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A STOCHASTIC INVESTMENT MODEL

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ABSTRACT

The purpose of this paper is to provide a method for calculating special contingency reserves for investment losses. The method is derived by first building a stochastic investment model and then utilizing its probabilistic structure. The model is essentially the collective risk model used in various ways with respect to insurance claims (both life and nonlife). Several examples are examined in considerable detail.

I. INTRODUCTION

ACTUARIES have become increasingly concerned with investment losses in recent years. The actuary's expectation of broad continuities in investment performance has been disrupted by events such as serious drops in stock market values accompanied by surrenders of equity-linked contracts. Because of the randomness of these events, it seems natural to build probabilistic models for investment losses and utilize these models for planning purposes. An advocate of research on such stochastic models is Edward A. Lew, who has written on the subject in his presidential address [17] as well as in [18].

During the annual meeting of 1975, Mr. Lew was the moderator for a panel discussion on "Reserves, Contingency Reserves, and Surplus for Life Insurance Companies" [23]. His opening remarks provide an incentive to seek methods for calculating special contingency reserves for investment losses. The thrust of those remarks may be paraphrased as follows:

An integrated system of reserves, contingency reserves, and unassigned surplus would be most helpful in difficult economic times. Companies should not freeze excessive portions of their total funds in reserves, which are not available to absorb sudden unexpected losses. The current unsettled economic climate emphasizes the need for contingency reserves and unassigned surplus to absorb such losses.

Provision should be made for losses resulting from excessive fluctuations in investment values, which may change the valuation basis of some securities from an amortized basis to a market-value basis, as well as for losses due to forced asset liquidation. Such provision cannot be accomplished by assuming a lower interest rate in the reserve calculations, because this increases the reserves, thus reducing funds available for surplus and contingency reserves. The determination of the contingency reserves should be on a prospective basis, and hence must recognize stochastic fluctuations in investment values.

The present form of the mandatory securities valuation reserve (MSVR) diminishes its usefulness when investment values take sharp drops or when losses occur because of the liquidation of assets. The MSVR may drop in size dramatically when surplus is most needed.

It is clear from the statements of the panelists and discussants at the 1975 session that they were not all in favor of special contingency reserves. However, a careful scrutiny of their remarks reveals a common concern regarding investment risk. Moreover, several of the discussants expressed a concern for basic research into methods of calculating such reserves.

The purpose of this paper is to provide one means of calculating special contingency reserves for investment losses. The paper does not attempt to convince readers of the need to set up such reserves.

II. A STOCHASTIC INVESTMENT MODEL

Assume that X_1, X_2, $X_3, \ldots$ are positive-valued random variables representing losses from investments, where the subscripts indicate time of loss. Gains from investments will not be considered. This is analogous to applying collective risk theory to life insurance, and ignoring the negative claims from deaths of annuitants. Let $\{N(t), 0 \leq t < \infty\}$ be a nonnegative, integral-valued, stochastic process, independent of the X_i's and with $N(0) = 0$. This process counts the random number of losses. The time variable t is to be thought of as calendar time, rather than operational time, which has been used in collective risk theory. The transformation of calendar time to operational time yields a stochastic process with independent increments. Although this has certain mathematical advantages, it distorts the user's normal perception of time. Investment losses occur suddenly and irregularly, to a greater degree than is true even for insurance claims.

Let $S(t) = \Sigma_{i=1}^{N(t)} X_i$ be a random sum of the random variables, representing the aggregate investment losses up to and including time t. Additionally, let

u = Initial amount in a special contingency reserve for the investment operation;

δ = Portion of the loading designated for investment contingencies; and

$$\begin{aligned} u(t) &= \text{Investment contingency reserve amount at time } t \\ &= u + \delta t - S(t), \quad 0 \leq t < \infty . \end{aligned}$$

We will be interested in the mean and variance functions for $S(t)$, the $u(t)$ stochastic process, and certain probability distributions. We will also be concerned with a method for computing an appropriate δ.

