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A PRACTICAL APPROACH TO GAINS ANALYSIS

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ABSTRACT

The pension actuary has always been on unsure ground in attempting to identify by source experience gains (and losses) in a pension plan. His approach has usually been more intuitive than scientific. The purpose of this paper is to describe an actuarially acceptable and mathematically definitive method for allocating gains by source.

THE pension actuary's interest in gains analysis has been heightened by the new requirements of the Employee Retirement Income Security Act of 1974 (ERISA). Not only must the experience gain or loss of a pension plan be reported (unless an aggregate cost method is used), but the enrolled actuary must certify that in his opinion the valuation assumptions "(i) are in the aggregate reasonably related to the experience of the plan and to reasonable expectations; and (ii) represent his best estimate of anticipated experience under the plan."[1] It is now more important than ever for him to evaluate and understand the effect on the plan experience of the actuarial assumptions, and, by implication, the effect of each assumption.

This paper will describe a general approach to gains determination that is not only mathematically correct but practical as well. The approach is essentially automatic and can be used for any pension plan, regardless of complexity or funding method. The approach also allows the pension actuary to determine the experience changes in a pension plan by source with such accuracy that any deviation between the sum of the individually determined changes and the total change invariably can be traced to faulty valuation technique.

The paper is organized in four parts. The first part deals with plans funded on an individual cost method, such as entry age normal or unit credit. The second part covers plans funded on an aggregate cost method, such as the frozen initial liability method. The third part describes the individual calculations that underlie the analysis of the change in values by source. The fourth part summarizes the first three parts.

[1] Employee Retirement Income Security Act of 1974, sec. 103(a)(4)(B).

Throughout the paper the period over which experience is measured is assumed to be one year, from time 0 to time 1. The theory can be extended readily to cover periods other than one year. In fact, the computer program developed by the author to implement the approach was designed to operate over any interval.

The effects of change in assumptions or funding method are not covered in the paper. However, they can be isolated by performing an additional valuation on the new basis after the approach described herein has been applied on the old basis.

I. INDIVIDUAL COST METHOD

The following definitions will be used.

A_t = Value of assets at time t;
AL_t = Accrued liability at time t;
UL_t = Unfunded liability at time t;
NC = Normal cost, employer and employee, value at time 0;
EE = Expected expense, value at time 0;
i = Annual rate of interest assumed;
C = Actual contributions, employer and employee;
B = Actual benefits and refunds paid;
E = Actual expenses paid;
I = Increment in assets from investment income and capital gains reflected in the valuation of assets;
${}^iC, {}^iB, {}^iE$ = C, B, and E, respectively, with interest at rate i to time 1.

For plans funded on an individual cost method, such as the entry age normal or the unit credit method, the gain or loss from each source is expressed as a monetary amount (in dollars, for example). The total gain, G, can be defined[2] by formula (1):

$$G = (UL_0 + NC + EE)(1 + i) - {}^iC - UL_1 . \quad (1)$$

We know that

$$UL_0 = AL_0 - A_0 ; \quad (2)$$

$$UL_1 = AL_1 - A_1 ; \quad (3)$$

and

$$A_1 = A_0 + C - B - E + I . \quad (4)$$

It can be demonstrated algebraically that

$$G = G_I + G_E + \sum_K G_K , \quad (5)$$

[2] Revenue Ruling 59-153.

where

$$G_I = A_1 - [A_0(1 + i) + {}^iC - {}^iB - {}^iE] ; \qquad (6)$$

$$G_E = EE(1 + i) - {}^iE ; \qquad (7)$$

and

$$\sum_K G_K = (AL_0 + NC)(1 + i) - AL_1 - {}^iB . \qquad (8)$$

Substituting expressions (6)–(8) in formula (5) produces formula (1).

It is customary and reasonable to consider G_I and G_E as the gains from the interest and expense assumptions. Formula (8), then, expresses the aggregate value of gains and losses from all sources except the interest and expense assumptions.

Formula (8) may be visualized as the sum of a matrix of two dimensions. One dimension is the universe of plan participants (as of time 0 or time 1 or both), and the other dimension is the universe of sources of gains and losses being studied.

