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A NEW APPROACH TO THE THEORY OF INTEREST

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ABSTRACT

This paper arose from an investigation into the concept of yield rate. The paper first outlines the difficulties involved in comparing financial transactions strictly on the basis of yield; it then develops an approach that does not depend on yield rates. This approach clarifies the true meaning of yield rates and helps explain such phenomena as multiple-valued and nonexistent yields. Various applications are discussed.

I. INTRODUCTION

THE theory of compound interest generally is considered to be among the most elementary topics in actuarial science. Most actuaries probably feel that they understand the subject completely. There are, however, certain paradoxes and ambiguities that suggest that some revision of the theory may be desirable.

In this paper we consider a new approach to the fundamental problem of comparing two financial transactions. If we are given, for example, a choice between two investments, or between two lenders, how do we decide which is preferable? Traditionally, we attempt to make the comparison on the basis of yield rates. We choose the investment with the higher rate of return, or borrow from the lender charging the lower rate of interest. Unfortunately, a complete reliance on yield rates for such comparisons leads to difficulties. One problem with yield rates is that there may be either too many of them or not enough. In other words, the yield rate on a given transaction may be multiple-valued or nonexistent. In addition, even in cases where a unique yield rate exists it may not adequately reflect the true value of the transaction.

Section IV provides a few simple examples that illustrate the problems mentioned above. The examples are not completely original; some of these phenomena are well known and have been noted by many actuaries. We hope, however, that all readers will find some new ideas in this section. Section V develops an approach to the comparison problem that avoids all the difficulties inherent in the use of yield rates. Sections VI and VII apply the ideas of Section V to produce a classification of financial transactions. We hope that these sections will clarify certain problems in the interpretation of yield rates and will help to explain such occurrences as multiple-valued and nonexistent yields. In Section VII, for example, we show that any financial transaction, even though it may not have a unique yield, can be considered as the composite of two transactions, each of which has a unique yield.

Additional extensions of the theory are discussed in Sections VIII, IX, and X. In Section VIII we give a generalization of a criterion of Kellison [5] for uniqueness of yield.

II. PSYCHOLOGICAL ASPECTS

There are psychological factors that help explain the overreliance on yield rates in judging financial transactions. The attraction of yield rates stems in part from an apparent compulsion to summarize complicated situations by a *single* real number. We see evidence of this same phenomenon in the theory of probability and statistics, where there seems to be an overwhelming urge to average. Given a probability distribution, a common initial reaction is to compute the mean. There is often a tendency to look only at this figure and to ignore other important features of the distribution. All students of statistics learn quite early that at least the standard deviation, as well as the expected value, is needed for any reasonable interpretation of a probability distribution. We will show that, analogously, the proper interpretation of a financial transaction requires more information than just the yield rate.

The concept of *order* may help explain why many people seek to describe situations by a single real number. Faced with a problem such as determining which financial transaction is preferable, our natural reaction is to attempt to find a solution. The more basic goal, however, should be first to decide whether or not a solution exists. We will show that, in general, there is no unqualified solution to the above problem. For some pairs of transactions, it is certainly possible to say that one is better than the other. In most cases, however, we cannot state that one transaction is preferable; the choice will depend on the particular individual contemplating the transaction. It is this idea and its ramifications that constitute one of the basic themes of the paper.

In mathematical terms, a set of financial transactions is a *partially ordered set*. This is to be distinguished from a *totally ordered set*, where it is possible to compare any two distinct quantities. The most common example of a totally ordered set is the real number system. Given any two distinct real numbers, we can always say that one is greater than the other. Our familiarity with the real number system may lead us to expect that this type of ordering holds universally. We use real-valued functions, such as yield rates, in an attempt to transfer the total-order property to other situations. We may, however, be led to invalid conclusions when the situation in question admits only a partial ordering. A more detailed discussion of order, including precise definitions, is given in Appendix II.

III. NOTATION AND TERMINOLOGY

In this section we list some basic notation and terminology that will be used in the remainder of the paper.

We work throughout with a fixed, but unspecified, time period and monetary unit. All interest rates will refer to effective rates for this period.

By a transaction T, we mean simply a finite sequence, $(c_0, c_1, \ldots, c_n)$, of real numbers. This represents the transaction that provides net payments of c_k units at the end of k periods. For example, consider a loan of 3 units that is to be repaid by 2 units at time 1 and 2 units at time 3. From the point of view of the lender, the transaction is the sequence $(-3, 2, 0, 2)$.

The subscript of the last nonzero entry will be called the *duration* of T. So, for example, the previous transaction has duration 3.

It is possible, of course, to consider more general types of transactions. We may wish to consider payments made at arbitrary times rather than just at the end of each period, or we may consider continuous payments. To simplify the presentation, however, we will work with the above definition of a transaction for most of the paper. Generalizations will be discussed in Section X.

For a transaction $T = (c_0, c_1, \ldots, c_n)$ and any real number $i > -1$, we let $P_i(T)$ denote the present value of the payments of T at interest rate i. That is,

$$P_i(T) = \sum_{k=0}^{n} c_k(1 + i)^{-k} .$$

Writing $P_i(T)$ in terms of $v = (1 + i)^{-1}$, we obtain a polynomial

$$f(v) = \sum_{k=0}^{n} c_k v^k , \tag{1}$$

which we will refer to as the *polynomial associated with* T.

We say that i is a *yield rate* of T if $P_i(T) = 0$. Note that a yield rate corresponds to a *positive* root of equation (1), while the more usual case of nonnegative yields corresponds to roots in the interval $(0, 1]$. It is, therefore, practical and mathematically convenient to include $i > -1$

in the definition of yield. Negative yields may apply, for example, if one considers the real value of monetary units in a time of inflation. We do not, however, consider values of $i \leq -1$, since it is difficult to visualize a practical interpretation for a negative or zero value of $1 + i$.

We can define addition and scalar multiplication of transactions in the obvious manner. If $S = (c_0, c_1, \ldots, c_n)$, $T = (d_0, d_1, \ldots, d_n)$, and r is any real number, then

$$S + T = (c_0 + d_0, c_1 + d_1, \ldots, c_n + d_n)$$

and

$$rS = (rc_0, rc_1, \ldots, rc_n) .$$

In particular, $(-1)T$ will be denoted by $-T$, and $S + (-T)$ will be denoted by $S - T$. The sum $S + T$ represents the composite transaction that results when both S and T are undertaken. In a two-party transaction T, such as one involving a lender and a borrower, $-T$ represents the transaction from the other party's point of view.

Note that for the definition of $S + T$ we can always consider the two transactions to be represented by sequences of the same length by including sufficient zero entries. We assume, of course, that two transactions are identical if they differ only by zero entries at the right-hand end.

It is easy to verify that, for fixed i, $P_i(T)$ is a linear function of T; that is, given two transactions S and T and a real number r,

$$P_i(S + T) = P_i(S) + P_i(T) , \quad P_i(rT) = rP_i(T) . \tag{2}$$

For the more technically minded reader, we can summarize the above discussion by saying that the mathematical framework of the paper consists of the vector space of real sequences with all but finitely many entries equal to zero and the family of linear functionals P_i defined on this space.

IV. LIMITATIONS ON THE USE OF YIELD RATES

Example 1. This example was suggested in part by example 5.12 of [5]. Suppose that a loan consists of payments to the borrower of 1 unit now and 6 units at the end of 2 periods. The total loan is to be repaid by a single payment of K at time t. In the case where $t \geq 2$ there is no problem, and it is not hard to verify that there is always a unique yield (it is, of course, negative for $K < 7$). Moreover, for fixed $t \geq 2$ the yield increases as K increases, so, as we expect, the higher the value of K, the higher the yield, and the more attractive the transaction is to the lender. Complications arise when $t < 2$. Consider, for example, the case $t = 1$. For $K = 7$, we get two very different yield rates: $i = 0$ and $i = 500$ percent. It is no longer clear how the yields behave when K changes. For $K = 5$, we again have two yields: $i = 100$ percent and $i = 200$ percent. For $K = 4$, there are no yield rates!

Example 2. Paradoxes also arise in cases not involving multiple-valued or nonexistent yields. Consider two loans, each of which consists of a payment of 1 to the borrower now, to be repaid by a single sum at time 1. In loan 1 the repayment is 2, for a yield of 100 percent, while in loan 2 the repayment is 1, for a yield of zero. It is obvious that loan 1 is more advantageous than loan 2 for the lender, while the reverse is true for the borrower. Consider a natural generalization of this simple example. Assume we are dealing with unique yield rate situations that avoid the complications of example 1. Is it always true that the higher the yield rate, the more favorable a loan is to the lender and the less favorable it is to the borrower? Most people would probably answer yes. However, consider the following two loans. Loan 3 consists of a payment to the borrower of 2 at time 1, to be repaid by 1 unit now. This is the transaction $(1, -2)$, and the yield is clearly 100 percent. Loan 4 consists of a payment to the borrower of 1 at time 1, to be repaid by 1 unit now. The yield is zero. Obviously the lender will prefer loan 4, the one with the *lower* yield!

One can easily anticipate the cries of protest that probably are arising in the mind of the reader. It will be claimed, no doubt, that I am simply playing with words, and that the so-called loans 3 and 4 are not really new loans at all but merely loans 1 and 2 from the borrower's viewpoint. (Indeed, for any transaction, T and $-T$ represent opposing viewpoints, but they have exactly the same yields.) It is true that in this simple case it is easy to distinguish between a loan transaction and a borrowing transaction. The question is whether we can do so in general. The example is intended to illustrate the importance of this question. Without an answer, we cannot even tell whether we desire a higher or a lower figure when comparing two transactions on the basis of yield. We will answer this question in Section VI.

Some readers may object to example 1 as well, noting that the same type of time-reversal trick was used. Both examples involve a repayment of a loan prior to the time the loan itself is completed. This is, of course, contrary to the normal pattern of loans, but it is possible. The fact that an example involves a highly unusual situation is not a sufficient reason to disregard it. Either a mathematical theory such as that of compound interest, which is designed to apply to real-life situations, should apply universally to all cases, or we must attempt to delineate clearly the cases to which it does not apply.

Example 3. Suppose that an investor begins with absolutely nothing. He borrows 100 units now, agreeing to repay the loan with 10 percent interest at the end of 1 period. He then lends the 100 units immediately to a third party, who agrees to repay the principal with 20 percent interest at time 1. What is his yield rate on the entire transaction? I am sure that many people have been perplexed by the problem of computing a yield for this familiar situation. The difficulty comes solely from our preconditioned belief that yield rates must always exist, whereas in this case they obviously do not.

We can easily verify this formally. The net payment at time 0 is 0 units, since the 100 received is immediately reinvested. The net payment at time 1 is 10 units (120 units received minus 110 units repaid). The transaction is given by the sequence (0, 10), and there is clearly no yield rate.

Some may wish to assign a yield of ∞ to the above situation, since the investor begins with nothing and has 10 units after 1 period. There are objections to this, however. For an investor interested in assessing the worth of a transaction, it is not very satisfactory simply to state that the yield is infinite. In addition, there does not appear to be any appropriate mathematical method of including ∞ as a possible yield rate. It may be tempting to define T as having a yield of ∞ if

$$\lim_{i \to \infty} P_i(T) = 0 .$$

This equation does, in fact, hold for the transaction $T = (0, 10)$ of the preceding example. Notice, however, that we have, for all T,

$$\lim_{i \to \infty} P_i(T) = c_0 . \tag{3}$$

This definition would result in an infinite yield whenever $c_0 = 0$, which is not a desirable situation.

We now turn to more mundane examples and avoid the complications of multiple-valued or nonexistent yields, unusual repayment patterns, and the like. It is well known that problems still arise in interpreting yields when we must consider reinvestments at different rates. Two simple examples follow.

Example 4. Consider two investment opportunities, each of which requires a present investment of 100 units. The first provides for a return of 150 units at time 1, the second for a return of 180 units at time 2. It would be naive, indeed, to suggest that the investor should choose the first simply because it has a yield of 50 percent as compared with 34.2 percent for the second. He would, of course, prefer the first investment if and only if he could reinvest the proceeds for one period at a rate greater than 20 percent.

Example 5. Consider the case of consumer loans. From the lender's standpoint the yield rate has some validity, since it is usually a reasonable assumption that the repayments will be invested in contracts similar in nature to the original one. However, it almost never will be true that the borrower can reinvest his money at the yield rate of the loan. This means that from the borrower's viewpoint, a comparison of loans strictly on the basis of yield rates is somewhat invalid.

For example, compare a 5 percent, 20-period loan, with level periodic repayments, with a 4.5 percent loan that requires only interest at the end of each period, the principal being due at the end of 20 periods. Assume that in the latter case the borrower decides to amortize the principal by level periodic deposits to his bank account, which earns $2\frac{1}{2}$ percent interest. For the 4.5 percent loan, the periodic cost per unit borrowed is $0.045 + (s_{\overline{20}|0.025})^{-1} = 0.0841$, compared with $(a_{\overline{20}|0.05})^{-1} = 0.0802$ for the 5 percent loan.

This example represents a typical situation. It may be preferable for a consumer to choose a loan with a higher yield if it also provides for faster repayment. The reason, of course, is that the repayments constitute an investment at the loan rate, which is usually higher than the rate the borrower could receive elsewhere.

Example 6. This example has nothing to do with compound interest or yield rates. We wish to pursue the analogy with probability theory and expected values, to which we alluded in Section II.

Suppose you are forced to choose between two games of chance, each costing $10,000 to play. The expected value of the winnings is $10,000 for game 1 and $20,000 for game 2. Which game is better? The "obvious" answer of game 2, based on the relative size of the expected winnings, may not be correct. Assume, for example, that both games consist of drawing a card randomly from ten cards, one red and nine black. In game 1 you win $10,000 regardless of which card is drawn. In game 2 you win $200,000 if the red card is drawn, but nothing if a black card is selected. Those with sufficient capital might find game 2 an attractive choice. I imagine, however, that most people would, if actually faced with this decision, choose game 1, which is the game we all play automatically when we elect not to gamble. There are not many who would risk a 90 percent chance of losing $10,000 by playing game 2.

The question of which game to play is like the question of which investment to choose in example 4. The answer depends on the circumstances of the individual making the decision.

Example 6 adds further insight into the analogy between expected values and yield rates. Just as the validity of the concept of yield rate depends on the opportunities to reinvest funds in similar transactions, so the validity of the concept of expected value depends on the opportunities to replay the same game many times. In game 2 of the above example, the high cost coupled with the low probability of a win would mean that most people would not have such an opportunity.

V. INTEREST PREFERENCE RATES

The solution to the problem of comparing two transactions is quite simple. The individual must assess a transaction by using quantities that depend solely on his particular circumstances and that are independent of the transaction itself, as opposed to using yield rates, which depend solely on the transaction and are independent of any particular individual.

We postulate that for a given individual we can find a real number $i > -1$ that is the payment required by that individual at time 1 in order to induce him to forgo 1 unit for 1 period. We call i the *interest preference rate* of the individual, since it measures his degree of preference for present as opposed to future capital. In essence we are adopting the basic definition of interest rate that an economist would give, allowing it to vary by individual.

For our purposes, we need not be concerned with the reason for an individual's choice of interest preference rate i. It may simply reflect his desire for present as opposed to future consumption. On the other hand, it may be that the individual (which could be some type of institution) has ready access to investments that yield i, and therefore will not forgo capital for a lesser rate. In the latter case, if T is a typical transaction undertaken by the individual, then the interest preference rate may correspond to the yield rate of T.

For an individual with interest preference rate i, a single unit at the end of t periods is worth $(1 + i)^{-t}$ at the present time. Hence, for any transaction T, the number $P_i(T)$ equals the present value of the gain accruing to such an individual should he undertake the transaction. To say that i is a yield rate of T means that an individual with interest preference rate i will neither gain nor lose on the transaction.

We now have an easy answer to the comparison problem. An individual with interest preference rate i will prefer transaction T to transaction S if and only if $P_i(T) \geq P_i(S)$. By virtue of the linearity equations (2), this is equivalent to

$$P_i(T - S) \geq 0 . \tag{4}$$

Example 7. In actual practice one does not need to make an *exact* calculation of an interest preference rate in order to compare two transactions. For example, let $T = (10, -9, -9)$ and $S = (10, -4.5, -14.5)$. This is a simplified version of example 5 involving two loans, one with level repayments and the other with payments of only interest until the end. The yields are 50 percent and 45 percent, respectively, but, as we have stressed before, this is immaterial. We have $T - S = (0, -4.5, 5.5)$, and $P_i(T - S) = v(5.5v - 4.5)$, which equals zero for $v = \frac{9}{11}$, or $i = \frac{2}{9}$. $P_i(T - S)$ is then greater or less than zero according as i is less than or greater than $\frac{2}{9}$. Using formula (4), we see that the individual need only decide whether his interest preference rate is greater than or less than $\frac{2}{9}$. If it is greater than $\frac{2}{9}$, he will choose transaction S; otherwise, he will choose transaction T.

It is of interest to note that the procedure for comparing T and S does involve the calculation of yield rates. The yield rates are not those of T or S, however, but rather of the difference, $T - S$.

Note that, as with yield rates, we have included values in the interval $(-1, 0)$ as possible interest preference rates. This may seem quite strange at first glance. An individual with a negative interest preference rate actually prefers future to present consumption. As compared with the "live it up today for tomorrow we may die" attitude of the person with a high positive interest preference, he takes the attitude of "saving for a rainy day at all costs." This does not appear quite so unusual, however, if we reflect on the fact that all of us who invest in fixed-return securities during a time of inflation, when it is almost certain that their real value will fall, exhibit some negative interest preference tendencies.

