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GEOMETRIC SOLUTIONS TO STATIONARY POPULATION PROBLEMS

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ABSTRACT

Conventional methods for solving stationary population problems require integration and multiple integration of mortality functions. Solutions to these problems customarily are expressed as algebraic combinations of certain basic functions that are defined by integrals.

This paper presents a method that uses three-dimensional graphs to solve stationary population problems. For each problem, a graph is prepared by using a systematic technique. Relevant quantities are then visualized on the graph as areas and volumes that correspond to the integrals of the conventional methods. Finally, such areas and volumes are cut up into areas and volumes that correspond to the basic functions customarily used to express solutions. Thus, solutions expressed as algebraic combinations of basic functions are obtained geometrically without resort to integrations.

Several examples illustrate the use of this geometric method on a broad range of typical problems. It appears to be more concrete and comprehensible than conventional methods.

I. AN ANIMATION OF THE DETERMINISTIC SURVIVORSHIP MODEL

CLASSICAL life contingency theory considers a hypothetical group of l_0 newborn lives. The number that survive to age x is denoted by l_x (see Fig. 1).

This paper is based on an interpretation of the graph of l_x, the survivorship curve, as illustrated in Figure 2. Visualize a group of l_x lives, now aged x, as a totem pole of height l_x. As time goes by and the lives in the group grow older, let the totem pole travel to the right (henceforth called *east*). At the same time, let the survivorship curve descend upon the group, so that they die in the order that the curve touches them; the top one dies first, the bottom one dies last. *This interpretation will be followed throughout this paper.*

An immediate consequence of this interpretation is that T_x, the area in

Figure 3 under the survivorship curve and east of the l_x pole, represents the total future lifetimes of the l_x lives now aged x. In the conventional approach to population theory,

$$T_x = \int_x^\infty l_t dt ,$$

which is visually evident in Figure 3.

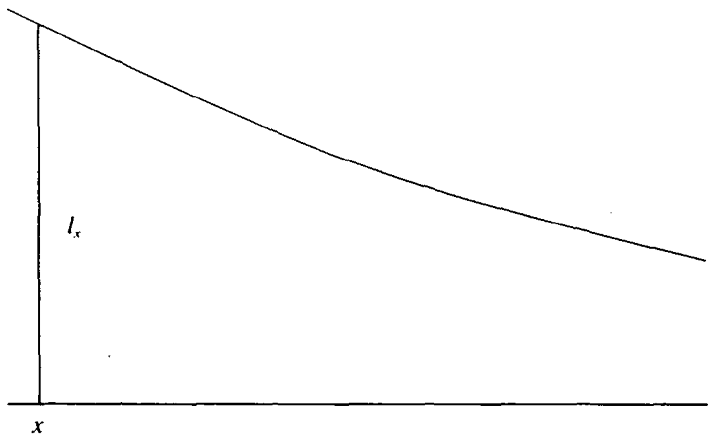

FIG. 1

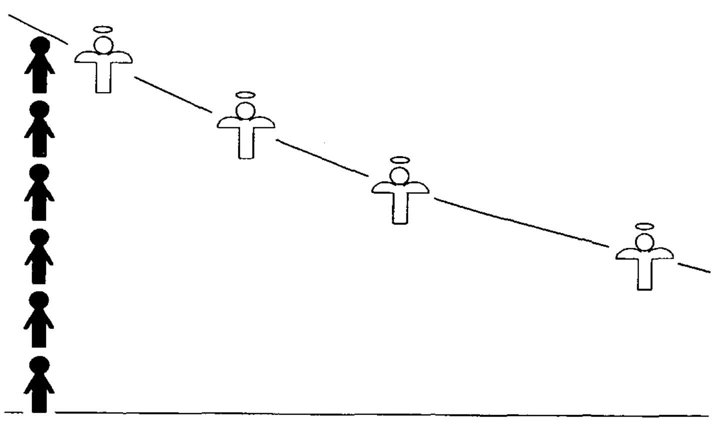

FIG. 2

EXAMPLE 1. *Find the average age at death for those who survive to age x but die before age $x + n$* [6, p. 176].

In Figure 4, of the l_x lives on AB, only those on the segment AI will die before age $x + n$. There are $l_x - l_{x+n}$ such lives. Their future lifetime after age x is represented by the area of AIC. In order to express this area in terms of functions customarily used in the solution of population problems,