Company records could be used to estimate $E\{X_i\}$, $E\{N(t)\}$, Var $\{X_i\}$, and Var $\{N(t)\}$. It would not be surprising if a sequence of past losses revealed an inflationary trend. Ideally, this could be handled by assuming different distributions for the X_i's. For example, if $\mu_1 = E\{X_1\}$, one could assume that $\mu_i = E\{X_i\} = \mu_1[1 + 0.005(i - 1)]$, for $i = 2, \ldots$. This would allow for a 0.5 percent inflationary trend in the expected values of losses. However, while this refinement is only marginally superior to assuming an inflationary trend for each year's aggregate claims, it greatly complicates obtaining distributions for $S(t)$ and functions of $S(t)$. Thus, we will assume that the X_i's are identically distributed, but replace the random sum $S(t)$ by $S(t) + f(t)$, where $f(t)$ can be thought of as $AE\{S(t)\}$. A value of A equal to 0.05 implies a 5 percent inflationary rate. One can thus view the aggregate random losses as being measured from the trend function $f(t)$, $t \geq 0$. At any point t_0, $S(t_0)$ is the random aggregate loss above the trend function value $f(t_0)$. Simultaneously, we will replace δt, the provision for investment contingencies, by $\delta t + f(t)$, a provision for investment contingencies and inflation.

In some situations, there is evidence of dependence among investment losses. It would be very difficult for our model to provide for all types of dependency. However, we will provide for "positive contagion" in the numbers-of-losses process. Bühlmann ([11], pp. 43 and 52–53) has shown that the $\{N(t), t \geq 0\}$ process has an intensity of loss frequency $\lambda_n = a + b(n - 1)$ for $n = 1, 2, 3, \ldots$ for $a \geq 0$, $b > 0$, where the index n represents the number of losses. Thus λ_n, the intensity of frequency of the nth loss (transition from $n - 1$ to n losses) grows, and losses become increasingly common, on the average. This form for λ_n leads to a negative binomial distribution for $N(t)$. By contrast, when $b = 0$ and $\lambda_n = a$, there is no contagion, and the intensity does not increase as the losses unfold. In this case, $N(t)$ has a Poisson distribution. If the random variable T represents the random time between losses, the Poisson assumption implies that $P\{T \leq t\} = 1 - e^{-at}$, $t \geq 0$, for some value of a. Andersen [2] generalized this interloss time distribution to $P\{T \leq t\} = 1 - 0.25e^{-0.4t} - 0.75e^{-2t}$, $t \geq 0$. In the examples that follow, $N(t)$ and T possess these various distributions.

The investment contingency reserve at time t can be expressed as

$$u(t) = u + [\delta t + f(t)] - [S(t) + f(t)] \,. \tag{1}$$

Since $f(t)$ is assumed to be monotonically increasing, the contribution to the special contingency reserve for the time period $[0, T]$ is $\delta T + f(T)$. It is possible to refine the provision for inflation to include a cyclic effect (see [4], p. 581, second example).

The mean and variance functions for $S(t)$ and $u(t)$, $t \geq 0$, are

$$E\{S(t)\} = E\{X_1\}E\{N(t)\} , \tag{2a}$$

$$E\{u(t)\} = u + \delta t - E\{X_1\}E\{N(t)\} , \tag{2b}$$

$$\begin{aligned} \text{Var } \{u(t)\} &= \text{Var } \{S(t)\} \\ &= E\{N(t)\} \text{ Var } \{X_1\} + \text{Var } \{N(t)\}(E\{X_1\})^2 . \end{aligned} \tag{3}$$

These results follow from Feller ([14], p. 301, problem 1) with minor changes.