The quantity $P_i(T)$, considered as a function of i for $-1 < i < \infty$, will be called the *interest preference function* of the transaction T. A basic principle of the interest preference function is that it completely determines the transaction. We state this precisely as follows:

THEOREM 1. *Let S and T be two transactions such that $P_i(S) = P_i(T)$ for all i. Then $S = T$.*

Proof. In our present context this follows immediately from the fact that $P_i(T)$ as a polynomial in v determines the coefficients. In fact, we can even give an explicit formula. This is just the usual identity,

$$c_n = \frac{f^{(n)}(0)}{n!}, \tag{5}$$

where $f(v) = P_i(T)$, is the polynomial (1) associated with T.

The principle illustrated by this theorem does not depend on the fact that $f(v)$ is a polynomial. For the more general types of transactions that we mentioned in Section III, $f(v)$ is not a polynomial and equation (5) does not hold. The theorem is nevertheless true, as we will see in Section X.

Theorem 1 provides further evidence of the importance of the interest preference concept. It says, in effect, that complete information about a financial transaction T is embodied in the present value of the gain under T accruing to the various interest preference classes.

We will call the graph of the interest preference function the *interest preference curve* of T. Given transactions S and T, it usually is the case, as we have seen, that their respective interest preference curves will cross at one or more points, namely, those corresponding to the yield rates of $S - T$. It may happen, of course, that the interest preference curve of T will lie above that of S; that is,

$$P_i(T) \geq P_i(S) \quad \text{for all } i\,. \tag{6}$$

We then say that T is *universally better* than S, and we can give an unqualified answer to the question of which transaction is preferable. (Note that we do not imply by our wording that T is strictly better for all i. It may happen that $P_i(T) = P_i(S)$ for some values of i. We are following the usual mathematical convention regarding the ordering of functions.)

It is of interest to derive criteria for such universal comparability. It seems difficult to find easily described sufficient conditions for formula (6), other than the obvious case where all the payments in one transaction are greater than the corresponding payments of the other. However, if we confine ourselves to nonnegative interest preference rates (the case of most general interest), we can give a simple sufficient condition for universal comparability.

Given $S = (b_0, b_1, \ldots, b_n)$ and $T = (c_0, c_1, \ldots, c_n)$, let

$$d_k = c_k - b_k \quad \text{and} \quad s_k = \sum_{j=0}^{k} d_j$$

for $k = 0, 1, \ldots, n$.

THEOREM 2. *If* $s_k \geq 0$ *for* $k = 0, 1, \ldots, n$, *then*

$$P_i(T) \geq P_i(S) \quad \textit{for all } i \geq 0\,. \tag{7}$$

Proof. We will show that, for all v in the interval $[0, 1]$,

$$\sum_{k=0}^{n} d_k v^k \geq s_n v^n\,. \tag{8}$$

This will imply that $P_i(S - T) \geq 0$ for all $i \geq 0$, and by formula (4) the theorem follows. We verify inequality (8) by induction on n. It is obvious for $n = 0$, since $d_0 = s_0$. Assume it is true for $n - 1$. Then

$$\begin{aligned} \sum_{k=0}^{n} d_k v^k &= \sum_{k=0}^{n-1} d_k v^k + d_n v^n \\ &\geq s_{n-1} v^{n-1} + d_n v^n \qquad (9) \\ &\geq s_{n-1} v^n + d_n v^n \qquad (10) \\ &= s_n v^n\,. \end{aligned}$$

In (9) we use the induction hypothesis, and in (10) the fact that $s_{n-1} \geq 0$ and $0 \leq v \leq 1$.

Theorem 2 becomes reasonably evident if we interpret the conditions verbally; they simply say that at the end of any time period, the total sum of the payments received under T is greater than or equal to the total sum received under S.

Example 8. Compare the transactions $T = (3, -1, 4, 0, 2)$ and $S = (1, 1, 3, -1, 4)$. We have $s_0 = 2, s_1 = 0, s_2 = 1, s_3 = 2, s_4 = 0$. The criterion of Theorem 2 holds, and we can say without further calculations that T is preferable to S for all individuals with nonnegative interest preference rates.

In order for inequality (7) to hold, we need both $s_0 \geq 0$ and $s_n \geq 0$, since

$$s_n = P_0(T - S) , \tag{11}$$

and, from equation (3),

$$s_0 = \lim_{i \to \infty} P_i(T - S) .$$

However, it is not necessary that $s_k \geq 0$ for all k, as the following example shows.

Example 9. Let $T = (2, 2, 2)$ and $S = (1, 6, -2)$. Then $P_i(T - S) = (1 - 2v)^2$, which is nonnegative for all $i \geq 0$ (in fact, for all $i > -1$), but $s_1 = -3$.

Combining Theorem 2 with formula (11) we obtain

COROLLARY 1. *Given $S = (b_0, b_1, \ldots , b_n)$ and $T = (c_0, c_1, \ldots , c_n)$, with $c_i \geq b_i$ for $i = 0, 1, \ldots , n - 1$, then* (7) *holds if and only if* $\Sigma_{i=0}^{n} c_i \geq \Sigma_{i=0}^{n} b_i$.

The situation given in the corollary occurs quite frequently. One example is when the transactions represent loans at the same interest rate and for the same amounts, but with the repayments under T higher than those under S. To compensate, the final payment of S must then be higher than that of T. We will use Corollary 1 in Section IX.

VI. NORMAL TRANSACTIONS

For the usual transaction that one encounters in practical situations there is a unique yield rate that, if properly interpreted, gives pertinent information about the transaction. Such transactions will be called *normal.* They will be defined precisely, and discussed, in this section.

We define a transaction T to be *L-normal* if there exists a number $i_o > -1$ such that

$$P_i(T) > 0 \quad \text{for } -1 < i < i_o$$

and

$$P_i(T) < 0 \quad \text{for } i_o < i .$$

By continuity of the interest preference function, the number i_o must be the unique yield of T.

Similarly, we define a transaction T to be *B-normal* if there exists a number $i_o > -1$ such that $P_i(T) < 0$ for $-1 < i < i_o$, and $P_i(T) > 0$ for $i_o < i$. It is clear that T is B-normal if and only if $-T$ is L-normal. The reader should note that any statement in the remainder of the paper about one type of normality implies a corresponding statement about the other type, which will not always be expressly given.

We will denote as *normal* a transaction that is of either of the above types.

Not all transactions with a unique yield are normal (see example 9). The following lemma will be useful in classifying unique-yield transactions.

LEMMA 1. *Let* $T = (c_0, c_1, \ldots, c_n)$, $c_n \neq 0$, *be a transaction that has at most one yield. Let* c_k *be the first nonzero entry. Let* $s = \Sigma_{j=0}^{n} c_j$. *Then*

(i) *If* $c_k < 0$ *and* $c_n > 0$, *then* T *is L-normal and the yield* i *is respectively positive, negative, or zero as* s *is positive, negative, or zero.*

(ii) *If* $c_k > 0$ *and* $c_n < 0$, *then* T *is B-normal and the sign of the yield is opposite to that of* s.

(iii) *If* c_k *and* c_n *are of the same sign, we have no information as to the existence or sign of the yield.*

Proof. We note first that

$$c_n = \lim_{i \to -1} (1 + i)^n P_i(T) \tag{12}$$

and

$$c_k = \lim_{i \to \infty} (1 + i)^k P_i(T) \ . \tag{13}$$

Suppose, for example, that $c_k < 0$ and $c_n > 0$. We know from equation (13) that, for sufficiently large values of i, $(1 + i)^k P_i(T)$ is negative and hence $P_i(T)$ is negative. Similarly, from equation (12), $P_i(T) > 0$ for sufficiently small values of i. This shows that T has at least one yield; this fact, coupled with the hypothesis that there is at most one yield, means there is exactly one yield i_o. It is clear that i_o satisfies the definition of L-normality. The condition on the sign of i_o follows immediately from the fact that $s = P_0(T)$. This proves (i), and the rest of the lemma follows by similar reasoning.

Our terminology is chosen to reflect the fact that L-normal transactions behave as the usual loan transaction, while B-normal transactions behave as the usual borrowing transaction. In the L-normal case, for example, a lender with an interest preference rate less than the yield will gain from the transaction, just as a lender would expect to gain by lending money at a rate higher than his usual one. Similarly, a lender with an interest preference rate greater than the yield will lose on the transaction. Hence, in the case of normal transactions, the unique yield, together with the type of normality, gives us the qualitative information needed to determine which individuals will gain and which will lose. We are now able to distinguish between loan transactions and borrowing transactions; thus, we have answered the question posed in example 2. If S and T are both L-normal transactions and T has the higher yield, then T is more desirable to the community of lenders as a whole, since more of them will profit from T than from S. The reverse is true for B-normal transactions.

Note that the above does *not* say that if S and T are L-normal, then the higher-yield transaction is universally better. In general, two L-normal transactions will not be universally comparable. However, suppose that S and T are transactions that we know *are* universally comparable. It is then true that if *either* S or T is L-normal, the higher-yield transaction is better for the lender, while if *either* S or T is B-normal, then the lower-yield transaction is better for the lender. Let i be a yield of S, j a yield of T, and suppose that $i \leq j$. If T is L-normal, we have $P_i(T) \geq 0 = P_i(S)$, showing that T must be the universally better transaction. If S is L-normal, then $P_j(T) = 0 \geq P_j(S)$, showing again that T is better. Similar reasoning handles the B-normal case.

If we confine ourselves to nonnegative yields, these conclusions obviously hold when S and T satisfy the restricted universal comparability as given by formula (7).

Some readers may perceive an apparent conflict in the above results if one of the transactions is B-normal and the other is L-normal with a higher yield. Our statement would seem to say that if these transactions are universally comparable, then each is better than the other. The answer is simply that a normal transaction can never be universally better than a transaction with the opposite type of normality. If, for example, T is L-normal and S is B-normal, then $P_i(T - S)$ is greater than zero for sufficiently small i and less than zero for sufficiently large i.

We define a transaction T to be ***strongly L-normal*** if

(a) T is L-normal with unique yield i; and
(b) There exists $i' > i$ such that $P_j(T) \geq P_k(T)$ for $-1 < j \leq k \leq i'$.

(In other words, the interest preference curve is decreasing, at least up to the point where it crosses the horizontal axis.)

The least upper bound of all i' satisfying (b) will be called the *critical value* of T. Note that in the case where $P_i(T)$ is decreasing over its entire domain, we will have a critical value of ∞.

We define strong B-normality in an analogous way, so that T is strongly B-normal if and only if $-T$ is strongly L-normal.

We will discuss the significance of these definitions later. We will give first a basic example of a strongly L-normal transaction; it will consist of negative payments followed by positive ones.

THEOREM 3. *Let* $T = (c_0, c_1, \ldots, c_n)$ *be such that there exists an index* k, *satisfying*

$$c_j \begin{cases} \leq 0\,, & 0 \leq j < k \\ < 0\,, & j = k \\ \geq 0\,, & k + 1 < j < n \\ > 0\,, & j = n\,. \end{cases} \tag{14}$$

Then T is strongly L-normal.

Note that the above pattern is typical of the usual loan from the lender's viewpoint. It consists of a series of expenditures, which constitute the loan, followed by a series of repayments beginning after the loan is completed.

To show the L-normality of such a transaction, it is sufficient by Lemma 1 to show that T has at most one yield rate. This problem has been considered before, and I believe that the results first appeared in [4]. The simplest approach is to apply Descartes's rule of signs ([2], Art. 547) to the associated polynomial $f(v)$. This rule states that the number of positive roots of a polynomial is bounded by the number of sign changes in the coefficients, which in this case is 1.

It is not difficult now to deduce the *strong* L-normality by applying Descartes's rule to the derivative $f'(v)$.

There are some defects in this approach. In the first place, the use of Descartes's rule does not seem to give one any intuitive feeling of why the result is true. A more pertinent objection is that this proof does not carry over to the more general types of transaction for which $f(v)$ is no longer a polynomial. We will present two other proofs here. Both of these are easily modified to apply to more general situations, and we will do so in Section X.

Alternate proof of Theorem 3. We will show that for any i_o such that $P_{i_o}(T) \geq 0$, we have $P_i(T) > P_{i_o}(T)$ for all $i < i_o$. Taking i_o to be the largest yield, it is then clear that it is the only yield, and that P_i is decreasing on $(-1, i_o)$.

Let

$$A = \sum_{j=0}^{k} c_j(1 + i_o)^{-j}, \quad B = \sum_{j=k+1}^{n} c_j(1 + i_o)^{-j},$$

and

$$q = \frac{1 + i_o}{1 + i}.$$

Then

$$\sum_{j=0}^{k} c_j(1 + i)^{-j} = \sum_{j=0}^{k} q^j c_j(1 + i_o)^{-j}$$

$$\geq q^k A, \tag{15}$$

since each term in the sum is negative or zero and $q^j \leq q^k$ for $j = 0, 1, \ldots, k$. Similarly,

$$\sum_{j=k+1}^{n} c_j(1 + i)^{-j} \geq q^{k+1}B > q^k B. \tag{16}$$

The result follows by adding expressions (15) and (16) and noting that $A + B = P_{i_o}(T) \geq 0$.

For those who prefer calculus to algebra, we present another proof. We first need the following mathematical result.

LEMMA 2. *Let g be a real-valued function with a continuous derivative g' defined on an open interval (a, b), $-\infty \leq a < b \leq \infty$. Suppose that there exist points s and t in (a, b) such that $g(x) < 0$ for $s < x < b$ and $g'(x) > 0$ for $t < x < b$. Then if g' has at most one zero in (a, b), the same is true for g. Moreover, the zero of g is less than that of g' when they both exist.*

Proof. Suppose g' has one zero c in (a, b). By continuity, g' cannot change sign in (c, b), so, using the condition for g', we must have $g'(x) > 0$ for all x in (c, b). Then g is increasing on this subinterval, and by the condition for g we must have $g(x) < 0$ for x in (c, b). Similarly, g is monotone on the interval (a, c) and can have at most one zero. This same fact is true on the entire interval (a, b) in the case where g' has no zeros.

Proof of Theorem 3 using Lemma 2. Consider the function $g(i) = P_i(T)$ and its derivative

$$g'(i) = \sum_{j=1}^{n} -jc_j(1 + i)^{-(j+1)}$$

defined on $(-1, \infty)$. We use induction on k to show that g has at most one zero. For $k = 0$ this is clear, since g' is negative, showing that g is decreasing. For $k > 0$, we see that $-g'(1 + i)^2$ is a function of the same form as g, but with an index of $k - 1$ in place of k. By the induction hypothesis, $-g'(1 + i)^2$, and hence g', has at most one zero. Moreover, we see from formulas (12) and (13) that the conditions of Lemma 2 are satisfied, so we may apply that lemma to conclude that g itself has at most one zero.

This shows that T is L-normal. From the last statement in Lemma 2, we know that g' has no zeros on the interval $(-1, i_o)$, where i_o is the unique yield, and the strong L-normality follows.

The first proof of Theorem 3 can be adapted to give a very quick method of estimating the yield in a strongly L-normal transaction.

LEMMA 3. *For T as in formula (14), let $s = |c_0 + c_1 + \ldots + c_k|$, and $t = c_{k+1} + \ldots + c_n$. Let i_o denote the unique yield.*

(a) *If* $t > s$, $\quad 0 < i_o \leq (t - s)/s$;

(b) *If* $t < s$, $\quad 0 > i_o \geq (t - s)/s$.

Proof. If $t > s$, Lemma 2 shows that $0 < i_o$. Now apply the first proof of Theorem 3 with $i = 0$. Using the first inequality in formula (16), we obtain

$$-s + \frac{t}{1 + i_o} \geq 0 ,$$

and part (a) of the lemma follows. If $t < s$, the yield is negative, and we argue similarly, noting, however, that with $i = 0 > i_o$, the inequalities in (15) and (16) are reversed.

It is quite easy to verify Lemma 3 by general reasoning. If $t > s$, for example, then the amount received is greater than the amount invested and the yield is obviously positive. The natural estimate of the yield is the ratio of the gain to the amount invested. This estimate generally will be higher than the true yield, since the estimate does not account for the loss of interest occasioned by the fact that receipts are deferred more than the expenditures.

The condition of strong L-normality gives us an additional feature that we expect from loan transactions: not only do those with interest preference rates less than the yield gain from the transaction, but the lower the interest preference rate, the higher the gain.

Note that there is a lack of symmetry here, since we do not require that $P_i(T)$ be decreasing for values of i greater than the yield. If we take T as in (14), then this will happen for $k = 0$, since $P_i'(T) \leq 0$ for all i in that case, but for $k > 0$ it is not necessarily true. We can explain this as follows. For $i_o < i < j$, where i_o is the yield, the j interest preference individual will incur a greater loss at the time the loan is repaid than the i interest preference individual. However, the higher rate of discounting may more than compensate for this when we consider present value. Consider the following example.

Example 10. Let $T = (0, -400, 800)$, which satisfies (14) with $k = 1$. T has a yield of 100 percent. An individual with an interest preference rate of 300 percent will incur a loss of 800 units at time 2, since he receives only 800 instead of the 1,600 his interest preference rate requires. This 800-unit loss is worth 50 units at time 0. The individual with a 400 percent interest preference rate incurs the greater loss of 1,200 units at time 2, but at time 0 this is worth only 48 units.