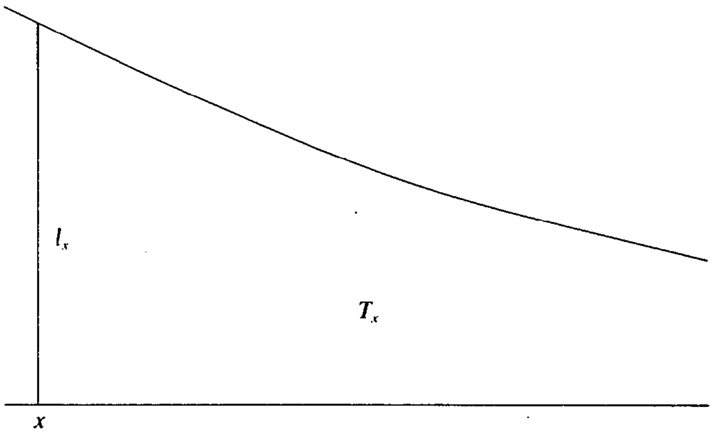

FIG. 3

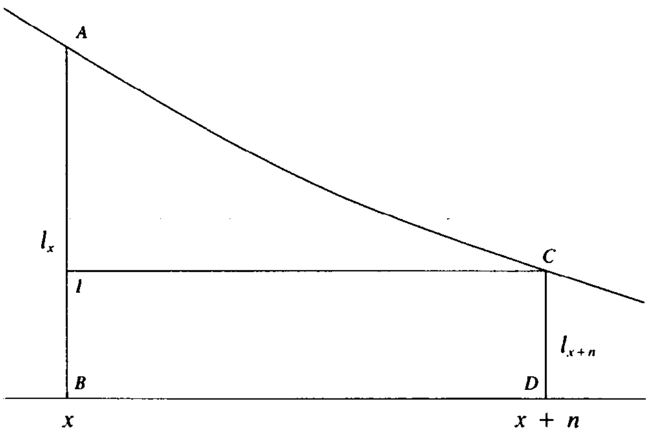

FIG. 4

the area of *AIC* is visualized as the area to the east of *AB*, minus the area to the east of *CD*, minus the area of *IBDC*; that is,

$$T_x - T_{x+n} - nl_{x+n} .$$

Thus,

$$\text{Average age at death} = x + \frac{\text{Lifetime after age } x}{\text{Number of lives}}$$

$$= x + \frac{T_x - T_{x+n} - nl_{x+n}}{l_x - l_{x+n}} .$$

II. VISUALIZATION OF THE STATIONARY POPULATION

The curve l_x can be used to build a population model consistent with the survivorship model described above. A population of lives reproducing and aging over time is said to be a *stationary population* if (1) there is no migration other than by birth or by death, (2) the relative age distribution in the population remains the same at all times, and (3) the total number of lives in the population is constant. It can be demonstrated that, in a stationary population, the curve l_x represents the age distribution histogram at any time as well as the survivorship experience of each cohort group. On this basis, one can visualize the stationary population in a way that is consistent with the survivorship model described in Section I.

Since the population is stationary, the number of lives now aged $x + n$ equals the number of lives that will be aged $x + n$ after n years have passed. The latter group will be the survivors after n years of those now aged x. Figure 5 presents these relationships visually. Figure 5 contains Figure 4 on plan $ABCD\infty$, which describes the future mortality of those now aged x. A totem pole, *EF*, of height l_{x+n}, and located n units due south of *CD*, represents lives now aged $x + n$.

Figure 6 systematically extends the process, representing lives now aged $x + t$ due south of, and t units away from, the lives on $ABCD\infty$ that will be aged $x + t$ after t years. Thus, plane $ABEF\infty$, which makes a 45° angle with plane $ABCD\infty$, represents the total population now aged x and above. For example, *ABFE* in Figure 6 represents lives now between ages x and $x + n$.

In these figures, units on the ground represent years, height above the ground represents numbers of lives, directions are named east, south, west, and north, and each cohort of lives moves east through time. Figures 1–4 can be regarded as east-west cross-sections of Figure 6.

In Figure 6 the curve AC is the l_x curve. The top surface is generated by moving the l_x curve south and cutting the surface off along the northwest-southeast plane passing through AB. The future mortality experience of the group of lives now aged x and above, represented by plane $ABEF\infty$, will be decided by the surface in Figure 6 as these lives move east through time in the manner described in Figure 2.

An immediate consequence of this interpretation is that Y_x, the wedge of volume under the surface in Figure 6 and east of $ABEF\infty$, represents the total future lifetimes of the lives now aged x and above. In the conventional approach to population theory,

$$Y_x = \int_x^\infty T_t dt \, .$$

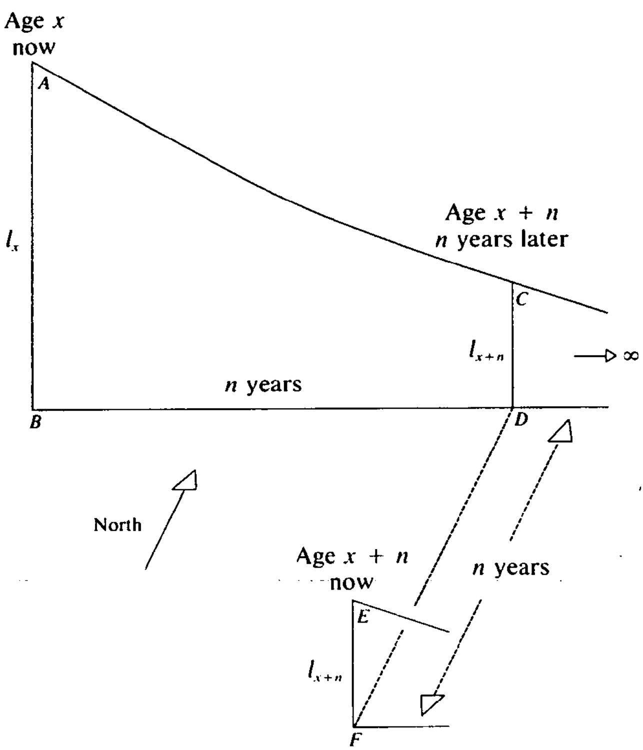

FIG. 5

Figure 6 makes this visually evident because $ABCD\infty$ represents T_x.

A second consequence of this interpretation is that the number of lives represented on a given part of plane $ABEF\infty$ equals the area of the corresponding part on plane $ABCD\infty$ after projection from $ABEF\infty$ due north to $ABCD\infty$. For example, the group of lives now between ages x and $x + n$, represented by $ABFE$, contains lives equal to the area of $ABDC$, or $T_x - T_{x+n}$. This projection principle follows from the dual role of the curve l_x in the stationary population model, representing both cohort survivorship and the age distribution histogram in the population.