Let the prefacing of the subscript j represent a value for a single participant. We can see that the basic component of our gains analysis is ${}_jG_K$, the gain attributable to participant j from cause K.

$${}_jG_K = [({}_jAL_0 + {}_jNC)(1 + i) - {}_jAL_1]_K - {}_j^iB_K . \qquad (9)$$

In practice, the actuary will determine ${}_j^iB_K$, the actual benefits and refunds paid, from the trustees' report on the fund transactions. Our remaining task, then, will be to determine each $[({}_jAL_0 + {}_jNC)(1 + i) - {}_jAL_1]_K$.

Let us now introduce the general concept of using EV and V to represent potential year-end value and actual year-end value for a single participant. In other words, under an individual cost method,

$$EV = ({}_jAL_0 + {}_jNC)(1 + i) ; \qquad (10)$$

$$V = {}_jAL_1 . \qquad (11)$$

Let ${}_jGV$ equal $EV - V$. The following relationships arise.

$${}_jGV = EV - V \qquad (12)$$

$$= ({}_jAL_0 + {}_jNC)(1 + i) - {}_jAL_1 \qquad (13)$$

$$= \sum_K [({}_jAL_0 + {}_jNC)(1 + i) - {}_jAL_1]_K \qquad (14)$$

$$= \sum_K (EV - V)_K \qquad (15)$$

$$= \sum_K {}_jGV_K . \qquad (16)$$

The individual calculations to produce ${}_jGV_K$ for participant j and cause K are described in Section III of this paper.

With reference to formula (8), it should be noted that, once the ${}_jGV_K$ values are calculated,

$$\sum_K G_K = \sum_K \sum_j {}_jGV_K - \sum_K \sum_j {}^i_jB_K . \tag{17}$$

II. AGGREGATE COST METHOD

The following definitions will be used in addition to those already given.

PVB_t = Value at time t of future benefits and refunds;

PVS_t = Value at time t of future salaries;

PVC_t = Value at time t of future employee contributions;

S = Value at time 0 of salaries expected to be paid during year;

EC = Value at time 0 of employee contributions expected to be made during year;

f_t = Normal cost accrual factor

$$= (PVB_t - PVC_t - AL_t)/PVS_t . \tag{18}$$

For plans funded on an aggregate cost method, the deviations from expected experience usually are expressed as changes in the normal cost accrual factor. Although the mathematics herein are based on the frozen initial liability method, the approach is sufficiently general to apply to any aggregate method.

To repeat formula (1), the total gain, G, for a pension plan is

$$G = (UL_0 + NC + EE)(1 + i) - {}^iC - UL_1 .$$

But under an aggregate cost method there is no gain. Therefore, G is equal to zero, and

$$UL_1 = (UL_0 + NC + EE)(1 + i) - {}^iC . \tag{19}$$

Formulas (2)–(4) will still apply, of course.

From formula (18) we know that

$$AL_0 = PVB_0 - PVC_0 - f_0PVS_0 ; \tag{20}$$

$$AL_1 = PVB_1 - PVC_1 - f_1PVS_1 . \tag{21}$$

We also know that

$$NC = f_0S + EC . \tag{22}$$

Substituting expressions (2), (3), and (20)–(22) in formula (19), we have

$$\begin{aligned} PVB_1 - PVC_1 - f_1PVS_1 - A_1 \\ = (PVB_0 - PVC_0 - f_0PVS_0 - A_0 \\ + f_0S + EC + EE)(1 + i) - {}^iC . \end{aligned} \tag{23}$$

Adding $f_0 PVS_1$ to both sides of equation (23) and rearranging terms produces

$$\begin{aligned} f_0 PVS_1 - f_1 PVS_1 & \\ = f_0 PVS_1 &+ (PVB_0 - PVC_0 - f_0 PVS_0 - A_0 \\ &+ f_0 S + EC + EE)(1+i) \\ &- {}^iC - (PVB_1 - PVC_1 - A_1)\,. \end{aligned} \tag{24}$$

This can be expressed as

$$f_0 - f_1 = \Big(G_I + G_E + \sum_K G_K\Big)\Big/ PVS_1\,, \tag{25}$$

where G_I and G_E are defined in formulas (6) and (7), and

$$\begin{aligned} \sum_K G_K = {} & [PVB_0(1+i) - PVB_1] \\ & - [(PVC_0 - EC)(1+i) - PVC_1] \\ & - f_0[(PVS_0 - S)(1+i) - PVS_1] - {}^iB\,. \end{aligned} \tag{26}$$

Substituting expressions (6), (7), and (26) in formula (25) produces formula (24).

