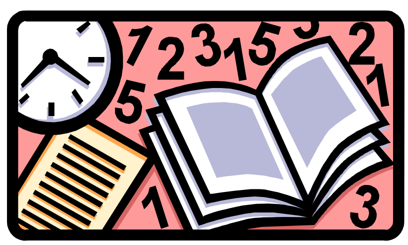
# Long-term Forecasting for Interest Rates

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## ABSTRACT

This paper develops a new technique, which allows the analyst to maximally use all the historical interest rate information available in forecasting interest rates. Interest rates' daily changes do not follow a stationary random process over long time periods. Instead, the process is comprised of stationary periods, each a few years long, with relatively short transitional periods between different stationary periods. This finding has two practical implications: (i) it is impossible to reliably forecast statistical properties of interest rates on a period longer than a few years using models based on stationary random processes, and (ii) one should use only a stationary piece of data for model calibration. We developed a technique for identification of stationary periods in historical data. This is useful for evaluating distribution parameters in parametric models. This technique leads to better evaluation of the historical distribution of random variables, such as interest rates, stock prices, currency exchange rates. Finally, we propose a technique for the identification and separation of different economic periods. Long-term forecasts need to consider mean reversion. Given an arbitrary distribution of daily changes and the mean reversion speed, we derived an analytic solution for the distribution of interest rates in *N* days. Our long-term technique yields better tail estimates than the Vasicek and Cox-Ingersoll-Ross models. The tail estimates are better for both interest rates and options on interest rates. There are efficient numerical algorithms suitable for implementation of the long-tem technique either as a separate software product or as part of a forecasting or risk management package. Finally, we applied our method to forecast payoffs of very high-risk instruments such as out-of-the-money options with good results.

## 1. Introduction

Interest rate forecasting is one of the most important and widely studied problems related to managing fixed income securities. Numerous researchers and practitioners have studied this topic and published many papers suggesting different approaches and methods. Among facts that have been established the most fundamental are that interest rates exhibit stochastic fluctuations, and that it has been impossible to find any deterministic pattern in these fluctuations.

Basic properties of the interest rates fluctuations are:

- The daily changes of interest rates are almost independent identically distributed (i.i.d.) random variables.
- The distribution of daily changes is not normal.
- The distribution has fat tails.
- The distribution is not universal, for example, it depends on the maturity.
- The interest rate changes exhibit mean reversion, i.e. they tend to stay near the mean value.

The tail areas of interest rate fluctuations are of special practical significance. They provide the probabilities of relatively infrequent but most dangerous big drops and jumps in interest rate values.

There are a number of different models underlying statistical methods used for the estimation of interest rate parameters. These models establish equations governing interest rate fluctuations. The equations are relatively simple, i.e. they contain only few unknown coefficients. Statistical methods based on these models facilitate finding values of the coefficients to secure the "best fit" of the formula with the data available.

The simplest model is the geometric Brownian motion with normal distribution of the interest rates increments. This approximation underestimates the effect of thick tails. More advanced models incorporate the effect of fat tails to some extent. However, they approximate the distribution of interest rates changes by a function with several fitting parameters.

The analytically solvable models provide advantages including clarity and the ability to trace the effect of input parameters on final results. As data from financial random processes accumulated, models have become more and more complex. Most recently, proposed models are not analytically solvable, and in many cases it is even impossible to show that there exist “good” solutions to those models. Models with a small number of fitting parameters based on normal random processes seldom yield good quantitative descriptions of real financial data. Meanwhile, the major purpose of the quantitative approach is to develop reasonably precise techniques for computing expected prices of financial instruments. The only reason to prefer one model is its better precision in describing market prices. The choice of a good model is based on the following criteria:

- Maximum information on the random process governing interest rates should be extracted from historical data.
- The validity of any simplification or approximation should be tested on historical data.
- Analytically treatable models *a priori* have no advantage over any other computable model or approximation. The only criterion of model goodness is its precision in description of historical data.

Our approach includes powerful techniques that have not been widely used in financial mathematics:

- Nonparametric statistic for the historical data.
- Statistical hypothesis testing to evaluate the properties of historical data and for model validation.
- The Fourier transforms.

The logic of our method is as follows.

1. Assume that the daily changes of interest rates are independent identical distributions (i.i.d.); random variables with an arbitrary distribution. We show that the *n-day* distribution can be expressed in terms of the Fourier transform of the daily probability density function (PDF).
2. Use the interest rate histogram as the daily PDF and compare the computed *n-day* distribution against the historical one. Namely, we test the hypothesis that the historical *N*-

day changes belong to the computed statistic. Since the distribution is arbitrary we use the nonparametric Kolmogorov-Smirnov and Kuiper tests.

3. Tests are separated into two groups.
   a. The model validation tests evaluate the degree of consistency between the model and historical data.
   b. The forecast tests show the model's forecasting ability.

   We found the approach based on i.i.d. daily interest rates increments yields good or acceptable short-term forecasts with time horizons up to half a month. The discrepancy for longer time scales is attributed to mean reversing.
4. Finally, we determine the *n-day* distribution from the distribution of daily changes with mean reversion. The general solution of the problem involves the Fourier. The long time horizon limit is important and so we reviewed it particularly and compare our results to historical data. In addition we performed the same tests on the popular Vasicek and Cox-Ingersoll-Ross models.

A considerable part of the paper discusses empirical research into the statistical properties of interest rates. Indeed, our nonparametric models as well as all other interest rates models assume that the interest rates increments exhibit a stationary random process. We tested the hypothesis that the historical series represent a stationary process and found that it is not always the case. The historical process consists of relatively long stationary periods separated by short transitional periods. The distributions of interest rates changes in different stationary periods typically differ significantly. So there are major limitations in the choice of historical series for calibration and in the expected time horizons for long-term forecasts.

### 1.1 Nonparametric approach

This paper presents a nonparametric technique for interest rates forecasting. In contrast to the popular parametric model approach where the distribution of daily increments of interest rates is approximated by some function with a few fitting parameters derived from historical data, this technique employs no model assumptions for the interest rates increment distribution.

We wanted to eliminate the assumption that the random process governing interest rates can be described by an analytical model with a few fitting parameters. In general, a nonparametric distribution yields better precision than any parametric one as the class of nonparametric distributions contains all parametric ones as a subset, typically of zero measure. Furthermore, we consider time to be discrete, reflecting the reality of financial data. Changes in interest rates and other financial variables such as stock returns or currency exchange rates are not independent. However, historical daily changes of interest rates are almost independent. Longer time frames require additional analysis.

We start with a series of arbitrarily distributed independent random variables. The distribution of the interest rates in *n-day* can be expressed in terms of the Fourier transform of the distribution of daily increments. This solution has an analytically closed form. What is even more important, the solution can be easily obtained numerically thanks to the efficiency of the fast Fourier transform. If this solution is more precise than other ones, it is certainly better from our pragmatic point of view.

To show how the nonparametric approach approximates market data, we compare predicted non-normal distributions of interest rates with historical data.

### 1.2 Market data as a stationary stochastic process and model calibration

All interest rates models including our nonparametric models presume that the random process governing the interest rates is stationary. This assumption is critical. If the process were not stationary, the models would need to be built differently. In checking whether the interest rates series do represent a stationary random process, we used the statistical hypothesis testing approach to determine whether a series of data represents a stationary random process. The methodology follows. Take a series of historical data and cut it into two halves. For each half, compute the distribution of daily changes. If the process is stationary, both samples belong to the same statistics. Therefore, we have to test the hypothesis that the samples from both halves belong to the same statistics. Since the distribution of historical daily changes is arbitrary we again applied the Kolmogorov-Smirnov and Kuiper tests.

We found that the hypothesis that the historical data represent a stationary process does not always hold. Typically, long series of interest rates data are not stationary. However, they are piece-stationary, i.e., they consist of pieces of stationary series separated by rather short transitional periods. The typical length of stationary periods is a few years; and the typical length of a transitional period is a few months. The distributions of interest rates changes in different stationary periods typically differ significantly. We interpret the stationary periods as periods of stable economic development. After a transitional period the economical situation changes, and therefore expectations of interest rates change. This results in different interest rates distributions in different stationary periods.

Our results can be summarized as follows:

- First we observe that the stationary periods based on the Kolmogorov-Smirnov criterion are of the same length or longer than those based on the Kuiper criterion.
- Since the Kuiper test is more sensitive to the discrepancy in the distribution tails than the Kolmogorov-Smirnov test, this implies that the mean area of the interest rate distributions is more stable in time than the tails. So, to forecast the mean behavior of interest rates, the longer historical periods based on the Kolmogorov-Smirnov criterion can be used. However, if big changes in the interest rates are the major concern, stationary periods based on the Kuiper criterion are better.
- Another important observation is that ***the longer the maturity, the longer the stationary periods.*** This is not surprising since the interest rates are the average estimates of the profit that the market participants expect to receive from lending money. Definitely, the long-term estimates are more stable than the short ones. Practically this implies that our chances of making a "good" forecast for the long-term instruments are higher than those for the short-term instruments.
- Stationary periods are not cyclical: they reproduce neither their lengths nor their fundamental statistical properties.

The existence of well-defined stationary periods in historical data leads us to two conclusions:

- One can use data from only one stationary period for calibration of a model. Contrary to the common belief, the use of long series of data that contain more than one stationary period reduces the calibration precision, because the data in such a long series represents a mixture of several different distributions.
- The time horizon of forecasts based on stationary models is limited by the length of a stationary period; it cannot exceed a few years.

In this paper, we always use one stationary period for model calibration and testing. The use of data from different stationary pieces would introduce inconsistencies. We believe restricting the construction of forecasting models to a single stationary period improves the precision of any model calibration. This approach is recommended for any risk management practice.

### 1.3 Model validation as a hypothesis testing problem

When we use a model to describe a random process, we implicitly accept the hypothesis that the model correctly describes the market data, even though it is never obvious that a model does yield a correct description of the market data. We are given a finite sample of historical market data and the theoretical distribution of interest rates that follows from the model. This distribution can be obtained either analytically or numerically. Whether a model provides a reasonable quantitative agreement with market data can be decided using statistical hypothesis testing. Namely, does the sample belong to the statistic obtained from the model given a reasonable confidence level? We either accept the model when the probability that the historical data fit the forecasted distribution exceeds the confidence level, or reject the model otherwise.

In this paper we use the nonparametric distribution-independent Kolmogorov-Smirnov and closely related Kuiper statistics for hypothesis testing. Both are distribution independent and their results are easy to interpret. The Kolmogorov-Smirnov test is more sensitive to the closeness of the sample to the central part of the distribution. The Kuiper test is more sensitive to the tail distribution.

We perform two statistical tests of mathematical models against historical data, model validation tests and forecast tests. Model validation tests are necessary to verify a model, i.e., to accept the statistical hypothesis that the model does correctly describe a stationary period of historical data. The forecast tests compare the forecasted results with historical data that are **not** in the data set used to construct the model, but are in the same stationary period.

First we perform test aimed at model validation.

Classifications of the model quality are:

- Excellent - Confidence level above 90%
- Good - Confidence level above 50%
- Possibly acceptable - Confidence level between 10% and 50%
- Questionable - Confidence level below 10%
- Poor - Confidence level below 1%

To perform a model validation test, take a series of historical data and find the empirical distribution of daily increments. Using this distribution, compute the distribution of *n-day* changes. From the same series of data, obtain the distribution of *n-day* changes of the interest rates. To validate the model, test the hypothesis that the historical sample of *n-day* changes belongs to the computed statistic. The model validation test indicates how well the model agrees with the historical series; however it does not tell about the predictive abilities of the model.

This method for hypothesis testing was used to validate a number of popular interest rates models. The results are rather remarkable. Confidence levels for the geometric Brownian motion, Vasicek, and Cox-Ingersoll-Ross models are poor, well below 1%. The confidence level of our nonparametric model is typically higher but still too low to be accepted as a good model. Only on time horizons from 2 to 10 days does the nonparametric model exhibit acceptable to good confidence levels.

### 1.4 Fat tails and forecast precision

Forecast tests evaluate the forecast precision. Start with hypothesis testing. First choose a calibration series of data as the basis for the forecast, and, using this series, forecast the distribution *n-day* (We used $N = 2, 5, 10, 15$ and 20.) after the end of the series. Next, consider a new calibration series one day longer than the initial calibration series, and forecast the distribution *n-day* after the end of the new calibration series. Continue this process and collect a large number of forecasts with an *n-day* time horizon. Compare these forecasts with the actual changes of interest rates in *n-day* after the end of each series. Using the Kolmogorov-Smirnov and Kuiper tests we compare the historical and predicted distributions. We found that the forecast results are close to those of model validation tests.

This project aspired to develop a forecasting technique for big jumps and falls in interest rates. Therefore, we concentrated on forecast tests of extreme events. Since the number of such events is small and we are not aware of distribution-independent tests for tails, we used a naive approach, counting the frequency of large changes and compared the frequency with the computed probability. First, we counted big positive and negative daily increments of interest rates and compared that frequency with those in the nonparametric and the normal distribution. Namely, we counted the increments greater than 1.65 standard deviations and smaller than –1.65 standard deviations. These increments should have the frequency of 0.05 if they follow the normal distribution. Next, we found the greatest positive and negative daily increments and compared the frequency of such events with those from the nonparametric and the normal distribution. The counting tests indicated that both the nonparametric and the normal distribution yield close estimates of the frequency of large daily increments, and these estimates are close to the historical frequency. The nonparametric probabilities of the greatest increments and decrements are systematically closer to the observed frequency, which is equal to 0.01 for our series of 100 daily forecasts.

In the case of long-term forecasts we compared the frequencies of extreme events obtained from the nonparametric, Vasicek, and Cox-Ingersoll-Ross models. We found the nonparametric model yields the best frequency estimates, the Vasicek model is second, and the Cox-Ingersoll-Ross model trails behind.

### 1.5 Mean reversing

Finally, we examine mean reversion, the tendency of interest rates to stay near the mean value. First, generalize the Vasicek model for the case of an arbitrary distribution of the interest rates increments. Then obtain a complete analytic solution in terms of the Fourier transform and the finite-interval Hilbert transform. In the long time horizon limit, when the resulting distribution does not depend on the initial condition, the solution is simple. Typically, the time horizon of several months or longer is sufficient for application of the long-term asymptotic distribution. Compare the asymptotic distribution with the long time horizon limits of the Vasicek and the Cox-Ingersoll-Ross models and with historical data. The major results show:

- The Cox-Ingersoll-Ross forecasts are inferior to those by the nonparametric and the Vasicek models both in the central area and in the tails.
- There is practically no difference between the model-free and Vasicek forecasts near the center of the distribution.
- The model-free forecasts are systematically more precise in the tail areas.

As an example, the expected payoffs of interest rates option traded on the Chicago Board of Option exchange are computed and compared against actual historical payoffs. The expected payoffs computed with the nonparametric technique are systematically closer to historical ones than those of the Vasicek and Cox-Ingersoll-Ross models.

## 2. Basic methodology and short-term forecasting

Our starting point is the empirical distribution of historical daily changes, the Empirical Distribution Function ($EDF_1$ or simply EDF). We use the EDF to obtain the *n-day* Nonparametric Distribution Function ($NPDF_n$). We also obtain the n-day Normal Distribution Function ($NDF_n$). We compare the $NPDF_n$ with the $NDF_n$ and each with the *n-day* Empirical Distribution Function, $EDF_n$. The latter is the distribution of historical *n-day* changes. Additional abbreviations are DF for Distribution Function and PDF for Probability Density Function.

We start with a brief description of the methodology. Then the calculated *n-day* $NPDF_n$s are compared with historical $EDF_n$s as well as with *n-day* $NDF_n$s computed under the hypothesis of normality of daily changes.

### 2.1 Theoretical description and numerical algorithm

We assume that daily values of interest rates $y(t)$ $(t=1,2,\ldots,T)$ follow the stochastic process $Y(t)$ $(t=1,2,\ldots,T)$, which is governed by the equation:

$$Y(t+1) - Y(t) = \xi(t), \qquad (2.1.1)$$

where $\xi(t)$ $(t=1,2,\ldots,T)$ are random variables describing the interest rate daily changes. Eq.(2.1.1) is one of the simplest ways to describe the interest rate dynamics for a short time interval when the mean reversion property is negligible. One may consider this equation as a finite difference form of the classical Vasicek model (cf. Vasicek [1977]) with the mean reversion coefficient equal to zero. In its original form, the Vasicek model stipulates that the random variables $\xi(t)$ $(t=1,2,\ldots,T)$ are independent identical normal variables. We replace this assumption with more realistic one that requires only that

**For all $t = 1,2,\ldots,T$ the random variables $\xi(t)$ of Eq.(2.1.1) are independent and identically distributed with a known probability density function $f(\xi)$.**

The function $f(\xi)$ is usually referred to as the probability density function of daily changes. We do not assume any parametric form or representation for the function $f(\xi)$. In this section of the paper, we derive an explicit formula for the probability distribution function for *n-day* changes in interest rates based on the function of daily changes $f(\xi)$.