EXAMPLE 2. *Find the average age at death of those in the stationary population who are now living between ages x and x + n* [6, p. 188, exercise 20(a)].

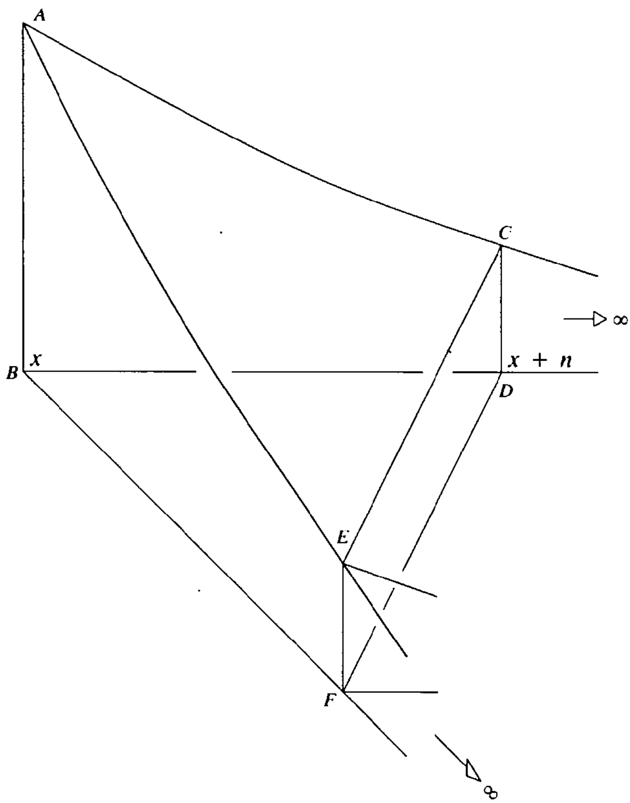

FIG. 6.—Y_x is the volume east of $ABEF\infty$

The group of lives now living between ages x and $x + n$ is represented by *ABFE* in Figure 7. By the projection principle, the number of lives equals the area of *ABDC*, which is $T_x - T_{x+n}$. All of these lives survive to age x. The past lifetime from age x until now for the group of lives now living between ages x and $x + n$ is represented by the volume *ABEFGH*, since *ABGH* is the plane of constant age x.

Figure 7 is symmetric with respect to the ABEF plane; the volume of *ABEFGH* equals the volume of *ABEFCD*. In order to express this volume in terms of functions customarily used in the solution of stationary population problems, *ABEFCD* can be visualized as the volume of the 45° wedge east and southeast of *AB*, minus the volume of the 45° wedge east and southeast of *EF*, minus the volume east of *EFCD*; that is,

$$Y_x - Y_{x+n} - nT_{x+n} .$$

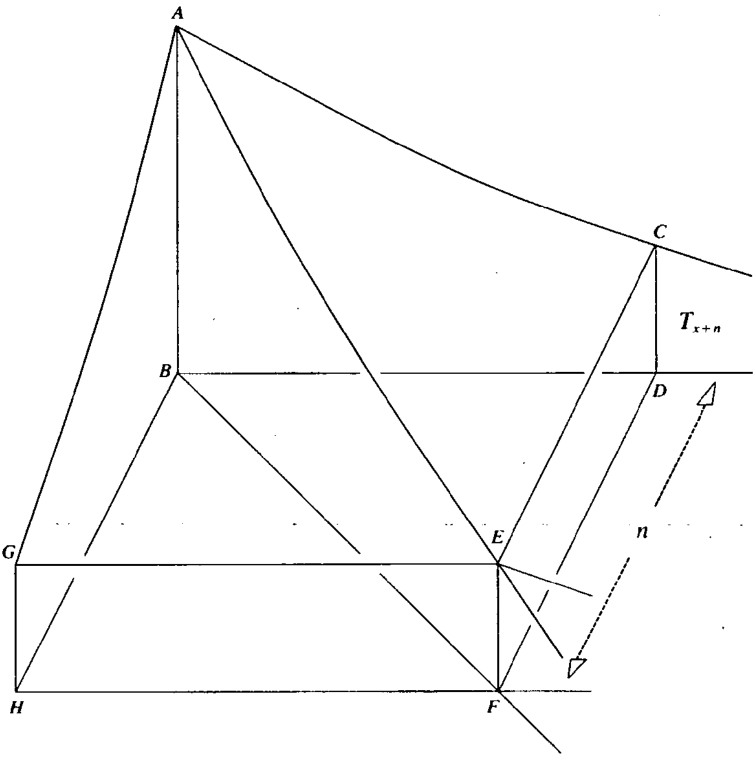

FIG. 7

The future lifetime of the group of lives now living between ages x and $x + n$ is represented by the volume to the east of $ABFE$. In order to express this volume in terms of functions customarily used in the solution of stationary population problems, it can be visualized as the volume of the 45° wedge east and southeast of AB, minus the volume of the 45° wedge east and southeast of EF; that is, $Y_x - Y_{x+n}$. Thus,

$$\text{Average age at death} = x + \frac{\text{Lifetime after age } x}{\text{Number of lives}}$$

$$= x + \frac{\text{Past lifetime since age } x + \text{future lifetime}}{\text{Number of lives}}$$

$$= x + \frac{Y_x - Y_{x+n} - nT_{x+n} + Y_x - Y_{x+n}}{T_x - T_{x+n}} .$$

EXAMPLE 3. *Find the average age at death of those in the stationary population now living between ages x and x + n and who will die before attaining age x + n* [6, p. 188, exercise 20(b)].

