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A UNIFIED APPROACH TO PENSION PLAN GAIN AND LOSS ANALYSIS

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ABSTRACT

The objective of this paper is twofold. First, a pedagogical approach is taken to derive the pension plan gain and loss formulas by extending the mathematical theory underlying the actuarial funding methods. It is the author's intent to emphasize the similarities between gain and loss formulas developed for individual funding methods and the formulas developed for aggregate funding methods. Indeed, the formulas are fundamentally identical.

The second objective is to afford the reader an opportunity to reinforce his understanding by providing two comprehensive numerical examples and analyzing actuarial experience by source.

I. INTRODUCTION

FOR the actuarial student learning pension mathematics, gain and loss analysis has tended to be somewhat esoteric, for two principal reasons. First, the theory underlying the analysis of actuarial experience generally has been separated from the mathematics of actuarial funding methods. An important feature of this paper is that the funding method that generates the costs for successive valuations is also the origin of the algebraic derivation of the gain and loss formulas. Second, except for Dreher's paper [2], the literature is lacking in sufficient numerical illustrations from which to learn.

Section II introduces the notation that will be used in the paper. Section III develops three fundamental identities. In Section IV a detailed algebraic derivation of the gain and loss formulas for an aggregate-cost funding method is presented, and in Section V a comprehensive problem is resolved so that the reader can demonstrate with numbers that the sum of the parts must equal the whole. Section VI treats the gain and loss formulas for other funding methods, with particular emphasis on the similarities among the resulting formulas. In Section VII an analysis is given of two successive valuations of a pension plan with ancillary benefits.

II. NOTATION

The notation used in this paper is consistent with that commonly accepted among pension actuaries. Where convenient, the notation used in Christopher Street's paper [4] has been adopted and is restated below.

i = Valuation interest rate;

i' = Actual experience rate of interest;

F_t = Value of fund assets at time t;

S_t = Salaries at time t;

PVB_t = Present value of benefits at time t;

$1 + s_x$ = Salary-scale factor at age x;

PVS_t = Present value of future salaries at time t;

NCR_t = Normal cost rate at time t, which may be expressed as a rate per dollar of salary or a rate per individual;

PVB_1^0 = Present value of benefits at time 1 as projected from time 0, assuming no projected benefit, plan, or status changes, but considering projected changes in salary;

PVS_1^0 = Present value of future salaries at time 1 as projected from time 0, assuming no status changes but considering projected changes in salary;

AL_t = Accrued liability at time t as defined by the actuarial cost method;

AL_1^0 = Accrued liability at time 1 as projected from time 0, assuming no projected benefit, plan, or status changes but considering projected changes in salary;

UL_t = Unfunded liability at time t;

EP^i = Expected benefit payments to be made from time 0 to time 1 accumulated with interest, i, to time 1;

P^i = Actual benefit payments made from time 0 to time 1 accumulated with interest, i, to time 1;

C^i = Employer pension contributions made from time 0 to time 1 accumulated with interest, i, to time 1;

ΔPVB (ΔAL) = Additional present value of benefits (accrued liability) arising from salary increases exceeding those expected according to salary scale;

ΔPVS = Additional present value of future salaries arising from salary increases exceeding those expected according to salary scale.

III. PRELIMINARY RELATIONSHIPS

Three fundamental identities are essential to the derivation of the gain and loss formulas that are presented in the next section. Two of the identities link the present value of benefits at time 1 as projected from time 0, and the third links the present value of future salaries at time 1 as projected from time 0.

In general, if an individual remains in the same status with salary at time 1 equal to his expected salary, then the following relationships must hold.

1. For an active participant eligible for a deferred benefit:

$$PVB_1^0 = PVB_0 \frac{D_x}{D_{x+1}} = PVB_0 \frac{1+i}{p_x},$$

or

$$PVB_0(1+i) = p_x PVB_1^0 . \tag{1}$$

2. For a retired participant currently receiving a life annuity:

$$\begin{aligned}
\frac{N_x^{(12)}}{D_x}(1+i) \\
&= \frac{N_x - (11/24)D_x}{D_x}(1+i) \\
&= \frac{N_{x+1} - (11/24)D_{x+1} + D_x + (11/24)D_{x+1} - (11/24)D_x}{D_x} \\
&\quad \times (1+i) \\
&= \frac{N_{x+1}^{(12)} + (13/24)D_x + (11/24)D_{x+1}}{D_{x+1}} \frac{D_{x+1}}{D_x}(1+i) \\
&= p_x \frac{N_{x+1}^{(12)}}{D_{x+1}} + (13/24)(1+i) + (11/24)p_x \\
&= p_x \frac{N_{x+1}^{(12)}}{D_{x+1}} + (13/24)(1+i) + (11/24)(1-q_x) \\
&= p_x \frac{N_{x+1}^{(12)}}{D_{x+1}} + [1 + (13/24)i - (11/24)q_x] ,
\end{aligned}$$

or

$$PVB_0(1+i) = p_x PVB_1^0 + EP^i , \tag{2}$$

where

$$EP^i = 1 + (13/24)i - (11/24)q_x .$$

3. For an active participant currently receiving a salary of S_0 who has a projected salary of $S_1 = S_0(1 + s_x)$:

$$\begin{aligned}
PVS_1^0 &= S_1[1 + (1 + s_{x+1})vp_{x+1} \\
&\quad + (1 + s_{x+1})(1 + s_{x+2})v^2\ {}_2p_{x+1} + \ldots] \\
&= S_0(1 + s_x)[1 + (1 + s_{x+1})vp_{x+1} \\
&\quad + (1 + s_{x+1})(1 + s_{x+2})v^2\ {}_2p_{x+1} + \ldots] , \\
vp_xPVS_1^0 &= S_0[(1 + s_x)vp_x + (1 + s_x)(1 + s_{x+1})v^2p_xp_{x+1} \\
&\quad + (1 + s_x)(1 + s_{x+1})(1 + s_{x+2})v^3p_x\ {}_2p_{x+1} \\
&\quad + \ldots] , \\
vp_xPVS_1^0 + S_0 &= S_0[1 + (1 + s_x)vp_x + (1 + s_x)(1 + s_{x+1})v^2\ {}_2p_x \\
&\quad + (1 + s_x)(1 + s_{x+1})(1 + s_{x+2})v^3\ {}_3p_x + \ldots] \\
&= PVS_0 ,
\end{aligned}$$

or

$$(PVS_0 - S_0)(1 + i) = p_xPVS_1^0 . \qquad (3)$$

IV. DERIVATIONS

Other papers on gain and loss analysis have tended to distinguish between individual actuarial funding methods and aggregate funding methods. This probably arises from the notion that a gain or loss can be more easily understood as the difference between an expected unfunded liability and an actual unfunded liability than as the difference between two normal cost percentages. As a result, other authors have had to give one treatment of gain and loss for entry age and accrued benefit funding methods and another treatment for frozen initial liability and aggregate funding methods. The distinguishing feature of this paper is that no distinction is made between the different funding methods.

This section presents a derivation of the gain and loss formulas for an aggregate-cost funding method.