It is apparent from formula (25) that the changes in the normal cost accrual factor arising from the interest and expense assumptions can be determined by formulas (27) and (28), respectively.

$$(f_0 - f_1)_I = G_I/PVS_1\,; \tag{27}$$

$$(f_0 - f_1)_E = G_E/PVS_1\,. \tag{28}$$

The change in the normal cost accrual factor from all causes except the interest and expense assumptions is then

$$\begin{aligned} \sum_K (f_0 - f_1)_K = {} & \{[PVB_0(1+i) - PVB_1] \\ & - [(PVC_0 - EC)(1+i) - PVC_1] \\ & - f_0[(PVS_0 - S)(1+i) - PVS_1] \\ & - {}^iB\}/PVS_1\,. \end{aligned} \tag{29}$$

Formula (29) may be visualized as the sum of a matrix of two dimensions. One dimension is the universe of plan participants, and the other is the universe of sources of change in the normal cost accrual factor. Again representing a value for a single participant by prefacing the subscript j, formula (29) can be expressed as

$$\sum_K (f_0 - f_1)_K = \sum_K \sum_j \{[{}_jPVB_0(1+i) - PVB_1]_K - [({}_jPVC_0 - {}_jEC)(1+i) - {}_jPVC_1]_K - f_0[({}_jPVS_0 - {}_jS)(1+i) - {}_jPVS_1]_K - {}^i_jB_K\}/PVS_1 \,. \tag{30}$$

Formula (30) looks more formidable than in fact it is, as may be seen when its components are reduced to general form. As earlier, introduce the general EV and V to represent, respectively, the potential and actual year-end values. Now, however, EV and V represent successively the present values of benefits and refunds (B), employee contributions (C), and salaries (S).

The following definitions will apply.

$$EV(B) = {}_jPVB_0(1+i); \tag{31}$$

$$EV(C) = ({}_jPVC_0 - {}_jEC)(1+i)\,; \tag{32}$$

$$EV(S) = ({}_jPVS_0 - {}_jS)(1+i)\,; \tag{33}$$

$$V(B) = {}_jPVB_1\,; \tag{34}$$

$$V(C) = {}_jPVC_1\,; \tag{35}$$

$$V(S) = {}_jPVS_1\,. \tag{36}$$

Generalizing within formulas (31)–(36), the following relationships will apply.

$${}_jGV = EV - V\,; \tag{37}$$

$${}_jGV_K = (EV - V)_K\,. \tag{38}$$

Formula (30) then becomes simple and manageable:

$$\sum_K (f_0 - f_1)_K = \sum_K \sum_j \{{}_jGV(B)_K - {}_jGV(C)_K - f_0\,{}_jGV(S)_K - {}^i_jB_K\}/PVS_1\,. \tag{39}$$

As noted earlier, in practice the actuary will determine ${}^i_jB_K$ from the trustees' report on the fund transactions.

The individual calculations to produce the generalized ${}_jGV_K$ from the generalized EV and V for participant j and cause K are described in Section III.

III. INDIVIDUAL CHANGES IN VALUES

General

The objective in this section is to determine the value of ${}_jGV_K$ for participant j and cause K. Once the value is determined, it can be summed within the single cause K over the universe of plan participants for use in formula (17) under an individual cost method or in formula (39) under an aggregate cost method. In short, the ultimate objective is $\Sigma_j\ {}_jGV_K$.

Consider again the following relationships which were developed previously.

$$
\begin{aligned}
{}_jGV &= EV - V \\
&= \sum_K (EV - V)_K \\
&= \sum_K {}_jGV_K \,.
\end{aligned}
$$

Summing ${}_jGV_K$ over all causes K for a single participant must produce that participant's $(EV - V)$. It is important to understand that $(EV - V)$ is a generalized concept for a single participant and may, according to circumstances, be referring to the participant's accrued liabilities, as in formulas (10) and (11), or to his present values of benefits and refunds, as in (31) and (34), or to his present values of employee contributions, as in (32) and (35), or to his present values of future salaries, as in (33) and (36).

To compute EV and V for a participant requires valuations at time 0 and time 1. In order to allocate the changes in values by cause, two or more valuations may be required at time 1.

Although the remainder of the paper applies specifically to active participants, the principles are identical for nonactive participants.