As in the previous examples, all quantities related to the solution of this problem will be visualized. The group is represented on Figure 8 by AIE; the IE boundary reflects death before age $x + n$. Projecting north, the number of lives in this group equals the area of AIC, which equals

$$T_x - T_{x+n} - nl_{x+n} .$$

The total lifetime of the group after age x is represented by the volume of $AGECI$. This can be visualized as twice the volume of $ABEFCD$ minus the volume of $IBGHEFCD$, that is,

$$2(Y_x - Y_{x+n} - nT_{x+n}) - n^2 l_{x+n} .$$

As usual,

$$\text{Average age at death} = x + \frac{\text{Lifetime after age } x}{\text{Number of lives}}$$

$$= x + \frac{2(Y_x - Y_{x+n} - nT_{x+n}) - n^2 l_{x+n}}{T_x - T_{x+n} - nl_{x+n}} .$$

EXAMPLE 4. *Find the average age at death of those persons now living between ages 20 and 70 who die between ages 60 and 80* [6, p. 184, example 2].

In Figure 9 the group is represented by *PQYXW.* Boundaries *PW* and *QY* reflect death between ages 60 and 80, while *PQ* and *XY* reflect ages now between 20 and 70. Projecting north, the number of lives in this group equals the area of *PQSTR,* which equals the area under *PRT* minus the area under *QS;* that is,

$$[40l_{60} + (T_{60} - T_{70}) - 50l_{80}] \ . \qquad \text{(i)}$$

The total lifetime of the group after age 20 could be represented by the volume of a certain block outlined in Figure 9. However, since the entire group will live to age 60, for the purpose of finding their average age at death it suffices to consider their total lifetime after age 60. This lifetime is represented by the volume to the east of *RZVUW* in Figure 9. To express the volume of this attic in customary terms, it can be visualized as the

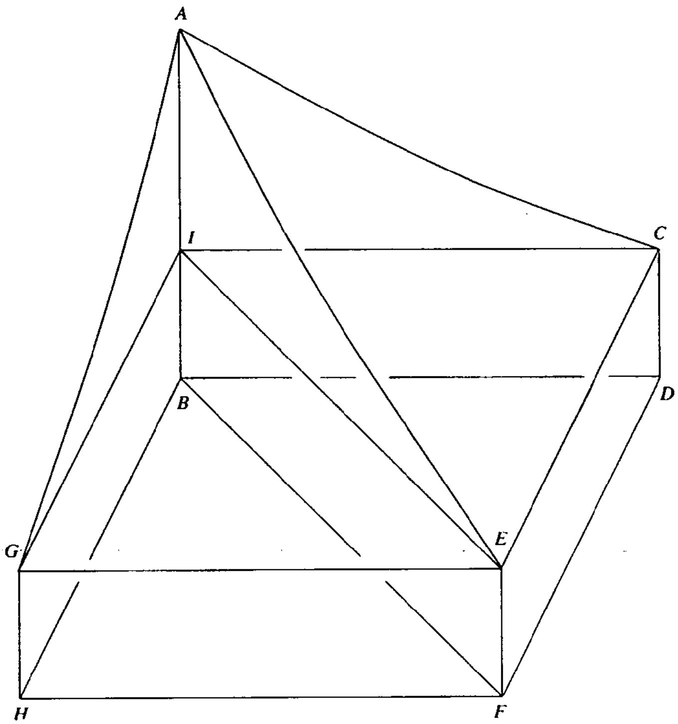

FIG. 8

volume of the house minus the volume of the ground floor; that is, the volume above *GDEF*, plus that above *BCDG*, plus that above *ABG*, minus the volume of the ground floor; that is,

$$[40(T_{60} - T_{80})] + [Y_{60} - Y_{70} - 10T_{80}] + [Y_{60} - Y_{70} - 10T_{70}] - [20(50l_{80})] . \qquad \text{(ii)}$$

As usual,

$$\text{Average age at death} = 60 + \frac{\text{Lifetime after age 60}}{\text{Number of lives}}$$

$$= 60 + \frac{\text{(ii)}}{\text{(i)}} .$$

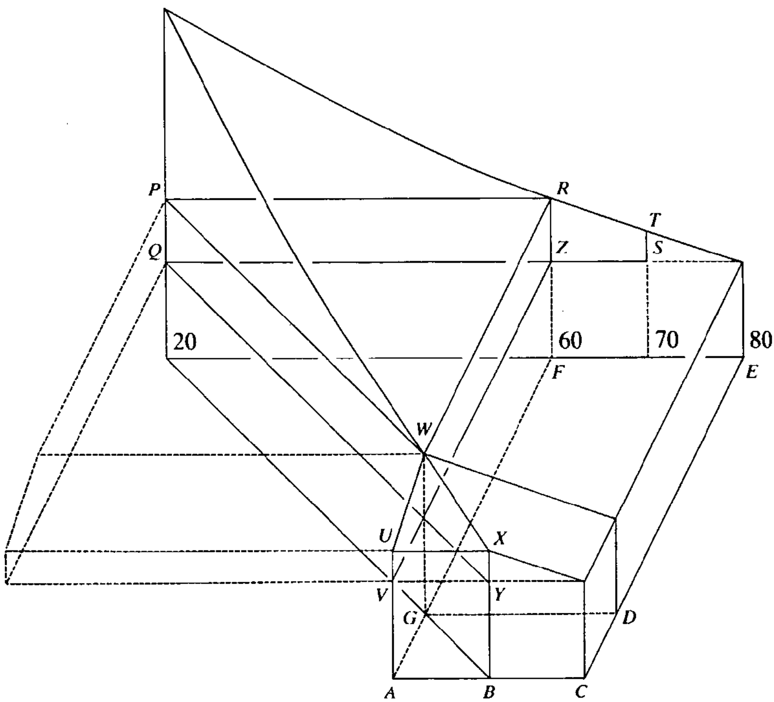

FIG. 9

EXAMPLE 5. *Find an expression for the average attained age of those persons in a stationary population now between ages 25 and 40 who will die between the ages of 30 and 50 within the next twenty years* [1, p. 55].