At time $t = 0$,

$$NCR_0 = \frac{PVB_0 - F_0}{PVS_0} .$$

Note for future reference that

$$F_0 = PVB_0 - NCR_0 PVS_0 \tag{4}$$

and

$$NC_0 = NCR_0 S_0 . \tag{5}$$

At time $t = 1$,

$$NCR_1 = \frac{PVB_1 - F_1}{PVS_1} .$$

Hence,

$$NCR_1 PVS_1 = PVB_1 - F_1 \tag{6}$$

and

$$NCR_0 PVS_1 = NCR_0 PVS_1 . \tag{7}$$

Subtracting (6) from (7) gives

$$(NCR_0 - NCR_1)PVS_1 = NCR_0 PVS_1 - PVB_1 + F_1 . \tag{8}$$

Let us look at the left-hand side of equation (8). If the value of the expression within the parentheses is positive, that is, if NCR_0 is greater than NCR_1, then there is an actuarial gain. Similarly, there would be an actuarial loss if the value were negative. The difference in normal cost rates multiplied by the present value of salaries is the change in present value of future normal costs and is the dollar amount of the gain or loss.

Because equation (8) is of very limited use, the following algebraic transformations are aimed at reducing PVS_1, PVB_1, and F_1 to their most elementary components. (The reader may want to glance ahead to eq. [15].)

We first note that

$$\begin{aligned} F_1 &= (F_0 + NC_0)(1 + i') - P^{i'} \\ &= [(F_0 + NC_0)(1 + i) - P^{i}] \\ &\quad + [(F_0 + NC_0)(i' - i) - (P^{i'} - P^{i})] \,. \end{aligned}$$

The expression $(F_0 + NC_0)(i' - i) - (P^{i'} - P^{i})$ is the gain from investment experience. If $i' > i$, there is a gain from investments and a loss from pension payments, since the trust was required to "unfund" at a more favorable investment return. This expression will be denoted by G_I; hence we may write

$$F_1 = (F_0 + NC_0)(1 + i) - P^{i} + G_I \,. \tag{9}$$

After substituting equations (4) and (5) in equation (9) we have

$$F_1 = (PVB_0 - NCR_0PVS_0 + NCR_0S_0)(1 + i) - P^{i} + G_I$$

or

$$F_1 = [PVB_0 - NCR_0(PVS_0 - S_0)](1 + i) - P^{i} + G_I \,. \tag{10}$$

Equations (1), (2), and (3) can be substituted in equation (10) to produce

$$F_1 = p_x(PVB_1^0 - NCR_0PVS_1^0) + EP^{i} - P^{i} + G_I \,. \tag{11}$$

This value of F_1 can be substituted back into equation (8), giving

$$\begin{aligned} (NCR_0 - NCR_1)PVS_1 &= NCR_0PVS_1 - PVB_1 \\ &\quad + p_x(PVB_1^0 - NCR_0PVS_1^0) + EP^{i} - P^{i} + G_I \,. \end{aligned} \tag{12}$$

Let us assume that in our pension plan there are

A_0 actives at time 0 and
B_0 retireds at time 0,

and that during the year there were

D_{A0} deaths from A_0,
D_{B0} deaths from B_0,
W withdrawals,

V vesteds among the withdrawals ($V \subset W$),
H disabilities,
R retirements, and
N new entrants.

At time 1 the pension population will consist of

$$A_1 = A_0 - D_{A_0} - W - H - R + N ,$$

$$B_1 = B_0 - D_{B_0} + R ,$$

$$V_1 = V .$$

The recursion relationship between time $t = 0$ and $t = 1$ is

$$A_1 + B_1 + V_1 = A_0 - D_{A_0} - W - H - R + N + B_0 - D_{B_0} + R + V .$$

From the recursion relationship, the following identity must hold:

$$\begin{aligned} PVB_1 = {} & \sum_{A_0} PVB_1^0 - \sum_{D_{A_0}} PVB_1^0 - \sum_{W} PVB_1^0 - \sum_{H} PVB_1^0 \\ & + \sum_{N} PVB + \sum_{V} PVB + \sum_{B_0} PVB_1^0 - \sum_{D_{B_0}} PVB_1^0 \qquad (13) \\ & + \sum_{A_0 \cap A_1} \Delta PVB . \end{aligned}$$

Equation (13) can be easily verbalized. The present value of benefits at time 1 equals the present value of benefits for actives at time 0 valued at time 1 assuming that they all remained in the same status and received their expected salary increases; less the release in liability due to deaths, withdrawals, and disabilities; plus the incurred liability for new entrants and vesteds. To this we add the liability for the retireds had they all remained alive to time 1, less the release due to deaths. The last term, $\Sigma_{A_0 \cap A_1} \Delta PVB$, is the change in liability due to salary experience and is significant only for those active at both $t = 0$ and $t = 1$.

A very similar identity exists for PVS_1, namely,

$$\begin{aligned} PVS_1 = {} & \sum_{A_0} PVS_1^0 - \sum_{D_{A_0}} PVS_1^0 - \sum_{W} PVS_1^0 \\ & - \sum_{H} PVS_1^0 + \sum_{N} PVS + \sum_{A_0 \cap A_1} \Delta PVS . \end{aligned} \qquad (14)$$

If we substitute equations (13) and (14) in equation (12), we have

$$
\begin{aligned}
&(NCR_0 - NCR_1)PVS_1 \\
&\quad = NCR_0\Bigg(\sum_{A0} PVS_1^0 - \sum_{D_{A0}} PVS_1^0 - \sum_{W} PVS_1^0 \\
&\qquad\qquad - \sum_{H} PVS_1^0 + \sum_{N} PVS + \sum_{A0\cap A1} \Delta PVS\Bigg) \\
&\qquad - \Bigg(\sum_{A0} PVB_1^0 - \sum_{D_{A0}} PVB_1^0 - \sum_{W} PVB_1^0 - \sum_{H} PVB_1^0 \\
&\qquad\quad + \sum_{N} PVB + \sum_{V} PVB + \sum_{B0} PVB_1^0 - \sum_{D_{B0}} PVB_1^0 \\
&\qquad\quad + \sum_{A0\cap A1} \Delta PVB\Bigg) \\
&\qquad + (1 - q_x)\left[\left(\sum_{A0} PVB_1^0 + \sum_{B0} PVB_1^0\right) - NCR_0 \sum_{A0} PVS_1^0\right] \\
&\qquad + EP^i - P^i + G_I \,.
\end{aligned}
$$

As unwieldy as this last expression might appear, it can be greatly simplified and easily retained after observing that $\Sigma_{A0} PVB_1^0$, $\Sigma_{B0} PVB_1^0$, and NCR_0 $\Sigma_{A0} PVS_1^0$ cancel out and that the remaining terms can be regrouped to arrive at

$$
\begin{aligned}
&(NCR_0 - NCR_1)PVS_1 \\
&\quad = \left(\sum_{D_{A0}} PVB_1^0 - NCR_0 \sum_{D_{A0}} PVS_1^0\right) \\
&\qquad - q_x\left(\sum_{A0} PVB_1^0 - NCR_0 \sum_{A0} PVS_1^0\right) \\
&\qquad + \left(\sum_{W} PVB_1^0 - NCR_0 \sum_{W} PVS_1^0\right) - \sum_{V} PVB \\
&\qquad + \left(\sum_{H} PVB_1^0 - NCR_0 \sum_{H} PVS_1^0\right) \\
&\qquad - \left(\sum_{N} PVB - NCR_0 \sum_{N} PVS\right)
\end{aligned}
\tag{15}
$$

$$+ \left(\sum_{D_{B_0}} PVB_1^0 - q_x \sum_{B_0} PVB_1^0\right) + (EP^i - P^i)$$

$$- \left(\sum_{A_0 \cap A_1} \Delta PVB - NCR_0 \sum_{A_0 \cap A_1} \Delta PVS\right)$$

$$+ G_l \, .$$

Formula (15) can be analyzed by source as follows:

Deaths:

$$\left(\sum_{D_{A_0}} PVB_1^0 - NCR_0 \sum_{D_{A_0}} PVS_1^0\right) - q_x\left(\sum_{A_0} PVB_1^0 - NCR_0 \sum_{A_0} PVS_1^0\right) .$$

The gain is the excess of the actual release in "accrued liability" over the expected release in "accrued liability." Although the term accrued liability is not universally defined for aggregate funding, the derivation of formula (15) suggests a reasonable definition, which will be used throughout this paper.