The random variable of *n-day* changes in interest rates $\zeta^{(n)}(t) = Y(t+n) - Y(t)$ can be presented as a sum of daily changes:

$$\zeta^{n}(t) = Y(t+n) - Y(t) = Y(t+n) - Y(t+n-1) + Y(t+n-1) - Y(t+n-2) + \ldots \\ + Y(t+1) - Y(t) = \sum_{k=0}^{N-1} \xi(t+k) \quad . \qquad (2.1.2)$$

Since the random variables $\xi(t)$ are identically distributed for all $t = 1,2,\ldots,T$, the random variables $\zeta^{(n)}(t)$ are also identically distributed for all $t$. Denote by $g_n[\zeta^{(n)}]$ the probability density function of the random variables $\zeta^{(n)}(t)$. Since we assume that $\xi(t)$ are independent, Eq.(2.1.2) shows the random variables $\zeta^{(n)}(t)$ is a sum of independent random variables and, therefore, (see, for example, Korn [1961], Sect. 18.5-7), the function $g_n$ is the $n$-fold convolution of the function $f$:

$$G_n = f \otimes f \otimes \ldots \otimes f,$$

and the Fourier transform $G_n$ of the function $g_n$ is the $n$-fold product of the Fourier transform $F$ of the functions $f$:

$$G_n(\lambda) = [F(\lambda)]^{n}. \qquad (2.1.3)$$

Let $\varphi(\lambda)$ be the logarithm of the Fourier transform $F(\lambda)$, $\varphi(\lambda) = \ln[F(\lambda)]$.

Then in the terms of Fourier images, formula (2.1.3) takes the form, which is remarkably simple and convenient for the numerical calculations:

$$G_n(\lambda) = \exp[n\varphi(\lambda)]. \qquad (2.1.4)$$

Formula (2.1.4) underlies the following algorithm for computation of the Nonparametric Distribution Function of *n-day* changes ($NPDF_n$) in interest rates:

***N-Day Changes Nonparametric Distribution Function ($NPDF_n$) Algorithm***

**Step 1.** Select a "proper" basic historical period and build the Empirical Distribution Function (EDF) of daily yield changes $f(\xi)$ using the data pertaining to that period.

**Step 2.** Calculate Fourier transform $F(\lambda)$ of $f(\xi)$ and, using formula (2.1.4), calculate the Fourier transform of the *n-day* yield changes $G_N(\lambda)$.

**Step 3.** Calculate the $NPDF_n$ of *n-day* yield changes $g_n(\xi)$ as the inverse Fourier transform of $G_N(\lambda)$.

The modern numerical methods featuring the Fourier transform (see, for example, Press [1992]), so called "Fast Fourier Transform Algorithms," are very efficient. Therefore, when the EDF of daily yield changes *f(ξ)* is found, the implementation of Steps 2 and 3 can be done in a relatively short period of time.

In the course of research we tested the $NPDF_n$ Algorithm on the daily data collected by the Federal Reserve Bank on the US Treasury Securities over the period January 3, 1983 through December 31, 1998. The following historical data were reviewed: yields on 3-Month, 6-Month, and 1-Year Treasury bills; yields on 2-year, 3-year, 5-year, and 10-year Treasury notes; and yields on 30-year Treasury bonds.

Exhibit 1 illustrates how the $NPDF_n$ Algorithm transforms the distribution of historical daily interest rate changes into the Nonparametric Distribution Function of 2-day, 5-day, 10-day, 15-day, and 20-day changes. For better graphical display, cumulative distribution functions are shown. Exhibit 1.1 shows the original and n-day changes distributions for 3-month Treasury rates; Exhibit 1.2 shows the original and n-day changes distributions for 10-year Treasury rates.

### 2.2 Selection of basic period for $NPDF_n$ Algorithm: notion of stationary period

The selection of basic period for the Nonparametric Distribution Function of n-day changes is the most important and trickiest part of the $NPDF_n$ Algorithm. The choice of a particular historical period of interest rate development is basic to the algorithm. The period defines the statistical characteristics of the underlying EDF of daily changes and, therefore, the statistical characteristics of $NPDF_n$. It is hard to expect that statistically different EDFs would yield statistically indistinguishable $NPDF_n$s. So how can one choose which historical period should be the base one for the $NPDF_n$ Algorithm?

Since $NPDF_n$ is to be used for forecasting, it is desirable for the basic period to be as long as possible. On the other hand, the major assumption underlying the $NPDF_n$ Algorithm is that the interest rate development follows a stochastic process with stationary independent increments.

An arbitrary long series of interest rate observations may not represent a stochastic process with the same distribution of daily increments. One of the earliest proofs of this is presented in the paper written by David Becker (cf. Becker [1991]) who showed that interest rates volatility is not constant in a series of consecutive years. This seems reasonable since interest rates express the market expectations on gains from lending money. These expectations depend on the economic situation as well as Federal Reserve Bank policies and the international situation. A long series of data can represent several very different economic periods. If one uses historical data for more than one economic period, the distribution of interest rate increments is a mixture of distributions corresponding to the different periods. Such a mixture is not appropriate for a short-term forecast while the most recent economic period is expected to continue. Forecasts based on arbitrarily chosen historical data periods often yield low precision or just wrong values.

We developed a method to establish the longest time period when interest rates exhibit stationary daily changes. Such a period is referred to as a **stationary period** in this paper.

We assume a stationary period consists of one or more calendar years, so the shortest stationary period is one year. We cannot provide a rigorous proof that a calendar year is the proper time unit for measuring the length of stationary periods. However, looking at the history of global changes in the economy, we have observed that a typical time interval when the economic environment is stable runs from one to three or four years; and that relatively long stable periods are divided by relatively short, about one to three month, transitional periods. See also Gwartney [1995], p. 507.

Based on this observation, the following method is used to construct a stationary period for each of the maturities studied.

1. For every year from 1983 to 1998 and every maturity – 3 month, 6 month, 1 year, 2 years, 3 years, 5 years, 10 years, and 30 years - we obtained the cumulative empirical distribution function of daily changes in the interest rates: ***CEDF(M,Y).*** In this notation, M stands for a maturity and Y stands for a year;
   $M \in \mathbf{M} = \{M_1 = 3$ month, $M_2 = 6$ month, $M_3 = 1$ year, $M_4 = 2$ year, $M_5 = 3$ year, $M_6 = 5$ year, $M_7 = 10$ year, $M_8 = 30$ year$\}$; $Y \in \mathbf{Y} = \{Y_1 = 1983, \ldots, Y_{16} = 1998\}$.

2. For each maturity $M \in \mathbf{M}$ and for every pair of years $Y, Y' \in \mathbf{Y}$, $Y \neq Y'$, we used *CEDF(M,Y)* and *CEDF(M,Y')* to test the hypothesis ***H(M; Y, Y')*** that for the Treasury Security with the maturity *M* the empirical distribution of daily changes in the year *Y* and the empirical distribution of daily changes in the year *Y'* are identical. If the hypothesis ***H (M; Y, Y')*** holds true, we conclude that for the maturity *M* the years *Y* and *Y'* belong to the same stationary period. If the opposite is true, we conclude the years *Y* and *Y'* belong to two different stationary periods.

Since the distributions of daily changes are not expected to have an analytic form, we applied two nonparametric criteria, the classical Kolmogorov-Smirnov criterion (Kolmogorov [1933], Smirnov [1939]) and its modification, the Kuiper criterion (Kuiper [1962]), to test the hypothesis ***H (M; Y, Y')***. (Please see, for example, Press [1992], pp. 623–628 or Von Mises [1964], pp. 490-493 for a detailed modern description of Kolmogorov-Smirnov and Kuiper criterions.) The reason for applying two similar tests to compare the same pairs of cumulative empirical distributions of daily changes lies in the ways Kolmogorov-Smirnov and Kuiper tests weight the discrepancy in the tail areas of the compared distributions. When analyzing the function employed by Kolmogorov-Smirnov criterion to measure the distance between two distributions, it is easy to see that as soon as two compared distributions are close in their "central" (i.e., near the means) areas it does not "feel" even significant discrepancy in the distribution tails. Kuiper test is more sensitive to the distribution tails and requires the compared distribution to be "uniformly" close in order to be accepted as identical.

This method allowed us to establish stationary periods for all maturities by accepting or rejecting hypothesis ***H(M; Y, Y')*** at a chosen significance level. Exhibits 2.1 and 2.2 present all stationary periods for the 3-month and 10-year Treasury securities at the 10% significance level. Each of these exhibits consists of two tables: the top table gives the stationary periods based on the Kuiper criterion, and the bottom table gives the stationary periods based on the Kolmogorov-Smirnov criterion.

In addition to indicating the correct historical periods to be used in the construction of empirical distribution functions of *n-day* interest rate changes, Exhibit 2 provides us with several important insights about the stability of the interest rate environment discussed in Introduction.

- The stationary periods based on the Kolmogorov-Smirnov criterion are of the same length or longer than those based on the Kuiper criterion.
- The longer the maturity, the longer the stationary periods. Comparing the stationary periods shown on Exhibit 2.1 (3-Month T-bill) with those shown on the Exhibit 2.2 (10-year T-note), the longest stationary period for 10-year T-note is 13 years (Kolmogorov-Smirnov) or 11 years (Kuiper), while for 3-Month T-bill the longest stationary period is 8 years (Kolmogorov-Smirnov) or 7 years (Kuiper). Again, this is an expected result. Interest rates are the average estimates of profit market participants expect to receive from lending money. Definitely, long-term estimates are more stable than the short ones. Practically this implies that the likelihood of making a "good" forecast for long-term instruments is higher than for short-term instruments.

We believe that the question of finding and analyzing the stationary periods of interest rates is important and deserves additional research.

### 2.3 Comparison tests for calculated nonparametric density function of *n*-day yield changes

The tests are separated into two groups, model validation tests and forecast tests. As described in the Introduction, model validation tests show how well a model describes a series of historical data. Forecast tests examine the predictive ability of a model. A better model is expected to outperform a worse one for both kinds of tests. However, a model that describes a series of historical data better is not necessarily also superior for forecasting. If one model systematically yields better model validation test results and another produces better forecasts, either at least one of the models is inadequate or the tests are insufficient.

The purpose of the model validation test is to check whether a model is compatible with the market data. For example, suppose that the historical returns distribution is far from normal and we employ the normal model. In this case, we would extract the mean value and the variance from the historical data. However, the comparison of the empirical distribution with the normal one would indicate a significant discrepancy, i.e., the model validation test would reject the normal model. The acceptance of the model validation test is a necessary condition for the model to be valid. In model validation tests the same series of data is used for calibration and testing.

The forecasting tests verify the quality of forecasting. Note that the success of a model validation test is a necessary but insufficient condition for the model to be accepted for any application, especially, for forecasting. For forecasting tests the historical data was always separated into two non-overlapping parts. The first series of data is used for calibration and validation; and the second (later) series for testing of the forecasts.

### 2.3.1 Model validation tests

We performed the following tests comparing results obtained by using our nonparametric technique with those obtained by using the traditional approach, i.e. assuming the daily yield changes are normally distributed:

1. Using the historical data, we obtained the historical EDFs of *n-day* yield changes, $EDF_{n.}$
2. We applied the convolution integral/Fourier transform technique (Steps 2 and 3 of the *N*-Day Changes PDF Algorithm above) to calculate the Nonparametric Distribution Function of *n-day* yield changes, $NPDF_n$.
3. We obtained the *n*-day Normal Distribution Function, $NDF_n$ assuming that the underlying EDF of daily changes are normal.
4. We calculated the difference between our $NPDF_n$ and the corresponding historical $EDF_n$ as well as the difference between the $NDF_n$ and the corresponding historical one.

Note that if the daily changes were identically distributed independent random variables governed by a stationary random process and if the series of available historical data were infinite, then the $NPDF_n$ would equal the corresponding $EDF_n$; and, in view of the Central Limit Theorem (see, for example, Breiman [1968]), $NPDF_n$ would converge to the normal distribution as $n$ tends to infinity.

We have applied the following measure $\mu(f,g)$ to find the difference between the calculated $f$ and historical $g$ distribution functions of $n$-day changes in the interest rates:

$$\mu(f,g)=\frac{1}{2}\int\left[\sqrt{f(x)}-\sqrt{g(x)}\right]^2dx=1-\int\sqrt{f(x)g(x)}dx$$

This measure is a slightly modified Lebesgue measure of the functional space $L_2(-\infty,+\infty)$ of square integrable functions (see, for example, Kolmogorov [1961]). Note that since a probability density function is non-negative and integrable, its square root belongs to $L_2(-\infty,+\infty)$. Measure $\mu(f,g)$ uniformly accumulates the difference between two distributions over the whole sample space and, therefore, well reflects the discrepancy in the tail areas. This distance is always between 0 and 1.

The model validation test described above was applied to the yields on 3-month T-bills and 10-year T-notes in the period of January 4, 1996 through January 2, 1998. Table 2.1 below presents the results of the comparison tests. In this table, the first column contains the time lag in business days, columns 2 and 4 show the difference between $NPDF_n$ and correspondent $EDF_n$, and columns 3 and 5 show the difference between $NDF_n$ and correspondent $EDF_n$.

TABLE 2.1 Lebesgue distance $\mu$ between indicated distributions and $EDF_n$

| Time lag in business days | 3-month T-bills | | 10-year T-note | |
|---|---|---|---|---|
| n | $NPDF_n$ | $NDF_n$ | $NPDF_n$ | $NDF_n$ |
| 1 | 0 | 0.124 | 0 | 0.099 |
| 2 | 0.059 | 0.116 | 0.046 | 0.081 |
| 5 | 0.078 | 0.114 | 0.057 | 0.069 |
| 10 | 0.095 | 0.121 | 0.057 | 0.065 |
| 15 | 0.104 | 0.122 | 0.060 | 0.065 |
| 20 | 0.105 | 0.119 | 0.063 | 0.066 |

Note that first we used a series of data for calibration of the one-day $NDF_1$ and for obtaining the one-day $EDF_1$. Then, we used the same series for comparison of the $NDF_n$ and $EDF_n$ with the historical *n-day* distribution $NPDF_n$.

### 2.3.2 Forecast tests

The purpose of this project is to develop a forecasting technique for big jumps and falls in interest rates. Therefore, we concentrated on forecast tests of the extreme events.

We performed several series of tests of the forecasting technique on historical data. For each time series started January 4, 1996 and ended January 2, 1998 we computed 100 forecasts with the time horizons 1, 2, 5, 10, and 20 business days. We considered the largest positive and negative daily changes in the 100-day series. The results are summarized in the Tables 2.2.1 – 2.4.2. Each table contains 3 rows. The first row represents results taken from historical series. The columns show the largest positive and negative daily increments in the series measured in units of the standard deviation. The second and the third rows show probabilities of corresponding events computed with NPDF and the Normal distribution of daily increments. There are 100 events in the series. The largest positive and negative daily changes happen only once, so the frequencies of the extreme events are equal to 1/100. Thus numbers of order of 1% in the second and third rows indicate good forecasts. The numbers well below 1% indicate that the tails are too light, and the numbers well above 1% indicate that the tails are too heavy.