The group now is representd by *KLTZYS* on Figure 10. The boundaries *KS* and *TZ* reflect death between ages 30 and 50, while *KL* and *YZ* reflect ages now between 25 and 40. The boundary *LT* reflects death within twenty years. It is a projection due west of *IJ*, the intersection of the surface with the plane of time twenty years from now. Projecting north, the number of lives in this group equals the area of *KLRXWQ*, which equals the area under *KQW*, minus the area under *LRX*; that is,

$$[5l_{30} + (T_{30} - T_{40})] - [(T_{45} - T_{50}) + 10l_{50}] . \qquad \text{(i)}$$

The total past lifetime of this group since age 25 is represented by the volume of the 45° wedge between *KLNPOM* and *KLTZYS*. To express this volume

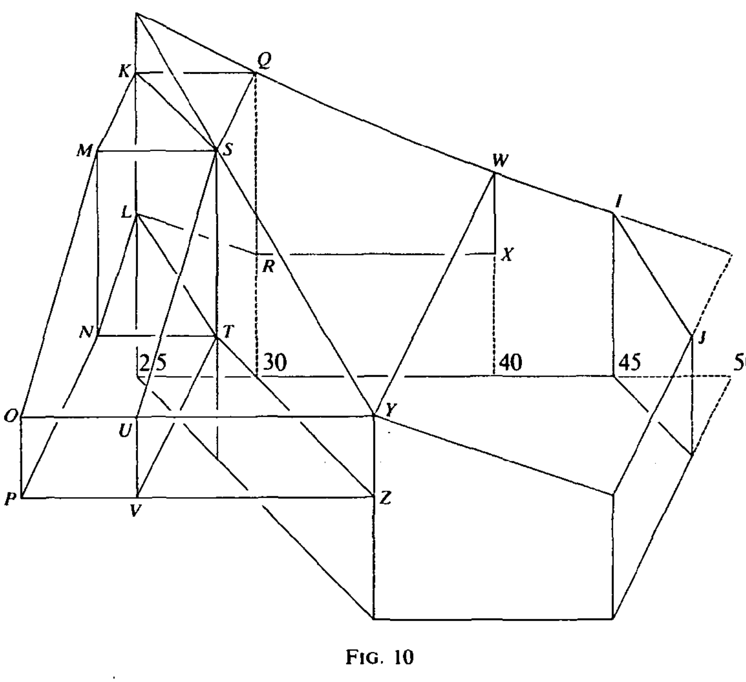

FIG. 10

in customary terms, it can be visualized as the sum of volumes between

$$\left.\begin{array}{l} KLNM \text{ and } KLTS\text{: } \frac{5\cdot 5}{2} l_{30} - (Y_{45} - Y_{50} - 5T_{50}) \\ MNPO \text{ and } STVU\text{: } +5(T_{30} - T_{40} - 10l_{50}) \\ STVU \text{ and } STZY\text{: } +Y_{30} - Y_{40} - 10\cdot T_{40} - \frac{10\cdot 10}{2} l_{50} \end{array}\right\} \quad \text{(ii)}$$

Thus,

$$\text{Average attained age} = 25 + \frac{\text{Past lifetime since age 25}}{\text{Number of members}}$$

$$= 25 + \frac{\text{(ii)}}{\text{(i)}} .$$

III. THE GEOMETRIC METHOD

In each of the preceding examples, visualization of the problem produces the answer directly upon inspection of the figure. The conventional method of integration, on the other hand, is based on the following formulas [6, pp. 170–71]:

$$\int_a^b l_x \mu_x dx = l_a - l_b , \quad \text{(i)}$$

$$\int_a^b l_x dx = T_a - T_b , \quad \text{(ii)}$$

$$\int_a^b T_x dx = Y_a - Y_b , \quad \text{(iii)}$$

$$\int_a^b x l_x \mu_x dx = a l_a + T_a - b l_b - T_b , \quad \text{(iv)}$$

$$\int_a^b x l_x dx = a T_a + Y_a - b T_b - Y_b . \quad \text{(v)}$$

The number of lives and their lifetime are expressed as integrals and evaluated by the above five formulas.

Can the geometric method solve all problems that can be solved by applying these integration formulas? The answer is *yes*. It is an easy exercise to visualize these formulas by using the method presented in the preceding sections. Formulas (iii) and (v) require the three-dimensional figures of Section II. The other three formulas require only the two-dimensional figures presented in Section I. Since these formulas, the foundation of the integration method, can be proved and visualized by the geometric method, applications of these integration formulas can be replaced by direct visualization on properly prepared figures.

Many authors have contributed methods for solving stationary population problems. One of the most useful methods, suggested by Grace and Nesbitt [5], requires the definition of two auxiliary functions,

$$F_x = xl_x + T_x , \quad G_x = xT_x + 2Y_x .$$

Here F_x represents the total past and future lifetime of the l_x lives now living at age x [6, p. 176, line 4], and G_x represents the total past and future lifetime of the T_x lives now living at age x and above [6, p. 181, line 6].