We now have seen that the set of all transactions T satisfying (14) is contained in the set of all strongly L-normal transactions, which in turn is contained in the set of all L-normal transactions. Following are some examples to show that these inclusions are strict; that is, there are strongly L-normal transactions that do not satisfy (14), and there are L-normal transactions that are not strongly L-normal.

We use the fact that for an L-normal transaction T with yield i_o, strong L-normality is equivalent to showing that the associated polynomial f satisfies

$$f'(v) \geq 0\,, \quad v > (1 + i_o)^{-1}\,. \tag{17}$$

Example 11. Let $T = (-3, 4, -18, 24)$. The associated polynomial, $f(v) = -3 + 4v - 18v^2 + 24v^3$, factors as $(4v - 3)(6v^2 + 1)$, showing by Lemma 1 that T is L-normal with a unique yield of $33\frac{1}{3}$ percent. Also, $f'(v) = 4(6v - 1)(3v - 1)$ is nonnegative for $v \geq \frac{1}{3}$. So T is strongly L-normal with critical value 200 percent; however, T does not satisfy (14).

Example 12. Let $T = (-15, 60, -76, 32)$. The associated polynomial, $f(v) = -15 + 60v - 76v^2 + 32v^3$, factors as $(2v - 1)(16v^2 - 30v + 15)$. The second factor has no real roots, so T has the unique yield rate of 1. By Lemma 1, T is L-normal. Now $f'(v) = 4(6v - 5)(4v - 3)$ is negative on the interval $\frac{3}{4} < v < \frac{5}{6}$, so $P_i(T)$ is increasing on the interval $\frac{1}{5} < i < \frac{1}{3}$. T is not strongly L-normal.

The above examples were found by using some criteria that relate strong L-normality to the nonpositive roots of the associated polynomial. Because these results are more of theoretical interest than of practical use, we will discuss them in Appendix I. In practice, the best way to test a transaction for strong L-normality is by using criterion (17).

VII. DECOMPOSITION OF TRANSACTIONS

Theorem 3 shows that transactions that are not normal can always be written as the sum of two normal transactions. For example, given $T = (c_0, c_1, \ldots, c_n)$ with $n > 0$, let a_0, b_0, a_n, and b_n be any numbers such that $a_0 < 0$, $a_n > 0$, $b_0 > 0$, and $b_n < 0$. Let $a_0 + b_0 = c_0$, and $a_n + b_n = c_n$. For $k = 1, 2, \ldots, n - 1$, we let

$$\begin{aligned} a_k &= c_k \quad \text{if } c_k \leq 0 \\ &= 0 \quad \text{if } c_k \geq 0 \end{aligned}$$

and

$$\begin{aligned} b_k &= c_k \quad \text{if } c_k \geq 0 \\ &= 0 \quad \text{if } c_k \leq 0 . \end{aligned}$$

Then by Theorem 3, $S = (a_0, a_1, \ldots, a_n)$ is L-normal (in fact, strongly L-normal), $R = (b_0, b_1, \ldots, b_n)$ is strongly B-normal, and $T = S + R$. Those who are still uncomfortable with negative yields should note, using Lemma 1, that by adding a sufficiently large positive constant to a_n and subtracting it from b_n, we can, if we want, ensure that the yields on both S and R are greater than zero.

A decomposition of T as the sum of normal transactions is not unique; it generally can be done in an infinite variety of ways. This observation can be used to give an explanation of multiple-yield transactions.

Example 13. Consider again the transaction $T = (-1, 7, -6)$ of example 1. Some of the ways of decomposing T are the following:

(a) $$T = (-1, 1) + (0, 6, -6) ;$$

(b) $$T = (-1, 6) + (0, 1, -6) ;$$

(c) $$T = (-1, 3) + (0, 4, -6) .$$

In (a) we view the transaction as one of lending 1 unit for 1 period at 0 percent interest, followed by borrowing 6 units at 0 percent interest, while in (b) we see it as lending 1 unit at 500 percent, followed by borrowing 1 unit at 500 percent. This explains the two yields of $i = 0$ and $i = 500$ percent.

In (c) we view the transaction as lending 1 unit at 200 percent and then borrowing 4 units at 50 percent. Those with an interest preference rate between 50 and 200 percent gain on both of these subtransactions. Those with an interest preference rate less than 50 percent lose on the borrowing but gain on the loan, while the reverse is true for those with an interest preference rate more than 200 percent. It is only those with an interest preference rate of 0 percent in the former group and 500 percent in the latter group for whom the gain and loss will balance each other.

Going through an analysis such as the foregoing should serve to dispel the surprise that some may feel at the existence of multiple yields.

In fact, it may even tend to make one surprised that there are actually transactions with unique yields!

We now consider decompositions into normal transactions of the same type. A natural question is whether the sum of two L-normal transactions is itself L-normal. Suppose that S and T are L-normal with yields of j and k, respectively, with $j \leq k$. An individual with interest preference rate r will gain on both transactions if $r < j$, and will lose on both if $r > k$. If, however, r is in the interval (j, k), the individual will lose on S and gain on T. Hence, $S + T$ may have multiple yields in this interval and will not then be L-normal (see example 14 below).

However, suppose that S and T are both *strongly* L-normal and that $k \leq j'$, the critical value of S. Then $P_i(S + T)$ is decreasing on the interval $(-1, k)$. Moreover, $P_j(S + T) = P_j(T) > 0$, and $P_k(S + T) = P_k(S) < 0$. So $S + T$ is itself strongly L-normal with yield h in the interval (j, k), and critical value $h' \geq k$.

Note that in the familiar situation of a loan consisting of a single payment, that is, $k = 0$ in formula (14), we have a critical value of ∞. The above result then implies that we can add any strongly L-normal transaction of a higher yield to preserve strong L-normality.

Example 14. Let $S = (0, 0, -40, 48)$ and $T = (-1, 11)$. Then $S + T = (-1, 11, -40, 48)$ has yields of $i = 2$ and $i = 3$. The difficulty here is that the yields are too far apart. T has a yield of 10, which is greater than 0.8, the critical value of S.

We can summarize the above discussion verbally as follows. Suppose we take two investments, each consisting of the usual pattern of payments and having a unique yield. Making both investments will result in a composite transaction with a unique yield between the two given yields, provided that these yields are sufficiently close. If the two yields are very far apart, the composite investment may have multiple yields.

The importance of the concept of the critical value of a strongly L-normal transaction is that it defines precisely what is meant by the phrase "sufficiently close" in this situation.

We now turn our attention to transactions T with nonexistent yields. For such a T, we must have either $P_i(T) > 0$ for all i, or $P_i(T) < 0$ for all i. We will call the former transactions *universally profitable*, and the latter *universally unprofitable*. Clearly, T is universally profitable if and only if $-T$ is universally unprofitable.

Examples of universally profitable transactions are (i) the transaction in example 3, which consisted of borrowing capital and then lending the proceeds at a higher rate, and (ii) the transaction $(1, -4, 6)$ (the negative of a transaction in example 1). This latter transaction is similar to the one in example 3, which we can see by writing it as $(1, -2) + (0, -2, 6)$. It consists of borrowing money at 100 percent and then lending it out at 200 percent.

It is easy to see that all universally profitable transactions can be considered in this way. Let T be universally profitable, and consider any decomposition $T = S + R$, where S and R are normal. S and R cannot both be L-normal, for as we have seen, those with a sufficiently high interest preference rate will lose on T. Similarly, they cannot both be B-normal. It must then be true that one of these subtransactions, say R, is B-normal with yield j, and S is L-normal with yield k. We must also have $j < k$; otherwise, we would have $P_i(T) \leq 0$ for i in the interval $[k, j]$. In other words, the only way to make money for everybody is by borrowing and then lending the proceeds at a higher rate.

Note, however, that not all transactions of the type described above are universally profitable. As we saw in example 13(c), if $T = S + R$, where R is B-normal with yield j and S is L-normal with yield $k > j$, we can have yield rates in the intervals $(-1, j)$ or (k, ∞). We will return to this situation in the next section, after first developing some more theory.

VIII. OUTSTANDING INVESTMENT ANALYSIS

In this section we show how the concept of outstanding investment unifies some of the results obtained in preceding sections.

Given a transaction $T = (c_0, c_1, \ldots, c_n)$, a number $i > -1$, and a nonnegative integer t, we let $B_i^t(T)$ denote the outstanding investment in T at time t for interest rate i; that is,

$$B_i^t(T) = \sum_{j=0}^{t} c_j(1 + i)^{t-j} ,$$

the value at time t of all payments up to that time. (We let $c_j = 0$ for j greater than the duration of T.)

It is clear that B_i^t is a linear function of T for i and t fixed, and is related to P_i by

$$P_i(T) = (1 + i)^{-n}B_i^n(T) \tag{18}$$

for all T of duration not greater than n.

We have the recursive relation

$$B_i^t(T) = B_i^{t-1}(T)(1 + i) + c_t . \tag{19}$$

Kellison ([5], sec. 5.7) shows that if i is a yield rate of T and if $B_i^t(T) \geq 0$ for $t = 0, 1, 2, \ldots, n$, then i is a unique yield. In the course of deriving this fact, Kellison actually shows that more and stronger results can be obtained. It is shown, in fact, using equation (19), that, given any $i > -1$ and a transaction T satisfying

$$B_i^t(T) \geq 0\,, \quad t = 0, 1, \ldots, n-1\,, \tag{20}$$

then for any interest rates $j < i < k$,

$$B_j^t(T) \leq B_i^t(T) \leq B_k^t(T) \tag{21}$$

for $t = 0, 1, 2, \ldots, n$. Moreover, the inequalities are strict unless $c_r = 0$ for all indices $r < t$. (It is important to note that the upper index of t is $n - 1$ in (20) and n in (21).)

From (21) we see that (20) will hold for all interest rates higher than i. Hence, (20) implies that, given $j < k$, $k \geq i$,

$$B_j^n(T) \leq B_k^n(T)\,, \tag{22}$$

with equality holding only if $c_r = 0$ for $r = 0, 1, \ldots, n-1$.

Suppose that c_k is the first nonzero payment of T. Then, for all i,

$$B_i^t(T) = 0\,, \quad t < k\,;$$

$$B_i^k(T) = c_k\,; \tag{23}$$

and

$$\lim_{i\to\infty} \frac{B_i^t(T)}{(1+i)^{t-k}} = c_k\,, \quad t > k\,.$$

From this we see that if $c_k > 0$, then (20) will hold for sufficiently high values of i.

We now use the concept of outstanding investment to discuss the question of speed of repayment. What does it mean to say that one loan is being repaid faster than another? Intuitively we think of this as occurring when, at the end of each time period prior to maturity, the amount owed by the borrower on one loan is less than the amount owed on the other. Hence, if the loans are represented by the B-normal transactions R and S of duration not greater than n, we would like to think of R as admitting faster repayment if

$$B_i^t(R) \leq B_i^t(S)\,, \quad t = 0, 1, \ldots, n-1\,. \tag{24}$$

This has no meaning without further qualification, since it depends on the value of i that is used. Indeed, condition (24) holds for sufficiently high values of i simply under the hypothesis that, for the first place where the payments of S and R differ, the one in S is higher. This follows from equations (23), taking $T = S - R$, and using the linearity. Similarly, taking $T = S - R$ in (20) and (21), we see that if (24) holds for some values of i, it will also hold for all higher interest rates. With these points in mind, we are led to the following formal definition.

Given B-normal transactions S and R, we say that *R is faster than S* if (24) holds for i equal to the maximum of the two yields and n equal to the maximum of the two durations.

Note that condition (24) applies only to times that are strictly prior to the maturity of one of the loans. We do not want to require (24) for $t = n$; in fact, it will never hold for the maximum yield if R and S are both of duration n and R has the lower yield. If S has yield i, which is greater than the yield of R, then by B-normality, $B_i^n(R) > 0 = B_i^n(S)$.

As an example of faster repayment, we have the following.

LEMMA 4. *Let $R = (b_0, b_1, \ldots, b_n)$ and $S = (c_0, c_1, \ldots, c_n)$ be B-normal transactions with yields of i and j, respectively, with $i \geq j$. Suppose that for some nonnegative $k \leq n$,*

$$b_t \leq c_t\,, \quad t = 0, \ldots, k$$

and

$$b_t \geq c_t\,, \quad t = k + 1, \ldots, n\,.$$

Then R is faster than S.

Proof. Let $T = S - R = (d_0, d_1, \ldots, d_n)$, where $d_t = c_t - b_t$. For $t = 0, 1, \ldots, k$, $d_t \geq 0$, and obviously $B_i^t(T) \geq 0$. For $t > k$, we use induction on $n - t$. Given that $B_i^{t+1}(T)$ is nonnegative for some value $t > k$, we can conclude that $B_i^t(T)$ is nonnegative from the fact that $d_t \leq 0$ and the relation

$$B_i^t(T) = [B_i^{t+1}(T) - d_t](1 + i)^{-1}\,.$$

The induction starts with the fact that $P_i(T) = P_i(S) \geq 0$, so that, from formula (18), $B_i^n(T) \geq 0$.

Note that if $k = n - 1$, we can drop the assumption that $i \geq j$, but in general we cannot do this, as the next example shows.

Example 15. Let $R = (20, -10, -5, -5)$ and $S = (20, 0, -27, -27)$. Then R and S satisfy the condition of Lemma 4 with $k = 1$, but they have respective yields of 0 and 0.5. We have $B_{0.5}^2(R) = 25$ and $B_{0.5}^2(S) = 18$, so R is not faster than S.

We must confess to one pathological feature in the definition of faster repayment. The terminology suggests that if R is faster than S, and S is faster than T, then R should be faster than T. However, this transitivity property does not always hold.

Example 16. Let $R = (16, -13, -3)$, $S = (20, -18, -18)$, and $T = (22, -20, -2)$. These are all B-normal with respective yields of 0, 0.5, and 0. Since $B^1_{0.5}(R) = 11$, $B^1_{0.5}(S) = 12$, and $B^1_{0.5}(T) = 13$, we see that R is faster than S and S is faster than T. But $B^1_0(R) = 3 > 2 = B^1_0(T)$, so R is not faster than T.

Note that the transitivity will hold in the above situation whenever the maximum yield of the three transactions occurs for either R or T rather than S. In particular, it holds when all transactions have the same yield.

One can develop an alternate definition for faster repayment by using the minimum yield in place of the maximum. This is more restrictive, however. As we saw in the preliminary discussion, using a lower interest rate means that there will be fewer cases for which we can state that one transaction is faster than another. Also, this definition will not solve the lack-of-transitivity problem. It is easy to find counterexamples similar to example 16 for transitivity in the minimum-yield definition. We would need to have the yield on either R or T strictly less than the yield on the other two. Note also that in example 15 the conclusion is the same using the minimum yield.

Still another definition arises if we vary i in (24), allowing it to assume the yield rate for each transaction. This may appear more natural, at first, and we do indeed obtain transitivity. With this definition, however, the corollary to Theorem 4 is no longer valid.

The speed of repayment of a B-normal transaction is dependent on the "size" of the loan involved as well as on the repayment pattern. For example, given any B-normal transaction T, the transactions T and $2T$ involve the same underlying pattern of repayment, but T is faster, since it represents a loan of one-half the amount. This feature of the definition is necessary for the validity of the corollary to Theorem 4. However, it may be of interest to introduce an auxiliary concept of *relative* speed of repayment to eliminate this dependency on units.

One approach to this would be first to define, for any nonzero B-normal transaction T, a positive number $s(T)$ that would reflect the size of the loan involved. There does not appear to be any completely obvious way to define $s(T)$ for the general B-normal transaction, and we will not pursue the matter further here. (One possibility would be to set $s(T)$ equal to the present value of the positive payments computed at the yield rate of T.) In any event, we would want the natural requirement that, for any positive r,

$$s(rT) = rs(T) .$$

We then define the B-normal transaction S to be ***relatively faster*** than the B-normal transaction T if

$$\frac{S}{s(S)} \text{ is faster than } \frac{T}{s(T)} .$$

It is clear that for any positive r, the transactions T and rT exhibit the same relative speed of repayment.

Note that the lack of transitivity still remains with relative speed of repayment. For example, let $R = (100, -30, -30, -40)$, $S = (100, -30, -30, -225)$, and $T = (100, -20, -45, -35)$. These are all B-normal with yields of 0, 0.5, and 0, respectively. In each case, there is a single loan payment of 100 units, so any reasonable definition of $s(T)$ should assign the same "size" to each transaction. Direct calculations show that R is faster than S, and S is faster than T, but R is not faster than T.

Additional comments on speed of repayment can be found in Appendix II.

We will use the concept of speed of repayment to complete the discussion begun at the end of Section VII. First, we state the main theorem on outstanding investments.

THEOREM 4. *Let T be a transaction of duration not greater than n with $c_t \neq 0$ for some $t < n$, and suppose that (20) holds for some i.*

(a) *If $B_i^n(T) = 0$, T is B-normal with unique yield i.*

(b) *If $B_i^n(T) < 0$, T is B-normal with unique yield $> i$.*

(c) *If $B_i^n(T) > 0$, $P_j(T) > 0$ for $j \geq i$.*

Proof. Assume that $B_i^n(T) < 0$. Let c_k be the first nonzero payment in T. By assumption, $k < n$, so, from (20), $B_i^k(T) > 0$. Then, from (23), we see first that $c_k > 0$ and then that $B_j^n(T) > 0$ for j sufficiently large. By the continuity of $B_j^n(T)$ as a function of j, there exists $i' > i$ with $B_{i'}^n(T) = 0$. Now, for $j < i'$, we see from (22) that $B_j^n(T) < 0$, and from (18) that $P_j(T) < 0$. Similarly, for $j > i'$, we will have $P_j(T) > 0$. Hence, T is B-normal with yield i'. This proves part (b), and the same reasoning demonstrates (a) and (c), taking $i' = i$.