Withdrawals:

$$\left(\sum_{W} PVB_1^0 - NCR_0 \sum_{W} PVS_1^0\right) - \sum_{V} PVB \, .$$

Disabilities:

$$\left(\sum_{H} PVB_1^0 - NCR_0 \sum_{H} PVS_1^0\right) .$$

New entrants:

$$- \left(\sum_{N} PVB - NCR_0 \sum_{N} PVS\right) .$$

Deaths among retirees:

$$\left(\sum_{D_{B_0}} PVB_1^0 - q_x \sum_{B_0} PVB_1^0\right) + (EP^i - P^i) .$$

Salaries:

$$- \left(\sum_{A_0 \cap A_1} \Delta PVB - NCR_0 \sum_{A_0 \cap A_1} \Delta PVS\right) .$$

Investments:

$$G_I = (F_0 + NC_0)(i' - i) - (P^{i'} - P^{i}) .$$

It might be observed that the only decrement considered in this derivation was mortality. This was an expedient taken to reduce the number of terms. If decrements for withdrawals and disabilities are included *and* if the decrements are additive, it is a simple matter to separate the expected release of accrued liability,

$$q_x^T\left(\sum_{A_0} PVB_1^0 - NCR_0 \sum_{A_0} PVS_1^0\right) ,$$

into its components.

Finally, the measure of an actuarial gain or loss for an aggregate or frozen initial liability funding method is the difference between normal cost rates rather than a dollar amount. In order to calculate this measure, we simply divide both sides of equation (15) by PVS_1.

V. EXAMPLE

One of the difficulties encountered in learning gain and loss analysis is that, except for the most trivial pension plan, the calculations quickly become unmanageable; hence, a gain and loss analysis is feasible only when it is computer-generated. Under these circumstances, the theory developed in the previous section tends to be replaced by summarizing computer output.

In this section we will perform two valuations on a pension plan and explain the difference in normal cost percentages by analyzing the plan experience by source.

BACKGROUND

Plan summary: The normal retirement benefit is 50 percent of the final year's salary. The plan provides a graded 5–15-year vesting schedule.

Funding method: Aggregate funding.

Actuarial assumptions:

Interest: 5 percent.
Preretirement mortality: 1971 GAM.
Postretirement mortality: 1971 GAM.
Salary scale: none.
Withdrawal: none.
Retirement age: 65.

Participants on January 1, 1981:

Age	No.	Salary
30	80	$12,000
45	75	20,000
50	100	36,000
65	10	12,000 (annual benefit)

Assets on January 1, 1981: $3,600,000.

During 1981 there were the following occurrences:

Age	Deaths	With-drawals	Vesteds*	New Entrants
30	1	7	1	...
35	0	0	0	25
45	2	5	3	...
50	3	1	1	...
65	1	...	...	...

* The number of people vested is a subset of those who withdrew during the year; for example, among the five people who withdrew at age 45, three were vested.

Additional information:

1. All of the terminated vested participants were 50 percent vested in their projected benefits.
2. The 65-year-old who died during the year received monthly pension payments totaling $3,000.
3. Assets earned 8 percent for 1981.

Active participants on January 1, 1982:

Age	No.	Salary
31	72	$14,000
35	25	16,000
46	68	24,000
51	96	42,000

Relevant actuarial factors:

Age (x)	$N_{65}^{(12)}/D_x$	$\ddot{a}_{x:\overline{65-x}\rceil}$	q_x
30	1.46912	16.71037	0.000809
31	1.54383	16.50929	...
35	1.88368	15.60657	...
45	3.12028	12.49736	0.002922
46	3.28590	12.10761	...
50	4.05874	10.38019	0.005285
51	4.28432	9.90153	...
65	9.94404	...	0.021260
66	9.62861	...	...

JANUARY 1, 1981 VALUATION

Age (x)	No.	Salary	Benefit	$N_{65}^{(12)}/D_x$	PVB_0	$\ddot{a}_{x:\overline{65-x}\rvert}$	PVS_0
30	80	$12,000	$ 6,000	1.46912	$ 705,178	16.71037	$16,041,955
45	75	20,000	10,000	3.12028	2,340,210	12.49736	18,746,040
50	100	36,000	18,000	4.05874	7,305,732	10.38019	37,368,684
65	10	......	12,000	9.94404	1,193,285	....	....
Total ..	.....	......	......	....	$11,544,405	....	$72,156,679

$$NCR_0 = (\$11{,}544{,}405 - \$3{,}600{,}000)/\$72{,}156{,}679 = 11.0099\% .$$

$$NC_0 = (0.110099)[(80)(\$12{,}000) + (75)(\$20{,}000) + (100)(\$36{,}000)]$$
$$= \$667{,}200 .$$

JANUARY 1, 1982 VALUATION

Age (x)	No.	Salary	Benefit	$N_{65}^{(12)}/D_x$	PVB_1	$\ddot{a}_{x:\overline{65-x}\rvert}$	PVS_1
31	72	$14,000	$ 7,000	1.54383	$ 778,090	16.50929	$16,641,364
35	25	16,000	8,000	1.88368	376,736	15.60657	6,242,628
46	68	24,000	12,000	3.28590	2,681,294	12.10761	19,759,620
51	96	42,000	21,000	4.28432	8,637,189	9.90153	39,922,969
31	1	......	3,000	1.54383	4,631	....	....
46	3	......	5,000	3.28590	49,289	....	....
51	1	......	9,000	4.28432	38,559	....	....
66	9	......	12,000	9.62861	1,039,890	....	....
Total ..	.....	......	......	....	$13,605,678	....	$82,566,581

$$\begin{aligned}
\text{Assets } (F_1) &= (\$3{,}600{,}000 + \$667{,}200)(1.08) \\
&\quad - 9(\$12{,}000)[1 + (13/24)(0.08)] \\
&\quad - \$1{,}000\{(1 + 0.08) + [1 + (11/12)(0.08)] + [1 + (10/12)(0.08)]\} \\
&= \$4{,}492{,}676 .
\end{aligned}$$

$$\begin{aligned}
NCR_1 &= (\$13{,}605{,}678 - \$4{,}492{,}676)/\$82{,}566{,}581 \\
&= 11.0372\% .
\end{aligned}$$