The following integration formula follows easily from (iii) and (v):

$$\int_a^b F_x dx = G_a - G_b . \qquad \text{(vi)}$$

Figure 11 provides a visual interpretation of F_x and G_x. With this interpretation, formula (vi) can be visualized. The geometric method begins with visualizing the relevant quantities as areas and volumes on the graph, and then recognizing these areas and volumes as combinations of l_x, T_x, and Y_x. Now the method can be strengthened by recognizing these areas and volumes as combinations of F_x and G_x. This point will be illustrated in Section IV.

The idea of analyzing stationary population problems with three-dimensional figures has been studied by Tino [10]. In her presentation, the figures are drawn over the Lexis diagram [8], the customary diagram of mathematical demography [2, 4, 7, 9]. Figure 12 compares the Lexis diagram with the ground layout used in this paper. Since the Lexis diagram and the ground layout of this paper are related by linear transformations, the figures in [10] and the figures in this paper are related by linear transformations on the ground; the two methods are theoretically equivalent. As an illustration of this equivalence, compare Figure 13, a visualization of Example 3 by Tino's method, with Figure 8.

In spite of their theoretical equivalence, there is a crucial practical difference between Figure 8 and Figure 13. While the volume of *AGEI* and the volume of *AECI* are equal in Figure 13 by *computation*, they are equal in

Figure 8 by *symmetry*. Blocks of equal volume are *seen* to be equal in the figures of this paper; solutions become *visible*.

The geometric method presented in this paper uses three-dimensional figures. Lifetimes are represented concretely by volumes (triple integrals), which can be expressed in customary terms simply by inspection of the figures.

The method of Veit [12] uses Lexis diagrams. Lifetimes are found by analyzing the two-dimensional Lexis diagrams and performing double integrations over them, or by general reasoning. The information contained in the suppressed third dimension is handled analytically (in the setting-up of double integrals) or mentally (by general reasoning).

The method of Grace and Nesbitt [5] implicitly uses one-dimensional diagrams (age) and reaches solutions by performing single integrations. The information contained in the suppressed dimensions must be manipulated mentally, unless it happens to be reflected directly in the auxiliary functions, F_x and G_x.

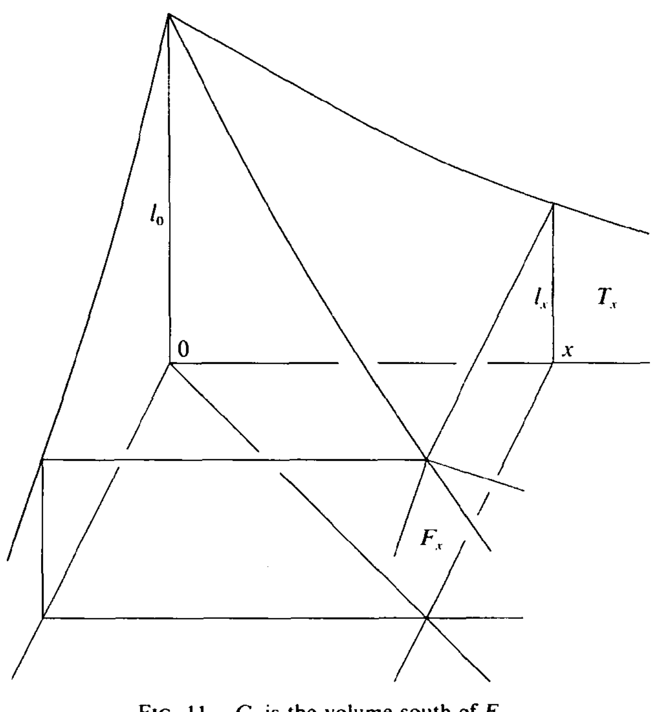

FIG. 11.—G_x is the volume south of F_x

The method of Batten [1] uses no diagrams (zero-dimensional diagram) and no integration. It reaches solutions solely by general reasoning. All information must be handled mentally.

The geometric method presented in this paper provides detailed, concrete representations of all groups and lifetimes involved in stationary population problems. It reaches the customary solution solely by inspection of the figures. Regardless of which method one prefers to use, this geometric method can be a valuable supplement. For problems complicated enough