Note that part (a) of the theorem contains Kellison's original result on uniqueness of yield. Part (c) generalizes Theorem 2. Noting that $B_0^t(T)$ is the sum of the payments up to time t, we see that Theorem 2 is just the particular case of part (c) for $i = 0$.

COROLLARY 2. *Suppose that S and Q are B-normal transactions with respective yields of i and j, $i \leq j$.*

(a) *If S is faster than Q, then $S - Q$ is L-normal with yield $\geq j$. In particular,*

$$P_k(S - Q) \geq 0\,, \quad -1 < k \leq j\,. \tag{25}$$

(b) *If Q is faster than S, then*

$$P_k(S - Q) \geq 0\,, \quad i \leq k\,. \tag{26}$$

Proof. Let $T = Q - S$, and let n be the duration of T.

For part (a) we have, by definition, that $B_j^t(T) \geq 0$ for $t = 0, 1, \ldots, n - 1$. Also, $B_j^n(T) = -B_j^n(S) < 0$, so we can apply part (b) of Theorem 4 to conclude that $S - Q = -T$ is L-normal.

To show part (b), it is clear that relation (26) holds for $i \leq k \leq j$. To see that it holds for $j \leq k$, we apply part (c) of Theorem 4 to $-T$.

This corollary applies to the situation discussed at the end of Section VII. $S - Q$ represents a composite transaction resulting from borrowing at interest rate i and then lending at the higher rate j. Part (a) says that if we do this and then repay the loan faster than our debtor repays us, the lending component of the transaction dominates, and the result is L-normal. In particular, the transaction is not universally profitable.

To interpret part (b), we recall from Section VII that those with interest preference rates not less than i will gain on the borrowing but lose on the lending. Part (b) says that if our debtor repays us faster than we repay our lender, then the gain will be more than the loss. In this situation, the transaction may or may not be universally profitable. We have no information about $P_k(T)$ for $k < i$.

We can also view part (a) as making a statement of comparison about the two loans S and Q. It reflects the general principle illustrated in example 5 that for those at the lower end of the interest preference range it is better to repay loans as fast as possible. Formula (25) says, in fact, that if one loan combines both a lower yield *and* faster repayment, it is necessarily more attractive to individuals with low interest preference rates.

IX. CONSUMER LOANS REVISITED

In this section we will consider transactions involving a loan that consists of a single payment made at the present time. We wish to compare various repayment schemes. In doing this, we will make use of the corollary to Theorem 4 (Corollary 2) and the resulting inequality (25), which deals with interest preference rates lower than the yield rates. This reflects the fact that we wish to regard the loans from the standpoint of the typical consumer borrower, who, as indicated in example 5, can be expected to have an interest preference rate lower than the yield rates on the loans he is offered. His objective is to minimize his loss. We do not concern ourselves with those having interest preference rates higher than the yield; they will gain from the transactions in any event.

One of the main objectives of this section is to clarify the point exhibited in examples 5 and 7 that we mentioned at the end of the previous section. In Theorem 5 we will show that there is some limitation on the extent to which a loan can be more favorable than one with a lower yield.

We wish to consider transactions of the form

$$S = (1, -c_1, -c_2, \ldots, -c_n)$$

with

$$c_k \geq 0\,, \quad k = 1, \ldots, n \quad \text{and} \quad \sum_{k=1}^{n} c_k \geq 1\,. \tag{27}$$

By Theorem 3, such a transaction is B-normal with a yield rate $i \geq 0$. (For simplicity, we confine ourselves in this section to the usual case of nonnegative yields.)

We will call a transaction S satisfying (27) an *I.P. transaction* (the initials standing for "interest paid") if $B_i^t(S) \leq 1$ for $t = 0, 1, \ldots, n$, where i is the yield rate of S. This simply says that at least the interest on the original loan of 1 unit is paid at the end of each period.

Examples satisfying (27) with duration n and yield i are

(a) $R_{n\,i}$, with $c_k = (a_{\overline{n}|i})^{-1}$, $k = 1, 2, \ldots, n$;

(b) $I_{n\,i}$, with $c_k = i$, $k = 1, 2, \ldots, n-1$, $c_n = 1 + i$;

(c) $E_{n,i}$, with $c_k = 0$, $k = 1, 2, \ldots, n-1$, $c_n = (1+i)^n$.

In (a) the loan is repaid by level amounts; in (b) interest only is paid until maturity; in (c) nothing is paid until maturity. Clearly (a) and (b) are I.P., while for $n > 1$, (c) is not.

LEMMA 5. *Suppose that S satisfies relations (27) and has duration not greater than n and yield not greater than j. Then*

(a) $$P_k(S) \geq P_k(E_{n,j})\,, \qquad -1 < k \leq j\,.$$

(b) *If, in addition, S is I.P., then*

$$P_k(S) \geq P_k(I_{n,j})\,, \qquad -1 < k \geq j\,.$$

Proof. In both cases, we simply apply part (a) of Corollary 2. It is clear that S is faster than $(E_{n,j})$ (an easy case of Lemma 4, with $k = n - 1$). If S is I.P., it is by definition faster than $I_{n,j}$, since $B_j^i(I_{n,j}) = 1$ for $i = 1, 2, \ldots, n - 1$.

We now want to consider the following functions of two variables. For any positive integer n, and $i > 0$, we define

$$g(n, i) = \left(\frac{n}{a_{\overline{n}|i}}\right)^{1/n} - 1$$

and

$$h(n, i) = \left(\frac{1}{a_{\overline{n}|i}} - \frac{1}{n}\right).$$

So if $i' = g(n, i)$, $i'' = h(n, i)$,

$$(1 + i')^n = 1 + ni'' = \frac{n}{a_{\overline{n}|i}}\,. \tag{28}$$

It is not hard to verify that

$$g(n, i) \leq h(n, i) \leq i\,, \tag{29}$$

with equality if and only if either $i = 0$ or $n = 1$. The second inequality in (29) comes from the familiar identity

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

and the fact that $s_{\overline{n}|} \geq n$.

The significance of these functions is shown by the following theorem.

THEOREM 5. *Given any positive integer n, and $i \geq 0$, let S satisfy (27) with duration not greater than n. Then*

(a) *If the yield of S is not greater than $g(n, i)$,*

$$P_k(S) \geq P_k(R_{n,i})\,, \qquad 0 \leq k \leq i\,. \tag{30}$$

(b) *If S is I.P., then (30) holds, provided that the yield of S is not greater than $h(n, i)$.*

Proof. (a) If $j \leq g(n, i)$, then $(1+j)^n \leq n/a_{\overline{n}|i}$. By Corollary 1, $P_k(E_{n,j}) \geq P_k(R_{n,j})$ for $k \geq 0$. We now apply part (a) of Lemma 5. (b) If $j \leq h(n, i)$, then $(1 + nj) \leq n/a_{\overline{n}|i}$. By Corollary 1, $P_k(I_{n,j}) \geq P_k(R_{n,j})$ for $k \geq 0$, and we apply part (b) of Lemma 5.

Some sample values of $g(n, i)$ and $h(n, i)$ are shown in Table 1 and Table 2, respectively. The following is an example to show how we may interpret these figures. Suppose that a lender who charges 20 percent interest on a 10-period level-payment loan is accused of charging an unjustifiably high rate. He may try to defend his position by citing examples (like example 5 or example 7) that show that some lower-yield loans may be even less favorable to the typical borrower. We see from Theorem 5, however, that there is some limitation to this defense. Any loan of duration 10 or less with an interest rate lower than $g(10, 0.2) =$ 9.1 percent is necessarily more favorable than a 10-period level-payment loan at 20 percent. Hence, if 9.1 percent is still considered an excessive rate, the lender's defense loses force.

The restriction is more drastic if we limit ourselves to I.P. loans. In this case, any loan of duration 10 or less with an interest rate lower than $h(10, 0.2) = 13.9$ percent is more favorable than the original

TABLE 1

$$g(n, i) = \left(\frac{n}{a_{\overline{n}|i}}\right)^{1/n} - 1$$

n	$i = 0.01$	$i = 0.05$	$i = 0.10$	$i = 0.20$	$i = 0.50$
5	.005	.029	.056	.108	.235
10	.005	.026	.050	.091	.177
20	.005	.023	.044	.073	.122
50	.005	.020	.032	.047	.066

TABLE 2

$$h(n, i) = \left(\frac{1}{a_{\overline{n}|i}} - \frac{1}{n}\right)$$

n	$i = 0.01$	$i = 0.05$	$i = 0.10$	$i = 0.20$	$i = 0.50$
5	.006	.031	.064	.134	.375
10	.005	.030	.063	.139	.409
20	.005	.030	.068	.155	.450
50	.006	.035	.081	.180	.480

level-payment loan. The restriction to I.P. loans seems a reasonable one to make in practice. We would expect that the typical lender in a consumer loan transaction would not allow the loan balance to increase.

We conclude this section with some estimates for values of $g(n, i)$ and $h(n, i)$. From Table 1, it appears that $g(n, i)$ increases with i and decreases with n. It seems to be close to $i/2$ in many cases, with the ratio of $g(n, i)$ to i decreasing as either i or n increases. We can prove a general inequality that bears out some of these observations.

For $0 \leq i < 1$, we have

$$(1 + i)^{-n} = 1 - ni + \frac{n(n+1)}{2} i^2 + \ldots, \tag{31}$$

from which we can derive the familiar expansions

$$\frac{a_{\overline{n}|}}{n} = 1 - \frac{n+1}{2} i + \frac{(n+1)(n+2)}{3!} i^2 + \ldots \tag{32}$$

and

$$\frac{n}{a_{\overline{n}|}} = 1 + \frac{n+1}{2} i + \frac{n^2 - 1}{12} i^2 + \ldots. \tag{33}$$

(See [5], formulas 3.24 and 3.25.) Let i' denote $g(n, i)$. Using the standard alternating series estimates in (31) and (32), we have

$$\begin{aligned} 1 - ni' \leq (1 + i')^{-n} &= \frac{a_{\overline{n}|i}}{n} \\ &\leq 1 - \frac{n+1}{2} i \left(1 - \frac{n+2}{3} i\right), \end{aligned}$$

from which it follows that

$$g(n, i) \geq \frac{i}{2}\left(1 - \frac{n+2}{3} i\right). \tag{34}$$

It is of interest to consider a continuous analogue of the function $g(n, i)$. For a positive integer n and $\delta > 0$, we define $\bar{g}(n, \delta)$ to be the number δ' such that

$$e^{n\delta'} = \frac{n}{\bar{a}_{\overline{n}|}},$$

where $\bar{a}_{\overline{n}|}$ is at a force of interest δ.

We now show that

$$\frac{\delta}{2}\left(1 - \frac{n\delta}{3}\right) \leq \bar{g}(n, \delta) \leq \frac{\delta}{2}. \tag{35}$$

The left-hand inequality in (35) follows in a way similar to (34), by use of the expansions

$$\frac{\bar{a}_{\overline{n}|}}{n} = 1 - \frac{n\delta}{2} + \frac{(n\delta)^2}{3!} + \dots$$

and

$$e^{-n\delta} = 1 - n\delta + \frac{(n\delta)^2}{2!} + \dots .$$

For the right-hand inequality in (35), we use the series

$$\frac{\bar{s}_{\overline{n}|}}{n} = \frac{e^{n\delta} - 1}{n\delta} = 1 + \frac{n\delta}{2!} + \frac{(n\delta)^2}{3!} + \dots + \frac{(n\delta)^k}{k+1!} + \dots \quad (36)$$

and

$$e^{n\delta/2} = 1 + \frac{n\delta}{2} + \frac{(n\delta)^2}{2^2 2!} + \dots + \frac{(n\delta)^k}{2^k k!} + \dots . \quad (37)$$

Since $2^k \geq k + 1$ for all k, a comparison of (36) and (37) shows that

$$e^{n\delta/2} \leq \frac{\bar{s}_{\overline{n}|}}{n} . \quad (38)$$

Multiplying (38) by $e^{-n\delta}$ and taking reciprocals, we have

$$e^{n\delta/2} \geq \left(\frac{n}{\bar{s}_{\overline{n}|}}\right) e^{n\delta} = \frac{n}{\bar{a}_{\overline{n}|}} = e^{n[\bar{g}(n,\delta)]} ,$$

and we take logarithms to complete the derivation of (35).

The behavior of $h(n, i)$ is somewhat different. We see from Table 2 that this function increases with i, as is immediate from the definition. For large values of n, it increases as n increases. In fact, it is clear from the definition that

$$\lim_{n \to \infty} h(n, i) = i .$$

We can derive a much higher lower bound for $h(n, i)$ than we did for $g(n, i)$. If $i'' = h(n, i)$, then, from (33),

$$1 + ni'' = \frac{n}{a_{\overline{n}|}} \geq 1 + \frac{n+1}{2} i ,$$

and it follows that

$$h(n, i) \geq \frac{i}{2} .$$

X. GENERALIZATIONS

In this section we will discuss some possible generalizations of the preceding ideas. As previously suggested, we can consider a more general definition of a transaction to provide for payments at arbitrary times. From this point of view, a transaction T is a function, $r \rightarrow c_r$, defined on $[0, \infty)$, such that $c_r = 0$ for all but finitely many points. We think of c_r as denoting the payment at time r. Our original definition was in fact similar to this, except that then we had the restriction that $c_r = 0$ for all noninteger values of r.

As before, we define the interest preference function

$$P_i(T) = \Sigma_r (1 + i)^{-r} c_r .$$

Theorem 2 holds in this setting with an almost identical statement and proof. The only difference is that we must define $s_k = \Sigma_{j \leq k} d_j$, and the induction is on the number of values of $r \leq n$ for which $c_r \neq 0$.

The statement of Theorem 3 and both alternate proofs hold with similar modifications. In the proof involving derivatives, the induction is again on the number of values of $r \leq k$ for which $c_r \neq 0$, and the term $-g(1 + i)^2$ must be replaced by $-g(1 + i)^{(r'+1)}$, where r' is the smallest such positive r.

As we mentioned before, Theorem 1 is still true; however, we need a different proof. Suppose that T and S are represented by the functions $r \rightarrow c_r$ and $r \rightarrow b_r$, respectively, and let $d_r = c_r - b_r$. Then $P_i(T) = P_i(S)$ for all i simply means that

$$\Sigma_r d_r v^r = 0 \tag{39}$$

for all positive v. We want to show that $d_r = 0$ for all r. If not, let s be the minimum value of r for which $d_r \neq 0$. Dividing by v^s in (39) results in

$$d_s = - \Sigma_r d_r v^{r-s} ,$$

and taking the limit as $v \rightarrow 0$, we have $d_s = 0$, a contradiction.

The reader familiar with vector-space theory may have noticed already that Theorem 1 follows immediately from the linear independence of the functions $\{x^r: r \geq 0\}$ in the space of real-valued functions on $[0, \infty)$. The derivation given above essentially provides a proof of this fact.

We now proceed to further generalizations. The discussion in the rest of this section will assume a familiarity with Stieltjes integrals. See Apostol ([1], chap. 7) for an exhaustive and reasonably elementary treatment. Also, see Huffman [3] for an application of Stieltjes integrals to actuarial questions.

The ultimate generalization is to define a transaction T by a distribution function α_T on $[0, \infty)$. By this we mean that $\alpha_T(t)$ represents the total of all units received up to and including time t. For technical reasons we impose the following conditions on α_T.

(a) $\alpha_T(0) = 0$.
(b) α_T is of bounded variation ([1], Definition 6.4).
(c) α_T is right-continuous on $(0, \infty)$; that is,

$$\lim_{t \to s^+} [\alpha_T(t)] = \alpha_T(s) \qquad \text{for all } s \text{ in } (0, \infty) .$$

With this definition we can handle continuous as well as discrete payments, and even transactions that have both continuous and discrete components.

Example 17. Let $T = (7, -2, -6)$ as in our original definition. Then

$$\begin{aligned} \alpha_T(t) &= 0 , && t = 0 \\ &= 7 , && 0 < t < 1 \\ &= 5 , && 1 \le t < 2 \\ &= -1 , && 2 \le t . \end{aligned}$$

Example 18. An annuity provides for continuous payments for 3 periods, the payment at time t being at the rate of t per period. In addition, there is a bonus of 1 unit paid at the end of $1\frac{1}{2}$ periods. Suppose the annuity is purchased for 5 units. Let T represent the entire transaction—the purchase payment and the annuity payments. We want to find α_T.

For the continuous portion, we use integration instead of addition. Since

$$\int_0^t s ds = t^2/2 ,$$

we have

$$\begin{aligned} \alpha_T(t) &= 0 , && t = 0 \\ &= -5 + t^2/2 , && 0 < t < 1\tfrac{1}{2} \\ &= -4 + t^2/2 , && 1\tfrac{1}{2} \le t < 3 \\ &= \tfrac{1}{2} , && 3 \le t . \end{aligned}$$

Some readers may wonder why we define $\alpha_T(0) = 0$ rather than $\alpha_T(0) = c$, if c is the payment at time 0. The reasons will be pointed out in the sequel as they arise.