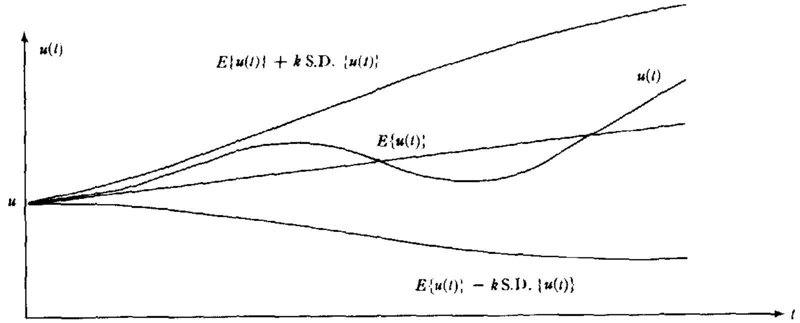

FIG. 1

The mean and standard deviation (S.D.) functions can be used, as shown in figure 1, to help visualize the sample function behavior of the $u(t)$ process. Note that $E\{u(0)\} = u$, and Var $\{u(0)\} = 0$. The actuary is interested in the proportion of the sample paths (portfolio histories) that dips below the lower confidence contour in the figure. This may be formulated as

$$P\{u(t) < L(k; t) \text{ for some } t \in [0, T]\} , \tag{4}$$

where

$$L(k; t) = E\{u(t)\} - k \text{ S.D. } \{u(t)\} . \tag{5}$$

The 0 and T in $[0, T]$ represent the initial and closing times for observation.

The probability (4) depends on the assumed distributions for $N(t)$, $0 \leq t \leq T$, and $\{X_i\}$. We will assume that $N(t)$ has one of three distributions: Poisson, negative binomial, or generalized Poisson. Our assumed distributions for $\{X_i\}$ will be uniform, or Paretian. Even though a pair of assumptions for $N(t)$ and $\{X_i\}$ may not apply to an entire investment portfolio, it could provide a reasonable model for a subset of the portfolio.

The various models for $N(t)$ have been used by many researchers in the field of casualty insurance (see [30], [28], [11], and [27]). The Weber paper involved a large amount of data from the 1964 California Driver Record Study. It used the negative binomial distribution for the number of accidents within the total population. This total population was subdivided into homogeneous groups, for each of which a Poisson distribution could be used to describe the number of accidents. The six criterion variables—sex, marital status, residence, age, conviction history, and accident history—were used to partition a sample of 148,000 individuals into 2,880 groups. There were 193 of these groups that contained 100 or more individuals. Poisson distributions were fitted to these groups, and were acceptable (at the 0.05 level of significance) in 167 of the groups.

There is a useful analogy between aggregate investment losses and aggregate casualty losses. With proper choice of criterion variables, an investment portfolio could be subdivided in a manner similar to that used in the Weber study. Two obvious choices for such variables are type of investment and measure of risk at time of investment. Since the Poisson distribution implies very small probabilities for multiple events, it can model the rare events of multiple investment losses. Positive contagion among investment losses could be handled through the negative binomial distribution as explained previously.

Assume either

$$P\{N(t) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \tag{6}$$

or

$$P\{N(t) = k\} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} \frac{c^b}{\Gamma(b)} \lambda^{b-1} e^{-c\lambda} d\lambda\,, \qquad c > 0, b > 0\,. \tag{7}$$

Equation (6) describes a Poisson distribution, while in equation (7) the parameter λ has a gamma distribution. After integration, equation (7) becomes

$$P\{N(t) = k\} = \frac{\Gamma(k+b)}{k!\Gamma(b)} \left(\frac{c}{c+t}\right)^b \left(\frac{t}{c+t}\right)^k. \tag{8}$$

This is the negative binomial distribution.

For the Poisson distribution, $E\{N(t)\} = \text{Var}\ \{N(t)\} = \lambda t$. For the negative binomial distribution, $E\{N(t)\} = bt/c$, and $\text{Var}\ \{N(t)\} = bt(c+t)/c^2$. It is easy to verify that for the former, $\text{Var}\ \{u(t)\} = p_2\lambda t$, whereas for the latter, Var $\{u(t)\} = btp_2/c + bt^2p_1^2/c^2$, where $p_i = E\{X^i\}$, $i = 1, 2, 3, \ldots$. Extensive tables from, and references to, works that fit these distributions to numbers of observed accidents of many types can be found in Seal [27].