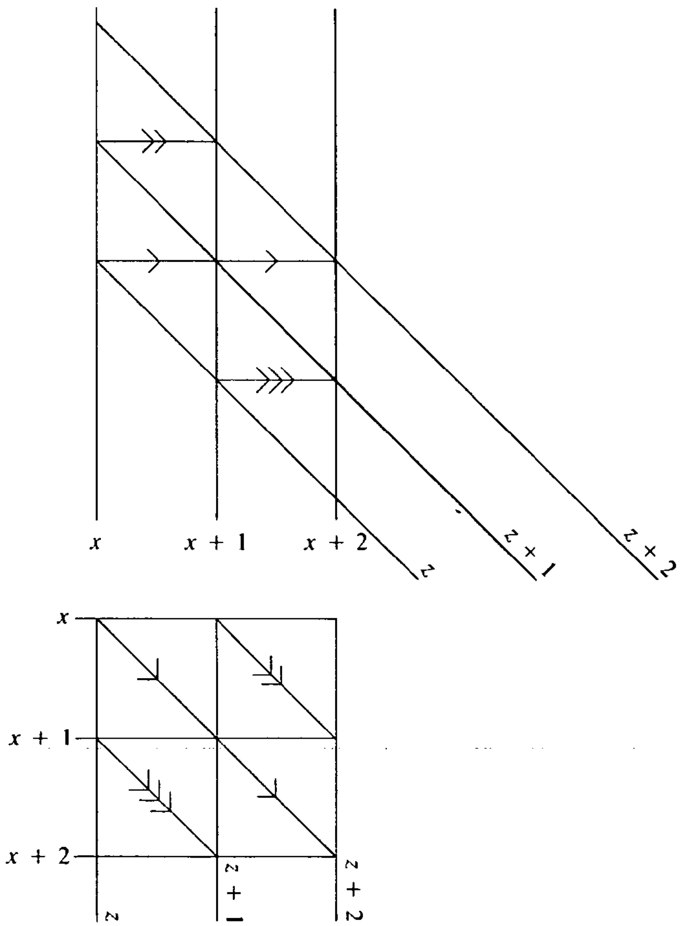

FIG. 12.—*Top:* Ground layout in this paper; *bottom:* Lexis diagram

to confound the mental or analytical manipulation of suppressed information, a full three-dimensional diagram can restore comprehension. Even for simpler problems, mental reference to the geometric method can help with the required manipulation of suppressed information.

IV. ADDITIONAL EXAMPLES

EXAMPLE 6. *Find the total lifetime of those in the stationary population now living between ages 20 and 70 who will die between ages 60 and 80 within the next fifty years* [6, p. 188, exercise 25].

The group is represented in Figure 14 by *POMYXW*. The boundary *OM* reflects death within fifty years. It is a projection due west of *TZ*, the intersection of the surface with the plane of time fifty years from now. The number of lives in this group equals the area of *PONSTR*, which equals the

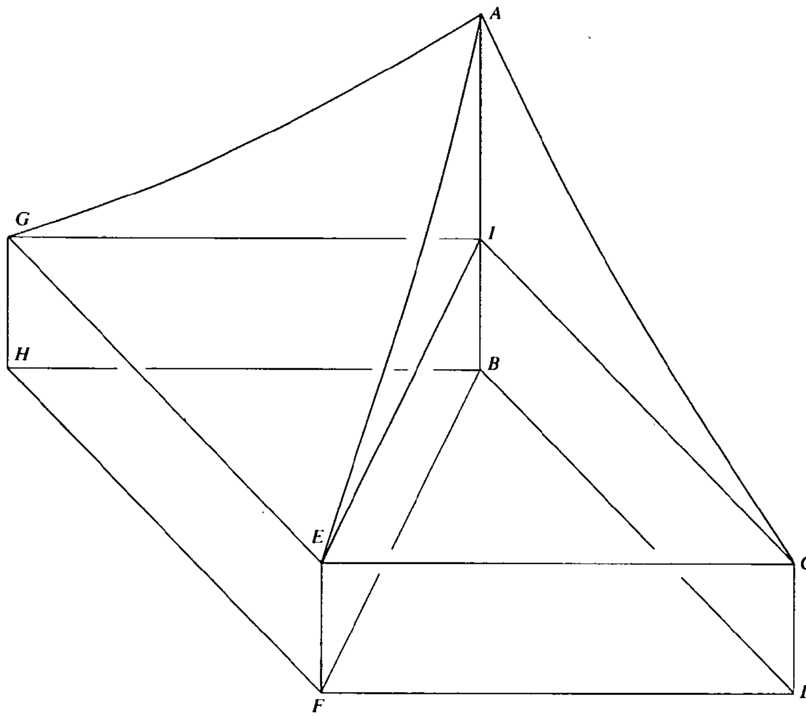

FIG. 13

area under *PRT* minus the area under *ONS*, that is,

$$[40 l_{60} + (T_{60} - T_{70})] - [(T_{70} - T_{80}) + 40 l_{80}] \,. \qquad \text{(i)}$$

Since the entire group will live to age 60, it suffices to consider their total lifetime after age 60. This lifetime is represented by the volume to the east of *RKLVUW*. As an exercise, one may express this volume in customary terms, following Example 4.

It is also possible to express the total lifetime of this group in terms of F_x and G_x. *At their sixtieth birthday,* this group is represented by *RKLVUW*. Their total lifetime equals

$$[40 F_{60} + (G_{60} - G_{70})] - [(G_{70} - G_{80}) + 40 F_{80}] \,, \qquad \text{(ii)}$$

that is, the total lifetime of those under *RWU*, minus the total lifetime of those under *KLV*.

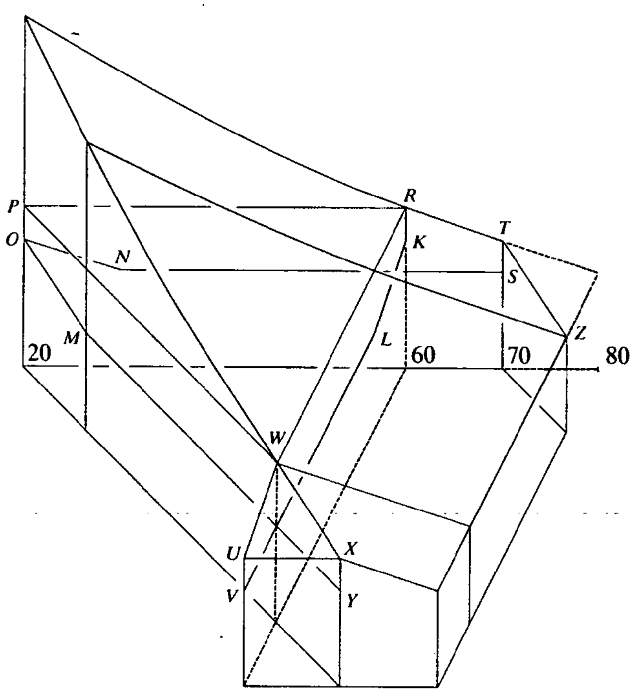

FIG. 14

Expression (ii) can also be obtained from expression (i) by replacing l_x with F_x, and T_x with G_x, as predicted by Grace and Nesbitt [5, p. 71]. In fact, Figure 14 serves as a geometric illustration of the Grace-Nesbitt principle [6, p. 185].