We can also generalize in another direction to allow for interest preference rates that vary with time. Let $r(t)$ denote the amount that an individual will forgo at the present time in return for 1 unit at time

t. We will call the function r the *interest preference scale* of the individual. We will assume that any interest preference scale satisfies the following natural conditions:

(a) r is continuous on $[0, \infty)$;
(b) $r(0) = 1$; and
(c) $r(t) > 0$ for all t.

In our original case of a constant interest preference rate i, the corresponding interest preference scale is $r(t) = (1 + i)^{-t}$.

For an individual with interest preference scale r, the gain resulting from undertaking transaction T is given by the Stieltjes integral

$$P^r(T) = \int_0^\infty r\,d\alpha_T ,$$

provided that the integral exists.

For the particular case $r(t) = (1 + i)^{-t}$ for some $i > -1$, we will denote $P^r(T)$ by $P_i(T)$, and it is not hard to verify that this coincides with our previous definition when T is as defined originally ([1], Theorem 7.11). (This is one place where we need the requirement that $\alpha_T(0) = 0$.)

We will say that a transaction T is of *finite duration* if there is a real number $n \geq 0$ such that $\alpha_T(t) = \alpha_T(n)$ for all $t \geq n$. (The interpretation of this is simply that there are no more payments after time n.) The greatest lower bound of all n satisfying the above will be called the *duration* of T. It is easy to see that this definition agrees with our previous definition of duration for T.

If T is of duration not greater than n, then

$$\int_0^\infty r\,d\alpha_T = \int_0^n r\,d\alpha_T ,$$

and, by the basic existence theorem for Stieltjes integrals ([1], Theorem 7.27), it follows that $P^r(T)$ exists.

We now consider the problem of generalizing some of the previous theorems. For simplicity we will consider only transactions of finite duration, and hence there will be no need to worry about the existence of the integrals. We expect that most transactions that one encounters in practice are of finite duration. There are of course some exceptions—perpetuities, for example.

THEOREM 2′. *Suppose that T and S are transactions of finite duration such that $\alpha_T(t) \geq \alpha_S(t)$ for all $t \geq 0$. Then for any decreasing interest preference scale r, $P^r(T) \geq P^r(S)$.*

Proof. Let n be the maximum of the durations of T and S. Let $\gamma = \alpha_T - \alpha_S$. From the integration-by-parts formula ([1], Theorem 7.6),

$$\begin{aligned} P^r(T) - P^r(S) &= \int_0^n r\,d\gamma \\ &= r(n)\gamma(n) - r(0)\gamma(0) - \int_0^n \gamma\,dr\,. \end{aligned} \tag{40}$$

Now

$$\int_0^n \gamma\,dr \leq 0\,,$$

since γ is nonnegative and r is decreasing. Moreover, $r(n)$ and $\gamma(n)$ are nonnegative, and $\gamma(0) = 0$ (another use of the fact that $\alpha_T(0) = 0$). This shows that the expression in (40) is nonnegative, completing the proof.

We come now to the generalization of Theorem 3.

THEOREM 3′. *Let T be a transaction of finite duration n. Suppose there is a number k, $0 \leq k \leq n$, such that*

$$0 \leq s < t < k \quad \textit{implies} \quad \alpha_T(s) \geq \alpha_T(t)$$

and

$$k \leq s \leq t \quad \textit{implies} \quad \alpha_T(s) \leq \alpha_T(t)\,.$$

(In other words, α_T is decreasing on the interval $[0, k]$ and increasing on $[k, n]$.) Let r and s be interest preference scales such that the function s/r is increasing on $[0, n]$ and $P^r(T) \geq 0$. Then

$$P^s(T) \geq P^r(T)\,.$$

Proof. The proof is almost identical with the first alternate proof of the original Theorem 3, except that integrals are used in place of sums. Let $q = s/r$. Then

$$\int_0^k s\,d\alpha_T = \int_0^k qr\,d\alpha_T \geq q(k)\int_0^k r\,d\alpha_T\,. \tag{41}$$

This follows directly from the definition of the integral, noting that $q(t) \leq q(k)$ for all t in $[0, k]$ and that α_T is decreasing on this interval. Similarly,

$$\int_k^n qr\,d\alpha_T \geq q(k)\int_k^n r\,d\alpha_T\,. \tag{42}$$

Adding (41) and (42) and noting that $q(k) \geq q(0) = 1$ gives the desired conclusion.

This theorem shows in particular that if r is a "yield" of T (in the sense that $P^r(T) = 0$), we cannot have another yield s for which s/r is increasing, or (applying the theorem to s) for which r/s is increasing. Now for $r(t) = (1 + i)^{-t}$ and $s(t) = (1 + j)^{-t}$, s/r is increasing if and only if $j \leq i$, which shows that we have indeed generalized the uniqueness part of Theorem 3.

The following result deals with the question of the sign of $P^r(T)$ and generalizes parts of Lemmas 2 and 3.

LEMMA 3′. *Let T be as in Theorem 3′.*

(a) *Suppose* $\alpha(n) > 0$.

(1) *If r is increasing on* $[0, n]$, *then* $P^r(T) \geq 0$.

(2) *If r is decreasing on* $[0, n]$ *and if for some c in the interval* $(0, k)$,

$$\frac{r(k)}{r(c)} \leq \frac{|\alpha(c)|}{\alpha(n) - \alpha(k)}, \tag{43}$$

then $P^r(T) \leq 0$.

(b) *Suppose* $\alpha(n) \leq 0$.

(1) *If r is decreasing on* $[0, n]$, *then* $P^r(T) \leq 0$.

(2) *If r is increasing on* $[0, n]$, *and if for some d in the interval* (k, n),

$$\frac{r(d)}{r(k)} > \frac{|\alpha(k)|}{\alpha(n) - \alpha(d)},$$

then $P^r(T) \geq 0$.

Proof. For simplicity we will let α_T be denoted by α. Since

$$\int_0^n 1 \cdot d\alpha = \alpha(n),$$

part (a)(1) follows immediately from Theorem 3′, with s replaced by r, and r replaced by the interest preference scale that takes the constant value 1 (that is, a constant interest preference rate of zero). Similarly, to prove (b)(1), we note that, if $P^r(T) > 0$, we can apply Theorem 3′ with $s = 1$, to conclude that $\alpha(n) > 0$, contradicting the hypothesis.

To prove (a)(2), we write

$$P^r(T) = \int_0^c r\,d\alpha + \int_c^k r\,d\alpha + \int_k^n r\,d\alpha.$$

Since α is decreasing on $[c, k]$, the second integral is negative or zero, and

$$P^r(T) \leq \int_0^c r\,d\alpha + \int_k^n r\,d\alpha\,;$$

since r is decreasing,

$$\begin{aligned} P^r(T) &\leq r(c)\alpha(c) + r(k)[\alpha(n) - \alpha(k)] \\ &\leq r(c)\left\{\alpha(c) + \frac{r(k)}{r(c)}[\alpha(n) - \alpha(k)]\right\} \\ &\leq r(c)[\alpha(c) + |\alpha(c)|] = 0\,. \end{aligned}$$

Similarly, to derive (b)(2), we note that in this case

$$\begin{aligned} P^r(T) &\geq \int_0^k r\,d\alpha + \int_d^n r\,d\alpha \\ &\geq r(k)\alpha(k) + r(d)[\alpha(n) - \alpha(k)] \\ &\geq r(k)\left\{\alpha(k) + \frac{r(d)}{r(k)}[\alpha(n) - \alpha(d)]\right\} \\ &\geq r(k)[\alpha(k) + |\alpha(k)|] = 0\,. \end{aligned}$$

At first glance, Lemma 3′ may not look much like Lemma 3, but it is indeed a reasonably close analogue. Suppose, for example, that $r(t) = (1 + i)^{-t}$ and T is as in Theorem 3. We apply (43) with $c = k$ on the right-hand side and $c = k - 1$ on the left. With these two changes, the statement of Lemma 3′(a)(2) says that $P_i(T) < 0$ for $(1 + i)^{-1} \leq s/(t - s)$, which is exactly equivalent to the statement of Lemma 3(a).

We have left for the end the generalization of Theorem 1, since it involves more advanced methods of mathematical analysis. A natural way to generalize this theorem would be to show that if S and T are two transactions of finite duration such that $P^r(S) = P^r(T)$ for all interest preference scales, then $S = T$. In fact, we will do better than this and show as we did before that a transaction is determined by the value of the resulting gain or loss for *constant* interest preference rates.

THEOREM 1′. *Let S and T be two transactions of finite duration such that $P_i(T) = P_i(S)$ for all $i > -1$. Then $S = T$.*

Proof. To say that $S = T$ means in our present context that $\alpha_S = \alpha_T$. Let n be the maximum of the durations of S and T, and let $\gamma = \alpha_T - \alpha_S$. We want to show that $\gamma = 0$, given that

$$\int_0^n v^t d\gamma = 0 \qquad \text{for all } v > 0\,. \tag{44}$$

Now consider the collection of functions $F = \{t \to v^t : v > 0\}$. We note that any continuous function on $[0, n]$ can be uniformly approximated to within any desired accuracy by a linear combination of functions in F. This follows from standard theorems on approximations. Perhaps the fastest method is to note that F is closed under multiplication—that is, $v_1^t v_2^t = (v_1 v_2)^t$—and use the Stone-Weierstrass theorem ([6], Theorem 7.32).

From the linearity of the integral and the continuity of the linear functional,

$$f \to \int_0^n f d\alpha$$

([7], sec. 4.32-10), we can then deduce from (44) that

$$\int_0^n f d\gamma = 0$$

for all continuous functions on $[0, n]$. Now, since $\gamma(0) = 0$ and γ is right-continuous, it is a well-known theorem that γ must equal zero ([7], p. 198).

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APPENDIX I

A CRITERION FOR STRONG NORMALITY

What significance can we attach to the nonpositive roots of the associated polynomial of a transaction? These do not represent yields, so one may feel that these quantities have no bearing on compound interest theory and should be ignored. This is not the case, however, since we can derive a criterion for strong normality and critical values in terms of these roots. The result is as follows.

THEOREM 6. *Suppose that* $T = (c_0, c_1, \ldots, c_n)$ *is L-normal with unique yield i. Let* $b = \max \{\text{Re}(z)\colon z \text{ a nonpositive root of the associated polynomial } f\}$ *and let* $r = (1+i)^{-1}$, *the unique positive root of f. If* $b \leq r$, *then T is strongly L-normal and the critical value* i' *satisfies*

$$\begin{aligned} (1+i')^{-1} &\leq \frac{(n-1)r+b}{n} \quad && \text{if } b > -(n-1)r\,, \\ i' &= \infty && \text{if } b \leq -(n-1)r\,. \end{aligned} \tag{45}$$

Proof. The associated polynomial can be written as $f(x) = (x-r)q(x)$, where $q(x)$ has no positive roots. By L-normality, $q(x) > 0$ for all $x > 0$. Differentiating,

$$\begin{aligned} f'(x) &= (x-r)q'(x) + q(x) \\ &= q(x)\left[(x-r)\frac{q'(x)}{q(x)} + 1\right], \quad x > 0\,. \end{aligned} \tag{46}$$

Suppose that the real nonpositive roots of $q(x)$ are $a_1, a_2, \ldots, a_m$, and the complex roots are $z_1, z_2, \ldots, z_l$ together with their conjugates. We have $m + 2l = n - 1$. Let $b_j = \text{Re}(z_j)$, $j = 1, 2, \ldots, l$. Then

$$q(x) = c(x-a_1)\ldots(x-a_m)(x-z_1)(x-\bar{z}_1)\ldots(x-z_l)(x-\bar{z}_l)\,,$$

where c is a positive constant. Now

$$\frac{q'(x)}{q(x)} = \frac{d}{dx}[\log q(x)] = \sum_{k=1}^{m}\frac{1}{x-a_k} + \sum_{j=1}^{l}\frac{2(x-b_j)}{|x-z_j|^2}. \tag{47}$$

Let

$$b' = \max\left\{0, \frac{(n-1)r+b}{n}\right\}.$$

We will show that, for $x > b'$,

$$f'(x) > 0\,, \tag{48}$$

which will complete the proof (see formula [17]). We do this in two steps, showing inequality (48) separately for the case $x \geq r$ and the case $r > x > b'$.

For $x \geq r$, each term in expression (47) is nonnegative, and inequality

(48) follows from equation (46). (At this point we have established the strong L-normality.)

For $x > b'$,

$$\frac{1}{x - a_k} \leq \frac{1}{x - b} \tag{49}$$

for $k = 1, 2, \ldots, m$; and

$$\frac{x - b_j}{|x - z_j|^2} \leq \frac{x - b_j}{[\mathrm{Re}\,(x - z_j)]^2} = \frac{1}{x - b_j} \leq \frac{1}{x - b} \tag{50}$$

for $j = 1, 2, \ldots, l$, so that from formulas (47), (49), and (50) we obtain

$$\left| (x - r)\,\frac{q'(x)}{q(x)} \right| \leq (r - x)\,\frac{n - 1}{x - b}\,. \tag{51}$$

Now for $b' < x < r$,

$$(n - 1)r + b < nx = (n - 1)x + x$$

and

$$(n - 1)(r - x) < x - b\,,$$

so that (48) follows from (46) and (51).

In particular, the hypothesis of the theorem is satisfied when $f(x)$ has no complex roots. We then have $b \leq 0$, so

$$b' \leq \frac{(n - 1)r}{n}\,,$$

and

$$i' \geq i + \frac{1 + i}{n}\,.$$

APPENDIX II

PARTIALLY ORDERED SETS

In this appendix we state the definition of a partially ordered set for the benefit of the reader not familiar with the concept, and we then illustrate the idea with examples from the paper.

> DEFINITION. A relation $\leq$ on a set X is called a *partial order* if it satisfies the following three axioms:
>
> (1) $x \leq x$ for all x in X.
>
> (2) If x, y, and z are elements of X such that $x \leq y$ and $y \leq z$, then $x \leq z$.
>
> (3) If x and y are elements of X such that $x \leq y$ and $y \leq x$, we must have $x = y$.

If, in addition, $\leq$ satisfies

(4) Given any two elements x and y in X, either $x \leq y$ or $y \leq x$, then $\leq$ is called a *total order*.

The typical example of a total order is, as we mentioned in Section II, the usual ordering on the set of real numbers. There are many familiar examples in mathematics of partial orders that are not total. One example occurs when X is the set of all subsets of a given set, and $A \leq B$ means A is a subset of B. For another example, take X to be the set of all real-valued functions defined on a set S, and define $f \leq g$ if $f(s) \leq g(s)$ for all s in S.

The basic example in this paper is a variation on the latter example. We let X be the set of all transactions and define $S \leq T$ if T is universally better than S. Then $\leq$ is a partial order. Axioms 1 and 2 are verified immediately, while Axiom 3 is a direct consequence of Theorem 1.

Let Y be the set of all B-normal transactions. If we define $S \leq T$ if S is faster than T, then $\leq$ is not a partial order, for, as we saw in example 16, Axiom 2 does not hold. (In fact, neither does Axiom 3. Consider $E_{n,i}$ and $E_{n,j}$ for two distinct interest rates i and j.) However, suppose we define $S \leq T$ to mean that S is faster than T *and* the yield of S is less than or equal to the yield of T. (Alternatively, we could require the yield of S to be greater than or equal to the yield of T.) In this case we do get a partial order. For the verification of Axiom 2, see the discussion following example 16. To verify Axiom 3, suppose that we have S and T in Y with $S \leq T$ and $T \leq S$. Then S and T must have the same yield i. If n is the maximum of the respective durations, we must have $B_i^t(S) = B_i^t(T)$ for $t = 0, 1, \ldots, n-1$, and also $B_i^n(S) = B_i^n(T) = 0$. Now, using formula (19), we can deduce recursively that $S = T$.

Let Y_i denote the set of all B-normal transactions with yield i. If we restrict the above relation on Y to the subset Y_i, we have the result that $S \leq T$ means simply that S is faster than T. Hence, for B-normal transactions of fixed yield, speed of repayment does constitute a partial order.

DISCUSSION OF PRECEDING PAPER

MARJORIE V. BUTCHER:

In this masterful paper, Dr. Promislow presents a new way of analyzing and comparing financial transactions. He develops his thesis through discussion and well-chosen examples that lead to his creation of the key concepts of the paper: the interest preference rate i of an individual; the interest preference function $P_i(T)$ of a transaction T; and normal, and strongly normal, lender- and borrower-type transactions having unique yields and depending on the behavior of $P_i(T)$. In practice, many if not most typical lender investments are strongly L-normal, and most borrower loans are strongly B-normal. Thus, the fresh ideas of the paper will be useful in financial analysis.

Dr. Promislow formulates a considerable number of insightful theorems, corollaries, and lemmas. He provides elegant, succinct proofs and, as needed, additional definitions, discussion, and examples. Especially noteworthy are the comparison of transactions by using interest preference rates rather than yield rates, and the development of outstanding investment and consumer loan analysis. Throughout, the mathematics is superb and crisp. I found it essential to read and consider the ideas very carefully, because they are so new and plentiful as to be rather elusive.

Curiously, a transaction T is *universally better* than itself, and, if it is B-normal, also *faster* than itself! Also, in the example preceding Theorem 4, different transactions R and S are faster than each other. These apparent anomalies are the result of definitions that are mathematically convenient and even necessary (as in the proof of Lemma 5). The problem is chiefly semantic. The paper provides a way around the first of these anomalies: Since the statement (Sec. VII) "$T - S$ is universally profitable" means "For all i, $P_i(T - S) > 0$," one could define the statement "T is universally more profitable than S" to mean "For all i, $P_i(T) > P_i(S)$." One should compare the similar, but different, concepts "universally better" and "universally more profitable" as they relate to two comparable transactions T and S.