TABLE 2.2.1 3 month T-bill, time horizon = 1 day

| | Greatest positive change in standard deviations | Greatest negative change in standard deviations |
|---|---|---|
| Observation | 4.087 | -2.196 |
| NPDF probability | 0.4% | 2.5% |
| Normal probability | 0.002% | 1.3% |

TABLE 2.2.2 10 year T-bond, time horizon = 1 day

| | Greatest positive change in standard deviations | Greatest negative change in standard deviations |
|---|---|---|
| Observation | 2.339 | -2.841 |
| NPDF probability | 1.4% | 1.2% |
| Normal probability | 1.0% | 0.2% |

Table 2.3.1 3 month T-bill, time horizon = 5 days

| | Greatest positive change in standard deviations | Greatest negative change in standard deviations |
|---|---|---|
| Observation | 2.650 | -2.084 |
| NPDF probability | 1.0% | 1.9% |
| Normal probability | 0.4% | 1.8% |

Table 2.3.2 10 year T-bond time horizon = 5 days

| | Greatest positive change in standard deviations | Greatest negative change in standard deviations |
|---|---|---|
| Observation | 1.330 | -2.095 |
| NPDF probability | 6.6% | 3.4% |

| | | |
|---|---|---|
| Normal probability | 9.1% | 1.9% |

Table 2.4.1 3 month T-bills, time horizon = 10 days

| | Greatest positive change in standard deviations | Greatest negative change in standard deviations |
|---|---|---|
| Observation | 1.59 | -2.069 |
| NPDF probability | 5.8% | 2.0% |
| Normal probability | 5.7% | 1.9% |

Table 2.4.2 10- year T-bond, time horizon = 20 days

| | Greatest positive change in standard deviations | Greatest negative change in standard deviations |
|---|---|---|
| Observation | 1.228 | -1.493 |
| NPDF probability | 0.1% | 0.1% |
| Normal probability | 0.1% | 0.1% |

These results indicate that the NPDF yield good forecasts for short time horizons, up to 1 week. For longer time horizons, NPDF overestimates the tails, i.e. large fluctuations happen less frequently. We believe these results from mean reversion, the attraction of the interest rate to its mean value. Mean reversion corrections to the NPDF are discussed in Part 3.

### 2.3.3 Discussion of test results

The results presented in the tables suggest the following:

1. For all time lags $n$ = 1, 2, 5, 10, 15, and 20, the Lebesgue distance between the nonparametric $NPDF_n$ and the observed $EDF_n$ is less than that between the normal

distribution based $NDF_n$ and $EDF_n$.

2. While for the short time lags ($n$ = 1, 2, and 5) the nonparametric $NPDF_n$ provides one with significantly better approximations of the observed $EDF_n$ than those yielded by the normal distribution based $NDF_n$, for a longer time lags of two to four weeks (10 to 20 business days) $NPDF_n$ does not have any real advantage over $NDF_n$.

3. The frequency of largest daily increments and decrements of interest rates (0.01) is always close to the NPDF forecast. The NDF systematically underestimates the frequency of large fluctuations. In other words, $NPDF_n$ always yields more precise approximation to the historical distribution $EDF_n$ in the tail areas, where most dangerous big jumps and falls occur.

Conclusions (1) and (2) might appear contradictory at first reading. However, both conform to the Central Limit Theorem; the PDF of *n-day* yield changes converges to a normal distribution when the time lag $n$ increases. The rate of convergence is faster in the vicinity of the distribution mean and slower in the tail areas.

## 3. Mean reversion and long-term forecasting

In Part 2 we developed a technique to anticipate the interest rates distribution in *n-day* using the historical distribution of daily changes. Our tests indicated that this technique yields very good forecasts for short time horizons up to 5 business days but it overestimates the tails for longer time horizons. We attribute this to mean reversion in interest rates.

Mean reversion corrections significantly improve forecasts for longer time horizons. This is due to the fact that the assumption that interest rates increments are i.i.d. random variables holds only for short time periods. For longer time periods, interest rates exhibit mean reversion. If it were not be the case, the range of $n$-day changes of interest rates would increase steadily while the time period $n$ gets longer. Analysis of historical data shows that the distribution of *200*-day changes has practically the same range as the distributions of *250*-day changes, which in its turn is very close to the distributions of *300*-day changes, and so on and so forth. The theory of stochastic processes suggests that this phenomenon can only be captured by a mean reverting stochastic process that exhibits time convergence to a time independent asymptotic distribution.

In Part 3 mean reversion is considered. In Section 3.1, we obtain mean reversing corrections to the Nonparametric Distribution Function (NPDF) for discrete time series with arbitrary distribution of the random factor. Also, in this section we obtain the asymptotic distribution for NPDF with mean reversing and show that this distribution can be effectively used to calculate the distribution of *n*-day changes of interest rates in the uniform fashion.

In Section 3.2.1 we compute interest rates forecasts using the most popular Vasicek and Cox-Ingersoll-Ross models and compare theses forecasts with historical data. These two models have analytic solutions that allow us to use them as benchmarks to perform detailed comparisons with our nonparametric approach. We presented the results of these comparisons in Sections 3.2.3 and 3.2.3.

In Section 3.3 we show how asymptotic NPDF can be employed to calculate interest rates option payoffs and compare our results with those obtained by using Vasicek and Cox-Ingersoll-Ross asymptotic distributions.

### 3.1 Distribution of *n*-day changes in interest rates under Vasicek type dynamic with nonparametric stochastic component

#### 3.1.1 Recursive method for finding distribution of *n*-day changes

We assume that daily values of interest rates $y(t)$ $(t=1,2,\ldots,T)$ follow the stochastic process $Y(t)$ $(t=1,2,\ldots,T)$, which is governed by the equation:

$$Y(t+1) - Y(t) = k[m - Y(t)] + \xi(t), \tag{3.1.1}$$

where $\xi(t)$ $(t=1,2,\ldots,T)$ are random variables describing the stochastic component of the interest rate daily changes. Eq.(3.1.1) is a finite difference form of the Vasicek model (cf. Vasicek [1977]) based on the Ornstein-Uhlenbeck process (cf. Uhlenbeck [1930]). The Vasicek model is one of the most common ways to describe the mean reversion property of the interest rate dynamics, and, in its original form, this model stipulates that the random variables $\xi(t)$ $(t=1,2,\ldots,T)$ are independent identical normal variables. We replace this assumption with more realistic one that requires only that

**For all $t=1,2,\ldots,T$ the stochastic components $\xi(t)$ of Eq.(3.1.1) are independent identically distributed random variables with a known probability density function $f(\xi)$.**

The function $f(\xi)$ is usually referred to as the probability density function of daily changes. We do not assume any parametric form or representation for the function $f(\xi)$: it is taken as it is. In this paper, we derive an explicit formula for the probability distribution function for n-day changes in interest rates based on the given function of daily changes $f(\xi)$.

The coefficient $k$ in Eq.(3.1.1) determines how fast the stochastic process $Y(t)$ converts to its mean $m$, usually referred to as the speed of mean reversion. The coefficient $k$ satisfies the following inequalities:

$$0 < k < 1. \qquad (3.1.2)$$

The left part of the formula (3.1.2) indicates the stochastic process does converge, not diverge. The right part of this formula indicates the mean reversion effect is observable only over a significant time interval.

Denote $x(t) = y(t) - m$. Then Eq.(3.1.1) implies that the stochastic process $X(t) = Y(t) - m$ is governed by the equation

$$X(t+1) - \lambda X(t) = \xi(t), \qquad \text{where } \lambda = 1 - k. \qquad (3.1.3)$$

Note that by its definition the parameter $\lambda$ satisfies the same inequalities (3.1.2) as the speed of mean reversion:

$$0 < \lambda < 1.$$

Eq. (3.1.3) means that

$$Pr\{X(t+1) = x(t+1) \mid X(t) = x(t)\} = f[\xi(t)], \tag{3.1.4}$$

where

$$\xi(t) = x(t+1) - \lambda x(t).$$

Using the method of mathematical induction it can be proved that for any $n \geq 1$

$$X(t+n) - \lambda^n X(t) = \sum_{k=0}^{n-1} \lambda^{n-1-k} \xi(t+k). \tag{3.1.5}$$

Really, for $n = 1$ Eq.(3.1.5) is the same as Eq.(3.1.3). Let us assume that the formula (5) holds true for $n = j\text{-}1$, i.e.,

$$X(t+j-1) - \lambda^{j-1} X(t) = \sum_{k=0}^{j-2} \lambda^{n-2-k} \xi(t+k), \quad j \geq 2. \tag{3.1.6}$$

Note that

$$X(t+j) - \lambda^j X(t) = X(t+j) - \lambda X(t+j-1) + \lambda X(t+j-1) - \lambda^j X(t).$$

Then due to Eq.(3.1.3) and the induction hypothesis (3.1.6), the equation above implies

$$X(t+j) - \lambda^j X(t) = \xi(t+j-1) + \lambda \sum_{k=0}^{j-2} \lambda^{j-2-k} \xi(t+k) = \sum_{k=0}^{j-1} \lambda^{j-1-k} \xi(t+k),$$

which concludes the proof of formula (3.1.5).

Similar to the interpretation (3.1.4) of Eq.(3.1.3), the formula (3.1.5) means that

$$Pr\{X(t+n) = x(t+n) \mid X(t) = x(t)\} = Pr\{X(t+n) - \lambda^n X(t) = x(t+n) - \lambda^n x(t)\},$$

and the random variable $X(t+n) - \lambda^n X(t)$ has the same distribution as the random variable $\zeta^{(n)}(t)$ defined by the formula:

$$\zeta^{(n)}(t) = \sum_{k=0}^{n-1} \lambda^{n-1-k} \xi(t+k). \tag{3.1.7}$$

In view of this, the probability density function of random variable $\zeta^{(n)}(t)$ is referred to as the probability density function of n-day changes in interest rates.

Note that if $Z_1, Z_2, \ldots, Z_n$ are independent random variables and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers, then $R_1 = \alpha_1 \bullet Z_1$, $R_2 = \alpha_2 \bullet Z_2, \ldots, R_n = \alpha_n \bullet Z_n$ are independent random variables.

Really, let $F_j$ be a distribution function of $Z_j$, $j= 1,2, \ldots, n$ and $F$ be a joint distribution function of the variables $Z_1, Z_2, \ldots, Z_n$. Denote by $G_j$ the distribution function of $R_j$, $j= 1,2, \ldots, n$ and let $G$ stand for the joint distribution function of the variables $R_1, R_2, \ldots, R_n$. Then because $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive

$$G(r_1, r_2, \ldots, r_n) = Pr\{R_1 \le r_1, R_2 \le r_2, \ldots, R_n \le r_n\} = Pr\{\alpha_1 \bullet Z_1 \le r_1, \alpha_2 \bullet Z_2 \le r_2, \ldots, \alpha_n \bullet Z_n \le r_n\}$$

$$= Pr\{Z_1 \le r_1 / \alpha_1, Z_2 \le r_2 / \alpha_2, \ldots, Z_n \le r_n / \alpha_n\} = F(r_1/\alpha_1, r_2/\alpha_2, \ldots, r_n/\alpha_n).$$

Since $Z_1, Z_2, \ldots, Z_n$ are independent random variables, the chain of equalities above implies that

$$G(r_1, r_2, \ldots, r_n) = F(r_1/\alpha_1, r_2/\alpha_2, \ldots, r_n/|\alpha_n) = F_1(r_1/\alpha_1) \bullet F_2(r_2/\alpha_2) \bullet \ldots \bullet F_n(r_n/\alpha_n) =$$
$$Pr\{Z_1 \le r_1/\alpha_1\} \bullet Pr(Z_2 \le r_2/\alpha_2\} \bullet \ldots \bullet Pr\{Z_n \le r_n/\alpha_n\} =$$
$$Pr\{R_1 \le r_1\} \bullet Pr\{R_2 \le r_2\} \bullet \ldots \bullet Pr\{R_n \le r_n\} = G_1(r_1) \bullet G_2(r_2) \bullet \ldots \bullet G_n(r_n),$$

which proves the independence of the random variables $R_1, R_2, \ldots, R_n$.

Denote by $\xi^{(j)}(t)$ the random variable defined by $\xi^{(j)}(t) = \lambda^j \xi(t)$. Since the random variables $\xi(t)$ are identically distributed for all $t = 1,2,\ldots,T$, the random variables $\xi^{(j)}(t)$ and $\zeta^{(n)}(t)$ are also identically distributed for all $t$. Denote by $f_j[\xi^{(j)}]$ the probability density function of the random variables $\xi^{(j)}(t)$ and let $g_n[\xi^{(n)}]$ stand for the probability density function of $\zeta^{(n)}(t)$.

An immediate corollary of the statement proved above is that since the random variables of daily changes $\xi(t), \xi(t+1), \ldots, \xi(t+n-1)$ are independent and $\lambda > 0$, the random variable $\zeta^{(n)}(t)$ is the sum of independent random variables $\xi^{(n-1)}(t), \xi^{(n-2)}(t+1), \ldots, \xi(t+n-1)$. Therefore (see, for example, Korn [1961], Sect. 18.5-7), the function $g_n$ is the convolution of the functions $f, f_1, f_2, \ldots, f_{n-1}$:

$$g_n = f \otimes f_1 \otimes f_2 \otimes \ldots \otimes f_{n-1},$$

and the Fourier transform $G_n$ of the function $g_n$ is the product of the Fourier transforms $F, F_1, F_2, \ldots, F_{n-1}$ of the functions $f, f_1, f_2, \ldots, f_{n-1}$ respectively:

$$G_n(\omega) = F(\omega) \bullet F_1(\omega) \bullet F_2(\omega) \bullet \ldots \bullet F_{n-1}(\omega). \quad (3.1.8)$$

In accordance with the formula for the linear change of variables and the definition of Fourier transform (see, for example, Korn [1961], Sect. 18.5-3 and Sect. 4.11),

$$f_j(\eta) = \frac{1}{\lambda^j} f\left(\frac{\eta}{\lambda^j}\right) \quad \text{and} \quad F_j(\omega) = F(\lambda^j \omega).$$

Hence we can rewrite (3.1.8) as

$$G_n(\omega) = F(\omega) \bullet F(\lambda\omega) \bullet F(\lambda^2\omega) \bullet \ldots \bullet F(\lambda^{n-1}\omega) \qquad (3.1.9)$$

and calculate the probability density function $g_n$ of n-day changes as the inverse Fourier transform of $G_n$:

$$g_n = F^{-1}[G_n].$$

### 3.1.2 Asymptotic distribution and its application to finding *n*-day distribution

Note that since $\lambda<1$ the product $\lambda^j\omega$ converges to zero for all $\omega$ as $j$ goes to infinity. This implies that

$$F_j(\omega) = F(\lambda^j\omega) \to 1 \text{ as } j\to\infty, \qquad (3.1.10)$$

and, therefore, if the time period $n$ is large, the factors $F_j(\omega) = F(\lambda^j\omega)$ in the formula (3.1.9) with $j$ exceeding some threshold $N$ do not contribute to the value of $G_n(\omega)$.

This observation suggests the existence of an asymptotic formula for the calculation of the probability distribution function $n$-day changes in interest rates for a large time period $n$. Namely, it implies that for any distribution of daily changes $f(\xi)$ there is a number $N$ such that for all $n > N$ the Fourier transform $G_n$ of the probability density function $g_n$ is well approximated by the function $G_a$ that can be formally defined as

$$G_a(\omega) = \prod_{j=1}^{\infty} F(\lambda^{j-1}\omega). \qquad (3.1.11)$$

The function $G_a(\omega)$ is well-defined by equality (3.1.11) since in accordance with the multiplicative version of Cauchy convergence criterion (see, for example, Theorem 68 at Schwartz [1967], Chapter II) formula (3.1.10) implies that the infinite product of $F(\lambda^j\omega)$ converges for any $\omega$.

Let $\varphi(\omega)$ be the natural logarithm of $F(\omega)$, $\gamma_n(\omega)$ be the natural logarithm of $G_n(\omega)$, and $\gamma_a(\omega)$ be the natural logarithm of $G_a(\omega)$ :

$$\varphi(\omega) = ln[F(\omega)], \qquad \gamma_n(\omega) = ln[G_n(\omega)], \quad \text{and} \quad \gamma_a(\omega) = ln[G_a(\omega)].$$

Since the convergence of the infinite product of $F(\lambda^j\omega)$ is equivalent to the convergence of infinite series of $\varphi(\lambda^j\omega)$ (see, for example, Theorem 69 at Schwartz [1967], Chapter II) formula (3.1.11) implies

$$\gamma_a(\omega) = \sum_{j=1}^{\infty} \varphi(\lambda^{j-1}\omega). \tag{3.1.12}$$

Also, the formula (3.1.9) can be rewritten as

$$\gamma_n(\omega) = \sum_{j=1}^{n} \varphi(\lambda^{j-1}\omega). \tag{3.1.13}$$

Comparing the formulae (3.1.12) and (3.1.13) one can observe that for any $\omega$ the functions $\gamma_n(\omega)$ and $\gamma_a(\omega)$ are bound by the equation:

$$\gamma_n(\omega) = \gamma_a(\omega) - \gamma_a(\lambda^n\omega). \tag{3.1.14}$$

Equation (3.1.14) means that if the asymptotic function $\gamma_a(\omega)$ is known, the distribution of $n$-day changes for any number of days $n$ can be found in the uniform fashion without going through laborious summation or multiplication process described by the equations (3.1.9) or (3.1.13).