Cause K Is Data Correction

It is essential that new values computed for time 0 agree with those used at the time, even if they were based on incorrect data. The change in values attributable to the correction of data is

$$
{}_jGV_K = EV - EV' \,, \tag{40}
$$

where EV is based on the original incorrect data and EV' is the same expression based on corrected data. Note that all of the subsequent calculations should use EV' in lieu of EV.

Cause K Is Decrement Assumption K

Decrement assumptions in a valuation are typically retirement, withdrawal, disablement, preretirement mortality, postretirement mortality, and postdisablement mortality. Several of these may be select and ultimate probabilities. The change in values for the participant attributable to decrement assumption K is

$$_{j}GV_{K} = AR_{K} - ER_{K}\,, \tag{41}$$

where AR_K is the participant's actual release (if any) and ER_K is his expected release from cause K. The determination of these values follows.

The valuation at time 0 required the use of a service table, which, if select and ultimate probabilities were used, could easily be unique for each participant. The service table is of the following form.

$$l_1 = l_0 - \sum_K d_0^K\,. \tag{42}$$

Define V' to be the expected year-end value contemplated at time 0 should the participant survive in the same status to time 1. By definition,

$$V' = EV + \sum_K ER_K\,. \tag{43}$$

Furthermore, define NL_K to be the year-end value contemplated at time 0 of expected benefits and new liabilities should the participant be subjected to decrement from cause K during the year. (Note that if EV represents present values of employee contributions or salaries, NL_K equals zero.)

According to the basic principle that the values at the beginning of the year were designed to provide adequate year-end values for survivors and nonsurvivors,

$$EV = \left(l_1 V' + \sum_K d_0^K NL_K\right)\Big/ l_0\,. \tag{44}$$

Solving equation (44) for V', we have

$$V' = \left(l_0 EV - \sum_K d_0^K NL_K\right)\Big/ l_1\,. \tag{45}$$

Solving equations (43) and (45) for $\Sigma_K\, ER_K$, we have

$$\sum_K ER_K = \sum_K d_0^K (EV - NL_K)/l_1\,, \tag{46}$$

from which we can deduce that

$$ER_K = d_0^K (EV - NL_K)/l_1\,. \tag{47}$$

We can define AR_K (if the participant actually decremented from cause K) to be

$$AR_K = V' - V\,, \tag{48}$$

where V' is defined in formula (43) and V is the value at time 1 of any actual postdecrement liabilities arising after decrement from cause K.

Of course, AR_K will be zero if the participant was not affected by decrement from cause K during the year. Note that if the participant was affected by decrement during the year, $V(C)$ and $V(S)$, defined in formulas (35) and (36), will be zero; and note further that V as defined in formula (11) or formula (34) will be zero unless the participant was included in the valuation at time 1 as a nonactive life with his new post-decrement benefit (if any).

It is important to note that if, during the year, the participant was a new entrant, or transferred into or out of the plan, ER_K in formula (47) and ${}_jGV_K$ in (41) should be defined to be zero.

Cause K Is New Entrant or Transfer In or Out

If during the year the participant was a new entrant, or transferred into the plan,

$$ {}_jGV_K = -V \,. \tag{49} $$

If during the year the participant transferred out of the plan,

$$ {}_jGV_K = EV \,. \tag{50} $$

Cause K Is Change in Benefits

For a continuing active participant, the effect of changes in benefits arising from such sources as salary changes, fractional service credits, and changes in social security benefits or wage bases can be isolated by performing multiple valuations at time 1. The expected year-end value for a continuing active participant is V', defined in formula (43). His actual year-end value, based on his circumstances at time 1, is V. The aggregate change in values for him arising from change in benefits is $(V' - V)$. The separate sources of change can be isolated by computing $V_1, V_2, \ldots, V_{n-1}$, altering only the single item being studied. Then

$$ {}_jGV_{K_1} = V_1 - V \,; \tag{51} $$

$$ {}_jGV_{K_2} = V_2 - V_1 \,; \tag{52} $$

$$ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot $$

$$ {}_jGV_{K_n} = V' - V_{n-1} \,. \tag{53} $$

We can see that

$$ \sum_{t=1}^{n} {}_jGV_{K_t} = V' - V \,. \tag{54} $$

As indicated below, each ${}_jGV_{K_t}$ arising from change in benefits is to be defined to be zero for the continuing active participant who was valued with a 100 per cent probability of retiring at time 0 but was not subject to decrement from any cause.