GAINS BY SOURCE

Interest:

$$G_I = (\$3{,}600{,}000 + \$667{,}200)(0.08 - 0.05) - (P^{0.08} - P^{0.05}) .$$

$$\begin{aligned}
P^{0.08} &= (9)(\$12{,}000)[1 + (13/24)(0.08)] + (\$1{,}000)\{(1 + 0.08) \\
&\quad + [1 + (11/12)(0.08)] + [1 + (10/12)(0.08)]\}. \\
&= \$115{,}900.
\end{aligned}$$

$$\begin{aligned}
P^{0.05} &= (9)(\$12{,}000)[1 + (13/24)(0.05)] + (\$1{,}000)\{(1 + 0.05) \\
&\quad + [1 + (11/12)(0.05)] + [1 + (10/12)(0.05)]\}. \\
&= \$114{,}063.
\end{aligned}$$

$$G_I = \$126{,}179.$$

New entrants:

$$-[\$376{,}736 - (0.110099)(\$6{,}242{,}628)] = \$310{,}571.$$

Withdrawals:

$$\begin{aligned}&(7)[(\$6{,}000)(1.54383) - (0.110099)(\$12{,}000)(16.50929)]\\&\quad + (5)[(\$10{,}000)(3.28590) - (0.110099)(\$20{,}000)(12.10761)]\\&\quad + (1)[(\$18{,}000)(4.28432) - (0.110099)(\$36{,}000)(9.90153)]\\&\quad - \$4{,}631 - \$49{,}289 - \$38{,}559\\&= -\$111{,}457.\end{aligned}$$

Preretirement deaths:

Actual release:

$$\begin{aligned}&(1)[(\$6{,}000)(1.54383) - (0.110099)(\$12{,}000)(16.50929)]\\&\quad + (2)[(\$10{,}000)(3.28590) - (0.110099)(\$20{,}000)(12.10761)]\\&\quad + (3)[(\$18{,}000)(4.28432) - (0.110099)(\$36{,}000)(9.90153)]\\&= \$113{,}465.\end{aligned}$$

Expected release:

$$\begin{aligned}&(0.000809)(80)[(\$6{,}000)(1.54383) - (0.110099)(\$12{,}000)(16.50929)]\\&\quad + (0.002922)(75)[(\$10{,}000)(3.28590) - (0.110099)(\$20{,}000)(12.10761)]\\&\quad + (0.005285)(100)[(\$18{,}000)(4.28432) - (0.110099)(\$36{,}000)(9.90153)]\\&= \$20{,}562.\end{aligned}$$

Gain = Actual release *minus* Expected release:

$$\$113{,}465 - \$20{,}562 = \$92{,}903.$$

Postretirement deaths:

Actual release *minus* Expected release:

$$\begin{aligned}&(1)(\$12{,}000)(9.62861) - (0.021260)(10)(\$12{,}000)(9.62861)\\&= \$115{,}543 - \$24{,}565\\&= \$90{,}978.\end{aligned}$$

$$\begin{aligned}EP^{0.05} &= (10)(\$12{,}000)[1 + (13/24)(0.05) - (11/24)(0.021260)]\\&= \$122{,}081.\end{aligned}$$

$$P^{0.05} = \$114{,}063.$$

$$\text{Gain} = \$90{,}978 + \$122{,}081 - \$114{,}063 = \$98{,}996.$$

Salaries:

$$\begin{aligned}&-\{(72)[(\$1{,}000)(1.54383) - (0.110099)(\$2{,}000)(16.50929)]\\&\quad + (68)[(\$2{,}000)(3.28590) - (0.110099)(\$4{,}000)(12.10761)]\\&\quad + (96)[(\$3{,}000)(4.28432) - (0.110099)(\$6{,}000)(9.90153)]\}\\&= -\$539{,}669.\end{aligned}$$

SUMMARY

Source	Gain	Points*
Interest	$126,179	0.1528%
New entrants	310,571	0.3761
Withdrawals	− 111,457	−0.1350
Preretirement deaths	92,903	0.1125
Postretirement deaths	98,996	0.1199
Salaries	− 539,669	−0.6536
Total	....	−0.0273%

*Gain (loss)/(PVS_1 = $82,566,581).

$$NCR_0 - NCR_1 = 11.0099\% - 11.0372\% = -0.0273\%.$$

VI. OTHER FUNDING METHODS

Thus far, gain and loss formulas have been derived for the aggregate-cost funding method. Because no new theory is required, it is a simple matter to extend our knowledge by deriving the formulas for other actuarial funding methods en masse.

A. *Frozen Initial Liability Funding*

$$NCR_0 = \frac{PVB_0 - F_0 - UL_0}{PVS_0};$$

hence,

$$F_0 = PVB_0 - UL_0 - NCR_0 PVS_0 . \tag{16}$$

$$NCR_1 = \frac{PVB_1 - F_1 - UL_1}{PVS_1};$$

hence,

$$(NCR_0 - NCR_1)PVS_1 = NCR_0 PVS_1 - PVB_1 + F_1 + UL_1 . \tag{17}$$

But

$$UL_1 = (UL_0 + NC_0)(1 + i) - C^i ,$$

and

$$\begin{aligned} F_1 &= F_0(1 + i') + C^{i'} - P^{i'} \\ &= [F_0(1 + i) + C^i - P^i] + G_I ; \end{aligned}$$

thus,

$$F_1 + UL_1 = (UL_0 + NC_0)(1 + i) - C^i + F_0(1 + i) + C^i - P^i + G_I = (UL_0 + NC_0 + F_0)(1 + i) - P^i + G_I .$$

Upon substituting equation (16) in the last equation and noting that $NC_0 = NCR_0S_0$, we arrive at

$$F_1 + UL_1 = [PVB_0 - NCR_0\,(PVS_0 - S_0)](1 + i) - P^i + G_I .$$

Finally, if we proceed to substitute this last expression in equation (17), we will produce a familiar equation:

$$(NCR_0 - NCR_1)PVS_1 = NCR_0PVS_1 - PVB_1 + [PVB_0 - NCR_0(PVS_0 - S_0)](1 + i) - P^i + G_I . \tag{18}$$

All that is necessary to complete the derivation is to substitute equations (1), (2), and (3) in equation (18) and expand PVB_1 and PVS_1, just as was done before in the aggregate-cost-funding derivation.