Note that for $n = 1$ equation (3.1.14) yields

$$\varphi(\omega) = \gamma_1(\omega) = \gamma_a(\omega) - \gamma_a(\lambda\omega). \tag{3.1.15}$$

This equation provides us with a simple connection between the original distribution of one-day changes and the asymptotic distribution and allows us to find the asymptotic function $\gamma_a(\omega)$ directly from the function $\varphi(\omega)$. Namely, let $\mu = \ln(\lambda)$ and $x = \ln(\omega)$. Then equation (3.1.15) takes the form:

$$\varphi(e^x) = \gamma_a(e^x) - \gamma_a(e^{x+\mu}),$$

or

$$v(x) = u(x) - u(x+\mu), \tag{3.1.16}$$

where $v$ is the composition of $\varphi$ and the exponential function and $u$ is the composition of $\gamma_a$ and the exponential function. Now let $U(\omega)$ be the Fourier transform of $u(x)$ and $V(\omega)$ be the Fourier transform of $v(x)$. Then equation (3.1.16) implies that $U(\omega)$ and $V(\omega)$ are bound by the equation:

$$V(\omega) = U(\omega)[1 - e^{i\omega\mu}].$$

Therefore, since $V(\omega)$ is known, the function $u(x)$ can be found as the inverse Fourier transform of the function $V(\omega)/[1 - e^{i\omega\mu}]$:

$$u(x) = \frac{1}{2\pi}\int \frac{V(\omega)e^{i\omega x}}{1 - e^{i\omega\mu}} d\omega. \qquad (3.1.17)$$

Note that in a small vicinity of zero both, the numerator $V(\omega)$ and denominator $1 - e^{-i\omega\mu}$ under the integral above, are well approximated by $\omega$:

$$V(\omega) \approx \omega \quad \text{and} \quad 1 - e^{i\omega\mu} \approx \omega .$$

This implies that $\omega=0$ is not a singular point and the function $u(x)$ calculated by the formula (3.1.17) is well-defined for any $x$.

The results of this section bring us to the following algorithm for the calculation of distribution of $n$-day changes given the distribution of one-day changes:

***Algorithm for Calculation of Asymptotic and N-Day Changes Nonparametric Distribution Functions ($NPDF_a$ and $NPDF_N$) with Vasicek Type Mean-Reversion***

**Step 1.** Select a "proper" basic historical period and build the Empirical Distribution Function (EDF) of daily yield changes $f(\xi)$ using the data pertaining to that period.

**Step 2.** Calculate Fourier transform $F(\omega)$ of $f(\xi)$ and the natural logarithm $\varphi(\omega)$ of $F(\omega)$.

**Step 3.** Calculate function $v(x)$ using the equation: $v(x) = \varphi(e^x)$.

**Step 4.** Calculate Fourier transform $V(\omega)$ of the function $v(x)$.

**Step 5.** Calculate inverse Fourier transform $u(x)$ of the function $V(\omega)/[1 - e^{i\omega\mu}]$ with $\mu = \ln(1-k)$, where $k$ is the speed of mean reversion.

**Step 6.** Calculate function $\gamma_a(\omega)$ using the equation: $\gamma_a(\omega) = u[\ln(\omega)]$.

**Step 7(A).** Calculate function $\Gamma_a(\omega)$ using the equation: $\Gamma_a(\omega) = \exp[\gamma_a(\omega)]$.

**Step 8(A).** Calculate the asymptotic distribution $g_a(\zeta)$ of interest rate changes by taking the inverse Fourier transform of the function $\Gamma_a(\omega)$.

**Step 7(N).** For any number of days *n* calculate function $\gamma_n(\omega)$ by applying the formula:

$\gamma_n(\omega) = \gamma_a(\omega) - \gamma_a(\lambda^n \omega)$ and $\Gamma_n(\omega)$ using the equation: $\Gamma_n(\omega) = \exp[\ \gamma_n(\omega)]$ .

**Step 8(N).** Calculate the distribution $g_n(\zeta)$ of *n*-day changes in interest rates by taking the inverse Fourier transform of the function $\Gamma_n(\omega)$.

One can observe that while the algorithm described above appears to be more complex than one, which was presented in Section 2.1, it does not involve anything that goes beyond simple algebraic equations and Fourier transforms. The numerical experiments we conducted showed that by employing software packages (such as Mathematica and MatLab) featuring Fast Fourier Transform Algorithms (see, for example, Press [1992]), the implementation of Steps 2 through 8 can be done in a relatively short period of time.

## 3.2 Comparison tests for Nonparametric Distribution Function with mean reversion

### 3.2.1 Benchmark models

We consider two popular interest rates models, Vasicek (cf. Vasicek [1977]) and the Cox-Ingersoll-Ross (cf. Cox [1985]) as benchmarks for testing our results. These models feature the mean reversion and have known analytic solutions that allow us to perform a detailed comparison of these models with our non-parametric approach.

The Vasicek model (a finite difference form) stipulates that daily values of interest rates $y(t)$ $(t=1,2,\ldots,T)$ follow the stochastic process $Y(t)$ $(t=1,2,\ldots,T)$, which is governed by the equation:

$$Y(t+1) - Y(t) = k\left[m - Y(t)\right] + \sigma \cdot \eta(t),$$

where $\eta(t)$ $(t=1,2,\ldots,T)$ are independent standard normal variables. The only difference between our technique (please see Eq.(3.1.1)) and the Vasicek model is that the latter assumes that the distribution of random variables $\xi(t) = \sigma \cdot \eta(t)$ is normal. The mean reversion level *m* and the speed of mean reversing *k* are the same.

We have tested the hypothesis that the distribution of $\xi(t)$ is normal by applying the Kolmogorov-Smirnov (KS) and Kuiper (KP) tests. The KS and KP probabilities (P-values) for the period from January 4, 1996 to January 2, 1998 are given in Table 3.2.1 below.

Table 3.2.1

P-values

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| KS | 0.019 | 0.014 | 0.045 | 0.165 |
| KP | 1.664E-6 | 9.796E-7 | 2.007E-3 | 4.346E-3 |

Table 3.2.1 shows that given 5% significance level the hypothesis of $\xi(t)$ normality should be rejected by the Kuiper criterion for all maturities. If the Kolmogorov-Smirnov is applied, this hypothesis can only be accepted for 30-year rates.

The results indicate that the hypothesis that the Vasicek model quantitatively describes historical data is less than 5% and unacceptable.

Even though the Vasicek model does not correctly approximate the distribution of the daily changes of interest rates it may yield reasonable forecasts for longer time horizons. It follows from the Ornstein-Uhlenbeck analytic solution (cf. [13]) that the values of

$$[Y(t+n) - m] - [Y(t) - m] \cdot e^{-k \cdot n} \; ; t=1, 2, \ldots, T$$

are normally distributed with zero mean and standard deviation $\tilde{\sigma}_n = \frac{\sigma\left(1 - e^{-2 \cdot k \cdot n}\right)}{2k}$.

This implies that for a large time period *n* the distribution of random variable *Y(t+n)* does not depend on the distribution of "initial" random variable *Y(t)* and can be well approximated by the normal distribution with mean *m* and standard deviation $\tilde{\sigma}_a = \frac{\sigma}{2k}$.

The Cox-Ingersoll-Ross (CIR) (a finite difference form) stipulates that daily values of interest rates *y(t) (t=1,2,…,T)* follow the stochastic process *Y(t) (t=1,2,…,T),* which is governed by the equation:

$$Y(t+1) - Y(t) = k\left[m - Y(t)\right] + \sqrt{Y(t)}\cdot\sigma\cdot\eta(t), \qquad (3.2.1)$$

where $\eta(t)$ *(t=1,2,…,T)* are independent standard normal variables. This implies that the random variable $\nu(t)$ defined by the formulae

$$\nu(t) = \frac{X(t+1) - \lambda X(t)}{\sqrt{Y(t)}}, \qquad X(t) = Y(t) - m, \quad \lambda = 1 - k \qquad (3.2.1)$$

is normally distributed.

We have tested the hypothesis that the distribution of $\nu(t)$ is normal by applying the Kolmogorov-Smirnov (KS) and Kuiper (KP) tests. The KS and KP probabilities (P-values) for the period from January 4, 1996 to January 2, 1998 are given in Table 3.2.2 below.

Table 3.2.2

P-values

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| KS | 8.181E-3 | 3.173E-4 | 0.051 | 0.052 |
| KP | 4.234E-7 | 9.811E-10 | 9.45E-4 | 4.847E-3 |

Table 3.2.2 shows that given 5% significance level the hypothesis of $\nu(t)$ normality should be rejected by the Kuiper criterion for all maturities. If the Kolmogorov-Smirnov is applied, this hypothesis should be rejected for 3-month and 3-year rates and can be accepted for 10-year and 30-year rates.

We concentrated on the most important case of the long time horizon where the distribution of interest rates does not depend on the initial value. In this limit interest rates obey the asymptotic $\Gamma$-distribution (cf. Feller [1951]. We performed a model validation test of the hypothesis that the interest rates distribution is the $\Gamma$-distribution. We used the Kolmogorov-Smirnov and Kuiper statistics on the 500 days time series from January 4, 1996 to January 2, 1998 for this test. The following table shows the hypothesis is always rejected at the 5% confidence level. One can see that the Kuiper tests yield especially low estimates for the asymptotic $\Gamma$-distribution.

Table 3.2.3

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| KS | 0.03 | 4.566E-3 | 6.07E-6 | 1.513E-8 |
| KP | 2.149E-5 | 3.979E-8 | <1E-15 | <1E-15 |

### 3.2.2 Model validation and forecasting tests

In the case of mean reversion, the distribution function depends on three variables: the time horizon, the initial interest rate and the final interest rate. Without mean reversing, the distribution depends only on two parameters, the time horizon and the difference between the final and initial interest rates. Therefore, one needs to carry out tests of two-dimensional distributions for every fixed time horizon in the case of mean reversing. It requires much more data to make a reliable comparison of two 2-dimensional distributions. Fortunately, the most important long-term distribution is the asymptotic distribution that depends on the final interest rates but does not depend on the initial ones. The asymptotic distribution is one-dimensional, so we concentrated our tests on it.

To perform the model validation test we take a series of historical data, find the distribution of daily increments, obtain the mean value $m$, determine the volatility $\sigma$ of daily increments, and the speed of mean reversion $k$. Using the daily increment distribution and the mean reversing parameter, we obtain the asymptotic NPDF, the asymptotic Vasicek distribution, and the asymptotic Cox-Ingersoll-Ross distribution. Test the hypothesis that the historical series used to obtain the daily distribution and the mean reversing parameter belong to the asymptotic distribution. Note that the length of the data series $T$ should be large so that the product $kT$ is significantly larger than 1. This allows one to apply the asymptotic distribution to the data.

The forecast tests use two series of data. The earlier historical data are used to obtain the speed of mean reversion and the asymptotic distribution from the daily distribution. The asymptotic distribution is compared with the historical distribution of the later data.

If the historical process is stationary and random the results of the model check and the forecasting check should not differ significantly. Any significant difference between these tests indicates that the underlying process is either not stationary or not entirely random. The latter is quite possible because of nonrandom activities of the Federal Reserve Bank that controls the interest rates according to certain policies.

Detailed data for the model validation tests are presented in Exhibit 3.A.

- For each maturity, we considered 15 series of 256 daily data starting from January 2, 1983.
- For each series, we obtained the daily PDF, the mean reversing parameter, and the asymptotic NPDF.
- We compared the historical interest rates in each series with the asymptotic NPDF using the Kolmogorov-Smirnov and Kuiper tests.
- We computed the average P-values of the 15 series of 256 days.
- The average results are presented in Tables 3.2.4 and 3.2.5.

Table 3.2.4

Average P-values of model validation tests based on the KS statistics

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| NPDF | 0.022824 | 0.015158 | 0.004085 | 0.003351 |
| Vasicek | 0.026747 | 0.0059 | 0.002835 | 0.018239 |
| CIR | 0.001828 | 0.000754 | 0.00228 | 9.47E-06 |

Table 3.2.5

Average P-values of model validation tests based on the Kuiper statistics

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| NPDF | 0.007269 | 0.000318 | 9.49E-06 | 9.49E-06 |
| Vasicek | 0.004371 | 0.000505 | 1.9E-05 | 1.9E-05 |
| CIR | 3.14E-07 | 1.18E-07 | 5.16E-07 | 5.16E-07 |

The tables indicate the asymptotic NPDF and the asymptotic Vasicek distribution better describe historical data than the Cox-Ingersoll-Ross model. However, there is no clear indication which of the two models is better. The asymptotic NPDF and the Vasicek distributions are practically indistinguishable except in the remote tails, where the NPDF is asymmetric and has thicker tails.

Exhibit 3.B displays a series of forecasting tests that correspond to the model validation tests.

- We chose 14 series of 256 historical daily data and computed the forecast distributions using the asymptotic NPDF, and Vasicek and CIR models.
- Next we found the historical distribution of interest rates in a series of 256 days following the first data series.
- We compared the computed and the historical distributions using the Kolmogorov-Smirnov and the Kuiper tests.
- A summary of the test results is presented in the following tables.

Table 3.2.6

P-values of forecasting tests based on the KS statistics

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| NPDF | 0.000225 | 0.007342 | 0.015098 | 0.01904 |
| Vasicek | 0.001221 | 0.008079 | 0.029572 | 0.034038 |
| CIR | 0.001959 | 5.2E-145 | 1E-147 | 1.01E-05 |

Table 3.2.7

P-values of forecasting tests based on the Kuiper statistics

| | 3-Month Rates | 3- Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| NPDF | 9.99E-09 | 5.28E-05 | 0.0044 | 0.001905 |
| Vasicek | 6.33E-08 | 0.000119 | 0.002893 | 0.002893 |
| CIR | 3.36E-07 | 6.8E-147 | 7.4E-150 | 9.65E-13 |

Please see Figure 1 for a typical picture of the historical and forecasted distributions of interest rates.

In general, the results of the forecasting tests are very close to those of the model validation tests: the Cox-Ingersoll-Ross model significantly overestimates the tails. The asymptotic NPDF and the Vasicek distribution are closer to the market data, and in the above graph are difficult to distinguish. There is no significant difference between these two models from the point of view of the Kolmogorov-Smirnov and the Kuiper tests.

### 3.2.3 Extreme events tests

The main advantage of our nonparametric approach over models based on the normal distribution of interest rates is better precision in the tail area, which produces better forecasts of extreme events. The tests shown in Exhibit 3.C were aimed at comparing the actual frequency of extreme deviations of interest rates from the mean value with the probabilities predicted by the NPDF, Vasicek, and Cox-Ingersoll-Ross models.

Consider pairs of consecutive time periods. The first period is used to estimate the distribution of daily changes of interest rates and the mean reversing parameters. These parameters are used to forecast the interest rate distribution for the next period. Then the forecast is compared with historical data, with special attention to the tails of the distributions.

The following test was used to compare the forecasted tail distribution against the observed one. We chose a small frequency of events $\eta$, which was equal to 0.01 in our tests. Then we found the deviation of interest rates from their mean value $y_0$, $\delta_-$ and $\delta_+$, such that the frequency of events with $y - y_0 < \delta_-$ and $y - y_0 > \delta_+$ is equal to $\eta$. In other words, $(y_0 + \delta_-)$ and $(y_0 + \delta_+)$ are $\eta$ and $(1- \eta)$ percentiles of the historical distribution. Next we computed the probabilities of large fluctuations with $y < y_0 + \delta_-$ and $y > y_0 + \delta_+$ according to the NPDF, Vasicek, and Cox-Ingersoll-Ross models.