Special Situation

The participant who was valued with a 100 per cent probability of retiring at time 0 presents a special situation. His service table shows that d_0^K equals l_0 for the retirement decrement, and d_0^K equals zero for all other decrements. Furthermore, his l_1 in the preretirement service table equals zero. A review of formulas (41)–(47) shows that they are not appropriate for him.

Returning to basic principles, we recall that the aggregate change in values for the participant totals $(EV - V)$. It is reasonable and practical to allocate this entire amount to the retirement assumption, and to exclude this participant from any other allocation. Therefore, for this participant

$$ {}_jGV_K = EV - V , \tag{55}$$

where cause K is the retirement assumption. For all other participants, the change in values attributable to the retirement assumption is covered in the section headed "Cause K Is Decrement Assumption K."

Allocation Hierarchies

The approach described herein presumes a certain hierarchy of allocations of changes in values by source. The changes in values arising from actual versus expected changes in benefits, for instance, are determined only for participants who do not change status during the year. Participants who exit during the year have their entire releases and any benefit payments or new liabilities allocated to the appropriate decrement, with no recognition given to actual versus expected changes in benefits between time 0 and date of exit.

IV. SUMMARY

Individual Cost Method

With respect to the gains analysis of a pension plan under an individual cost method, the author has endeavored to establish the following points.

1. The total gain is determined by formula (1).
2. The portions of the gain attributable to the interest and expense assumptions can be determined by formulas (6) and (7) from information in the trustees' report on fund transactions.
3. The total gain from all other sources is shown in formula (8), which can be construed as a matrix over participants and sources of gain, as illustrated in formula (17).
4. The allocation of actual benefits and refunds against sources of gain within formula (17) can be done from information in the trustees' report on fund transactions.
5. The allocation of the individual values of ${}_jGV_K$ for formula (17) is covered in Section III of this paper. These calculations are a byproduct of the plan valuation and are independent of the fund transactions.

When these steps have been completed, the experience gain for the pension plan has been allocated by source with such accuracy that any deviation of the sum of the individually determined gains from the total gain may be traced to faulty valuation technique.

Aggregate Cost Method

With respect to the analysis of the change in the normal cost accrual factor of a plan under an aggregate cost method, the author has attempted to establish the following points.

1. The total deviation from expected experience is reflected in the change in the normal cost accrual factor, that is, $(f_0 - f_1)$.
2. The changes in the normal cost accrual factor attributable to the interest and expense assumptions can be determined by formulas (27) and (28).
3. The total change from all other sources is shown in formula (29), which may be construed as a matrix over participants and sources of change, as illustrated in formulas (30) and (39).
4. The allocation of actual benefits and refunds against sources of change within formula (39) can be done from information in the trustees' report on fund transactions.
5. The determination of the individual values of ${}_jGV_K$ for formula (39) is covered in Section III of this paper. These calculations are a byproduct of the plan valuation and are independent of the fund transactions.

When these steps have been completed, the change in the normal cost accrual factor has been allocated by source with such accuracy that any deviation of the sum of the individually determined changes from the total change may be traced to faulty valuation technique.

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DISCUSSION OF PRECEDING PAPER

PAULETTE TINO:

In his paper Mr. Lynch develops a logical and cohesive presentation of the analysis of gains and losses. The following remarks reflect my aesthetic inclinations.

In general, a gain is the difference between the expected value of an item and its actual value. Any mathematical expression of a gain should be readily interpretable in the light of that principle; to that effect the following modifications of some of the equations in Mr. Lynch's paper are offered.

Individual Cost Method

Equation (8) of the paper expresses the aggregate gain exclusive of the interest and expense gains. I propose the following expression:

$$\sum_K G_K = [(AL_0 + NC)(1+i) - {}^iEB] - AL_1 + ({}^iEB - {}^iB)\,,$$

where ${}^iEB$ represents the expected benefit payments and the liabilities expected to be established on account of a decrement, and ${}^iB$ the actual benefit payments and the actual liabilities established on account of a decrement. The reason for introducing ${}^iEB$ can best be explained by taking as an example the vesting component V_x of the liability calculated as the discounted value of future term costs ($TC_x = q_x^w B_x N_{65}^{(12)}/D_x$). For employees aged x at the beginning of the year,