B. *Accrued Benefit Cost Method (ABCM)*

The actuarial gain (G) under ABCM is defined as the excess of the expected unfunded liability, UL_1^E, over the actual unfunded liability, UL_1, where

$$UL_1^E = (UL_0 + NC_0)(1 + i) - C^i = (AL_0 - F_0 + NC_0)(1 + i) - C^i$$

and

$$UL_1 = AL_1 - F_1 .$$

But

$$F_1 = F_0(1 + i') + C^{i'} - P^{i'} = F_0(1 + i) + C^i - P^i + G_I ,$$

and, therefore,

$$UL_1 = AL_1 - [F_0(1 + i) + C^i - P^i + G_I] ;$$

thus,

$$
\begin{aligned}
G &= UL_1^E - UL_1 \\
&= (AL_0 - F_0 + NC_0)(1 + i) - C^i - AL_1 \\
&\quad + F_0(1 + i) + C^i - P^i + G_I \\
&= (AL_0 + NC_0)(1 + i) - AL_1 - P^i + G_I .
\end{aligned} \tag{19}
$$

Now, under ABCM we have

$$NC_0 = \sum_{A_0} b_0 \frac{N_r^{(12)}}{D_x} ,$$

where b_0 is the benefit accrual. For active participants, we have

$$AL_0 = \sum_{A_0} AB_0 \frac{N_r^{(12)}}{D_x} ,$$

and, for retirees,

$$AL_0 = \sum_{B_0} AB_0 \frac{N_x^{(12)}}{D_x} ,$$

where AB_0 is the accrued benefit at time 0. Thus, we may write

$$
\begin{aligned}
&(AL_0 + NC_0)(1 + i) \\
&\quad = \left(\sum_{A_0} AB_0 \frac{N_r^{(12)}}{D_x} + \sum_{B_0} AB_0 \frac{N_x^{(12)}}{D_x} + \sum_{A_0} b_0 \frac{N_r^{(12)}}{D_x}\right)(1 + i) \\
&\quad = \left[\sum_{A_0} (AB_0 + b_0) \frac{N_r^{(12)}}{D_x} + \sum_{B_0} AB_0 \frac{N_x^{(12)}}{D_x}\right](1 + i) .
\end{aligned}
$$

Noting that $AB_1 = AB_0 + b_0$, and referring back to equations (1) and (2), we can rewrite this last equation as

$$(AL_0 + NC_0)(1 + i) = p_x AL_1^0 + EP^i . \tag{20}$$

Substituting this in equation (19) yields

$$G = p_x AL_1^0 - AL_1 + EP^i - P^i + G_I . \tag{21}$$

Once again, AL_1 can be expressed as the sum of its components, and we can collect terms to produce

$$\begin{aligned} G = & \left(\sum_{D_{A_0}} AL_1^0 - q_x \sum_{A_0} AL_1^0\right) \\ & + \left(\sum_{W} AL_1^0 - \sum_{V} AL\right) \\ & + \left(\sum_{H} AL_1^0 - \sum_{N} AL\right) \qquad (22) \\ & + \left(\sum_{D_{B_0}} AL_1^0 - q_x \sum_{D_{B_0}} AL_1^0\right) + EP^i - P^i \\ & - \sum_{A_0 \cap A_1} \Delta AL + G_I \,. \end{aligned}$$

C. *Entry Age Normal (EAN)*

The actuarial gain under entry age normal is also measured as the excess of the expected unfunded liability over the actual unfunded liability, that is,

$$G = UL_1^E - UL_1 \,.$$

Just as we did under the accrued benefit cost method, we can transform this equation into (19):

$$G = (AL_0 + NC_0)(1 + i) - AL_1 - P^i + G_I \,.$$

Under EAN,

$$AL_0 = PVB_0 - NCR_0 PVS_0 \,,$$

and

$$NC_0 = NCR_0 S_0 \,.$$

Also, by definition,

$$AL_1 = PVB_1 - NCR\, PVS_1 \,.$$

The subscript for the normal cost rate was intentionally omitted in the definition of AL_1. Classically, a normal cost rate was computed for each individual, and this rate, once determined, would remain constant with respect to that individual in the absence of any plan changes. A very convenient simplification of this approach was to assume an average age at entry, hence a single normal cost rate was applicable to all participants. This approach forces all actuarial experience into the change in unfunded liability.

Today, most valuations are handled by computers, and a valuation program usually will recompute the entry age normal cost for an individual each year. Under this scheme, the recomputed normal cost rate, NCR_1, may not equal the previous year's normal cost rate, NCR_0. This would be true in a social security–integrated plan, or in a pay-related benefit formula where normal costs are determined as a flat dollar amount per individual rather than as a percentage of salary. (The reader may want to satisfy himself that, in a pay-related plan, normal cost rates when determined as a percentage of salary will remain constant irrespective of salary experience as long as a percentage increase in salary produces the same percentage increase in projected benefit.) When the normal cost rate is recomputed each year, part of the actuarial experience is absorbed into the present value of future normal costs. To understand why this is so, we can substitute the above expressions for AL_0, NC_0, and AL_1 in equation (19):

$$\begin{aligned} G &= [PVB_0 - NCR_0(PVS_0 - S_0)](1 + i) \\ &\quad - (PVB_1 - NCR_1PVS_1) - P^i + G_I \\ &= [PVB_0 - NCR_0(PVS_0 - S_0)](1 + i) \\ &\quad -(PVB_1 - NCR_0PVS_1) - P^i + G_I \\ &\quad - (NCR_0 - NCR_1)PVS_1 \, . \end{aligned} \tag{23}$$

We readily observe that, if $NCR_0 > NCR_1$, actuarial gains will be smaller under this approach because some of the experience does not flow into the unfunded accrued liability but instead is reflected in lower future contributions. Conversely, if $NCR_0 < NCR_1$, then the actuarial gain will be larger because it will be offset by larger future contributions.

In actual practice, the last term of equation (23) is absorbed into the salary experience because it is relevant only for those individuals in the group $A_0 \cap A_1$. The completion of equation (23) to produce the gain and loss formulas of equation (22) should be familiar. We first write

$$\begin{aligned} &[PVB_0 - NCR_0(PVS_0 - S_0)](1 + i) \\ &\qquad = p_x(PVB_1^0 - NCR_0PVS_1^0) = p_xAL_1^0 \, . \end{aligned}$$

PVB_1 and PVS_1 then must be split into their components.

VII. MULTIPLE DECREMENTS

The purpose of this section is to expand upon the theory already developed by escalating the pension plan model to include multiple decrements and provide benefits resulting from each decrement. In Sections IV and V, we saw the effect of an individual's withdrawing from a plan with a vested benefit—a liability was released and a liability was incurred. When a withdrawal decrement is included, the model requires a value for the liability expected to be incurred. If the actual release in liability exceeds the expected release in liability, there will be an actuarial gain. Similarly, if the anticipated or expected liability to be incurred resulting from individuals withdrawing with vested benefits is greater than the liability actually incurred from such withdrawals, there will be an actuarial gain.

That is,

$$\text{Gain} = (\text{Actual release} - \text{Expected release}) + (\text{Expected incurred} - \text{Actual incurred}) .$$

Notationally, this will be written

$$G = (AR - ER) + (EI - AI) .$$

The problem that follows is intended to be comprehensive in that, in addition to a retirement benefit, a subsidized early-retirement benefit and a surviving spouse's benefit are valued; the valuation model assumes that decrements occur in the middle of the year.

Several abbreviations that may be unfamiliar to the reader are defined in the Appendix.

BACKGROUND

Plan summary: The normal retirement benefit is 1 percent of the final year's salary multiplied by the number of years of service at retirement. The plan provides a graded 5–15 vesting schedule. Upon attaining age 60 and completing twenty years of participation in the plan, a participant may retire with a reduced benefit payable upon his early-retirement date. The reduction is 0.5 percent per month for each month before age 65. A participant is eligible for a spouse's benefit upon attaining early-retirement eligibility. The amount payable to the surviving spouse upon the participant's death while still actively employed is equal to the amount that would have been payable to the spouse had the participant retired on his date of death and elected a qualified 50 percent joint and survivor annuity.

Funding method: Aggregate funding.