In Section VI, in the third paragraph following Lemma 1, the author correctly states that if S and T are universally comparable (in the sense of one being the better), and if either one is L-normal, then the higher-yield transaction is better for the lender. In the supporting argument, the inequalities all should be strict. For instance, suppose that T is L-normal with yield j, and S has yield i. If $i < j$, then $P_i(T) > 0 = P_i(S)$, so T is the better transaction at rate i and universally. But if $i = j$, then $P_i(T) = P_i(S) = 0$, and either S or T may be universally the better. I found it instructive to construct an example in which S is universally the better, based on $j = i =$ 100 percent. Let the associated polynomial f of T be

$$f(v) = 4(v - \tfrac{1}{2}) = -2 + 4v = -2 + 4(1 + i)^{-1} = P_i(T) ;$$

then $T = (-2, 4) = (-2, 4, 0)$ is L-normal (as stipulated) with yield $i = 1$. For S to be universally better than T, we want

$$\begin{aligned} P_i(S - T) = P_i(S) - P_i(T) = g(v) - f(v) &> 0 , \quad i \neq 1 (v \neq \tfrac{1}{2}) \\ &= 0 , \quad i = 1 , \end{aligned}$$

where g is the associated polynomial of S. A suitable choice is

$$P_i(S - T) = 4(v - \tfrac{1}{2})^2 = 1 - 4v + 4v^2 ,$$

and so $S - T = (1, -4, 4)$. Thus,

$$S = T + (S - T) = (-2, 4, 0) + (1, -4, 4) = (-1, 0, 4) .$$

Clearly, S is also L-normal with yield 1, and, by our construction, it is universally better than T. (Given the L-normality of T, S had to be either L-normal or nonnormal.)

The first proof of Theorem 3 (based on Descartes's rule) requires some additional analysis, using calculus, to deduce *strong* L-normality. It is fairly easy to show that the critical value i' is greater than or equal to the yield rate i, but it is not so readily apparent that i' is greater than i. In any event, Dr. Promislow provides an alternative proof of the theorem.

The proof of Corollary 2 proceeds by the separate cases $i < j$ (of the paper) and $i = j$, and produces stronger conclusions under (a) than those stated, namely,

(a) If S is faster than Q, then *any nonzero* [my addition] $S - Q$ is L-normal with yield greater than j if $i < j$ and equal to j if $i = j$. In particular,

$$P_k(S - Q) > 0, \quad -1 < k \leq j,$$

except that $P_k(S - Q) = 0$ if $i = j$ and $k = j$, or if $S = Q$.

I was able to convince myself of the proof of Theorem 5, as outlined, for $0 \leq k \leq g(n, i) = i'$ in (a) and for $0 \leq k \leq h(n, i) = i''$ in (b). To show the result in (a) for $i' < k \leq i$, one need only recall that if S is

B-normal and k is greater than the yield of S, then $P_k(S) > 0$; similarly, $P_k(R_{n,i}) \leq 0$, so $P_k(S) > P_k(R_{n,i})$. The case $i'' < k \leq i$, in (b), is similar.

In general, throughout this complicated paper, the author's choice of notation seems good. Occasionally, however, the same symbol is used simultaneously in two senses. For example, in Theorem 3′, s is used to denote both time and an interest preference scale.

I had a little trouble with the author's comparison of Lemma 3′ and Lemma 3. However, application of formula (43) with $c = k$ on the right-hand side and $c = k - 1$ on the left, $r(l) = (1 + i)^{-l}$, and insertion of the notation of Lemma 3 (e.g., $\alpha(n) = t - s > 0$), give

$$(1 + i)^{-1} \leq \frac{s}{(t - s) + s} = \frac{s}{t} ,$$

and $i \geq (t - s)/s$. For such i, Lemma 3′(a)(2) (as manipulated) suggests $P_i(T) \leq 0$. By Theorem 3, the T of Lemma 3(a) is strongly L-normal, so for all $i > i_0$, $P_i(T) < 0$. Since $(t - s)/s \geq i_0$ in Lemma 3(a), clearly $P_i(T) \leq 0$ for all $i \geq (t - s)/s$, and the comparison is complete.

Dr. Promislow states that his paper arose from an investigation into the concept of yield rate, and he mentions various difficulties, such as the phenomena of multiple-valued and nonexistent yield rates. In our 1971 book,[1] Dr. Cecil J. Nesbitt and I also explore certain questions of uniqueness of rates. For instance, problem 139 (p. 154) concerns the behavior of the basic interest functions—$(1 + i)^n$, v^n, $s_{\overline{n}|i}$, and $a_{\overline{n}|i}$—as functions of n (for fixed $i > 0$) and as functions of $i \geq 0$ (for fixed $n > 0$). We point out: "In solution of equations of value for i or n, an important question is whether there is a unique (exact) result." Following the analysis, we conclude: "Thus, if f denotes one of the basic interest functions (those above or $s_{\overline{n}|}^{-1}$ or $a_{\overline{n}|}^{-1}$), the result of, say, inverse interpolation for i or n, based on $f =$ a constant, is an approximation to the ***unique*** exact result, except in the cases $s_{\overline{1}|i} = 1$ and $s_{\overline{1}|i}^{-1} = 1$ (which are satisfied by all $i \geq 0$)." Again, in section 5.6, discussing the yield rate of a bond having a given book value at purchase and held to a fixed redemption date, we show that there is a unique, exact yield rate.

Our principal effort in this direction, however, is in section 6.6, the mathematical setting of the interest rate problem. We take the lender's or investor's viewpoint, but our orientation is the opposite of Dr. Promislow's; we want the outstanding principal to be positive while the borrower is in a debtor position, so we consider payments from the lender

[1] M. V. Butcher and C. J. Nesbitt, *Mathematics of Compound Interest* (Ann Arbor, Mich.: Ulrich's Books, Inc., 1971).

to the borrower to be positive. To us, transaction T is $(W_0 > 0, W_1, \ldots, W_n \neq 0)$, and "the interest rate problem" consists of solving for v the equation

$$\sum_{k=0}^{n} W_k v^k = 0 \,, \tag{1}$$

or, as we prefer it, solving for $u = 1 + i$ the equivalent equation

$$\sum_{k=0}^{n} W_k u^{n-k} = 0 \,, \tag{2}$$

in order to obtain i. Since equation (2) is a polynomial equation of degree n, it has n solutions, real or imaginary, and we want to identify cases having unique positive solutions for the yield rate i.

First, we consider the (retrospective) outstanding principal at the end of each period, namely,

$$S_h(u) = W_0 u^h + W_1 u^{h-1} + \ldots + W_h \quad (h = 0, 1, \ldots, n) \,, \tag{3}$$

and deduce that

$$i \sum_{h=0}^{n-1} S_h(u) = S_n(u) - \sum_{h=0}^{n} W_h \,. \tag{4}$$

If u is a solution of equation (2), then $S_n(u) = 0$ and

$$i \sum_{h=0}^{n-1} S_h(u) = - \sum_{h=0}^{n} W_h \,, \tag{5}$$

each of these expressions representing the total interest at rate i on the successive principals outstanding. The investor wants $i > 0$, and normally this is accomplished by having both

$$\sum_{h=0}^{n-1} S_h(u) > 0 \quad \text{and} \quad \sum_{h=0}^{n} W_h < 0 \,. \tag{6), (7}$$

In fact, in lieu of formula (6), we impose the more restrictive condition

$$S_h(u) \geq 0 \quad (h = 0, 1, \ldots, n - 1) \,, \tag{8}$$

because if some $S_h(u) < 0$, there is the question of what rate of interest should be credited when the original investor becomes, in effect, a borrower during the transaction. We call any transaction in which formula (8) holds a *pure investment* (at rate $i = u - 1$). It typifies the usual investment.

Now we come to our main theorem.

Let W_0 be greater than 0, W_n be not equal to 0, and let relation (7) *hold. Then there is at least one change of sign among the $W_h(h = 0, 1, \ldots, n)$.*

Case 1: If there is only one such change of sign, then $S_n(u) = 0$ has a unique solution $u_1 > 1$, $i_1 = u_1 - 1 > 0$, and at interest rate i_1, transaction T is a pure investment.

Case 2: If there is a positive even number of such changes of sign, then no solution $u_1 > 1$ of $S_n(u) = 0$ exists such that T is a pure investment at rate i_1.

Case 3: If there is an odd number, exceeding 1, of such changes of sign, then there is a positive number u_0 such that for all $u \geq u_0$ the transaction is a pure investment, and for $u > u_0$ the $S_h(u)$ $(h = 1, 2, \ldots, n)$ are strictly increasing functions of u. Further, $S_n(u) = 0$ has at most one solution u_1 such that $u_1 \geq u_0$ and $u_1 > 1$. If $u_0 \leq 1$, or if $\min u_0 = u_0^* > 1$ and $S_n(u_0^*) \leq 0$, then there is a unique such u_1 $(\geq u_0^*)$. If $u_0^* > 1$ and $S_n(u_0^*) > 0$, then there is no such $u_1(\geq u_0^*)$.

Since case 1 of this theorem includes the commonly encountered *single* advance W_0 by the lender to the borrower, it contains the usual amortization and bond transactions, all of which therefore have unique yield rates. Only in case 1 and the first two subcases of case 3 is there a pure investment at a unique positive rate of interest.

It is interesting to compare Dr. Promislow's ideas and ours. Allowing for opposite orientations, our case 1 is described by his relations (14); by Theorem 3, the T of case 1 is strongly L-normal. His Lemma 1 concerns transactions T with at most one yield. Our case 1 (his Theorem 3) identifies some such T's, namely, those with just one change of sign among the W_h, with $W_0 = -c_0 > 0$, and $W_n = -c_n < 0$; then (i) of Lemma 1 holds, and T is L-normal with unique yield (again) $i > 0$, since $\Sigma_{h=0}^{n} W_h = -s < 0$. At first it appears that case 3 identifies other transactions with at most one yield, but it is limited to solutions $i_1 > 0$, whereas the lemma permits yields greater than -1. Case 2, as well, does not pertain.

If a transaction T satisfies Promislow's inequalities (20), then $-T$ (to the lender) is a pure investment at rate i. By means of his relations (23), we see that, for a lender, any transaction with its first payment to a borrower is a pure investment at a sufficiently high interest rate. In comparing our theorem and Promislow's Theorem 4, we note that both are concerned with yield rates for pure investments. On the one hand, we seek information about the number of yield rates for pure investments at those yield rates. On the other hand, Theorem 4 hypothesizes that $-T$ is a pure investment at rate i and seeks information about yield rates for T, and hence $-T$, in relation to that rate i.

Like Dr. Promislow, we cite Jean's 1968 paper. Among other papers on the interest rate problem, we were intrigued by two 1965 papers by Teichroew, Robichek, and Montalbano,[2] which offer an extensive general discussion of the problem and, in the context of capital budgeting, analogies to some of Promislow's work. Actuaries may recall the lively and often amusing probing of the interest rate problem, featuring examples, in several issues of *The Actuary*, beginning in February, 1968.

Dr. Promislow has made his own significant contribution to the theory of interest with his creation of interest preference rates, normal transactions, and so on, and their mathematical ramifications. I commend him highly for his truly remarkable, stimulating, and thorough paper.

JAMES C. HICKMAN:

To many actuarial students, the theory of interest has been a sterile subject. Unlike subjects such as statistics, risk theory, and investment management, which have been profoundly changed by a series of new ideas and rocked by intellectual controversy, the solid old theory of interest remains as it was in the nineteenth century, untouched by new ideas. Mr. Promislow has helped change that. His achievement is considerable.

In his "new approach," the theory of interest is to be more than a description of commercial practice; it is to provide a framework for financial decision making. Consequently, he bases his theory on the time preferences of individuals. This is clearly the place to build a theory that will have decision consequences.

Next, he introduces actuaries to the two decision criteria developed within management science for ordering or selecting transactions. In management science this process is called "capital budgeting." The two decision criteria are (1) ordering or selecting on the basis of yields (called "internal rates of return" elsewhere), and (2) ordering or selecting on the basis of discounted present values, as indicated by equation (4). In Sections IV–VII, the author defines normal transactions and shows how this definition can provide insights into the use of the theory of interest as a decision tool.

The main part of this discussion will center on an elaboration of Section

[2] D. Teichroew, A. A. Robichek, and M. Montalbano, "Mathematical Analysis of Rates of Return under Certainty," *Management Science* (Ser. A) XI (January, 1965), 395–403, and "An Analysis of Criteria for Investment and Financing Decisions under Certainty," *Management Science* (Ser. A) XII (November, 1965), 151–79.

VIII. The extension is due to Teichroew, Robicek, and Montalbano [2, 3].

We start with a modification of equation (19), the definition of the outstanding investment at time t. We have

$$\begin{aligned} B_{i,r}^{t}(T) &= c_0\,, & t &= 0 \\ &= (1+i)B_{i,r}^{t-1}(T) + c_t\,, & B_{i,r}^{t-1}(T) &\geq 0 \\ &= (1+r)B_{i,r}^{t-1}(T) + c_t\,, & B_{i,r}^{t-1}(T) &< 0\,, \quad t = 1, 2, \ldots, n\,. \end{aligned}$$

We note that $B_{i,i}^{t}(T) = B_{i}^{t}(T)$, the symbol used in the paper. This richer model allows the outstanding balance to depend on two rates of interest. The rate i is applicable if the outstanding balance is nonnegative and a loan or financing exists. The rate r applies if the outstanding balance is negative and investment exists.

The economic justification for this model is that the financing or loan rate (i) associated with a transaction or project is not necessarily the same as the investment rate (r). This more elaborate model provides insights into issues that remain somewhat puzzling in models using only one rate. First, it will tell us something about the multiple roots that may occur for internal rate of return equations. Second, it will lead to a comprehensive selection rule.

We can learn a great deal about this model by examining $B_{i,r}^{n}(T)$, the final value of transaction T. Figure 1 traces typical contours of the form $B_{i,r}^{n}(T) = a_j$: on the left-hand side, i_0 is the smallest interest rate such that $B_{i,r}^{t}(T)$ is a function only of i; on the right, r_0 is the lowest investment rate such that $B_{i,r}^{t}(T)$ is a function only of r. That is, if $c_0 > 0$, there is an interest rate i_0 such that, for $i \geq i_0$, the transaction balances remain nonnegative and the value of $B_{i,r}^{n}(T)$ depends only on i. Likewise,

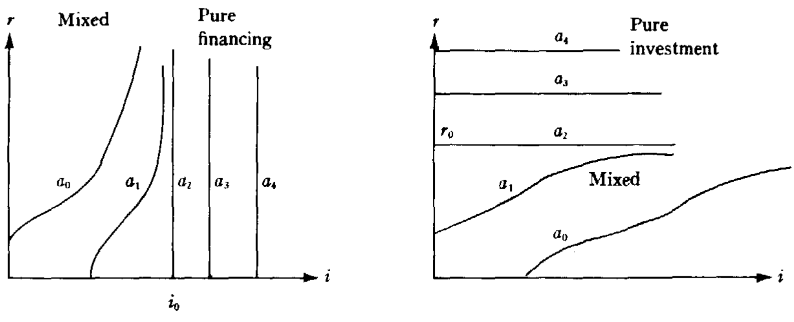

FIG. 1.—Typical contours of $B_{i,r}^{n}(T) = a_j$, $a_0 < a_1 < a_2 < a_3 < a_4$

if $c_0 < 0$, there is an investment rate r_0 such that the transaction balances remain nonpositive and $B^n_{i,r}(T)$ depends only on r for $r \geq r_0$.

A region in the (i, r) plane where $B^n_{i,r}(T)$ is a function of i alone is called a ***pure financing region***, and a region where $B^n_{i,r}(T)$ is a function of r alone is called a ***pure investment region***. Where $B^n_{i,r}(T)$ depends on both i and r, the region is called a ***mixed region***. We note that an increase in the loan rate i will cause an increase in the final balance $B^n_{i,r}(T)$, and an increase in the investment rate r will cause a decrease in the final balance $B^n_{i,r}(T)$.

Within the pure investment and the pure financing regions, a result by Kellison [1], quoted by Promislow, ensures a unique internal rate of return. Within the mixed region, multiple internal rates of return occur. To see this requires additional development. The equation

$$B^n_{i,r}(T) = 0$$

implicitly defines two functions $i = i(r)$ and $r = r(i)$. Since $B^n_{i,i}(T) = (1 + i)^n P_i(T)$, solving the equation $P_i(T) = 0$ for internal rates of return is equivalent to solving $B^n_{i,i}(T) = 0$. The equation $B^n_{i,i}(T) = 0$ will have solutions when $i = r(i)$. In the pure financing region, where the final balance does not depend on r and $B^n_{i,r}(T)$ increases with i, there can be only one solution to the equation $B^n_{i,i}(T) = 0$. Within the mixed region, $i = r(i)$ and $B^n_{i,r}(T) = 0$ for several values of i. One can see that if a transaction has a mixed region, solving the equation $B^n_{i,r}(T) = 0$ for i is restricting examination of the function to the line $i = r$.