$$V_x = \frac{D_xTC_x + D_{x+1}TC_{x+1} + \ldots + D_{64}TC_{64}}{D_x}\,.$$

When decrements are restricted to interest, death, and withdrawal,

$$V_{x+1} = V_x(1+i) - TC_x(1+i) + (q_x^d + q_x^w)\,V_{x+1}\,, \qquad (1)$$

where V_{x+1} is the vesting liability at the end of the year for all active employees included in the computation at the beginning of the year. V_{x+1} is distributed according to the various statuses acquired by these employees at the end of the year. (All decreasing liability functions develop along the same pattern. The negative term, similar to $-TC_x(1+i)$, is often called the "dropout." This term is absent for increasing liability functions.)

If all assumptions are realized, the expected value of V_x as of the end of the year is

$$EV_{x+1} = V_x(1+i) - TC_x(1+i)\,.$$

$TC_x(1+i)$, the liability expected to be established on account of the terminated vested employees, is a component of ${}^iEB$. The difference between this liability and the liability actually established, ${}^iB_w$, is a component of $({}^iEB - {}^iB)$.

If equation (10) in Mr. Lynch's paper is modified to read

$$EV = (AL_0 + NC)(1+i) - {}^iEB\ ,$$

then his equation (17) becomes homogeneous and reads

$$\sum_K G_K = \sum_K \sum_j {}_jGV_K\ ,$$

of which

$$\sum_K \sum_j ({}^i_jEB_K - {}^i_jB_K)$$

is a component. Note that ${}^iEB_K$ is introduced under the form $q_0^K NL_K$ in equations (44)–(47) of the paper.

Aggregate Cost Method

I propose that ${}^iEB$ be introduced into equations (26), (29), (30), and (39) in the same manner as it was introduced in equation (8) and that equation (31) be written

$$EV(B) = {}_jPVB_0(1+i) - {}^i_jEB\ .$$

When this is done, equations (31), (32), and (33) (which represent the expected value of three functions that develop similarly from the beginning to the end of the year) become compatible. Moreover, equations (26), (29), and (30) become directly interpretable.

1. $[PVB_0(1+i) - {}^iEB] - (PVC_0 - EC)(1+i) - f_0(PVS_0 - S)(1+i)$ is the *expected accrued liability* at the end of the year (i.e., the accrued liability at the end of the year when the actuarial assumptions are realized) and can be written
$$(AL_0 + f_0S + EC)(1+i) - {}^iEB\ .$$
2. $(PVB_1 - PVC_1 - f_0PVS_1)$ is the accrued liability AL_1 at the end of the year measured with the normal cost rate f_0 (in order to reflect the gain in the accrued liability).
3. $({}^iEB - {}^iB)$ is the difference between expected "dropouts" and the actual benefit payments or the liability newly established on accont of a decrement.

Then equations (26) and (8) are identical and uniformity has been demonstrated in the calculation of the dollar aggregate liability gain for all actuarial cost methods.

Individual Changes in Value

The purpose of the analysis of gains and losses is to allocate the difference between expected and actual liabilities to the various items that cause the discrepancy. To that effect, two valuations are performed.

The first valuation is made on the basis of the *expected salaries.* Developing the functions PVB, PVC, and PVS in the manner illustrated with the vesting liability function V_x, we write the equation linking the total accrued liability AL_0 at the beginning of the year to the accrued liability aL_1 calculated at the end of the year for the same employees, disregarding all changes in status (except for the sorting of the liability):

$$(AL_0 + f_0S + EC)(1+i) - {}^iEB + (q_0^d + q_0^w)aL_1 = \sum_s aL_1^s = aL_1\,, \quad (2)$$

where s represents the statuses under which the employees included in the valuation at the beginning of the year are distributed at the end of the year.

The second valuation uses the reported data (current salary for active employees, computed pension for new-retired employees, and so on) and produces the cost and liabilities stated in the report. The accrued liability aL_1 described above, which includes the liability actually established for new-vested employees (${}^iB_w$), is a by-product of this valuation.

