_0 - NC_1)](1+i)$ represents that portion of the gain ${}_0G_1$ not yet amortized.

At this point it is interesting to go through the same process again, assuming that it had been decided to take immediate advantage of the gain by making a cash contribution equal to $(S_1NC_0 - {}_0G_1)$ at time $t = 1$. We now have equations (7) and (8), similar to equations (5) and (6), respectively.

$$\begin{aligned} NC_2 &= \frac{N_2}{D_2} \\ &= \frac{[N_1 - (S_1NC_0 - {}_0G_1)](1+i)}{(D_1 - S_1)(1+i)} \\ &= \frac{[D_1NC_1 - S_1NC_1 - S_1(NC_0 - NC_1) + (NC_0 - NC_1)D_1](1+i)}{(D_1 - S_1)(1+i)} \\ &= NC_1 + (NC_0 - NC_1) = NC_0 . \end{aligned} \tag{7}$$

All gains having been used at $t = 1$, the cash contribution rate again is NC_0 at $t = 2$.

Equation (8) reflects the total contribution S_1NC_0, which is made up of cash contribution $(S_1NC_0 - {}_0G_1)$ and of the full gain ${}_0G_1$.

$$\begin{aligned} NC_0' &= \frac{[N_1 + {}_0G_1 - (S_1NC_0 - {}_0G_1) - {}_0G_1](1+i)}{(D_1 - S_1)(1+i)} \\ &= \frac{(D_1NC_0 - S_1NC_0)(1+i)}{(D_1 - S_1)(1+i)} = NC_0 , \end{aligned} \tag{8}$$

or, more explicitly,

$$NC_0 = \frac{N_2}{D_2} = \frac{TL_2 - A_2}{D_2} .$$

In the above equation the assets are now fully utilized.

Finally, consider the case where gains equal to ${}_1G_2$ have been experienced in the second year, in order to show how this occurrence modifies the amortization schedule of the gains ${}_0G_1$. Assume here that the cash contribution was equal to S_1NC_1. We now establish equations (9) and (10), similar to equations (1) and (3).

$$\begin{aligned} NC_2 = \frac{N_2}{D_2} &= \frac{(N_1 - S_1NC_1)(1+i) - NG_2}{(D_1 - S_1)(1+i) - DG_2} \\ &= NC_1 - \frac{NG_2 - NC_1DG_2}{D_2} \\ &= NC_1 - \frac{{}_1G_2}{D_2} \,. \end{aligned} \tag{9}$$

Taking the full funding position point of view, we develop equation (9) from equation (3), $NC_0 = (N_1 + {}_0G_1)/D_1$, remembering that the total contribution from cash and previous gains is S_1NC_0 and that the expected rate under those conditions is NC; because gains were realized, a normal cost rate NC_0' is calculated.

$$\begin{aligned} NC_0' &= \frac{[N_1 + {}_0G_1 - S_1NC_1 - S_1(NC_0 - NC_1)](1+i) - NG_2}{(D_1 - S_1)(1+i) - DG_2} \\ &= \frac{(D_1 - S_1)NC_0(1+i) - NG_2}{(D_1 - S_1)(1+i) - DG_2} \\ &= NC_0 - \frac{NG_2 - NC_0DG_2}{D_2} \,. \end{aligned}$$

Therefore, since $(N_1 - S_1NC_1)\,(1+i) - NG_2 = N_2$,

$$\begin{aligned} NC_0 &= \frac{1}{D_2}\,\{N_2 + [{}_0G_1 - S_1(NC_0 - NC_1)](1+i) \\ &\qquad + (NG_2 - NC_0DG_2)\} \,. \\ &= \frac{1}{D_2}\,\{N_2 + [{}_0G_1(1+i) - S_1(NC_0 - NC_1)(1+i) \\ &\qquad - (NC_0 - NC_1)DG_2] + (NG_2 - NC_1DG_2)\} \,. \end{aligned} \tag{10}$$

Examination of the term in brackets in the numerator reveals that, in addition to the regular amortization term $S_1(NC_0 - NC_1)$, we have a term, $(NC_0 - NC_1)DG_2$, equal to the present value of future amortization amounts on account of unexpected terminations.