Actuarial assumptions:

Interest: 6 percent.
Mortality: 1971 GAM (males); 1971 GAM set back six years (females).
Salary scale: 4 percent.
Withdrawal: T-1.
Retirement:

Age	q_x^r	Age	q_x^r
60	0.1	63	0.1
61	0.1	64	0.1
62	0.3	65	1.0

Other: 85 percent of the participants are married; husband is three years older than wife.

Participants on January 1, 1981:

Age	No.	Sex	Salary	Service
60	1,000	Male	\$20,000	30 years

Assets on January 1, 1981: \$45,800,000.

JANUARY 1, 1981 VALUATION

Age x	Year t	$\overset{r}{q}_x$	$\overset{d}{q}_x$	$\overset{T}{q}_x$	$\overset{T}{p}_x$	${}_{x-60}\overset{T}{p}_{60}$	$v^t\ {}_{x-60}\overset{T}{p}_{60}$
60	0.5	0.1	0.013119	0.113119	0.886881	1.000000	0.971286
61	1.5	0.1	0.014440	0.114440	0.885560	0.886881	0.812656
62	2.5	0.3	0.015863	0.315863	0.684137	0.785386	0.678920
63	3.5	0.1	0.017413	0.117413	0.882587	0.537312	0.438183
64	4.5	0.1	0.019185	0.119185	0.880815	0.474224	0.364844
65	5.0	1.0	0.021260	1.000000	0.000000	0.417704	0.312133

$$\begin{aligned} {}^s\ddot{a}_{60:\overline{5}|} &= 1 + (1.04/1.06)(0.886881) + (1.04/1.06)^2(0.785386) \\ &\quad + (1.04/1.06)^3(0.537312) + (1.04/1.06)^4(0.474224) \\ &= 3.573078\ . \end{aligned}$$

Age x	Year t	Salary S_x	YOS	$(0.01)S_x$ (YOS)	ERF	PB_x	$v^t\ {}_{x-60}\overset{T}{p}_{60}$
60	0.5	\$20,400	30.5	\$6,222.00	0.73	\$4,542.06	0.971286
61	1.5	21,216	31.5	6,683.04	0.79	5,279.60	0.812656
62	2.5	22,065	32.5	7,171.01	0.85	6,095.36	0.678920
63	3.5	22,947	33.5	7,687.32	0.91	6,995.46	0.438183
64	4.5	23,865	34.5	8,233.46	0.97	7,986.46	0.364844
65	5.0	24,342	35.0	8,519.85	1.00	8,519.85	0.312133

Age x	Year t	q_x^r	$PB_x v^t \, {}_{x-60}p_{60}^T q_x^r$	$\ddot{a}_{x+\frac{1}{2}}^{(12)}$	$PVRB$
60	0.5	0.1	\$ 441.16	10.46427	\$ 4,616.46
61	1.5	0.1	429.05	10.20650	4,379.10
62	2.5	0.3	1,241.48	9.94391	12,345.14
63	3.5	0.1	306.53	9.67656	2,966.15
64	4.5	0.1	291.38	9.40502	2,740.45
65	5.0	1.0	2,659.33	9.26833	24,647.50
Total ...	....	....	....	....	\$51,694.80

Age x	Year t	q_x^d	$f_{50:xy}$	$PB_x(0.5)(0.85) \times f_{50:xy} v^t \, {}_{x-60}p_{60}^T q_x^d$	$f\ddot{a}_{x-2\frac{1}{2}}^{(12)}$	$PVDB$
60	0.5	0.013119	0.87363	\$21.49	12.51186	\$ 268.87
61	1.5	0.014440	0.86831	22.86	12.30886	281.42
62	2.5	0.015863	0.86283	24.07	12.10030	291.28
63	3.5	0.017413	0.85716	19.44	11.88595	231.12
64	4.5	0.019185	0.85132	20.23	11.66553	235.94
65	5.0	0.021260	0.84836	....	11.55376	0
Total	......	....	....	....	....	\$1,308.63

Liability for

Retirement benefits (*PVRB*)	\$51,694.80
Death benefits (*PVDB*)	1,308.63
Total *PVB*	\$53,003.43

$$PVB_0 = (1{,}000)(\$53{,}003.43) = \$53{,}003{,}430.$$

$$PVS_0 = (1{,}000)(\$20{,}000)(3.573078) = \$71{,}461{,}560.$$

$$F_0 = \$45{,}800{,}000.$$

$$NCR_0 = (\$53{,}003{,}430 - \$45{,}800{,}000)/\$71.461{,}560.$$
$$= 10.0801\%.$$

$$NC_0 = (0.100801)(1{,}000)(\$20{,}000) = \$2{,}016{,}020.$$

During the year:

1. Seventy-five employees retired, seventy of whom were married; two subsequently died leaving spouses.
2. Ten actives died, nine leaving spouses; one spouse subsequently died.
3. The annual benefit for everyone who retired or died was \$4,563.00.
4. Each spouse was two years younger than her husband; the appropriate reduction factor, $f_{50:60:58}$, was 0.87816, producing the following: (0.87816) (\$4,563.00) = \$4,007.04 payable as a 50 percent joint and survivor, with (0.5) (\$4,007.04) = \$2,003.52 payable to the contingent annuitant.
5. The balance of active participants received a 6 percent salary raise to \$21,200.

JANUARY 1, 1982 VALUATION

A trustees' statement indicates that the assets as of December 31, 1981, are \$51,039,587. This amount includes a contribution on January 1, 1981, of \$2,016,020, and considers benefit payments for each month as indicated below.

Month	Payments
January	$ 2,003.52
February	4,174.00
March	6,390.81
April	8,607.62
May	10,611.14
June	12,614.66
July	14,785.14
August	16,788.66
September	19,005.47
October	21,602.53
November	23,939.97
December	26,277.41
	$166,800.93

An analysis of the assets reveals that the fund has earned 7.1 percent over the last twelve months.

For future reference, we will note that $P^{0.071}$ = $171,370 and $P^{0.060}$ = $170,662.

If we were to perform a valuation as of January 1, 1982, for each active participant, we would determine the following:

Retirement liability	$57,350.10
Death benefit liability	1,266.62
Total active *PVB*	$58,616.72

Also, the following annuity factors would be determined:

$$\ddot{a}_{61}^{(12)} \text{ (male)} = 10.33659.$$

$$\ddot{a}_{59}^{(12)} \text{ (female)} = 12.20599.$$

$$\ddot{a}_{61}^{(12)} + (0.5)({}^{f}\ddot{a}_{59}^{(12)} - \ddot{a}_{61:59}^{(12)}) = 11.80551.$$

$${}^{s}\ddot{a}_{61:\overline{4|}} = 2.957060.$$

$$\ddot{a}_{58}^{(12)} \text{ (female)} = 12.41173.$$

$$q_{57}^{d} \text{ (female)} = 0.005867.$$

Total *PVB:*

(915)($58,616.72)	$53,634,299
(68)($4,007.04)(11.80551)	3,216,750
(5)($4,563.00)(10.33659)	235,829
(10)($2,003.52)(12.20599)	244,549
	$57,331,427