Denote the forecast cumulative distribution function of interest rates as $F(y - y_0)$. If this distribution function coincided with the observed historical distribution, we would get

$$F(\delta_-) = \eta \text{ and } 1 - F(\delta_+) = \eta.$$

Let's define $P_- = F(\delta_-)$ and $P_+ = 1 - F(\delta_+)$. The deviations of $P_-$ and $P_+$ from $\eta$ measure the errors of the forecasts in the left and right tails. The value of $(P_\pm - \eta)$ always lies between $-\eta$ and $1 - \eta$. If this value is negative, the forecast underestimates the frequency of large deviations of interest rates. If it is positive, the forecast overestimates the frequency of the tail events. Both errors are dangerous. Underestimates of interest rate fluctuations can cause big losses. Overestimates can force the investor to buy unnecessary insurance against possible losses. Each investor should weight these errors according to his/her investment strategy and risk preferences. In this paper, we applied a weight function that places the errors in the interval from –1 to 1:

$$w(P) = \frac{P - \eta}{(1 - 2\eta)P + \eta}.$$

This measure weights both kinds of errors roughly equally. Distributions that overestimate tails have positive values $w(P)$, underestimates result in negative $w(P)$. A good forecast should yield small values of $w$. In a series of tests, a better forecast has a smaller mean value of $w$ and smaller fluctuations of this value. This suggests a better forecast is one with a smaller mean square value of $w$ for a series of tests.

The length of each time period is 256 business days, around 1 year. As we showed in Part 1, longer periods do not represent a single stationary process and the mean reversing is much weaker for longer periods. It appears a single mean reversion time is between one and two months. No single mean value during a long time period is representative. Instead, there are several different values that attract the interest rates on different subsets of the period. This results in smaller values of the mean reversing parameter and in a wider interest rate distribution for longer time periods and these distributions overestimate the tails. Shorter time periods yield a strong tendency to the mean value, but they are less interesting from the practical point of view. Furthermore, application of asymptotic formulas to a period shorter than a few mean reversion times is questionable.

Table 3.2.8 and 3.2.9 show summary results from Exhibit 3.C.

The mean squares of the measures $w_- = w(P_-)$ (left) and $w_+ = w(P_+)$ (right) averaged over 14 series of forecasts for four maturities are given. One can see that NPDF has the smallest value of $w_-$ for each maturity, i.e., NPDF gives the best estimate for the probability of large negative fluctuations of interest rates.

Table 3.2.8

| | | 3-Month Rates | 5-Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|---|
| NPDF | $w_-$ | 0.493381 | 0.462539 | 0.463719 | 0.482224 |
| | $W_+$ | 0.609075 | 0.467221 | 0.513906 | 0.570651 |
| Vasicek | $w_-$ | 0.756885 | 0.63032 | 0.593758 | 0.617438 |
| | $W_+$ | 0.612291 | 0.498277 | 0.517129 | 0.568891 |
| CIR | $w_-$ | 0.566857 | 0.669731 | 0.701424 | 0.712453 |
| | $w_+$ | 0.697422 | 0.741977 | 0.767592 | 0.770756 |

The following presents the sums of $w_-$ and $w_+$, which reflect the total precision of each model and for each maturity.

Table 3.2.9

| | 3-Month Rates | 5-Year Rates | 10-Year Rates | 30-Year Rates |
|---|---|---|---|---|
| NPDF | 1.102456 | 0.929761 | 0.977625 | 1.052875 |
| Vasicek | 1.369175 | 1.128597 | 1.110887 | 1.18633 |
| CIR | 1.26428 | 1.411708 | 1.469017 | 1.483208 |

One can see that the sum w = $w_- + w_+$ for NPDF is the smallest for each maturity. Thus, the asymptotic NPDF yields significantly more precise forecasts of large interest rates fluctuations compared to the Vasicek and CIR models.

### 3.3 Example: forecasting of interests options payoffs

To test the technique we have applied it to forecasting payoffs of interest options. We chose options as test instruments because they amplify the risk and they are more sensitive to underlying fluctuations. Particularly, the expected payoffs very are sensitive to the tails of the distributions.

There are standardized options on US treasuries yields traded on the Chicago Board of Option exchange (CBOE). We consider hypothetical long-term options (LEAPS) created according to the CBOE standard. They are European style options that expire on the third Saturday of June and December each year from 1984 to 1998. The underlying assets are the 3-month, 5-year, 10-year, and 30-year yields. The strikes change from 3% to 12% by 0.25%. The main difference between the hypothetical instruments and the real ones is that strikes of interest rate LEAPS traded at CBOE have a much narrower spectrum near the current yield value. We need such a wide strike spectrum in order to compare the precision of forecasts over the 15-year period when the interest rates changed from 10% to 3%.

#### 3.3.1 Rough tests

We chose a starting date 120 business days before each expiration date, i.e., the time horizon of the forecast, $T$, is 120 days. We computed the expected value of options payoffs based on NPDF, the Vasicek model, and the Cox-Ingersoll-Ross model. We used historical data available on the starting date for calibration and used the period of 128 days preceding the starting date for calibration in order to increase the forecast precision. As discussed in Part 1, longer time periods may include shorter periods, each with a different distribution, calibration based on longer series can yield less precise or even completely incorrect estimates. Since the mean reversing parameter of interest rates,

$\alpha$, is typically between 0.03 and 0.04, the mean reversion time is between 25 and 33 days. Using 128 days for calibration means and 120 for the time horizon means both these periods are significantly longer than the mean reversion time and we can apply asymptotic formulas for the forecast. We employed the asymptotic mean reversed NPDF, the Vasicek model, and the Cox-Ingersoll-Ross model to compute the expected options payoffs. The expected payoffs computed according the asymptotic NPDF, Vasicek, and Cox-Ingersoll-Ross models are presented in Exhibit 4. A.

To estimate the forecast precision we computed distances between the expected and historical payoffs. We defined the distance for each expiration date and for each type of options, put and call, averaging over the strikes. If the expected and historical payoffs for the strike $y_K$ are $E_K$ and $H_K$ the distance is

$$d = \left[ \frac{1}{M} \sum_{K=1}^{M} (E_K - H_K)^2 \right]^{1/2}$$

The distances for each expiration and for each kind of options are presented in Exhibit 4.B. In Tables 3.3.1 and 3.3.2 we present the distances averaged over all expiration dates.

Table 3.3.1

Averaged distances for call payoffs.

| | NPDF | Vasicek | CIR |
|---|---|---|---|
| 30-Year Rates | 0.795735 | 0.79318 | 2.895947 |
| 10-Year Rates | 0.578861 | 0.578269 | 0.955465 |
| 5-Year Rates | 1.108692 | 1.107851 | 1.316207 |
| 3-Month Rates | 0.668122 | 0.666409 | 1.895664 |

Table 3.3.2

Averaged distances for put payoffs.

| | NPDF | Vasicek | CIR |
|---|---|---|---|
| 30-Year Rates | 0.58967 | 0.589507 | 0.965161 |
| 10-Year Rates | 0.987179 | 0.98753 | 1.049722 |
| 5-Year Rates | 0.634321 | 0.634214 | 1.198192 |
| 3-Month Rates | 0.838893 | 0.8385 | 1.781132 |

The CIR forecast is worse than the NPDF and Vasicek forecasts. This is a result of highly overestimated tails in the CIR model as it is shown in Fig. 3.2 and discussed in Section 3.2.

The NPDF and Vasicek forecast results are very close, and the small differences between these two distances probably are due to the limited precision of the asymptotic NPDF computation rather than to model differences. The distances for these two models are almost identical because their distributions practically coincide near the center, significant differences occur only in the tails. The main contribution to the expected payoffs of in-the-money and at-the-money options comes from the center of the distribution. Only the expected payoffs out-of-the money options depend entirely on the tail behavior. However, the out-of-the money expected payoffs are small compared to those of in-the-money and at-the-money options. Therefore they do not significantly contribute to the distances in Tables 3.3.1 and 3.3.2.

Thus, the major conclusion of Section 3.3.1 is that the Cox-Ingersoll-Ross model yields significantly worse forecasts than the forecasts by NPDF and Vasicek distributions. To compare the last two forecasts additional detailed tests are needed.

### 3.3.2 Detailed tests

To compare NPDF and Vasicek forecasts we computed expected payoffs of out-of the-money options. More precisely, we considered European put and call options with the 120-business day maturity whose strikes are 1.00 % out-of the money with respect to the historical mean value. For calibration, we took the same 128 days period preceding the starting date. First, we have to find a suitable measure for the out-of-the money options.

The $L_2$ measure is useful for comparison of in-the-money and at-the-money options payoffs. It is inappropriate for out-of the-money options. Historical payoffs of out-of-the-money options are typically equal to zero, but sometimes they are substantial. The expected payoffs, $E_K$, are small. So the main contribution to the $L_2$ measure comes from the terms with non-zero historical payoffs, $H_K$. Consequently, the $L_2$ measure does not necessarily reflect the relationship between the expected payoff and the mean payoff.

There is no universal measure to compare the payoffs of out-of-the-money options. One approach is to use several data points; we compare the historical payoffs and the expected payoffs averaged over many expiration dates. This reflects profits and losses of an investor who systematically purchases the same amount of options for insurance. The average historical payoffs and the average expected payoffs computed with the right distribution must coincide when the interest rate distribution is stationary and the number of expiration dates tends to infinity.

In Tables 3.3.3 and 3.3.4, we present the averaged historical and expected payoffs of out-of-the-money options. Detailed results can be found in Exhibit 4.C.

Table 3.3.3

Averaged actual and expected payoffs of out-of-the-money call options.

| | Averaged payoff | NPDF | Vasicek |
|---|---|---|---|
| 30-Year Rates | 0.024333 | 0.005377 | 0.004985 |
| 10-Year Rates | 0.053 | 0.012911 | 0.012304 |
| 5-Year Rates | 0.079667 | 0.019202 | 0.018462 |
| 3-Month Rates | 0.076667 | 0.00087 | 0.000699 |

Table 3.3.4

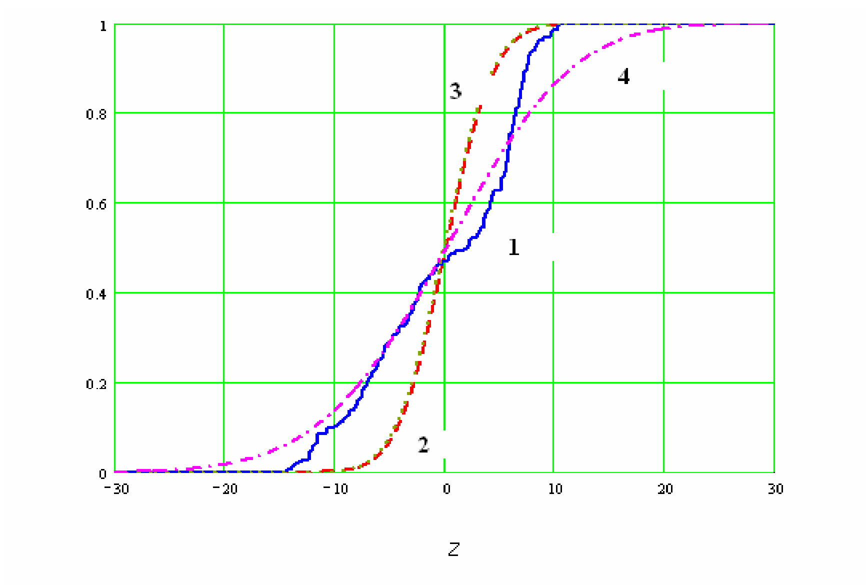
Averaged actual and expected payoffs of out-of-the-money put options.

| | Averaged payoff | NPDF | Vasicek |
|---|---|---|---|
| 30-Year Rates | 0.202333 | 0.042274 | 0.040068 |
| 10-Year Rates | 0.167 | 0.018377 | 0.016806 |
| 5-Year Rates | 0.213 | 0.031608 | 0.030481 |
| 3-Month Rates | 0.158 | 0.043321 | 0.042438 |

The NPDF forecasts are always closer to the historical payoffs because the NPDF has thicker tails.

**Figure 1**

Typical picture of the historical and forecasted distributions of interest rates




1 (solid line) – historical data

2 (dashed line) – asymptotic NPDF

3 (dotted line) – the Vasicek model

4 (dot-dashed line) – the Cox-Ingersoll-Ross model

$z$ is the deviation of the interest rates from the mean value measured in volatility units;

$z = (y - y_0)/\sigma$, $y_0$ is the mean interest rate.

# EXHIBIT 1

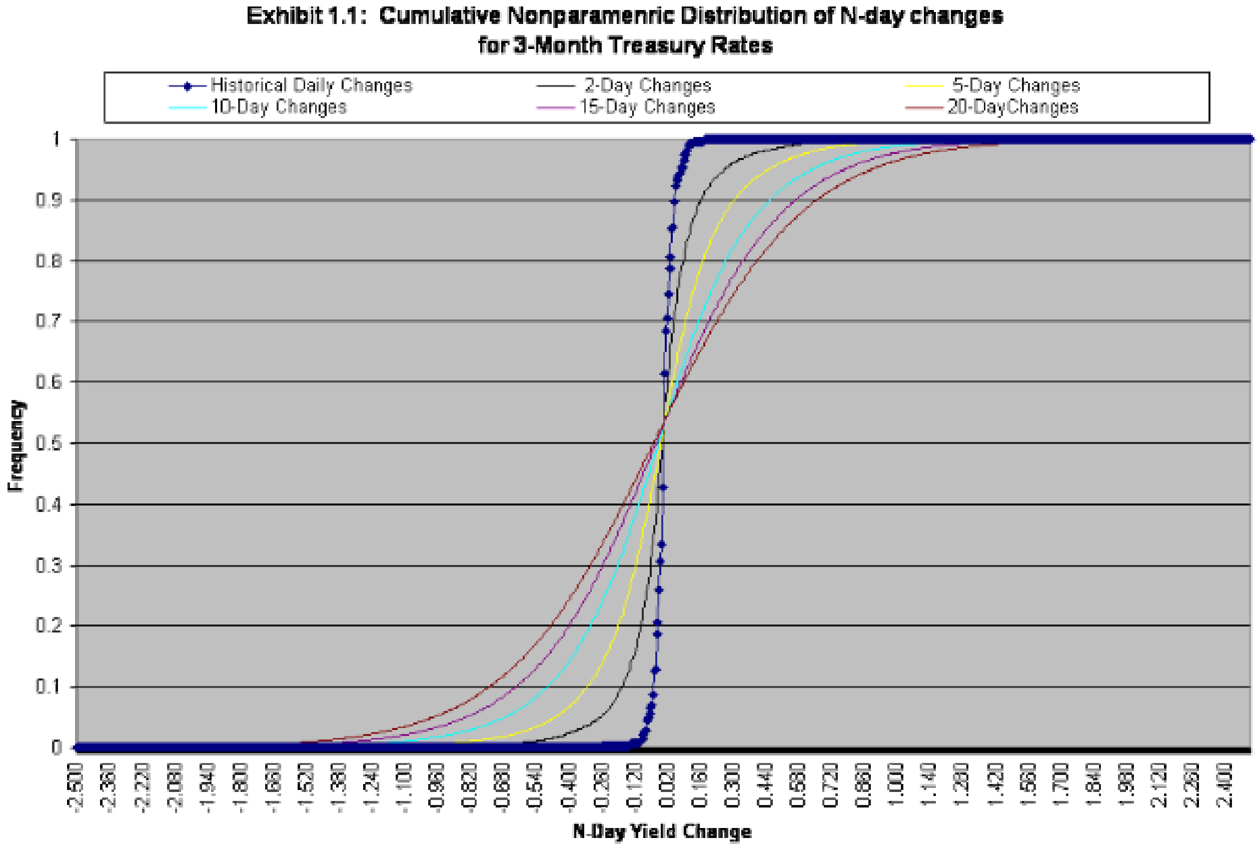
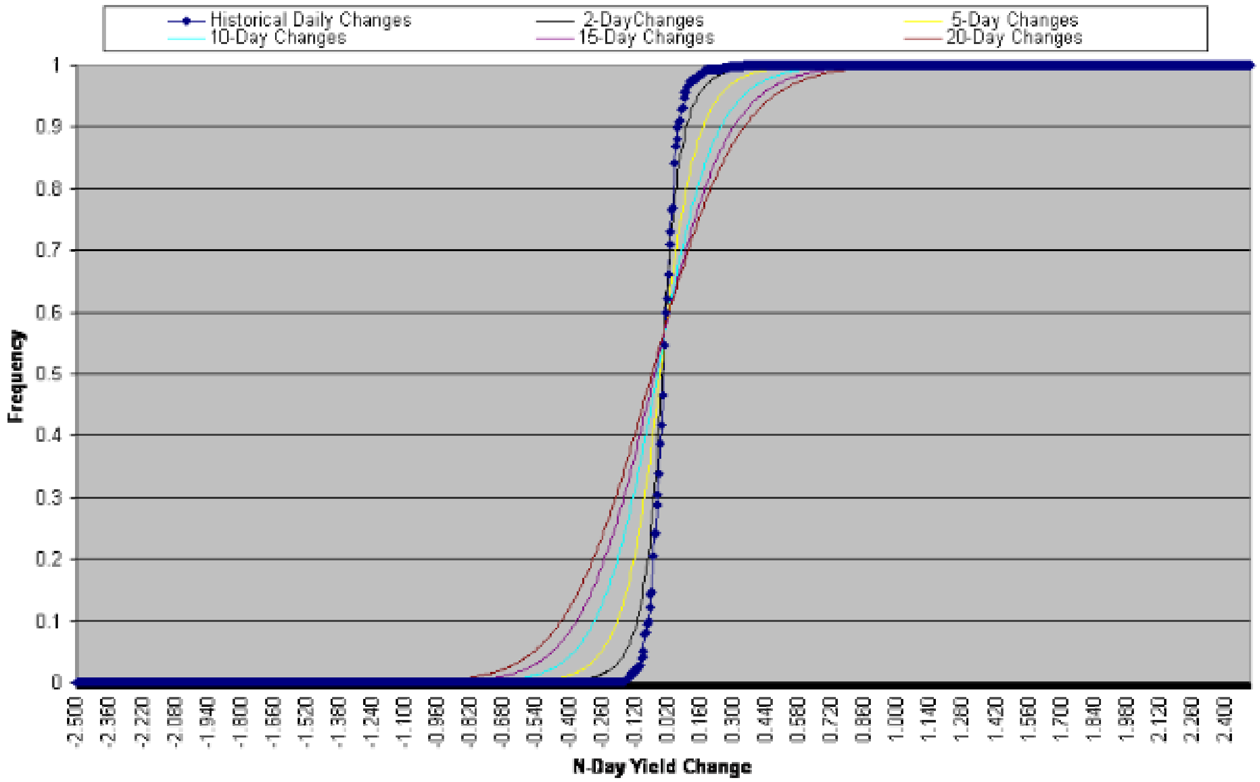