Remembering that ${}_0G_1 = D_1(NC_0 - NC_1)$, and that $D_2 = (D_1 - S_1)(1+i) - DG_2$, then

$$NC_0 = \frac{N_2 + (NC_0 - NC_1)D_2 + (NC_1 - NC_2)D_2}{D_2} \,. \tag{11}$$

In the absence of any further gains, we are back on the regular amortization schedule. If further gains do occur, the same type of modification as that just described will be made, and, at time t we should have

$$NC_0 = \frac{1}{D_t}[N_t + (NC_0 - NC_1)D_t + (NC_1 - NC_2)D_t + \ldots + (NC_{t-1} - NC_t)D_t] \,. \tag{12}$$

At time t, the unamortized portion of the gain of the nth year is therefore $(NC_{n-1} - NC_n)D_t$.

II. THE OPEN GROUP—SALARY INCREASES

It now remains to examine the case of the open group receiving salary increases. We focus our attention on the type of equation shown in (3). New entrants are admitted at time t. Their liability was expected to be funded at the rate NC_0. Therefore, the gain on their account is

$$D_t^{\mathrm{NE}} NC_0 - N_t^{\mathrm{NE}} \,.$$

The loss due to actual salary increases exceeding the expected increases equals

$$\Delta^S N_t - NC_0 \Delta^S D_t \,.$$

Both expressions can be broken into parts. For example, the new-entrant gain can be written

$$D_t^{\mathrm{NE}}[(NC_0 - NC_1) + (NC_1 - NC_2) + \ldots + (NC_{t-2} - NC_{t-1})] + (D_t^{\mathrm{NE}} NC_{t-1} - N_t^{\mathrm{NE}}) \,.$$

The gain of the tth year, computed with NC_{t-1}, is equal to

$${}_{t-1}G_t + (D_t^{\mathrm{NE}} NC_{t-1} - N_t^{\mathrm{NE}}) - (\Delta^S N_t - NC_{t-1}\Delta^S D_t) \,.$$

The denominator D_t is developed from D_{t-1} as follows:

$$D_t = (D_{t-1} - S_{t-1})(1 + i) - DG_t + \Delta^S D_t + D_t^{\mathrm{NE}} \,.$$

Therefore, equation (12) holds, with the difference that the outstanding gain measured by $(NC_{n-1} - NC_n)D_t$ includes elements emerging after $t = n$ from new entrants and salary experience. If it is necessary to isolate that part of the gain determined at $t = n$ still unamortized at time t, then a new temporary annuity D_t must be calculated excluding all new entrants after $t = n$ and using the expected salaries obtained from the salaries at $t = n$ by application of the salary scale.

A final comment should be added with respect to a change in benefits and/or in actuarial assumptions. Assume that such an event happens at time $t = 3$. Before the change,

$$NC_3 = \frac{N_3}{D_3} \quad \text{and} \quad NC_0 = \frac{N_3 + {}_0G_3^3}{D_3},$$

where ${}_0G_3^3$ represents the unamortized gains at $t = 3$. After the change,

$$NC_3' = \frac{N_3'}{D_3'} \quad \text{and} \quad NC_0^3 = \frac{N_3' + {}_0G_3^3}{D_3'}.$$

NC_0^3 is the new total rate of contribution from both cash and gains. (Note that ${}_0G_3^3 = (NC_0^3 - NC_3')D_3'$.)

At time $t = 4$, with a cash contribution rate NC_4, equation (12) reads

$$NC_0^3 = (NC_0^3 - NC_3') + (NC_3' - NC_4') + NC_4'.$$

The unamortized portion of ${}_0G_3^3$ is equal to

$$(NC_0^3 - NC_3')D_4',$$

which can be separated, if necessary, into parts proportionate to

$$(NC_0 - NC_1), \quad (NC_1 - NC_2), \quad (NC_2 - NC_3).$$

REFERENCE

ANDERSON, ARTHUR W. "A New Look at Gain and Loss Analysis," *TSA*, XXIII (1971), 7-47; 182.