Total *PVS:*

(915)($21,200)(2.957060)	$57,361,050

$$NCR_1 = (\$57{,}331{,}427 - \$51{,}039{,}587)/\$57{,}361{,}050$$
$$= 10.9688\%.$$

$$NCR_0 - NCR_1 = 10.0801\% - 10.9688\% = -0.8887\%.$$

GAINS BY SOURCE

First, we will need to determine PVB_1^0 and PVS_1^0.

$$PVB_1^0 = (\$53{,}003.43 - \$4{,}616.46 - \$268.87)(1.06/0.886881)$$
$$= \$57{,}510.74.$$

$$PVS_1^0 = (\$20{,}800)(2.957060) = \$61{,}506.85.$$

The gains are then as follows:

Interest:

$$(\$45{,}800{,}000 + \$2{,}016{,}020)(0.011) - (\$171{,}370 - \$170{,}662) = \$525{,}268.$$

Deaths:

$$\begin{aligned}
AR &= [\$57{,}510.74 - (0.100801)(\$61{,}506.85)](10) \\
&= \$513{,}108. \\
ER &= [\$57{,}510.74 - (0.100801)(\$61{,}506.85)](0.013119)(1{,}000) \\
&= \$673{,}146. \\
AR - ER &= \$513{,}108 - \$673{,}146 = -\$160{,}038. \\
AI &= (8)(\$2{,}003.52)(12.20599) = \$195{,}640. \\
EI &= (0.013119)(1{,}000)(\$4{,}542.06)(0.87363)(0.85)(0.5) \\
&\quad \times (12.41173)(1 - 0.005867/2) \\
&= \$273{,}796. \\
EI - AI &= \$273{,}796 - \$195{,}640 = \$78{,}156. \\
\text{Gain} &= -\$160{,}038 + \$78{,}156 = -\$81{,}882.
\end{aligned}$$

Retirements:

$$\begin{aligned}
AR &= (75)\,[\$57{,}510.74 - (0.100801)(\$61{,}506.85)] \\
&= \$3{,}848{,}309. \\
ER &= (0.1)(1{,}000)[\$57{,}510.74 - (0.100801)(\$61{,}506.85)] \\
&= \$5{,}131{,}079. \\
AR - ER &= \$3{,}848{,}309 - \$5{,}131{,}079 = -\$1{,}282{,}770. \\
AI &= (68)(\$4{,}007.04)(11.80551) + (2)(\$2{,}003.52)(12.20599) \\
&\quad + (5)(\$4{,}563.00)(10.33659) \\
&= \$3{,}501{,}489. \\
EI &= (0.1)(1{,}000)(\$4{,}542.06)(10.33659)(1 - 0.013119/2) \\
&= \$4{,}664{,}145. \\
EI - AI &= \$1{,}162{,}656. \\
\text{Gain} &= -\$1{,}282{,}770 + \$1{,}162{,}656 = -\$120{,}114.
\end{aligned}$$

Salaries:

$$\begin{aligned}
\Delta PVB &= PVB_1 - PVB_1^0 \\
&= \$58{,}616.72 - \$57{,}510.74 \\
&= \$1{,}105.98. \\
\Delta PVS &= PVS_1 - PVS_1^0 \\
&= (\$21{,}200 - \$20{,}800)(2.957060) = \$1{,}182.82. \\
\text{Gain} &= -(915)[\$1{,}105.98 - (0.100801)(\$1{,}182.82)] = -\$902{,}877.
\end{aligned}$$

Payments: The actuarial experience resulting from benefit payments is treated separately because an additional complication is introduced when our valuation model assumes that decrements occur in the middle of the year. In particular, we must determine the value of $EP^{0.06}$, the accumulated value of expected benefit payments. We can see from the January 1, 1981, valuation that $\ddot{a}^{(12)}_{60\frac{1}{2}} = 10.46427$, and from the January 1, 1982, valuation that $\ddot{a}^{(12)}_{61} = 10.33659$. Upon referring back to preliminary relationship (2), we can write the following analogous equation:

$$\ddot{a}^{(12)}_{60\frac{1}{2}}(1.06)^{\frac{1}{2}} = {}_{\frac{1}{2}}p_{60\frac{1}{2}}\ddot{a}^{(12)}_{61} + EP^{i} ,$$

$$(10.46427)(1.06)^{\frac{1}{2}} = (1 - 0.013119/2)(10.33659) + EP^{0.06} ,$$

which implies $EP^{0.06} = 0.504838$.

Similarly, the value of $EP^{0.06}$ resulting from benefit payments to surviving spouses can be derived from the equation

$$(12.51186)(1.06)^{\frac{1}{2}} = (1 - 0.005867/2)(12.41173) + EP^{0.06} ,$$

which yields $EP^{0.06} = 0.506428$.

The expected benefit payments can now be computed as follows:

$$\begin{aligned}&(0.1)(1{,}000)(\$4{,}542.06)(0.504838)\\&\quad + (0.013119)(1{,}000)(0.85)(0.5)(0.87363)(\$4{,}542.06)(0.506428)\\&= \$240{,}505 .\end{aligned}$$

Actual benefit payments have been determined to be \$170,662; thus

$$\begin{aligned}\text{Gain} &= \$240{,}505 - \$170{,}662\\ &= \$69{,}843 .\end{aligned}$$

SUMMARY

Source	Gain	Points*
Interest	\$525,268	0.9157%
Deaths	− 81,882	−0.1427
Retirements	− 120,114	−0.2094
Salaries	− 902,877	−1.5740
Payments	69,843	+0.1218
Total	....	−0.8886%

* Gain (loss)/57,361,050.

$NCR_0 - NCR_1 = 10.0801\% - 10.9688\% = -0.8887\%$.

The explanation presented in this section has been mostly intuitive and has been illustrated by example. Although it would be repetitious to provide a complete derivation of the gain and loss formulas for the multiple-decrement model, it will be instructive to reexamine equation (10). Specifically, the expression $PVB_0(1 + i)$ should be analyzed in light of the numerical illustration already presented.

$$PVB_0 = PB_{60}v^{1/2}\ {}_0p_{60}^T q_{60}^r \ddot{a}_{60\,1/2}^{(12)}$$

$$+\sum_{k=1}^{5} PB_{60+k}v^{k+1/2}\ {}_kp_{60}^T q_{60+k}^r \ddot{a}_{60\,1/2+k}^{(12)}$$

$$+PB_{60}(0.5)(0.85)f_{50:60:57}v^{1/2}\ {}_0p_{60}^T q_{60}^d\ {}^f\ddot{a}_{57\,1/2}^{(12)}$$

$$+\sum_{k=1}^{4} PB_{60+k}(0.5)(0.85)f_{50:60+t:57+k}v^{k+1/2}\ {}_kp_{60}^T q_{60+k}^d\ {}^f\ddot{a}_{57\,1/2+k}^{(12)}\ .$$

(Note: In first summation, when $k = 5$ use v^5 and $\ddot{a}_{65}^{(12)}$).