Exhibit 1.1: Cumulative Nonparamenric Distribution of N-day changes
for 3-Month Treasury Rates
Historical Daily Changes
2-Day Changes
5-Day Changes
10-Day Changes
15-Day Changes
20-DayChanges
Frequency
N-Day Yield Change

Exhibit 1.2: Cumulative Nonparamenric Distribution of N-day changes for 10-Year Treasury Rates



# EXHIBIT 2

**Exhibit 2.1 3-Month Maturity Significance Level = 10%**

| Base Year | Kuiper Stationary Periods | | | | | |
|---|---|---|---|---|---|---|
| **1983** | 1983 | 1984 | 1985 | 1987 | 1989 | | |
| **1984** | 1983 | 1984 | 1985 | 1987 | 1989 | | |
| **1985** | 1983 | 1984 | 1985 | 1987 | 1988 | 1989 | |
| **1986** | 1986 | 1988 | 1990 | 1991 | 1994 | 1998 | |
| **1987** | 1983 | 1984 | 1985 | 1987 | 1989 | | |
| **1988** | 1985 | 1986 | 1988 | | | | |
| **1989** | 1983 | 1984 | 1985 | 1987 | 1989 | | |
| **1990** | 1986 | 1990 | 1991 | 1994 | 1997 | 1998 | |
| **1991** | 1986 | 1990 | 1991 | 1995 | 1996 | 1997 | 1998 |
| **1992** | 1992 | 1993 | 1996 | | | | |
| **1993** | 1992 | 1993 | | | | | |
| **1994** | 1986 | 1990 | 1994 | 1998 | | | |
| **1995** | 1991 | 1995 | 1996 | 1997 | 1998 | | |
| **1996** | 1991 | 1992 | 1995 | 1996 | 1997 | | |
| **1997** | 1990 | 1991 | 1995 | 1996 | 1997 | 1998 | |
| **1998** | 1986 | 1990 | 1991 | 1994 | 1995 | 1997 | 1998 |

| Base Year | Kolmogorov-Smirnov Stationary Periods | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **1983** | 1983 | 1984 | 1985 | 1987 | 1989 | | | |
| **1984** | 1983 | 1984 | 1985 | 1987 | 1989 | | | |
| **1985** | 1983 | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1998 |
| **1986** | 1985 | 1986 | 1989 | 1990 | 1991 | 1995 | 1998 | |
| **1987** | 1983 | 1984 | 1985 | 1987 | 1989 | | | |
| **1988** | 1985 | 1988 | 1994 | | | | | |
| **1989** | 1983 | 1984 | 1985 | 1986 | 1987 | 1989 | | |
| **1990** | 1986 | 1990 | 1991 | 1994 | 1995 | 1997 | 1998 | |
| **1991** | 1986 | 1990 | 1991 | 1995 | 1997 | 1998 | | |
| **1992** | 1992 | 1993 | 1995 | 1996 | 1997 | | | |
| **1993** | 1992 | 1993 | | | | | | |
| **1994** | 1988 | 1990 | 1994 | 1998 | | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **1995** | 1986 | 1990 | 1991 | 1992 | 1995 | 1996 | 1997 | 1998 |
| **1996** | 1992 | 1995 | 1996 | 1997 | | | | |
| **1997** | 1990 | 1991 | 1992 | 1995 | 1996 | 1997 | 1998 | |
| **1998** | 1985 | 1986 | 1990 | 1991 | 1994 | 1995 | 1997 | 1998 |

**Exhibit 2.2 10-Year Maturity Significance Level = 10%**

| Base Year | Kuiper Stationary Periods |
|---|---|
| 1983 | 1983 1984 1985 1987 1994 1996 |
| 1984 | 1983 1984 1985 1986 1987 |
| 1985 | 1983 1984 1985 1987 |
| 1986 | 1984 1986 |
| 1987 | 1983 1984 1985 1987 1990 1994 |
| 1988 | 1988 1989 1990 1992 1993 1994 1995 1996 1998 |
| 1989 | 1988 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1990 | 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1998 |
| 1991 | 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1992 | 1988 1989 1990 1991 1992 1993 1994 1995 1996 1998 |
| 1993 | 1988 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1994 | 1983 1987 1988 1990 1992 1994 1996 |
| 1995 | 1988 1989 1990 1991 1992 1993 1995 1996 1997 1998 |
| 1996 | 1983 1988 1990 1992 1994 1995 1996 1998 |
| 1997 | 1989 1991 1993 1995 1997 1998 |
| 1998 | 1988 1989 1990 1991 1992 1993 1995 1996 1997 1998 |

| Base Year | Kolmogorov-Smirnov Stationary Periods |
|---|---|
| 1983 | 1983 1984 1987 1988 1990 1994 1996 |
| 1984 | 1983 1984 1985 1986 1987 1988 1990 1994 |
| 1985 | 1984 1985 1986 1987 |
| 1986 | 1984 1985 1986 |
| 1987 | 1983 1984 1985 1987 1988 1990 1994 1996 |
| 1988 | 1983 1984 1987 1988 1989 1990 1991 1992 1993 1994 1996 1998 |
| 1989 | 1988 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1990 | 1983 1984 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1998 |
| 1991 | 1988 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1992 | 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 |
| 1993 | 1988 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1994 | 1983 1984 1987 1988 1990 1992 1994 1996 |
| 1995 | 1989 1990 1991 1992 1993 1995 1997 1998 |
| 1996 | 1983 1987 1988 1990 1992 1994 1996 1997 1998 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **1997** | 1989 | 1991 | 1992 | 1993 | 1995 | 1996 | 1997 | 1998 | | |
| **1998** | 1988 | 1989 | 1990 | 1991 | 1992 | 1993 | 1995 | 1996 | 1997 | 1998 |

## EXHIBIT 3

### Exhibit 3.A

### Table A1. Results of model validation tests for 3-month interest rates

| Start | Length | KS test | | | Kuiper test | | |
|---|---|---|---|---|---|---|---|
| | | NPDF | Vasicek | CIR | NPDF | Vasicek | CIR |
| 0 | 256 | 2.33E-07 | 8.06E-08 | 2.92E-08 | 7.33E-26 | 2.75E-25 | 2.81E-29 |
| 256 | 256 | 8.57E-10 | 8.46E-08 | 2.39E-13 | 2.45E-19 | 9.65E-17 | 3.41E-40 |
| 512 | 256 | 2.09E-16 | 1.58E-19 | 9.06E-21 | 4.79E-33 | 2.34E-36 | 5.75E-53 |
| 768 | 256 | 3.45E-18 | 1.02E-19 | 0.026512 | 3.71E-51 | 4.50E-50 | 4.70E-06 |
| 1024 | 256 | 0.000109 | 1.15E-05 | 2.13E-16 | 1.70E-09 | 4.38E-11 | 1.65E-49 |
| 1280 | 256 | 5.08E-23 | 4.43E-24 | 0.000165 | 1.25E-76 | 1.38E-78 | 1.45E-11 |
| 1536 | 256 | 1.18E-07 | 3.06E-09 | 2.51E-21 | 1.26E-20 | 2.23E-22 | 3.40E-45 |
| 1792 | 256 | 1.39E-27 | 2.88E-26 | 2.52E-06 | 3.23E-71 | 1.07E-68 | 2.41E-16 |
| 2048 | 256 | 4.34E-38 | 4.55E-36 | 1.69E-11 | 1.99E-111 | 2.49E-108 | 2.20E-39 |
| 2304 | 256 | 1.17E-18 | 9.56E-19 | 2.00E-06 | 3.04E-62 | 2.49E-64 | 4.74E-22 |
| 2560 | 256 | 5.98E-09 | 2.37E-10 | 5.87E-11 | 1.73E-22 | 2.84E-23 | 9.69E-20 |
| 2816 | 256 | 3.61E-29 | 1.03E-28 | 2.78E-10 | 1.97E-103 | 1.34E-106 | 4.26E-27 |
| 3072 | 256 | 1.99E-12 | 4.48E-14 | 0.000742 | 1.60E-42 | 6.29E-38 | 1.48E-10 |
| 3328 | 256 | 0.00681 | 0.002284 | 3.05E-10 | 1.35E-07 | 3.29E-08 | 1.69E-30 |
| 3584 | 256 | 0.335449 | 0.398905 | 2.28E-13 | 0.109042 | 0.065559 | 4.32E-48 |
| Average | | 0.022824 | 0.026747 | 0.001828 | 0.007269 | 0.004371 | 3.14E-07 |

**Table A2. Results of model validation tests for 10-year interest rates**

| Start | Length | KS test | | | Kuiper test | | |
|---|---|---|---|---|---|---|---|
| | | NPDF | Vasicek | CIR | NPDF | Vasicek | CIR |
| 0 | 256 | 2.59E-17 | 5.30E-16 | 7.39E-10 | 3.18E-58 | 1.56E-57 | 2.10E-34 |
| 256 | 256 | 0.002029 | 0.000185 | 1.22E-24 | 3.93E-09 | 4.90E-10 | 1.89E-73 |
| 512 | 256 | 5.71E-11 | 1.76E-10 | 1.11E-05 | 3.69E-27 | 6.25E-27 | 3.81E-19 |
| 768 | 256 | 1.08E-10 | 8.12E-11 | 1.14E-10 | 1.53E-23 | 5.42E-23 | 4.46E-30 |
| 1024 | 256 | 0.000851 | 0.001189 | 3.40E-13 | 1.64E-11 | 1.73E-08 | 5.24E-40 |
| 1280 | 256 | 2.00E-06 | 5.13E-05 | 1.68E-08 | 1.64E-15 | 3.86E-13 | 1.09E-30 |
| 1536 | 256 | 6.51E-06 | 5.13E-07 | 1.47E-20 | 4.21E-18 | 6.30E-22 | 6.10E-46 |
| 1792 | 256 | 0.000434 | 0.001339 | 5.97E-11 | 3.46E-11 | 1.32E-09 | 1.51E-35 |
| 2048 | 256 | 8.18E-10 | 2.06E-07 | 8.62E-13 | 2.04E-17 | 8.77E-19 | 1.18E-40 |
| 2304 | 256 | 7.78E-12 | 1.60E-11 | 0.03418 | 5.38E-34 | 4.07E-35 | 7.74E-06 |
| 2560 | 256 | 0.021169 | 0.007496 | 1.92E-07 | 7.98E-06 | 3.43E-06 | 1.80E-26 |
| 2816 | 256 | 0.004431 | 0.008498 | 7.33E-23 | 3.71E-06 | 3.24E-07 | 1.40E-74 |
| 3072 | 256 | 0.029788 | 0.010663 | 6.51E-11 | 4.45E-06 | 1.34E-05 | 1.86E-36 |
| 3328 | 256 | 0.002562 | 0.013105 | 1.18E-13 | 0.000126 | 0.000267 | 9.51E-40 |
| 3584 | 256 | 1.08E-08 | 4.16E-10 | 1.67E-06 | 1.85E-27 | 2.31E-27 | 2.01E-13 |
| Average | | 0.004085 | 0.002835 | 0.00228 | 9.49E-06 | 1.9E-05 | 5.16E-07 |

**Exhibit 3.B**

Table B.1 Results of forecast tests for 3-month interest rates.

| Start | Length | KS test | | | Kuiper test | | |
|---|---|---|---|---|---|---|---|
| | | NPDF | Vasicek | CIR | NPDF | Vasicek | CIR |
| 0 | 256 | 1.82E-21 | 8.82E-20 | 2.92E-08 | 5.24E-56 | 1.26E-54 | 2.81E-29 |
| 256 | 256 | 8.12E-13 | 1.74E-15 | 2.39E-13 | 1.48E-33 | 4.11E-35 | 3.41E-40 |
| 512 | 256 | 0.00012 | 8.35E-06 | 9.06E-21 | 2.43E-09 | 7.83E-11 | 5.75E-53 |
| 768 | 256 | 0.000215 | 2.45E-06 | 0.026512 | 2.04E-11 | 1.83E-14 | 4.70E-06 |
| 1024 | 256 | 9.51E-24 | 1.01E-23 | 2.13E-16 | 6.03E-78 | 1.18E-78 | 1.65E-49 |
| 1280 | 256 | 1.63E-10 | 3.15E-12 | 0.000165 | 2.06E-29 | 2.02E-31 | 1.45E-11 |
| 1536 | 256 | 5.93E-16 | 5.00E-15 | 2.51E-21 | 1.03E-43 | 9.24E-43 | 3.40E-45 |
| 1792 | 256 | 7.99E-40 | 8.25E-39 | 2.52E-06 | 3.43E-118 | 2.11E-116 | 2.41E-16 |
| 2048 | 256 | 2.78E-09 | 2.03E-08 | 1.69E-11 | 3.95E-24 | 1.32E-27 | 2.20E-39 |
| 2304 | 256 | 2.20E-05 | 2.51E-10 | 2.00E-06 | 2.31E-12 | 6.71E-21 | 4.74E-22 |
| 2560 | 256 | 1.60E-44 | 4.43E-45 | 5.87E-11 | 8.85E-173 | 6.93E-172 | 9.69E-20 |
| 2816 | 256 | 3.64E-05 | 0.001506 | 2.78E-10 | 7.29E-12 | 3.58E-11 | 4.26E-27 |
| 3072 | 256 | 1.08E-11 | 3.15E-08 | 0.000742 | 2.42E-24 | 7.90E-23 | 1.48E-10 |
| 3328 | 256 | 0.002761 | 0.015579 | 3.05E-10 | 1.37E-07 | 8.86E-07 | 1.69E-30 |
| Average | | 0.000225 | 0.001221 | 0.001959 | 9.99E-09 | 6.33E-08 | 3.36E-07 |

**Table B.2 Results of forecast tests for 10-year interest rates.**

| Start | Length | KS test | | | Kuiper test | | |
|---|---|---|---|---|---|---|---|
| | | NPDF | Vasicek | CIR | NPDF | Vasicek | CIR |
| 0 | 256 | 3.21E-17 | 7.84E-18 | 2.80E-226 | 6.53E-48 | 9.47E-49 | 1.80E-224 |
| 256 | 256 | 0.000165 | 0.001242 | 1.32E-146 | 6.59E-09 | 8.36E-09 | 9.64E-149 |
| 512 | 256 | 2.07E-05 | 7.85E-07 | 6.70E-226 | 8.61E-17 | 4.62E-18 | 4.33E-224 |
| 768 | 256 | 2.99E-14 | 3.30E-13 | 2.52E-226 | 1.10E-35 | 4.77E-34 | 1.63E-224 |
| 1024 | 256 | 1.81E-10 | 4.04E-09 | 2.52E-226 | 9.70E-33 | 3.02E-31 | 1.63E-224 |
| 1280 | 256 | 1.90E-25 | 6.02E-26 | 2.52E-226 | 3.13E-64 | 1.73E-65 | 1.63E-224 |
| 1536 | 256 | 0.000863 | 0.034786 | 2.52E-226 | 1.03E-05 | 0.000343 | 1.63E-224 |
| 1792 | 256 | 3.63E-13 | 1.22E-11 | 2.52E-226 | 1.39E-31 | 2.87E-32 | 1.63E-224 |
| 2048 | 256 | 0.000987 | 0.000502 | 2.52E-226 | 1.01E-08 | 3.79E-09 | 1.63E-224 |
| 2304 | 256 | 6.11E-07 | 4.71E-06 | 2.52E-226 | 1.64E-21 | 8.86E-21 | 1.63E-224 |
| 2560 | 256 | 0.189648 | 0.343285 | 2.52E-226 | 0.057189 | 0.037272 | 1.63E-224 |
| 2816 | 256 | 0.004589 | 0.004617 | 2.52E-226 | 5.10E-08 | 4.96E-08 | 1.63E-224 |
| 3072 | 256 | 5.81E-11 | 7.41E-12 | 2.52E-226 | 5.95E-34 | 2.95E-33 | 1.63E-224 |
| Average | | 0.015098 | 0.029572 | 1E-147 | 0.0044 | 0.002893 | 7.4E-150 |

**Exhibit 3.C**

Detailed results of the comparison of the tail forecasting are presented in Exhibit. The starting day of a test is shown in the first column. In the second two columns the lengths of the calibration period and of the forecast period are shown. Next, the value of the parameter $\eta$ is given. In two columns under the title NPDF, the values of $w_{\pm}$ for the NPDF forecast are given for each pair of historic periods. Below those columns, the mean value, the standard deviation, the variance, and the mean square of data are shown. Finally, the mean square of both columns, Total Mean, is presented. The next two pairs of columns represent the values of $w_{\pm}$, their mean values, and mean squares for the Vasicek and Cox-Ingersoll-Ross models the same way as it is done for NPDF.