Let the observation period be from 0 to T. We would like to find k such that $P\{u(t) \geq L(k;\ t)$ for all $t \in [0,\ T]\} = 0.99$, or some other suitably large number. This is similar to $1 - \psi(u, T) = 0.99$ in collective risk theory. Excellent fast Fourier transform techniques are now available for obtaining $\psi(u, T)$. Considerable research has been conducted in this area by Bohman, Seal, Thorin, and Wikstad. References to much of their work will be found in the papers of the forthcoming *Proceedings of the Brown University Actuarial Research Conference.* Although fast Fourier transform techniques would be excellent methods for determining k, they have not been implemented by many actuaries. Therefore, we will select a simpler method that is familiar to more actuaries.

We will seek a constant k such that

$$P\{u(T) \geq L(k; T)\} = 0.99\,. \tag{9}$$

Note that the focus is now on the single time point T, rather than all time points t in the interval $0 \leq t \leq T$.

By using the definitions of $u(T)$ and $L(k;\ T)$, equation (9) can be restated as

$$P\{S(T) \leq p_1E\{N(T)\} + k\sigma\} = 0.99\,. \tag{10}$$

This is similar to $F(p_1E\{N(T)\} + k\sigma,\ T) = 0.99$ in collective risk theory. Thus the contribution to the special contingencies reserve for the period $[0,\ T]$ is

$$\delta T + f(T) = p_1E\{N(T)\} + k\sigma + f(T) \tag{11}$$

for k determined as above.

Consider the standardized process

$$u^*(t) = \frac{u(t) - E\{u(t)\}}{\text{S.D. } \{u(t)\}}\,, \qquad 0 \leq t < \infty\,.$$

Equation (9) is equivalent to

$$P\{u^*(T) \geq -k\} = 0.99\,. \tag{12}$$

In the Poisson case, where

$$E\{u(t)\} = u + \delta t - p_1\lambda t$$

and

$$\text{Var } \{u(t)\} = p_2\lambda t\,, \quad \text{for } 0 \leq t < \infty\,,$$

equation (10) becomes $P\{S(T) \leq p_1\lambda T + k(p_2\lambda T)^{1/2}\} = 0.99$. From equation (26) of Cramér [12],

$$P\{S(T) > p_1\lambda T + k(p_2\lambda T)^{1/2}\} = \Phi(-k) + \frac{c_3}{3!(\lambda T)^{1/2}}\Phi^{(3)}(-k)$$
$$+ \frac{c_4}{4!\lambda T}\Phi^{(4)}(-k) + \frac{10c_3^2}{6!\lambda T}\Phi^{(6)}(-k) + O(T^{-3/2}) . \tag{13}$$

Here,

$$\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-t^2/2)dt , \qquad \Phi^{(j)}(x) = d^j\Phi(x)/dx^j ,$$

$$c_n = p_n/p_2^{n/2} ,$$

and $O(T^{-3/2})$ indicates that $|\text{Remainder}| < AT^{-3/2}$ for large T and some positive constant A.

In the negative binomial case,

$$u^*(t) = \frac{p_1bt/c - S(t)}{[btp_2/c + p_1^2b^2t^2/c^2]^{1/2}} , \qquad 0 \leq t < \infty .$$

As stated on page 41 of Seal [27], the asymptotic distribution of $u^*(t)$, as $t \to \infty$, is also given by the right-hand side of equation (13) with λT replaced by bT/c, where the c's are defined by the cumulants of $S(T)$. However, we will use the incomplete gamma distribution to approximate the distribution of $u^*(T)$, as various authors have done (see [19], [9], [3], [10], [5], [7], [24], and [21]).