The first and third terms are essentially immediate annuities multiplied by constants, while the second and fourth terms are deferred annuities. Now, by application of preliminary relationships (1) and (2), we have

$$PVB_0(1 + i) = (PB_{60}q_{60}^r)({}_{1/2}p_{60\,1/2}^d \ddot{a}_{61}^{(12)} + {}^rEP^i)$$
$$+ (PB_{60})(0.5)(0.85)f_{50:60:57}(q_{60}^d)$$
$$\times ({}_{1/2}^fp_{57\,1/2}^d\ {}^f\ddot{a}_{58}^{(12)} + {}^dEP^i)$$
$$+p_x^T PVB_1^0\ .$$

The first term is the expected incurred liability for retired lives plus the expected benefit payments to the retirees, ${}^rEP^i$, accumulated with interest from midyear to the end of the year. The second term is the expected incurred liability arising from deaths plus the expected accumulated benefit payments, ${}^dEP^i$, to the surviving spouses. The last term, $p_x^T PVB_1^0$, has been presented earlier. It will be rewritten as $(1 - q_x^T)PVB_1^0$ and, as suggested at the end of Section IV, will provide the expected release in accrued liability by source.

As the reader may have perceived, one "trick" was employed to ensure that the gain and loss components add up to the total. The analogue to preliminary relationship (2) was forced because standard approximations will not permit each term to be explicitly defined when decrements occur in the middle of the year. The obvious expedient was to define either the expected incurred liability or the expected accumulated benefit payments and then solve for the other term.

VIII. ACKNOWLEDGMENTS

The author wishes to express appreciation to Improved Funding Techniques, Inc., for resources and support in preparing this paper. Particular gratitude goes to Christine Rao for typing the draft.

APPENDIX

YOS = Years of service.

ERF = Early-retirement reduction factor.

PB_x = Projected benefit payable at age x.

${}^f\ddot{a}_x^{(12)}$ = Female life annuity factor.

$f_{50:xy}$ = 50 percent joint and survivor option factor with a male principal annuitant aged x and a female contingent annuitant aged y.

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DISCUSSION OF PRECEDING PAPER

JAMES E. HOLLAND, JR.:

First, I would like to state that the following views are my own and do not necessarily represent those of the Internal Revenue Service.

Mr. Small's paper is a helpful addition to the growing body of literature on the analysis of actuarial experience by source. The numerical examples particularly add to his development of the analysis.

There is one inconsistency in the investment gain used for the various methods. For the aggregate method, the equation for the value of fund assets at time $t = 1$ is given as

$$\begin{aligned} F_1 &= (F_0 + NC_0)(1 + i') - P^{i'} \\ &= [(F_0 + NC_0)(1 + i) - P^{i'}] + [(F_0 + NC_0)(i' - i) \\ &\quad - (P^{i'} - P^{i})] . \end{aligned}$$

The gain from investment experience, G_I, is then stated as

$$G_I = (F_0 + NC_0)(i' - i) - (P^{i'} - P^{i}) .$$

However, for all the other funding methods the value of the assets at time $t = 1$ is developed as

$$F_1 = F_0(1 + i') + C^{i'} - P^{i'} = [F_0(1 + i) + C^{i} - P^{i}] + G_I .$$

For the other methods this produces an investment gain of

$$G_I = F_0(i' - i) + (C^{i'} - C^{i}) - (P^{i'} - P^{i}) .$$

The same investment gain results only if $NC_0(i' - i) = C^{i'} - C^{i}$. The equation for the aggregate method assumes, in effect, that a contribution equal to NC_0 is made at time $t = 0$. However, this may not be the typical case.

It is interesting to see the result when the equation $F_1 = F_0(1 + i') + C^{i'} - P^{i'}$ is used for the aggregate method. Expanding this results in

$$\begin{aligned} F_1 &= F_0(1 + i) + F_0(i' - i) + C^i + (C^{i'} - C^i) - P^i - (P^{i'} - P^i) \\ &= [F_0(1 + i) - P^i] + C^i + [F_0(i' - i) + (C^{i'} - C^i) - (P^{i'} - P^i)] \\ &= [(F_0 + NC_0)(1 + i) - P^i] + [(C^i - (1 + i)NC_0)] \\ &\quad + [F_0(i' - i) + (C^{i'} - C^i) - (P^{i'} - P^i)] \,. \end{aligned}$$

Using $G_I = [F_0(i' - i) + (C^{i'} - C^i) - (P^{i'} - P^i)]$, we may rewrite Mr. Small's equation (9) as

$$F_1 + [(1 + i)NC_0 - C^i] = [(F_0 + NC_0)(1 + i) - P^i] + G_I \,.$$

If the equation for NCR_1 is revised to be

$$NCR_1' = \frac{PVB_1 - [F_1 + (1 + i)NC_0 - C^i]}{PVS_1.} \,.$$

Mr. Small's equation (6) becomes

$$NCR_1'PVS_1 = PVB_1 - [F_1 + (1 + i)NC_0 - C^i] \,,$$

and his equation (8) becomes

$$\begin{aligned} &(NCR_0 - NCR_1')PVS_1 \\ &\qquad = NCR_0PVS_1 - PVB_1 + [F_1 + (1 + i)NC_0 - C^i] \,. \end{aligned}$$

Taking equation (9) as revised above, and making the substitution of equations (4) and (5), followed by the substitution of equations (1), (2), and (3), produces a revised equation (11), or

$$\begin{aligned} &F_1 + [(1 + i)NC_0 - C^i] \\ &\qquad = p_x(PVB_1^0 - NCR_0PVS_1^0) + EP^i - P^i + G_I \,. \end{aligned}$$

Substituting this in the revised equation (8) again gives Mr. Small's equation (12), or

$$\begin{aligned} (NCR_0 - NCR_1')PVS_1 &= NCR_0PVS_1 - PVB_1 \\ &\quad + p_x(PVB_1^0 - NCR_0PVS_1^0) + EP^i - P^i + G_I \,. \end{aligned}$$

The only change is in the values of G_I and of NCR_1; however, the change in the total gain is equal to the change in G_I.

The term $[(1 + i)NC_0 - C']$ has a verbal interpretation. If NC_0 is the contribution required at time $t = 0$ in order to satisfy the minimum funding standards of section 412 of the Internal Revenue Code, the term $[(1 + i)NC_0 - C']$ represents the credit balance (if negative) or funding deficiency (if positive) at time $t = 1$.

The above revision, NCR_1', eliminates any "gain" arising from a contribution that differs in amount or in timing from the normal cost NC_0. This gain is unique for aggregate methods. Under the frozen initial liability method, for example, any difference in a contribution from the amount computed will simply affect the values of the unfunded liability and the assets at time $t = 1$, but NCR_1 will not change. Also, under the regulations relating to reasonable funding methods under the Employee Retirement Income Security Act of 1974 (ERISA) (sec. 1.412(c)(3)-1(b)(1) of the Income Tax Regulations), a credit balance must be subtracted from the assets, or a funding deficiency added to the assets, when the normal cost is computed under the aggregate method. The result is the revised normal cost rate, NCR_1'.

(AUTHOR'S REVIEW OF DISCUSSION)

HOWARD J. SMALL:

I would like to thank Mr. Holland for commenting on the inconsistency of the investment gain under the aggregate method. His analysis is correct. It is interesting that the "textbook" definition of the normal cost rate under aggregate funding must be revised in light of the regulations relating to reasonable funding methods, which require that the assets be adjusted to reflect a credit balance or funding deficiency. One final note worth mentioning is that, under section 404 of the Code regarding maximum deductible contributions, the assets are adjusted by any contribution carryover. Hence, for this determination, NCR_1 *will* include a gain arising from contributions that differ in amount or timing from normal cost NC_0.