**Table C.1. Distances of extreme events for 3-month interest rates.**

| Start | Percentile | NPDF | | Vasicek | | CIR | |
|---|---|---|---|---|---|---|---|
| | | W- | W+ | W- | W+ | W- | W+ |
| 0 | 0.01 | -0.84464 | -1 | -0.99757 | -1 | 0.762404 | 0.33863 |
| 256 | 0.01 | -0.14123 | 0.826081 | -0.26346 | 0.811018 | 0.926958 | 0.956959 |
| 512 | 0.01 | -0.69648 | 0.332512 | -0.95578 | 0.285532 | 0.820004 | 0.928898 |
| 768 | 0.01 | -0.8179 | -0.52195 | -0.99999 | -0.58502 | 0.016822 | 0.869181 |
| 1024 | 0.01 | -0.90888 | -0.99823 | -0.99978 | -0.9982 | 0.241921 | 0.64851 |
| 1280 | 0.01 | -0.77502 | 0.632665 | -0.98972 | 0.573741 | 0.747667 | 0.939051 |
| 1536 | 0.01 | -0.0253 | -0.99832 | -0.03014 | -0.99103 | 0.925026 | 0.810965 |
| 1792 | 0.01 | -0.76742 | -1 | -0.99904 | -0.99999 | 0.640017 | 0.467109 |
| 2048 | 0.01 | -0.54419 | -0.13736 | -0.93054 | -0.16083 | 0.712201 | 0.880277 |
| 2304 | 0.01 | -0.38729 | 0.595982 | -0.87108 | 0.630984 | 0.523552 | 0.905759 |
| 2560 | 0.01 | -0.90727 | -1 | -1 | -1 | -1 | -1 |
| 2816 | 0.01 | 0.512029 | 0.636578 | 0.573816 | 0.567995 | 0.927446 | 0.924396 |
| 3072 | 0.01 | 0.819402 | 0.805867 | 0.844017 | 0.773665 | 0.953429 | 0.944107 |
| 3328 | 0.01 | -0.89486 | -0.79973 | -0.98687 | -0.91523 | 0.624061 | 0.771865 |
| Mean | | -0.45565 | -0.18756 | -0.61472 | -0.21481 | 0.558679 | 0.670408 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| St. dev. | | 0.554752 | 0.786155 | 0.638868 | 0.78083 | 0.523765 | 0.51677 |
| Var. | | 0.30775 | 0.61804 | 0.408153 | 0.609696 | 0.27433 | 0.267051 |
| Mean sq. | | 0.493381 | 0.609075 | 0.756885 | 0.612291 | 0.566857 | 0.697422 |
| Total mean | | 0.554254884 | | 0.688394584 | | 0.63550183 | |

**Table C.2 Distances of extreme events for 10-year interest rates.**

| Start | Percentile | NPDF | | Vasicek | | CIR | |
|---|---|---|---|---|---|---|---|
| | | W- | W+ | W- | W+ | W- | W+ |
| 0 | 0.01 | -0.79966 | -0.9107 | -0.99835 | -0.92999 | 0.821558 | 0.896011 |
| 256 | 0.01 | -0.00053 | 0.018801 | -0.09046 | -0.03051 | 0.943539 | 0.938731 |
| 512 | 0.01 | -0.77351 | 0.850316 | -0.97983 | 0.839128 | 0.864039 | 0.960761 |
| 768 | 0.01 | -0.95779 | -1 | -1 | -1 | -0.86516 | -0.09341 |
| 1024 | 0.01 | 0.897715 | 0.639526 | 0.907516 | 0.65071 | 0.969026 | 0.946782 |
| 1280 | 0.01 | -0.91547 | -0.98528 | -1 | -0.97711 | -0.04793 | 0.850501 |
| 1536 | 0.01 | 0.604395 | 0.466615 | 0.639615 | 0.393041 | 0.950766 | 0.937463 |
| 1792 | 0.01 | -0.22042 | -0.98109 | -0.28161 | -0.98062 | 0.919002 | 0.839497 |
| 2048 | 0.01 | -0.35406 | -0.68404 | -0.57437 | -0.72148 | 0.897406 | 0.888051 |
| 2304 | 0.01 | -0.83112 | -0.61372 | -0.99996 | -0.67576 | 0.364102 | 0.878771 |
| 2560 | 0.01 | -0.39145 | 0.303809 | -0.24684 | 0.063594 | 0.890335 | 0.907383 |
| 2816 | 0.01 | 0.05143 | 0.771056 | 0.024857 | 0.733669 | 0.922797 | 0.948294 |
| 3072 | 0.01 | 0.797023 | 0.603301 | 0.824449 | 0.520765 | 0.955666 | 0.931346 |
| 3328 | 0.01 | -0.81964 | -0.43188 | -0.98214 | -0.6204 | 0.757874 | 0.881039 |
| Mean | | -0.26522 | -0.13952 | -0.33979 | -0.19535 | 0.667359 | 0.836516 |
| St. dev. | | 0.650873 | 0.729708 | 0.717698 | 0.718198 | 0.525122 | 0.270281 |
| Var. | | 0.423636 | 0.532474 | 0.515091 | 0.515809 | 0.275753 | 0.073052 |
| Mean sq. | | 0.463719 | 0.513906 | 0.593758 | 0.517129 | 0.701424 | 0.767592 |
| Total mean | | 0.48945624 | | 0.556763483 | | 0.735253179 | |

# EXHIBIT 4

## Exhibit 4.A

In Tables 4.A.1 and 4.A.2, we present the results of expected payoff according asymptotic NPDF, Vasicek, and Cox-Ingersoll-Ross formulas, and the historical payoffs of interest 30-year rates puts and calls expired in December 1998.

Table 4.A.1 Actual and expected payoffs of calls expired in December 1998.

| Strike | Actual Payoff | NPDF | Vasicek | CIR |
|---:|---:|---:|---:|---:|
| 3 | 2.01 | 2.879571 | 2.872791 | 1.971955 |
| 3.25 | 1.76 | 2.629412 | 2.622791 | 1.800457 |
| 3.5 | 1.51 | 2.379261 | 2.372791 | 1.62896 |
| 3.75 | 1.26 | 2.129119 | 2.122791 | 1.457462 |
| 4 | 1.01 | 1.878988 | 1.872791 | 1.285964 |
| 4.25 | 0.76 | 1.628869 | 1.622791 | 1.114467 |
| 4.5 | 0.51 | 1.378764 | 1.372791 | 0.942969 |
| 4.75 | 0.26 | 1.128677 | 1.122791 | 0.771472 |
| 5 | 0.01 | 0.878611 | 0.872791 | 0.599991 |
| 5.25 | 0 | 0.628573 | 0.622791 | 0.42902 |
| 5.5 | 0 | 0.378587 | 0.372805 | 0.263747 |
| 5.75 | 0 | 0.135653 | 0.130755 | 0.124837 |
| 6 | 0 | 0.008316 | 0.00737 | 0.040141 |
| 6.25 | 0 | 1.62E-05 | 1.25E-05 | 0.007972 |
| 6.5 | 0 | 0 | 2.75E-10 | 0.000931 |
| 6.75 | 0 | 0 | 5.90E-17 | 6.30E-05 |
| 7 | 0 | 0 | 1.10E-25 | 2.49E-06 |
| 7.25 | 0 | 0 | 1.69E-36 | 5.83E-08 |
| 7.5 | 0 | 0 | 2.06E-49 | 8.30E-10 |
| 7.75 | 0 | 0 | 1.98E-64 | 7.36E-12 |

| | | | | |
|---|---|---|---|---|
| 8 | 0 | 0 | 1.47E-81 | 4.16E-14 |
| 8.25 | 0 | 0 | 8.45E-101 | 1.53E-16 |
| 8.5 | 0 | 0 | 3.71E-122 | 3.78E-19 |
| 8.75 | 0 | 0 | 1.25E-145 | 6.36E-22 |
| 9 | 0 | 0 | 3.18E-171 | 7.44E-25 |
| 9.25 | 0 | 0 | 6.15E-199 | 6.18E-28 |
| 9.5 | 0 | 0 | 9.03E-229 | 3.70E-31 |
| 9.75 | 0 | 0 | 1.00E-260 | 1.62E-34 |
| 10 | 0 | 0 | 8.41E-295 | 5.30E-38 |
| 10.25 | 0 | 0 | 0 | 1.31E-41 |
| 10.5 | 0 | 0 | 0 | 2.47E-45 |
| 10.75 | 0 | 0 | 0 | 3.60E-49 |
| 11 | 0 | 0 | 0 | 4.12E-53 |

Table 4.A.2 Actual and expected payoffs of puts expired in December 1998.

| Strike | Actual Payoff | NPDF | Vasicek | CIR |
|---|---|---|---|---|
| 3 | 0 | 0.000375 | 3.38E-145 | 1.09E-40 |
| 3.25 | 0 | 0.000439 | 9.25E-122 | 4.34E-33 |
| 3.5 | 0 | 0.000511 | 1.93E-100 | 1.14E-26 |
| 3.75 | 0 | 0.000593 | 3.09E-81 | 2.89E-21 |
| 4 | 0 | 0.000685 | 3.80E-64 | 9.64E-17 |
| 4.25 | 0 | 0.00079 | 3.64E-49 | 5.45E-13 |
| 4.5 | 0 | 0.000909 | 2.73E-36 | 6.49E-10 |
| 4.75 | 0 | 0.001045 | 1.64E-25 | 1.95E-07 |
| 5 | 0 | 0.001203 | 8.07E-17 | 1.75E-05 |
| 5.25 | 0.24 | 0.001389 | 3.46E-10 | 0.000544 |
| 5.5 | 0.49 | 0.001626 | 1.45E-05 | 0.006769 |
| 5.75 | 0.74 | 0.008915 | 0.007964 | 0.039356 |
| 6 | 0.99 | 0.131802 | 0.13458 | 0.126158 |
| 6.25 | 1.24 | 0.373726 | 0.377222 | 0.265487 |
| 6.5 | 1.49 | 0.623933 | 0.627209 | 0.429943 |
| 6.75 | 1.74 | 0.874157 | 0.877209 | 0.600573 |
| 7 | 1.99 | 1.12438 | 1.127209 | 0.77201 |
| 7.25 | 2.24 | 1.374604 | 1.377209 | 0.943505 |
| 7.5 | 2.49 | 1.624827 | 1.627209 | 1.115003 |
| 7.75 | 2.74 | 1.875051 | 1.877209 | 1.2865 |
| 8 | 2.99 | 2.125274 | 2.127209 | 1.457998 |
| 8.25 | 3.24 | 2.375498 | 2.377209 | 1.629496 |
| 8.5 | 3.49 | 2.625721 | 2.627209 | 1.800993 |
| 8.75 | 3.74 | 2.875945 | 2.877209 | 1.972491 |
| 9 | 3.99 | 3.126168 | 3.127209 | 2.143989 |
| 9.25 | 4.24 | 3.376392 | 3.377209 | 2.315486 |
| 9.5 | 4.49 | 3.626616 | 3.627209 | 2.486984 |

| | | | | |
|---|---|---|---|---|
| 9.75 | 4.74 | 3.876839 | 3.877209 | 2.658482 |
| 10 | 4.99 | 4.127063 | 4.127209 | 2.829979 |
| 10.25 | 5.24 | 4.377286 | 4.377209 | 3.001477 |
| 10.5 | 5.49 | 4.62751 | 4.627209 | 3.172975 |
| 10.75 | 5.74 | 4.877733 | 4.877209 | 3.344472 |
| 11 | 5.99 | 5.127957 | 5.127209 | 3.51597 |
| 11.25 | 6.24 | 5.37818 | 5.377209 | 3.687468 |
| 11.5 | 6.49 | 5.628404 | 5.627209 | 3.858965 |
| 11.75 | 6.74 | 5.878627 | 5.877209 | 4.030463 |
| 12 | 6.99 | 6.128851 | 6.127209 | 4.20196 |

**Exhibit 4.B**

**Table 4.B.1 Distances between actual and expected payoffs of calls on 3-month interest rates.**

| Expiration | NPDF | Vasicek | CIR |
|---|---|---|---|
| 8406 | 0.740432 | 0.748196 | 1.953743 |
| 8412 | 1.312698 | 1.305306 | 4.570649 |
| 8506 | 1.919081 | 1.910406 | 7.696159 |
| 8512 | 0.538621 | 0.531483 | 3.038471 |
| 8606 | 0.641953 | 0.639308 | 0.419936 |
| 8612 | 0.588279 | 0.581926 | 1.029046 |
| 8706 | 0.097049 | 0.103299 | 0.168238 |
| 8712 | 0.144562 | 0.147276 | 0.157765 |
| 8806 | 0.238829 | 0.240935 | 0.739738 |
| 8812 | 1.414019 | 1.418904 | 1.107198 |
| 8906 | 0.609771 | 0.612491 | 0.378241 |
| 8912 | 0.633124 | 0.623992 | 2.916685 |
| 9006 | 0.067388 | 0.063057 | 0.542994 |
| 9012 | 0.776348 | 0.774049 | 0.291459 |
| 9106 | 1.008339 | 1.005931 | 0.732979 |
| 9112 | 0.855862 | 0.85163 | 0.736539 |
| 9206 | 0.497805 | 0.495332 | 0.468988 |
| 9212 | 0.147983 | 0.146008 | 0.066842 |
| 9306 | 0.012956 | 0.011257 | 0.010411 |
| 9312 | 0.001711 | 0.002073 | 0.003059 |
| 9406 | 0.256755 | 0.257297 | 0.260451 |
| 9412 | 0.757865 | 0.759409 | 0.775866 |
| 9506 | 0.328931 | 0.331094 | 0.326436 |
| 9512 | 0.241405 | 0.238594 | 0.166548 |
| 9606 | 0.098769 | 0.097139 | 0.294817 |

| | | | |
|---|---|---|---|
| 9612 | 0.040568 | 0.038737 | 0.317289 |
| 9706 | 0.048373 | 0.04626 | 0.287964 |
| 9712 | 0.062887 | 0.066352 | 0.21966 |
| 9806 | 0.01768 | 0.016195 | 0.325042 |
| 9812 | 0.280043 | 0.278219 | 0.10406 |
| Average | 0.668122 | 0.666409 | 1.895664 |