Convincing discussions of the accuracy of this approximation are contained in [24] and on pages 121–22 of [19]. Equation (3) of [24] states that

$$P\left\{\frac{S(T) - T}{\sqrt{K_2}} \leq z\right\} = \frac{1}{\Gamma(\alpha)}\int_0^{\alpha+z\sqrt{\alpha}} e^{-y}y^{\alpha-1}dy \tag{14}$$

for

$$\alpha = 4K_2^3/K_3^2 , \tag{15}$$

where K_2 and K_3 are cumulants of $S(T)$. This assumes that $E\{N(T)\} = T$ and that $p_1 = 1$, but with minor changes $E\{N(T)\} = \lambda T$ can be handled, and if $p_1 \neq 1$, it can be set equal to 1, provided the original p_n's are then divided by p_1^n.

III. EXAMPLES

Assume that t counts weeks, and the observation period is from 0 to T. We will let $T = 52$, 104, 156, and 208. Also assume that the distribution of losses is uniform on $[a, b]$, that is,

$$P\{X_i \leq x\} = \frac{x-a}{b-a}, \quad a \leq x \leq b .$$

Then

$$p_1 = (a+b)/2 , \quad p_2 = (a^2+ab+b^2)/3 ,$$

$$p_3 = (b^4 - a^4)/[4(b-a)] , \quad p_4 = (b^5 - a^5)/[5(b-a)] .$$

Let the provision for inflation be

$$f(t) = 0.05 p_1 E\{N(t)\} , \quad t \geq 0 .$$

In the Poisson case, the standard deviation of aggregate investment losses is $\sigma = (\lambda p_2 T)^{1/2}$. In the negative binomial case, $\sigma = (bTp_2/c + bT^2p_1^2/c^2)^{1/2}$.

To apply equation (13), we first compute $c_3 = p_3/p_2^{3/2}$ and $c_4 = p_4/p_2^2$. We will obtain values for k from equation (13) and the tables in [15], and then show values for $\delta T + f(T) = 1.05 p_1 E\{N(T)\} + k\sigma$. The error magnitudes in equation (13) should be reassuring to readers, since $T^{-3/2} = 0.00267$, 0.00094, 0.00051, and 0.00033, when $T = 52$, 104, 156, and 208, respectively.

To apply equation (14), we first compute α using equation (15). We will do this first for the Poisson case for comparative purposes, and then for the negative binomial case. For the Poisson case, we have (using [27], p. 35, eq. [2.41])

$$\alpha = 4(\lambda T p_2)^3/(\lambda T p_3)^2 = 4\lambda T p_2^3/p_3^2 . \tag{16}$$

For the negative binomial case, $K_2 = Tbp_2/c + T(b/c)^2$; $K_3 = Tbp_3/c + 3Tp_2(b/c)^2 + 2T(b/c)^3$ (using formulas on p. 41 of [27], with the monetary unit of p_1 [i.e., $p_1 = 1$]). The resulting α is

$$\alpha = \frac{4(bT/c)(p_2 + b/c)^3}{[p_3 + 3p_2 b/c + 2(b/c)^2]^2} . \tag{17}$$

The example will now be made more specific. Let a equal 1 and b equal 11, where 1 represents a loss of \$20,000, and 11 represents a loss of \$220,000. Then $p_1 = 6$, $p_2 = 133/3$, $p_3 = 366$, and $p_4 = 3{,}221$. Thus $c_3 = 1.23990$, and $c_4 = 1.63882$. We will assume that the Poisson parameter λ is equal to 0.1; thus, one loss is expected every 10 weeks, which allows for some suddenness and irregularity in time of loss. It is assumed that the negative binomial constants b and c are 1 and 10, respectively. These values are comparable to m and r in Tables 1 and 2 of [30].

A detailed discussion of the use of the tables in [15] will be found on page 67 (exercise 3) of [6].

Table 1 is the computational table for the two cases. For the Poisson case, k is determined by using equation (13). The coefficients are given in Table 2.