Teichroew, Robicek, and Montalbano also use their model in the selection of capital budgeting projects. Promislow uses his decision models to create a partial ordering of possible transactions or projects. The selection model assumes that the firm or individual selecting projects can obtain funds at rate k. In addition, it is assumed that the financial goal of the firm or individual is to increase its present value. Then the rule is to accept transaction or project T if

$$r(k) > k \qquad \text{or} \qquad k > i(k)\,. \tag{1}$$

Recall that the functions $r(i)$ and $i(r)$ were defined implicitly by the equation $B^n_{i,r}(T) = 0$.

The detailed justification of this selection rule is rather intricate. However, that it is plausible may be seen from an examination of the following expressions:

$$\begin{aligned} B^n_{k,k}(T) &> B^n_{k,r(k)}(T) = 0\,, \\ B^n_{k,k}(T) &> B^n_{i(k),k}(T) = 0\,. \end{aligned} \tag{2}$$

Inequalities (2) hold if inequalities (1) prevail. They may be verified by recalling the impact on $B^n_{i,r}(T)$ of a change in i or r. Since

$$B^n_{k,k}(T) = \sum_{t=0}^{n} (1 + k)^{n-t} c_j = P_k(T)(1 + k)^n ,$$

which may be interpreted as the final balance at the firm or individual's cost of money rate k, the rule is equivalent to accepting a project if $P_k(T) > 0$.

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PIERRE CHOUINARD:

Papers on the theory of interest are too rare. Mr. Promislow must be thanked for his contribution to the field.

I used to teach the theory of interest, and I found that the concept of yield rate was made difficult by the reinvestment question. Students learn from the beginning that an investment of \$1,000 at compound rate i will accumulate to \$1,000$(1 + i)^n$ at the end of n periods. In their words, they understand that \$1,000 will grow to \$1,000$(1 + i)^n$ if invested in a fund whose *yield* is i. When they get to the chapter on yield rates, a few months later, they learn that the yield rate of a transaction is the rate at which the present value of inflows equals the present value of outflows, that is, the rate i satisfying

$$\sum_{t=0}^{n} I_t v_i^t = \sum_{t=0}^{n} O_t v_i^t , \quad \text{or} \quad P_i(T) = 0 , \tag{1}$$

where I is an inflow and O an outflow. Because students are used to thinking of yield rate retrospectively, they inevitably raise the question, "what rate of reinvestment is implied by that definition?" They want to evaluate how much money the investor has in his hands by the end of the transaction. The easy answer to that question is, "No rate of reinvestment is implied by that definition; apply it without question."

Another answer is, "That definition implies that the investor reinvests his money at this same rate i." To reinforce this last point, we may rewrite (1) as

$$\sum_{t=0}^{n} I_t(1 + i)^{n-t} = \sum_{t=0}^{n} O_t(1 + i)^{n-t} . \tag{2}$$

Taking the example of a loan transaction, we observe that the left-hand side of equation (2) represents the repayments of the borrower reinvested by the lender at rate i until the end of the transaction. The right-hand side of the equation represents the lender's original disbursements with interest at the steady "yield" rate of i.

The second answer generally is more satisfactory to students, because they are able to see by general reasoning what may have happened. For most finance people, however, the first answer seems more satisfactory and less troublesome. Even if we could convince students of the reasonableness of the first answer, they would not remain convinced after encountering a loan problem ending with the question, "What is the yield rate of this loan assuming that Mr. A can replace his capital at a rate of 4 percent?" where 4 percent is not the lending rate. You then would have to explain that a portion of each borrower's repayment is reinvested in a fund earning less than the lending rate, so that the fund will have accumulated, by the end of the transaction, the capital disbursed at the outset of the transaction. The portion of the borrower's payment left over after the sinking-fund disbursement is the pure interest portion and leads to the yield when divided by the original price of the transaction.

"Make up your mind, Prof. Is the definition of yield rate an objective one, independent of reinvestment, or do we have to make assumptions as to reinvestment? And, if we have to make assumptions, why do we assume that only a portion of the repayment is reinvested at a different rate? What about the reinvestment of the interest portion?"

What I have always theorized (but never dared to say out loud because I thought I was the only one to worry about this silly reinvestment) is that there should be two different yield rates: a prospective one, objective as it is now and aimed at problems not involving reinvestment, and a retrospective one, calculated as of the end of the transaction and necessitating hypotheses or facts as to reinvestment. The retrospective yield would be the rate i satisfying

$$\sum_{t=0}^{n} I_t(1 + j)^{n-t} = \sum_{t=0}^{n} O_t(1 + i)^{n-t} , \tag{3}$$

where j is the rate at which the inflows are reinvested. Actually, the retrospective yield i turns out to be the rate of growth of the amounts disbursed in the course of the transaction. If we assume $j = i$, we obtain equations (2) and (1) and conclude that the retrospective yield equals the prospective yield when $j = i$.

By analogy with Mr. Promislow's approach based on individual preference rates, we might develop a retrospective yield approach in order, for example, to determine the most profitable among a series of transactions that are not readily comparable. Let us define $F_{i,j}(T)$ as

$$F_{i,j}(T) = \sum_{t=0}^{n} I_t(1+j)^{n-t} - \sum_{t=0}^{n} O_t(1+i)^{n-t}, \qquad (4)$$

where the symbols have the same meanings as previously. The retrospective yield rate of T is the rate i_T satisfying

$$F_{i_T,j}(T) = 0$$

for a given rate j. Given two transactions T and S, we may, analogically with the critical individual preference rate (defined as the rate corresponding to the point of intersection of the respective interest preference curves of T and S), calculate a critical reinvestment rate, which is the rate j' at which $i_T = i_S$, that is, the rate of reinvestment that makes an investor indifferent in choosing between T and S. For all reinvestment rates exceeding j', we will then say that either i_T exceeds i_S or i_S exceeds i_T (but not both!), and vice versa for reinvestment rates below j'.

The graph of the retrospective yield rate i against the reinvestment rate j, for various transactions, provides a visual check of what has just been said. For transactions T and S, for example, the graph may take the form shown in Figure 1. The points of intersection y_S and y_T of line Y

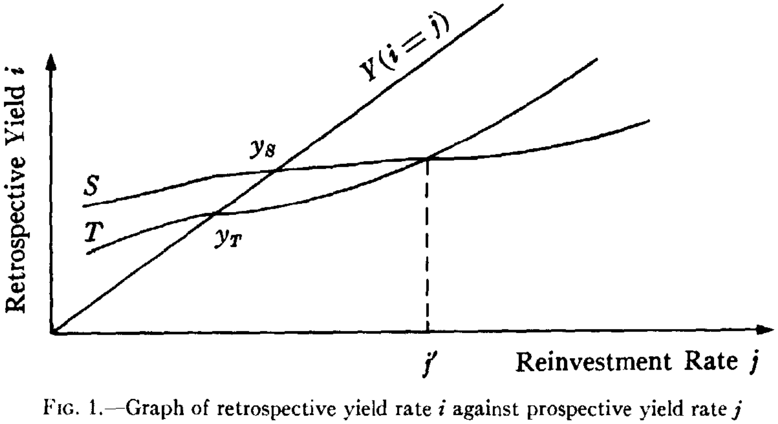

FIG. 1.—Graph of retrospective yield rate i against prospective yield rate j

$(i = j)$ with curves S and T are the traditional prospective yield rates of S and T, respectively, since, at these points, the rates of reinvestment equal the yield rates $(i = j)$, and, as seen before, the retrospective yield equals the prospective yield when this condition is met. The point of intersection of S and T corresponds to the critical reinvestment rate j', at which rate the retrospective yields of both transactions are equal $(i_T = i_S)$.

At this point, it is worthwhile to mention an interesting fact. Suppose we wish to compare two usual loan transactions T and S (each consisting of a single outflow at time 0 repaid by a series of inflows). We reach exactly the same conclusions using either the interest preference rate approach or the retrospective yield approach, provided that the amount lent at the outset is the same. In other words, two loan transactions involving the same single disbursement at time 0 will produce a critical interest preference rate identical with the critical reinvestment rate mentioned above. Other interesting facts certainly will be brought to light with more research on this subject of retrospective yields and reinvestment rates.

I will end my discussion by redoing example 7 of the paper with the retrospective yield approach and from the lender's viewpoint. The two loan transactions were $T = (-10, +9, +9)$ and $S = (-10, +4.5, +14.5)$. The critical reinvestment rate is the rate j' at which the two series of repayments will accumulate to the same amount after two years. Mathematically, it is rate j' such that

$$9(1 + j') + 9 = 4.5(1 + j') + 14.5 .$$

It is no surprise to observe that j' equals $\frac{2}{9}$, that is, the critical interest preference rate found in example 7. Using well-known terms, $\frac{2}{9}$ is the sinking-fund rate that will make an investor indifferent in choosing between the two transactions. If he can reinvest his money in a sinking fund earning more than $\frac{2}{9}$, he will choose transaction T, because the retrospective yield of T is then higher; otherwise, he will choose transaction S.

WARREN R. LUCKNER:

Professor Promislow has written an excellent paper that is worthy of study by any student of interest theory. Although I do not consider myself in the group of actuaries who "probably feel that they understand the subject completely," I do feel that I have a fairly good understanding of the subject. Professor Promislow has helped to present a new perspective on the concept of yield rate with his discussion of the concept of interest preference rate. My discussion is centered primarily on the definition of yield rate, but also comments on some of the examples and attempts to expand upon some of the practical interpretations presented in the paper.

Yield Rate

Professor Promislow defines yield rate in a theoretical manner as follows: "We say that i is a yield rate of T if $P_i(T) = 0$." In this definition, $P_i(T) = \Sigma_{k=1}^{n} c_k(1+i)^{-k}$ denotes the familiar present value of a sequence of payments. The transaction T is denoted by $(c_0, c_1, \ldots c_n)$, where c_k is the payment at time k.

Having taught interest theory from Mr. Kellison's text and having placed substantial emphasis on the definition of yield rate, I am particularly interested in comparing Kellison's definition with the above. On page 117 of Kellison's text, yield rate to an investor is defined as follows: "The yield rate is that effective rate of interest at which the present value of his expenditures is equal to the present value of his returns." Mathematically, the two definitions are equivalent.

It should be noted that the Kellison definition is presented in the context of an investor making a series of expenditures at various points in time and receiving payments in return at various points in time; but, as Professor Promislow points out, yield rates "depend solely on the transaction and are independent of any particular individual."

The Kellison definition has some advantage in being more intuitively understandable than the Promislow definition, which is presented in strictly mathematical terms. However, the Kellison definition may be deficient in not discussing yield rate from the point of view of the borrower in the standard loan transaction. If one tries to apply the Kellison definition to the borrower, one may become somewhat confused.

For example, consider a loan of L being repaid by equal annual payments of P for n years, with the first payment due one year from the time of the loan. From the lender's point of view, the present value of expenditures is L and the present value of returns is $Pa_{\overline{n}|i_0}$. The yield rate is determined by the equation "present value of expenditures equals present value of returns," or $L = Pa_{\overline{n}|i_0}$.

From the borrower's point of view the present value of expenditures is $Pa_{\overline{n}|i_0}$, and the present value of returns is L. Then the yield rate is determined by the equation "present value of expenditures equals present value of returns," or $Pa_{\overline{n}|i_0} = L$. Both viewpoints yield the same equation. Does that mean that the borrower receives the same yield rate as the lender? Of course not. It supports Professor Promislow's point that yield rates depend solely on the transaction, and it points out the importance of distinguishing between what Professor Promislow defines as L-normal and B-normal transactions.

From the lender's point of view, the loan transaction just discussed is given by $(-L, P, P, \ldots, P)$ and thus is L-normal. From the borrower's point of view, the transaction is given by $(L, -P, -P, \ldots, -P)$ and thus is B-normal.

As Professor Promislow points out, once the normality of the transaction is determined, the interest preference rate concept enables an individual to determine whether or not the transaction is to his advantage.

Examples

The following comments briefly address the question of how the theory presented in the paper resolves the difficulties presented in some of the examples in Section IV ("Limitations on the Use of Yield Rates").

Example 1: The theory does not resolve the problem, but helps to analyze the reason for the problem. (Sec. VII: "Decomposition of Transactions").

Example 2: Of all the examples in this section, this is the one that is perhaps most directly addressed by the basic theory presented in the paper. The categorization of transactions into L-normal and B-normal is undertaken specifically because of the difficulties presented in this example.

Example 3: The paper introduces the concept of "universally profitable" to address the problem presented in this example. This example is similar to policy loan borrowing, which perhaps should remind one not to forget the practical questions of tax implications and the like when evaluating alternative transactions.

Example 4: This example serves to remind one that, in considering alternative investments, it is important to use comparable time periods.

Example 5: The 4.5 percent loan arrangement in this example is interesting because the yield rate from the lender's point of view is 4.5 percent, while from the borrower's point of view the yield is not 4.5 percent. This problem is similar to example 5.4 in Mr. Kellison's text. Using the definition of yield rate in that text, the equation to be solved is the following: $1 = [0.045 + (s_{\overline{20}|0.045})^{-1}]a_{\overline{20}|i_0}$; which results in $i_0 \fallingdotseq 5.56$ percent. Thus, the borrower, by choosing the 5 percent loan (which has a 5 percent yield rate), is still choosing the loan with the lower yield rate.

Practical Interpretations

In the discussion of the question of speed of repayment in Section VIII, the test given is as follows: "Given B-normal transactions S and R, we say that R *is faster than* S if (24) holds for i equal to the maximum of the two yields and n equal to the maximum of the two durations." The author states that condition (24) $[B_i^t(R) \leq B_i^t(S), t = 0, 1, \ldots, n - 1]$

"holds for sufficiently high values of i simply under the hypothesis that, for the first place where the payments of S and R differ, the one in S is higher." One may have some difficulty in trying to interpret that statement intuitively. Considering $B_i^t(T)$ as the outstanding balance, it would seem that if, for the first place where the payments of S and R differ, the one in S is *higher*, then the outstanding balance in S should be *lower*, and hence $B_i^t(S) \leq B_i^t(R)$ rather than $B_i^t(R) \leq B_i^t(S)$. However, in considering B-normal transactions, payments on a loan appear as negative values and, in the context of the paper, the phrase "the payment is higher" does not mean that the loan repayment is higher but rather that the net payment (loan less loan repayment) is numerically greater under S at the first place where the net payments differ.

For example, consider the following R and S transactions:

$$R = (20, -10, -10)\,, \qquad S = (20, -5, -15)\,.$$

One would like to say that R is faster than S. Note that at the first place where the payments differ, the payment in S (-5) is greater than the payment in R (-10). Thus, the statement quoted above would suggest that R is faster than S.

What does the test indicate? Note that both transactions have a yield rate of zero. Thus, we only need to consider $B_0^t(S)$ and $B_0^t(R)$ for $t = 0$, 1, 2. The table below summarizes the results:

	$t=0$	$t=1$	$t=2$
$B_0^t(R)$	20	10	0
$B_0^t(S)$	20	15	0

Thus, $B_0^t(R) \leq B_0^t(S)$ for $t = 0, 1, 2$, and R is faster than S. This simple example also points out the importance of considering speed of repayment as well as yield rate.

Theorem 4 in Section VIII lends itself fairly easily to practical interpretation. Part (a) says that, under the conditions given ($B_i^t(T) \geq 0$, $t = 0, 1, \ldots, n-1$, and $c_t \neq 0$ for some $t < n$), if at the end of the transaction period (i.e., at time n), the outstanding balance, based on a certain interest rate i, is zero, then the transaction is a borrowing transaction with unique yield rate i. This is not surprising.

Part (b) says that if at the end of the transaction period the outstanding balance, based on i, is less than zero, then the transaction is a borrowing transaction with unique yield rate greater than i. Does that make sense? Yes, because $B_i^n(T) < 0$ means that the borrower paid too much for the transaction to have a yield rate of i; that is, he paid enough to give the lender a somewhat higher yield rate.

One might like to be able to say more in part (c). However, all that one can say is that if at the end of the transaction period the outstanding balance, based on i, is greater than zero, then all individuals with interest preference rate greater than or equal to i, will gain on the transaction. But that actually says quite a bit and gives some useful information.

Conclusion

In concluding this brief and elementary discussion of Professor Promislow's paper, I would like to make two observations.

First, perhaps the main point of the paper is that when attempting to compare transactions, one should keep in mind that in most cases the choice will depend on the particular individual contemplating the transaction and that an analysis on the basis of yield rates is not always possible or conclusive.

Second, some may say that for the usual cases, yield rate analysis is relatively simple and works well. However, even if that is true, it is important to specify which cases are the "usual" cases. Professor Promislow aids us in this area with the L-normal and B-normal categorizations. Moreover, example 3 is an illustration of an important situation in which yield rate analysis is deficient.

MICHAEL A. BENNETT:

I wish to thank Professor Promislow for his efforts to clarify the yield rate problems that sometimes arise in practical situations. I always have believed that it is very important to specify clearly what a yield rate is and is not supposed to be. If Professor Promislow is providing a new approach to the fundamental problem of comparing two financial transactions, as is claimed in his introductory comments, then I think that his mathematical exposition would be made much clearer with a definition of exactly what he means by one transaction as distinct from several transactions. A rigorous definition, both in mathematical terminology and in words, would be helpful, at least to me, in determining whether his examples involve one transaction, from which I would expect one yield rate, or several transactions combined, for which I would not be surprised at a result showing several possible answers.

Kellison, on page 119 of his text *The Theory of Interest,* states: "Thus, if the outstanding investment is positive at all points throughout the life of the investment, then a yield rate will be unique. However, if the outstanding investment ever becomes negative at any one point, then a yield rate is not necessarily unique."