**Table 4.B.2 Distances between actual and expected payoffs of puts on 3-month interest rates.**

| Expiration | NPDF | Vasicek | CIR |
|---|---|---|---|
| 8406 | 0.472037 | 0.474354 | 1.424551 |
| 8412 | 0.997968 | 0.996201 | 0.479137 |
| 8506 | 1.55194 | 1.550021 | 0.765843 |
| 8512 | 0.534338 | 0.531628 | 1.613037 |
| 8606 | 0.781921 | 0.781271 | 1.059265 |
| 8612 | 0.819515 | 0.817267 | 0.441554 |
| 8706 | 0.163133 | 0.164889 | 0.179759 |
| 8712 | 0.22594 | 0.227136 | 1.229338 |
| 8806 | 0.335617 | 0.336812 | 2.963845 |
| 8812 | 1.580336 | 1.582078 | 2.686177 |
| 8906 | 0.585545 | 0.586446 | 0.922864 |
| 8912 | 0.550018 | 0.54713 | 0.802062 |
| 9006 | 0.05907 | 0.05768 | 0.35524 |
| 9012 | 0.833964 | 0.833485 | 1.387583 |
| 9106 | 1.297393 | 1.29686 | 1.591106 |
| 9112 | 1.803948 | 1.80289 | 2.115999 |
| 9206 | 1.229703 | 1.229011 | 1.40017 |
| 9212 | 0.608378 | 0.608631 | 2.770261 |
| 9306 | 0.043471 | 0.043565 | 1.706929 |
| 9312 | 0.056375 | 0.055433 | 3.589005 |
| 9406 | 1.035527 | 1.033833 | 2.905249 |
| 9412 | 1.758607 | 1.758756 | 0.709733 |
| 9506 | 0.597129 | 0.597925 | 0.576412 |
| 9512 | 0.389665 | 0.389097 | 1.587599 |
| 9606 | 0.16813 | 0.168026 | 1.915096 |
| 9612 | 0.073185 | 0.073116 | 2.275675 |

| | | | |
|---|---|---|---|
| 9706 | 0.085399 | 0.085192 | 2.023568 |
| 9712 | 0.119209 | 0.119919 | 0.838789 |
| 9806 | 0.027684 | 0.027508 | 1.989126 |
| 9812 | 0.574627 | 0.574473 | 2.341326 |
| Average | 0.838893 | 0.8385 | 1.781132 |

**Table 4.B.3 Distances between actual and expected payoffs of calls on 10-year interest rates.**

| Expiration | NPDF | Vasicek | CIR |
|---|---|---|---|
| 8406 | 1.757745 | 1.757605 | 4.284245 |
| 8412 | 2.414662 | 2.409176 | 1.511155 |
| 8506 | 1.006825 | 1.003712 | 2.235486 |
| 8512 | 1.143259 | 1.138716 | 1.312025 |
| 8606 | 1.044693 | 1.040973 | 0.446065 |
| 8612 | 0.200154 | 0.19487 | 0.176241 |
| 8706 | 0.857899 | 0.859797 | 0.373244 |
| 8712 | 0.396911 | 0.402316 | 0.185487 |
| 8806 | 0.154773 | 0.158232 | 0.847866 |
| 8812 | 0.249286 | 0.256284 | 0.74063 |
| 8906 | 0.567797 | 0.565716 | 0.219072 |
| 8912 | 0.284003 | 0.281317 | 0.430486 |
| 9006 | 0.510445 | 0.512625 | 0.52681 |
| 9012 | 0.179934 | 0.176169 | 0.218583 |
| 9106 | 0.129145 | 0.134563 | 0.521073 |
| 9112 | 0.889213 | 0.887917 | 0.832218 |
| 9206 | 0.27517 | 0.277687 | 0.078257 |
| 9212 | 0.260897 | 0.257896 | 0.725897 |
| 9306 | 0.437146 | 0.432941 | 1.287895 |
| 9312 | 0.021269 | 0.018645 | 1.156334 |
| 9406 | 0.844373 | 0.848282 | 0.674354 |
| 9412 | 0.402122 | 0.407633 | 0.69331 |
| 9506 | 1.06127 | 1.055847 | 1.384658 |
| 9512 | 0.208829 | 0.206534 | 0.747458 |
| 9606 | 0.791025 | 0.794549 | 0.359252 |
| 9612 | 0.241217 | 0.237679 | 1.399288 |

| | | | |
|---|---|---|---|
| 9706 | 0.039528 | 0.041747 | 0.879127 |
| 9712 | 0.430872 | 0.428243 | 1.311545 |
| 9806 | 0.15418 | 0.151613 | 1.304387 |
| 9812 | 0.411184 | 0.408794 | 1.801504 |
| Average | 0.578861 | 0.578269 | 0.955465 |

**Table 4.B.4 Distances between actual and expected payoffs of puts on 10-year interest rates.**

| Expiration | NPDF | Vasicek | CIR |
|---|---|---|---|
| 8406 | 4.074393 | 4.082543 | 1.493311 |
| 8412 | 0.649841 | 0.648645 | 1.197004 |
| 8506 | 0.864427 | 0.862164 | 0.520571 |
| 8512 | 0.202274 | 0.196971 | 0.127022 |
| 8606 | 0.823196 | 0.82527 | 0.342289 |
| 8612 | 1.958674 | 1.947573 | 1.859354 |
| 8706 | 0.31272 | 0.31684 | 0.187679 |
| 8712 | 0.130165 | 0.130415 | 0.852951 |
| 8806 | 0.118475 | 0.121014 | 0.732411 |
| 8812 | 0.180935 | 0.185693 | 0.602237 |
| 8906 | 0.429567 | 0.428287 | 0.198277 |
| 8912 | 0.252158 | 0.249995 | 0.455341 |
| 9006 | 0.446062 | 0.448028 | 0.468886 |
| 9012 | 0.148036 | 0.145181 | 0.209403 |
| 9106 | 0.113102 | 0.117506 | 0.498207 |
| 9112 | 0.867011 | 0.866021 | 0.813471 |
| 9206 | 0.307456 | 0.310353 | 0.095939 |
| 9212 | 0.291452 | 0.288303 | 0.746472 |
| 9306 | 0.566814 | 0.561789 | 1.437482 |
| 9312 | 0.028778 | 0.024688 | 1.371464 |
| 9406 | 1.083993 | 1.090536 | 0.645733 |
| 9412 | 0.407503 | 0.413301 | 0.537063 |
| 9506 | 1.178504 | 1.173861 | 1.434322 |
| 9512 | 0.294406 | 0.291214 | 0.948585 |
| 9606 | 1.054007 | 1.06017 | 0.367311 |
| 9612 | 0.295108 | 0.29089 | 1.411169 |

| | | | |
|---|---|---|---|
| 9706 | 0.052858 | 0.055817 | 0.964407 |
| 9712 | 0.593916 | 0.590569 | 1.547352 |
| 9806 | 0.23713 | 0.233095 | 1.624015 |
| 9812 | 0.766464 | 0.762235 | 2.487316 |
| Average | 0.987179 | 0.98753 | 1.049722 |

**Exhibit 4.C**

**Table 4.C.1 Actual and expected payoffs of out-of-the-money calls on 3-month yields.**

| Expiration | Strike | Payoff | NPDF | Vas. |
|---|---|---|---|---|
| 9812 | 6.03 | 0 | 1.54E-06 | 0 |
| 9806 | 6.06 | 0 | 0 | 0 |
| 9712 | 6.05 | 0 | 0 | 0 |
| 9706 | 6.04 | 0 | 0 | 0 |
| 9612 | 5.97 | 0 | 0 | 0 |
| 9606 | 6.32 | 0 | 1.56E-05 | 0 |
| 9512 | 6.66 | 0 | 0 | 0 |
| 9506 | 5.79 | 0 | 0.004885 | 0.003981 |
| 9412 | 4.57 | 0.99 | 0.000982 | 0.000491 |
| 9406 | 4.03 | 0.09 | 0 | 0 |
| 9312 | 3.97 | 0 | 0 | 0 |
| 9306 | 4.1 | 0 | 0 | 0 |
| 9212 | 4.8 | 0 | 0 | 0 |
| 9206 | 6.02 | 0 | 0 | 0 |
| 9112 | 6.82 | 0 | 0 | 0 |
| 9106 | 8.27 | 0 | 0 | 0 |
| 9012 | 8.75 | 0 | 0 | 0 |
| 9006 | 8.78 | 0 | 0 | 0 |
| 8912 | 9.47 | 0 | 0 | 6.18E-09 |
| 8906 | 8.27 | 0 | 0.013557 | 0.011024 |
| 8812 | 6.94 | 1.22 | 0.003746 | 0.003186 |
| 8806 | 6.95 | 0 | 7.22E-06 | 1.85E-06 |
| 8712 | 6.6 | 0 | 0 | 0 |
| 8706 | 6.46 | 0 | 0 | 1.57E-09 |
| 8612 | 7.54 | 0 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| 8606 | 8.13 | 0 | 0 | 0 |
| 8512 | 8.82 | 0 | 0 | 4.77E-09 |
| 8506 | 10.65 | 0 | 8.91E-05 | 0 |
| 8412 | 10.47 | 0 | 0.002807 | 0.002273 |
| 8406 | 9.98 | 0 | 8.13E-06 | 4.17E-06 |
| | | | | |
| Average | | 0.076667 | 0.00087 | 0.000699 |

**Table 4.C.2 Actual and expected payoffs of out-of-the-money calls on 10-year yields.**

| Expiration | Strike | Payoff | NPDF | Vas. |
|---|---|---|---|---|
| 9812 | 6.6 | 0 | 0 | 0 |
| 9806 | 7.1 | 0 | 0 | 0 |
| 9712 | 7.62 | 0 | 0 | 0 |
| 9706 | 7.59 | 0 | 0 | 0 |
| 9612 | 7.3 | 0 | 0.001676 | 0.000835 |
| 9606 | 7.12 | 0 | 0 | 0 |
| 9512 | 8.11 | 0 | 0 | 0 |
| 9506 | 8.54 | 0 | 0.000111 | 0.000128 |
| 9412 | 7.51 | 0.3 | 0.037491 | 0.032943 |
| 9406 | 6.62 | 0.52 | 0 | 7.63E-08 |
| 9312 | 7.17 | 0 | 0 | 0 |
| 9306 | 7.71 | 0 | 0 | 0 |
| 9212 | 8.32 | 0 | 0 | 0 |
| 9206 | 8.7 | 0 | 0 | 0 |
| 9112 | 9.08 | 0 | 0 | 0 |
| 9106 | 9.56 | 0 | 0 | 0 |
| 9012 | 9.54 | 0 | 0 | 0 |
| 9006 | 9.03 | 0 | 0 | 0 |
| 8912 | 10.02 | 0 | 0 | 0 |
| 8906 | 10.02 | 0 | 0 | 0 |
| 8812 | 9.67 | 0 | 0.00041 | 0.000478 |
| 8806 | 9.97 | 0 | 4.59E-07 | 1.72E-05 |
| 8712 | 8.73 | 0.16 | 0.03944 | 0.034709 |
| 8706 | 8.3 | 0 | 0 | 0 |
| 8612 | 9.12 | 0 | 0 | 3.37E-07 |
| 8606 | 11.09 | 0 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| 8512 | 12.22 | 0 | 0 | 2.59E-09 |
| 8506 | 13.42 | 0 | 0 | 0 |
| 8412 | 13.54 | 0 | 0.308206 | 0.299998 |
| 8406 | 12.6 | 0.61 | 0 | 3.29E-09 |
| | | | | |
| Average | | 0.053 | 0.012911 | 0.012304 |

**Table 4.C.3 Actual and expected payoffs of out-of-the-money puts on 3-month yields.**

| Expiration | Strike | Payoff | NPDF | Vas. |
|---|---|---|---|---|
| 9812 | 4.03 | 0 | 0.000116 | 0 |
| 9806 | 4.06 | 0 | 0.001411 | 0 |
| 9712 | 4.05 | 0 | 0.001614 | 0 |
| 9706 | 4.04 | 0 | 0.00053 | 0 |
| 9612 | 3.97 | 0 | 0.001136 | 0 |
| 9606 | 4.32 | 0 | 0.00024 | 0 |
| 9512 | 4.66 | 0 | 0.002235 | 0 |
| 9506 | 3.79 | 0 | 0.001436 | 0 |
| 9412 | 2.57 | 0 | 0.002581 | 0 |
| 9406 | 2.03 | 0 | 0.001989 | 0 |
| 9312 | 1.97 | 0 | 0.002199 | 0 |
| 9306 | 2.1 | 0 | 0.001427 | 0 |
| 9212 | 2.8 | 0 | 0.002025 | 0 |
| 9206 | 4.02 | 0.37 | 0.170162 | 0.176799 |
| 9112 | 4.82 | 1.08 | 0.001684 | 1.09E-09 |
| 9106 | 6.27 | 0.7 | 0.005396 | 0.00409 |
| 9012 | 6.75 | 0.22 | 0.000564 | 0 |
| 9006 | 6.78 | 0 | 0.000144 | 0 |
| 8912 | 7.47 | 0 | 0.003984 | 0.000792 |
| 8906 | 6.27 | 0 | 0.001071 | 0 |
| 8812 | 4.94 | 0 | 0.002139 | 0 |
| 8806 | 4.95 | 0 | 0.001733 | 0.000395 |
| 8712 | 4.6 | 0 | 0.000108 | 0 |
| 8706 | 4.46 | 0 | 0.003202 | 0 |
| 8612 | 5.54 | 0.03 | 0.004516 | 0.001246 |
| 8606 | 6.13 | 0.03 | 0.000228 | 0 |

| | | | | |
|---|---|---|---|---|
| 8512 | 6.82 | 0 | 0.107898 | 0.109186 |
| 8506 | 8.65 | 1.61 | 0.972095 | 0.98059 |
| 8412 | 8.47 | 0.7 | 0.002533 | 2.59E-07 |
| 8406 | 7.98 | 0 | 0.003229 | 5.23E-05 |
| | | | | |
| Average | | 0.158 | 0.043321 | 0.042438 |

**Table 4.C.4 Actual and expected payoffs of out-of-the-money puts on 10-year yields.**

| Expiration | Strike | Payoff | NPDF | Vas. |
|---|---|---|---|---|
| 9812 | 4.6 | 0.02 | 0.000196 | 0 |
| 9806 | 5.1 | 0 | 0.001 | 5.66E-09 |
| 9712 | 5.62 | 0 | 0.002456 | 1.82E-09 |
| 9706 | 5.59 | 0 | 0.002568 | 1.26E-05 |
| 9612 | 5.3 | 0 | 0.002056 | 1.11E-10 |
| 9606 | 5.12 | 0 | 0.004094 | 0.000613 |
| 9512 | 6.11 | 0.36 | 0.081897 | 0.083493 |
| 9506 | 6.54 | 0.33 | 0.002803 | 1.42E-09 |
| 9412 | 5.51 | 0 | 0.002991 | 3.91E-09 |
| 9406 | 4.62 | 0 | 0.003113 | 2.77E-10 |
| 9312 | 5.17 | 0 | 0.001869 | 4.05E-07 |
| 9306 | 5.71 | 0 | 0.001661 | 3.81E-10 |
| 9212 | 6.32 | 0 | 0.002029 | 1.36E-08 |
| 9206 | 6.7 | 0 | 0.019685 | 0.018597 |
| 9112 | 7.08 | 0.11 | 0.000623 | 0 |
| 9106 | 7.56 | 0 | 0.004059 | 0.00052 |
| 9012 | 7.54 | 0 | 0.001555 | 1.89E-09 |
| 9006 | 7.03 | 0 | 0.000813 | 0 |
| 8912 | 8.02 | 0.22 | 0.029849 | 0.028444 |
| 8906 | 8.02 | 0 | 0.000905 | 0 |
| 8812 | 7.67 | 0 | 0.004861 | 7.33E-06 |
| 8806 | 7.97 | 0 | 0.004118 | 0.000468 |
| 8712 | 6.73 | 0 | 0.003923 | 6.05E-07 |
| 8706 | 6.3 | 0 | 0.00033 | 0 |
| 8612 | 7.12 | 0.02 | 0.059338 | 0.059113 |
| 8606 | 9.09 | 1.49 | 0.147389 | 0.150037 |

| | | | | |
|---|---|---|---|---|
| 8512 | 10.22 | 1.18 | 0.068747 | 0.067894 |
| 8506 | 11.42 | 1.08 | 0.094177 | 0.094989 |
| 8412 | 11.54 | 0.2 | 0.002026 | 0 |
| 8406 | 10.6 | 0 | 0.000177 | 0 |
| | | | | |
| Average | | 0.167 | 0.018377 | 0.016806 |

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