TABLE 1

T	$1.05p_1E\{N(T)\}$	Poisson σ	Negative Binomial σ
52	32.76	15.18332	34.69832
104	65.52	21.47246	65.99111
156	98.28	26.29829	97.22428
208	131.04	30.36665	128.44132

TABLE 2

T	$c_3/6(T/10)^{1/2}$	$c_4/(2.4T)$	$10c_3^2/(72T)$
52	0.09062	0.01313	0.00411
104	0.06408	0.00675	0.00205
156	0.05232	0.00438	0.00137
208	0.04531	0.00328	0.00103

The logical trial value of k is 2.33. For this value, the right-hand side of equation (13) equals the sum of four quantities Q_i as displayed in Table 3. In evaluating Q_i, use was made of the facts that

$$\Phi'(x) = f(x) = (2\pi)^{-1/2} \exp(-x^2/2);$$

$$\Phi^{(6)}(x) = -x\Phi^{(5)}(x) - 4\Phi^{(4)}(x) = -xf^{(4)}(x) - 4f^{(3)}(x);$$

$$f^{(4)}(-x) = f^{(4)}(x);$$

$$f^{(3)}(-x) = -f^{(3)}(x);$$

$$\Phi^{(6)}(-x) = -\Phi^{(6)}(x).$$

More accurate values of k and the corresponding Q_i's are given in Table 4.

The contribution to the special contingency reserve can now be calculated from equation (11). The results are summarized in Table 5. Recall that 1 unit stands for $20,000. Comparing the columns of Table 5 with that labeled $1.05p_1E\{N(T)\}$ in Table 1 reveals the portions of the contributions that should provide for deviations from expected values.

We now recalculate values of k and contributions to the special contingency reserve by using equations (14) and (16). For $T = 52$, 104, and 156, the probabilities were obtained by two-way linear interpolation in the tables of [22]. The parameters u and p are determined from the equations $u = \alpha^{1/2} + k$ and $p = \alpha - 1$. The final values of u and p are shown in Table 6. For $T = 208$, the probability was obtained from the

TABLE 3

T	Q_1	Q_2	Q_3	Q_4	Sum
52	0.00990	0.01060	0.00196	−0.00249	0.01997
104	0.00990	0.00750	0.00098	−0.00124	0.01714
156	0.00990	0.00612	0.00066	−0.00083	0.01585
208	0.00990	0.00530	0.00049	−0.00062	0.01507

TABLE 4

T	k	Q_1	Q_2	Q_3	Q_4	Sum
52	2.73	0.00320	0.00562	0.00153	−0.00043	0.00992
104	2.60	0.00470	0.00501	0.00087	−0.00050	0.01008
156	2.55	0.00540	0.00445	0.00060	−0.00042	0.01003
208	2.53	0.00570	0.00398	0.00046	−0.00034	0.00980

TABLE 5

T	52	104	156	208
Contribution in Poisson case	74.21049	121.34840	165.34064	207.86762

TABLE 6

T	α	k	u	p	Prob $\{S^*(T) \leq k\}$	Contribution to Reserve
52	13.52983	2.72	6.398	12.530	0.99006	74.05863
104	27.05966	2.60	7.802	26.060	0.98991	121.34840
156	40.58949	2.55	8.921	39.589	0.98988	165.34064
208	54.11932	2.48	$N(2.33422)$		0.99021	206.34929

normal distribution tables and approximation (26.4.13) on page 941 of [1]. Let $S^*(T)$ denote the standardized $S(T)$ random variable. In the probability calculation for $T = 208$, $\chi^2 = 144.72728$ and $\nu = 108.23864$. Note that the k's and contributions for $T = 104$ and 156 are the same as were previously calculated (see Tables 4 and 5), and are not much different for the other two T-values.

Let us now consider the negative binomial case, where $b = 1$ and $c = 10$. Setting $p_1 = 1$ yields $p_2 = 1.2314815$ and $p_3 = 1.6944444$. Table 7 was derived using equations (14) and (17).