I suggest that this condition forms the basis of the definition required for "a transaction" and that, if the outstanding investment (including the interest consideration) ever does change its sign, then more than one transaction is being considered.

Suppose we define the outstanding balance at any point r, just after the payment c_r has been made, as $B_r = \Sigma_{k=0}^{r} c_k(1+i)^{r-k}$ (following Kellison's notation). Taking c_0, the initial part of the loan payment, as a negative amount and the repayments as positive amounts, as the author has done, then the condition suggested is that $B_r < 0$ for $0 \leq r < n$ and $B_r = 0$ for $r = n$.

Taking $(1+i)^r$ outside the summation sign yields

$$B_r = (1+i)^r \sum_{k=0}^{r} c_k v^k ,$$

which we require to be negative.

That is, since $(1+i)^r > 0$ for $i > -1$, Professor Promislow's polynomial $f(v)$ is less than zero for $0 \leq r < n$, and *the first time* it becomes zero is at time n.

To make an analogy with life contingencies, if a survival function $s(x)$ is equal to $[(25-x)(100-x)]/2{,}500$, then the value of omega is 25 and is unique. The fact that 100 also produces a value $s(x) = 0$ does not mean that two satisfactory values of omega exist.

I suggest that the author may have a different definition of one transaction. If so I hope he will help me to understand his paper by stating what it is.

ROBERT L. BROWN:

I would like to thank Professor Promislow for presenting this paper for publication in the *Transactions* at this time, since it brings to the surface the importance and complexity of the theory of interest at a time when the Society is, unfortunately, deemphasizing that same topic.

As Professor Promislow says, "The theory of compound interest generally is considered to be among the most elementary topics in actuarial science. Most actuaries probably feel that they understand the subject completely. There are, however, certain paradoxes and ambiguities that suggest that some revision of the theory may be desirable." The author hits a very important nail squarely on the head.

Professor Promislow's paper may not be the best example of those paradoxes or ambiguities, however. If the paper is written for the actuarial community, then it is of limited value, since, as the author points out, "for the usual transaction that one encounters in practical situations there is a unique yield rate that, if properly interpreted, gives pertinent information about the transaction." If the paper is written for the mathematical community, then it is of limited value, since the mathematical theory contained therein is familiar to any capable mathematics undergraduate.

The real value of the paper, to me, lies in the fact that it illustrates in a clear and most interesting way just how varied and complex the theory of interest can be. As Professor Promislow states, most actuaries probably feel that they understand the topic of compound interest completely. Perhaps that is little more than proof of the adage "the more you know the more you realize how little you know."

Certainly, there is a very real danger that actuaries in the future will be desperately limited in their knowledge of the theory of interest. In an effort to satisfy the enrolled actuaries, the Society has reduced the requirement of demonstrable knowledge in the area of compound interest, at least superficially, to a level comparable to Life Office Management Association Part 6. Students of today's theory of interest need not be capable of taking a derivative or handling an integral. Continuous interest theory is virtually nonexistent. This is at a time when interest rates have dropped from around 20 percent to around 11 percent in three months and when the majority of North American financial institutions are introducing daily interest assumptions somewhere in their systems. This is also at a time when compound interest theory is becoming even more important in areas such as immunization, pension funding, and asset valuation.

It is regrettable that future actuaries will have little feel for the continuous case in the theory of interest.

Furthermore, there are several compound interest topics that have never been explored properly in the Society's education material. These include the "rule of 78," which is used widely throughout North America; the nice rule of thumb for the doubling time of investments (or the halving time because of inflation), sometimes called the "rule of 70" (i.e., time to double $= t \approx 70/i$); the method of equated time, which takes on a new level of importance when applied to a daily interest setting; and applications of interest theory to noninterest topics (e.g., exponential growth of cells).

In conclusion, the timing of this well-written paper is fortunate because it reinforces the variety and complexity of compound interest problems. Let us hope that the Society sees fit to return the theory of interest to its proper "cornerstone" position in our educational process.

RALPH E. EDWARDS:

A subject so mathematically determinable as the theory of compound interest gives the impression that by now every avenue for possible further development has been thoroughly explored and found unproductive. This excellent paper by Professor Promislow shows the extent to which that feeling departs from the truth. A serious fault in the education of actuarial candidates may be our failure to disclose how far we are from the ultimate frontiers of knowledge even in subjects of this kind.

A particular virtue of this paper is its introduction of the concept of interest preference rate. That I found it difficult to comprehend fully is not a criticism. We have colleges where this subject is new to the student and where the instruction process will determine whether or not this approach deserves perpetuation. Meanwhile, we have the advantage that something new and different has been made available.

I did find a contradiction between the statement in the paper that "the typical lender in a consumer loan transaction would not allow the loan balance to increase" and, from *TSA*, XXVI, 255, "the loan amount increased during the first four months." While the latter seems inconsequential, I hope Professor Promislow will express his view in his reply to the discussions.

MARK D. J. EVANS:

Dr. Promislow has made a valuable contribution to the theory of interest. Perhaps a few minor comments will be helpful.

Theorem 3 does not handle the common situation where the first payment is negative and all subsequent payments are zero or positive. One might put a zero at the front of T and view the transaction as if it had been initiated one time unit in the past, but this would change the critical value. Alternatively, one might propose a new theorem, Theorem 3A.

THEOREM 3A. *Let $T = (c_0, c_1, \ldots, c_n)$ be such that there exists an index k satisfying*

$$\begin{aligned} c_j &< 0\,, \quad 0 \le j \le k \\ &\ge 0\,, \quad k+1 < j < n \\ &> 0\,, \quad j = n\,. \end{aligned}$$

Then T is strongly L-normal.

In introducing Lemma 3, the author states: "The first proof of Theorem 3 can be adapted to give a very quick method of estimating the yield in a strongly L-normal transaction." Consider $T = (0, -9.59049, 1, 1, 1, 1,$

11). The yield is $i_0 = \frac{1}{9} = 0.1111$. Lemma 3 shows that $0 < i_0 < 0.5640$. Kellison's formulas (6.13) and (6.14) give $i_0 = 0.1109$ and $i_0 = 0.1105$, respectively. Lemma 3, however, should be helpful in obtaining a very broad interval containing the yield of less typical transactions.

The proof of Lemma 5, part (b), is difficult to follow. It would appear that one could define S such that $S = I_{n,j}$ without violating any of the conditions stated in the lemma. In such case one would have $P_k(S) = P_k(I_{n,j})$.

(AUTHOR'S REVIEW OF DISCUSSION)

S. DAVID PROMISLOW:

I would like to express my sincere appreciation to all the discussants of my paper.

I am indeed grateful to Mrs. Butcher for her several comments and suggestions and an illuminating analysis of some of her work with Dr. Nesbitt. Dr. Hickman has prepared an extensive discussion and provided a valuable supplement to the paper through his excellent summary of the ideas of Teichroew, Robicek, and Montalbano. In fact, the discussions of Butcher and Hickman have helped me to clarify my own thoughts on some aspects of outstanding investment analysis, on which I will elaborate at the end of the review. Mr. Chouinard has provided a useful addition to the paper by describing some of his own ideas on interest theory. Mr. Luckner has done an excellent job of illustrating and interpreting many of the concepts in the paper. Messrs. Bennett, Brown, Edwards, and Evans have all made interesting comments and asked stimulating questions.

I would now like to consider some of these points in greater detail.

It is always gratifying for the author of an actuarial paper to feel that his work may have some effect on the education and examination process. I am therefore pleased that two of the discussants have made comments along these lines. I share Mr. Brown's concern over the downgrading of interest theory on the Society examinations. Mr. Edwards also has made some very pertinent remarks concerning education.

As Mr. Evans notes, it is important to include the case $c_0 < 0$ in Theorem 3. I have done so, but possibly in a disguised form that may not be readily apparent. This occurs when the index k is equal to 0. There are now no indices satisfying the first inequality and therefore no zero payments at the beginning.

I agree with Mr. Evans that the estimate of the yield as given by Lemma 3 can be quite crude.

Mr. Edwards refers to his discussion of the paper "An Analysis of the

Rule of 78" by James H. Hunt (*TSA*, Vol. XXVI). I am not sure that I understand the contradiction to which he alludes. In the example referred to, the loan balance increases over the first four months when interest is credited according to the approximate "rule of 78." However, since the contract is one with level repayments, it certainly forms an I.P. transaction as defined in the paper. This phenomenon would seem to provide further evidence that for long-term contracts the rule of 78 may not be very accurate, one of the main points of Mr. Hunt's paper.

I would like to thank Mrs. Butcher, Mr. Evans, and Mr. Luckner for pointing out various errors in the original proofs. Mrs. Butcher notes that in the comparison of Lemma 3′ and Lemma 3 we must take $c = k - 1$ on the left side of (43) rather than $c = k + 1$ as originally printed. Mr. Evans's difficulty with part (b) of Lemma 5 arises from the fact that it was originally printed incorrectly with a strict inequality sign.

There is possible confusion concerning the form of the definition of *strong normality*. This was first brought to my attention by Donald Sondergeld. It might have been preferable to define strong L-normality by simply requiring the interest preference curve to be decreasing up to i, the unique yield. This in effect postulates that the critical value i' is $\geq i$ rather than $> i$. As Mrs. Butcher notes, it is this conclusion that is actually shown in the proof of Theorem 3. This alternative formulation is in fact equivalent to the one given. To see this, choose $i' > i$ but small enough so that the derivative of the interest preference function has no zero in the interval (i, i'). The fact that the interest preference function has a value of zero at i and is less than zero after i means that its derivative cannot be positive throughout this interval. Hence the derivative is negative and the function is decreasing on (i, i'). This shows that even with the alternate formulation the critical value is always strictly greater than the yield.

Mr. Luckner uses the phrase "yield from the borrower's point of view." It is not clear to me how to give a precise definition of this concept that would apply in all cases. Consider the interest-only option of example 7 in the paper, where the yield from the borrower's point of view is computed to be 5.66 percent. It is true that the yield on a new transaction obtained from the original one by incorporating the proposed sinking-fund arrangements is 5.66 percent, and in this case this provides an effective means of comparing the two options. It is possible, however, that in some cases our information could be insufficient to determine a new transaction. Suppose, in the same example, that instead of being given the detailed sinking-fund plans, we were told only that the borrower had an interest preference rate of 2.5 percent. I do not believe that we could then compute a yield from the borrower's point of view. However, we still could compare interest preference curves to see that the level payment option is preferable to the borrower.

In any event, replacing a transaction with another one in order to reflect the particular circumstances of some individual is an interesting concept and merits further investigation. We can look upon Mr. Chouinard's ideas in precisely this way. He takes a transaction T and an interest rate j and obtains a new transaction by accumulating all positive payments to the end of the term at rate j. Let us denote this new transaction by T'. It is always B-normal, so it has a unique yield i_T, which Mr. Chouinard calls the retrospective yield. Note that

$$P_j(T) = P_j(T') .$$

I would now like to give a generalization and extension of Mr. Chouinard's remarks on comparison. Let S and T be two transactions with exactly the same negative payments. Suppose that j' is a critical reinvestment rate. Then S' and T' have the same yield, but since they differ only in the final payment, we must have $S' = T'$. Using the above formula, we see that $P_{j'}(S - T) = P_{j'}(S' - T') = 0$. That is, as noted by Mr. Chouinard in a special case, j' is also a critical interest preference point. Consider now the question of what we can say about $-S$ and $-T$ in this case. They will have exactly the same *positive* payments. From Chouinard's formula (3) we deduce an interesting reciprocity property. If i is a retrospective yield of T at reinvestment rate j, then j is a retrospective yield of T at reinvestment rate i. Combining this with our previous result, we see that $-S$ and $-T$ will have a critical reinvestment rate equal to the common value of i_S and i_T at rate j', and, at this reinvestment rate, the common value of i_{-S} and i_{-T} is j'. We illustrate with the given example. Let

$$T = (-10, 9, 9) , \qquad S = (-10, 4.5, 14.5) .$$

Then S and T have a critical reinvestment rate of $\frac{2}{9}$, as shown by Mr. Chouinard, and at that rate $i_S = \sqrt{2} - 1$; similarly, $-S$ and $-T$ have a critical reinvestment rate of $\sqrt{2} - 1$, and at that rate $i_{-S} = i_{-T} = \frac{2}{9}$.

Mr. Bennett has raised a number of points to which I would like to reply. In the first place, I do not agree with the survivor function analogy. In that case, ω is clearly unique, since by definition it is the *smallest* zero of the survivor function. This is precisely the definition we want for the use we make of ω. One could achieve uniqueness of yield similarly by simply choosing the smallest such value, but in this case there would be no meaning or purpose to the result. In any event, the given example of a survivor function needs modification, as it produces negative values of $s(x)$ for $25 < x < 100$.

In reply to Mr. Bennett's question concerning the definition of a transaction, I do not believe that there is any other reasonable alternative. I view a transaction as simply a sequence of payments made at various points of time. If we simplify the definition by assuming finiteness of duration and periodicity, we necessarily are led to the definition given in Section II of the paper, where a transaction is represented by a vector.

I do not see how in general we can distinguish between a single transaction and a combination of several transactions as Mr. Bennett suggests. If we simultaneously undertake each of a set of several transactions, the result of this combined operation is itself a single transaction, represented mathematically as the vector sum of its components.

It is true, however, that one intuitively feels that there are certain basic transactions, typical of those normally encountered, having unique yields and such that every transaction is a combination of basic ones. The problem is to give a precise definition of this class. I have attempted to do so in the paper by defining normal transactions. There are, however, other possibilities. Mr. Bennett suggests considering transactions for which the outstanding investment does not change sign. This idea needs elaboration, for, as indicated in the paper, outstanding investments depend on a particular value of i.

To simplify the terminology we will call a transaction T *pure at rate* i if the outstanding investment computed at that rate does not change sign; that is, formula (20) of the paper holds for either T or $-T$. (Depending on which alternative holds and on one's orientation, T would be called a contract of pure financing or a contract of pure investment in the terminology used by Mrs. Butcher and Dr. Hickman).

Modifying Mr. Bennett's suggestion, we can consider transactions T satisfying the following property (property $*$):

T is pure at rate i, where i is a yield rate of T.

Such transactions are precisely the content of the Butcher-Nesbitt analysis. Their work, together with Theorem 4 of the paper, gives us considerable information about this class, which we will now describe.

Let T be any transaction with at least one yield, and let i_1 be the maximum such yield. Let i_0 be the minimum value of i for which T is pure. Section VIII of the paper shows that i_0 exists (formula [23]) and that T is pure for all $i \geq i_0$ (formula [21]). (Note that $(1 + i_0)$ is the u_0^* of the Butcher-Nesbitt case 3. Moreover, in the terminology used by Dr. Hickman, $\{i \geq i_0\}$ is the pure financing region in the case of nonnegative outstanding investments, while $\{r \geq i_0\}$ is the pure investment region in the case of nonpositive outstanding investments. From Theorem 4 of the paper, it is not hard to see that $i_1 > i_0$ implies that T is normal. In the Butcher-Nesbitt theorem, this occurs in case 1 and the first two subcases of case 3. If T is not normal, we must have $i_1 < i_0$ and there can be no yield at which T is pure; that is, T does not satisfy property * above. In particular, this occurs in the Butcher-Nesbitt case 2. An even number of sign changes clearly implies nonnormality, as $P_i(T)$ has the same sign for both high and low values of i.

Consider now the converse question. If T is normal, is i_0 less than i_1? The answer is yes for the case of a single sign change (Butcher-Nesbitt case 1), but not in general. We can take, for example, the transaction

$$T = (7, -24, 24, -8) ,$$

which exhibits three sign changes. Since

$$P_i(T) = 8[1 - (1 + i)^{-1}]^3 - 1 ,$$

we see that T is B-normal (in fact, strongly B-normal) with a unique yield of 100 percent. On the other hand, we have $i_0 = \frac{17}{7}$.

To summarize, we see that the class of transactions satisfying property * is a proper subclass of the normal transactions. These transactions will be L-normal or B-normal, respectively, depending on whether the outstanding investments at the yield rate are less than zero or greater than zero. This class includes the prototype normal transactions involving a single sign change. From Section VII of the paper, it then follows that every transaction can be written as a sum of transactions satisfying property *.

It may be instructive to compute the quantities $B^n_{i,r}(T)$ and $r(i)$, as defined in Dr. Hickman's discussion, for the example T given above.

For $i \geq \frac{17}{7}$ (the pure region),

$$B^3_{i,r}(T) = 7i^3 - 3i^2 - 3i - 1 ,$$

which we note is independent of r.

For $i < \frac{17}{7}$ (the mixed region),

$$\begin{aligned} B^3_{i,r}(T) &= (7i - 17)(1 + i)(1 + r) + 24i + 16 && \text{if} \quad (1 + r) \leq 24/(17 - 7i) \\ &= (7i - 17)(1 + r)^2 + 24r + 16 && \text{if} \quad (1 + r) > 24/(17 - 7i) . \end{aligned}$$

There is no value of r for which the second expression above equals zero and such that $(1 + r) > 24/(7 - 7i)$. Setting the first expression equal to zero and solving for r, we obtain

$$r(i) = \left[\frac{24}{17 - 7i} - \frac{8}{(17 - 7i)(1 + i)} - 1\right].$$

In this particular case, $[r(i) - i]$ is increasing and so $r(i) = i$ only for $i = 1$. In general, however, as noted by Dr. Hickman, the curve $r(i)$ can cross the line $r = i$ in several places within the mixed region.

In conclusion, I would like to thank again all those who discussed my paper.