If we designate by σ the statuses, among the statuses s, for which liabilities are included in the report at the end of the year, we can consolidate the results of the two valuations:

$$\sum_K G_K = (AL_0 + f_0S + EC)(1+i) + (q_0^d + q_0^w)aL_1 + ({}^iEB - {}^iB) - \sum_\sigma aL_1^\sigma + \sum_\sigma (aL_1^\sigma - AL_1^\sigma)\,. \quad (3)$$

From this equation the analysis of gains and losses is as follows:

Mortality gain	$= aL_1^d - q_0^d aL_1$;
Turnover gain[1]	$= aL_1^w - q_0^w aL_1$;
Excess of liability expected to be established on account of turnover over actual liability established	$= ({}^iEB - {}^iB)_w$;
Excess of expected pension payments over actual pension payments[2]	$= ({}^iEB - {}^iB)_{pp}$;

[1] The accrued liability aL_1^w includes the liabilities for terminated vested and nonvested employees.

[2] If the valuation assumes that all employees will retire at 65, this item is equal to the excess of expected pension payments to employees aged 66 and over on the valuation date over payments actually made.

Gain due to salary experience computed for employees active at the beginning and at the end of the year $= aL_1^A - AL_1^A$;
Gain on account of new-retired employees $= aL_1^R - AL_1^R$;
Gain on account of new entrants $= -AL_1^{NE}$.

This rather lengthy preamble has been presented in order to shed some light on the following comments:

1. In his paper Mr. Lynch does not derive his analysis of gains and losses by pursuing the line of thought which led to equations such as (29). Instead, he introduces new concepts that I find both disturbing and interesting, and by so doing he has caused the importance of equation (44), the pivotal point of the analysis, to be lost. This equation can be written

$$l_0 EV - \sum_K d_0^K NL_K = l_1 V' = l_0 V' - \sum_K d_0^K V' ,$$

or

$$V' = V_0(1 + i) - \sum_K q_0^K NL_K + \sum_K q_0^K V' .$$

This equation expresses the general pattern of the development of a function applied in this discussion to V_x, PVB, PVC, and PVS.
2. In Mr. Lynch's paper the expected release ER_K is an algebraic result which includes the expected dropout $q_0^K NL_K$. It could have been defined from the expression of V' as $q_0^K V' - q_0^K NL_K$. This form has the advantage over that used in the paper, $d_0^K(EV - NL_K)/l_1$, in that it introduces the probability factor under the form q_0^K, as expected, rather than under the challenging form d_0^K/l_1.
3. The two previous paragraphs present preferences in approach and in form. The results of the analysis, however, are the same. For example, the turnover gain written $AR_w - ER_w$ in the paper can be written $(aL_1^w - {}^iB_w) - (q_0^w aL_1 - {}^iEB_w)$ in the notation of my preamble, combining two lines of the above analysis.

(AUTHOR'S REVIEW OF DISCUSSION)

JOSIAH M. LYNCH, JR.:

Mrs. Tino's perceptive discussion is a welcome addition to the growing literature on gain analysis.

With respect to her first point, that expected benefits can be added and subtracted in formula (8), there is complete agreement. Many actuaries feel more comfortable comparing actual benefits with expected benefits. Such comparisons can be quite useful in evaluating cash-flow and liquidity requirements.

The determination of expected benefits and their allocation by source, however, are not required in gain analysis. The difference between actual and expected benefits is never cited as a source of gain or loss; rather, the event or assumption is named—retirement, death, disablement, or other termination.

Mrs. Tino is quite correct when she cites formula (44) as the pivotal point of the analysis of the individual changes in values attributable to the decrement assumptions. While her subsequent analysis derives expected releases that are algebraically equivalent to formula (47), the approach in the paper is to be preferred for the following reason. All the values on the right-hand side of formula (47) are readily available during the valuation at time 0, whereas V' is not. V' for the purpose of formula (48) is derived from formula (43), after the expected releases have been computed by formula (47).

The approach described in the paper is called "practical" because it works. The paper followed the development of the approach in a computer program created by the author. At time 1 the computer program cycles the participant data three times, performing three consecutive valuations, as follows:

1. A valuation as of time 0 is performed (which must reproduce the valuation results actually used at time 0). The various values of EV are computed, as are the expected releases and the actual releases.
2. A valuation as of time 1 is performed, using expected salaries. V' for formulas (53) and (54) is computed.
3. A valuation as of time 1 is performed, using actual salaries. The various values of V are computed, as are any other values needed for formulas (51)–(53).

The computer program automatically accumulates the appropriate values and displays all the components necessary for the gain analysis, with the exception of G_I, G_E, and ${}^iB$.