TABLE 7

T	α	k	u	p	Prob $\{S^*(T) \leq k\}$	Contribution to Reserve
52........	11.30636	2.75	6.112	10.306	0.98995	128.18038
104.......	22.61272	2.63	7.385	21.613	0.98997	239.07662
156.......	33.91908	2.58	8.404	32.919	0.99009	349.11864
208.......	45.22544	2.55	9.275	44.225	0.99012	458.56537

The contributions to reserves for the negative binomial example are considerably larger than for the Poisson case. This is consistent with our earlier remarks that the negative binomial distribution allows for dependence among investment losses. This was done through positive contagion in the numbers-of-losses process, which, on the average, allows for increasingly frequent losses.

IV. PRACTICAL IMPLEMENTATION

Suggestions for the practical implementation of the method are given in this section. Company records could be used to estimate the mean and variance of the distribution of individual investment losses. Such records also could be used for estimating the means and variances of the distributions of the numbers of losses for various time periods. These quantities would be sufficient to estimate the means and variances of aggregate investment losses for various time periods. Formulas (2a) and (3) would be used for these calculations. A provision for inflation could be determined as a fraction of the expected aggregate investment losses for the various time periods.

Recall that $p_i = E\{X^i\}$, $i = 1, 2, \ldots$. These are the theoretical moments for the distribution of individual investment losses. It has been suggested that p_1 and Var $\{X\}$ could be estimated from company records. This must also be done for p_3 and p_4. We now have the quantities needed to compute k from equation (13), or equation (14), assuming that the numbers of losses follow either the Poisson or the negative binomial law. The discussion following equation (5) and the references cited should be helpful when fitting the two distributions to observed data.

Given k, one can calculate the contribution to the special contingency reserve for the period $[0, T]$. In words, it is the sum of (1) expected aggregate investment losses, (2) k times the standard deviation of aggregate investment losses, and (3) provision for inflation to time T. Such calculations would be made for various values of T.

V. EXTENSIONS TO PARETO DISTRIBUTIONS

The model developed in Section II assumed that Var $\{X_1\} < \infty$. This is not true if we assume that the distribution of an individual loss follows a typical Pareto distribution law, and such an assumption would be appropriate for certain sets of data. Hickman [16] provided actuaries with a very useful guide to the work of Mandelbrot ([20]; see also [13]), who used the stable Paretian class of distributions to describe the family of distributions for stock price changes. The only member of the stable Paretian class that has a finite variance is the normal distribution. As explained by Professor Hickman, Mandelbrot used the Paretian class for three reasons: (1) to allow for the heavy extreme tails in the distributions (i.e., they contain more of the total probability than they would in the case of a normal distribution); (2) because a sum of independent Paretian random variables also has a Pareto distribution; and (3) because random price changes over days, weeks, months, years, and so on, have distributions that belong to the same class.

Both Mandelbrot [20] and Fama [13] recommend stable Paretian distributions for use in models of speculative markets, such as commodity and securities markets. The stable Paretian hypothesis "implies that there are a larger number of abrupt changes in the economic variables that determine equilibrium prices in speculative markets than would be the case under a Gaussian hypothesis" ([13], p. 303). Both researchers acknowledge that the infinite variances prohibit the use of many statistical techniques. However, they do give some suggestions on estimating the needed density parameters ([13], pp. 299–300, and [20], pp. 310–11). Moreover, a number of actuarial researchers now have used Paretian densities for modeling claim losses (see [8], [26], [25], and [29]). Fortunately, some excellent numerical inversion work of Laplace-Stieltjes transforms of Pareto-type collective risk probabilities has been done, and we will utilize tables so produced to illustrate our ideas.

If the loss distribution is Paretian, the technique involving the standard deviation does not work. However, from the earlier formulation, one would seek a value for δ such that $P\{u + \delta t - S(t) \geq 0 \text{ for } 0 \leq t \leq T\} = 0.99$, or some other suitably large number. Again note that a provision for inflation, $f(t)$, can be added without affecting the probability statement, since $u + \delta t - S(t) = u + [\delta t + f(t)] - [S(t) + f(t)]$, for $0 \leq t \leq T$.

Consider an example in which the claim distribution is Paretian. A family of Paretian distributions is given by $P\{X \leq x\} = 1 - \{1 + x/A\}^{-B}$, $x \geq 0$, $A > 0$, $B > 0$. If A is chosen equal to $B - 1$, the mean value is 1. As B increases, the "dangerousness" or extreme variability of the distribution diminishes.

Thorin and Wikstad [29] used a Paretian distribution in two of their nine tables. In Table 5 of [29], the distribution of individual losses is

TABLE 8

u	$\delta = 1.05$	$\delta = 1.10$	$\delta = 1.15$	$\delta = 1.20$	$\delta = 1.25$	$\delta = 1.30$	$\delta = 2.00$
100.....	0.03805	0.03639	0.03488	0.03351	0.03226	0.03112	0.02130
1,000...	0.00114	0.00113	0.00113	0.00112	0.00112	0.00111	0.00105

assumed to be $P\{X \leq x\} = 1 - (1 + 2x)^{-3/2}$, $x > 0$. The variance of this distribution is infinite and the mean is 1. Thus $p_1 = 1$, our unit of money. The interloss time distribution $P\{T \leq t\} = 1 - e^{-t}$, $t \geq 0$, also has a mean of 1. This distribution is equivalent to a Poisson distribution for numbers of losses, $N(t)$, with $E\{N(t)\} = t$. Using the values for $T = 100$ from Table 5 of [29], we obtain the probabilities that $P\{u + \delta t - S(t) \geq 0 \text{ for } 0 \leq t \leq 100\}$, for various values of δ and u. These probabilities are shown in Table 8.

To illustrate the use of Table 8, assume $u = 100$ units and $f(t) = 0.05 p_1 E\{N(t)\}$. A further contribution to the special contingency reserve of 205 units would hold the probability of excessive losses to 0.02130. An extrapolation based on δ-values and $u = 100$ indicates that δ would have to be approximately 2.55 to reduce the probability to 0.01. Thus, a total special contingency reserve of 360 units is required to reduce the probability of excessive losses to 0.01. This is considerably more than the 121.35 units needed for the uniform distribution example in which $T = 104$. This result is consistent with the greater "dangerousness" of the Pareto distribution.

As a second example, assume the same Pareto distribution for losses, but now assume that the interclaim time distribution is $P\{T \leq t\} = 1 - 0.25e^{-0.4t} - 0.75e^{-2t}$, $t \geq 0$. This distribution for time between losses also has a mean value of 1 and was used by Andersen [2] as a generalization of the Poisson distribution for $N(t)$. Again assume that $T = 100$. This dual assumption about the individual loss distribution and the interclaim time distribution was made by Thorin and Wikstad in obtaining probability values contained in Table 6 of [29]. We utilize that table to obtain the set of complementary probabilities for various values of δ and u shown in Table 9.

If $u = 100$ units, and $f(t) = 0.05p_1E\{N(t)\}$, Table 9 shows that a further contribution to the special contingency reserve of 205 units will hold the probability of excessive losses to 0.02219. An extrapolation based on four points indicates that δ would have to be approximately

TABLE 9

u	$\delta=1.05$	$\delta=1.10$	$\delta=1.15$	$\delta=1.20$	$\delta=1.25$	$\delta=1.30$	$\delta=2.00$
100	0.04057	0.03870	0.03702	0.03549	0.03411	0.03284	0.02219
1,000	0.00115	0.00114	0.00114	0.00113	0.00113	0.00112	0.00106

2.70 to reduce the probability to 0.01. Thus, a total special contingency reserve of 375 units is required to reduce the probability of excessive losses to 0.01. Note that this is 15 units greater than in the previous Pareto example.

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