EXAMPLE 7. *Find the number of deaths in 1966–70 for those between ages 50 and 60 on January 1, 1960, who die between ages 65 and 75 in the given five-year span; and find the total of their ages at death* [3, p. 91, problem 8(c), (d)].

Figure 15 provides the ground layout of the problem. Since all lives in the group survive to age 65, it suffices to consider their lifetime after age 65. Figure 16 illustrates the mortality experience of the group *only after their sixty-fifth birthday.* At their sixty-fifth birthday, this group is represented by *PQRS*. The boundary *PQ*, which is parallel to the ground at a height of l_{65}, reflects death after age 65; *QR* reflects death after the start of 1966 and is a projection due west from *QT*. Similarly, *PS* reflects death before the start of 1971 and is a projection from *PU*. The number of lives in this group equals the area under *PQR*, minus the area under *PS;* that is,

$$[5l_{65} + (T_{65} - T_{66})] - [(T_{65} - T_{71})] \,. \qquad \text{(i)}$$

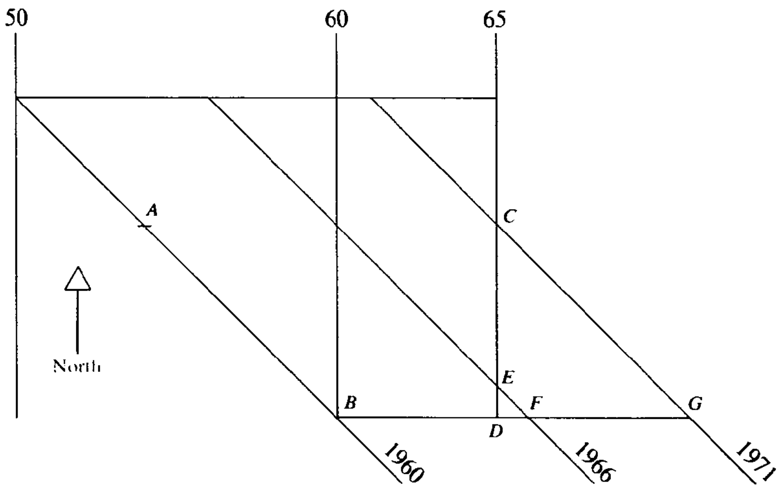

FIG. 15

Their total lifetime equals the total lifetime of those under PQR, minus the total lifetime of those under PS; that is,

$$[5F_{65} + (G_{65} - G_{66})] - [(G_{65} - G_{71})] \,. \qquad \text{(ii)}$$

Expression (ii) can also be obtained from expression (i) by replacing l_x with F_x, and T_x with G_x. Thus, Figure 16 gives another geometric illustration of the Grace-Nesbitt principle. In fact, six out of the seven examples (all except Example 5) in this paper can be handled efficiently by the Grace-Nesbitt principle.

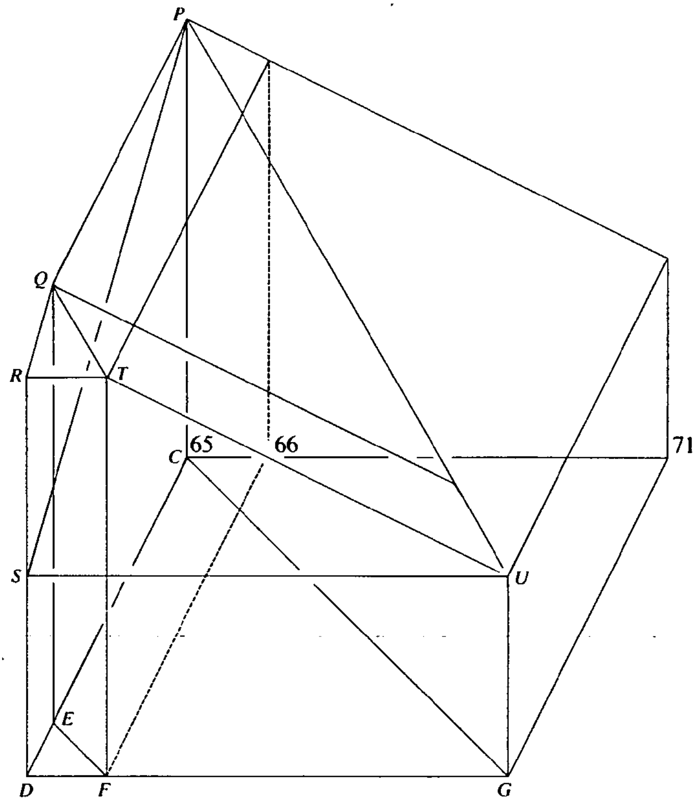

FIG. 16

V. CONCLUSION

Although the stationary population is an idealized model, it provides a starting point for analysis in population theory [7, 9], and in the theory of pension funding [11]. The geometric method presented in this paper provides a visual aid to the comprehension of population problems.

Under stationary assumptions, inspection of the figures alone provides the solutions to such problems. Considering the time and effort incurred in the preparation of the figures, however, the geometric method seems unlikely to be the most efficient method. It could be the most concrete method and is more of an educational and analytical tool than a computational wizardry.

Under nonstationary assumptions, the three-dimensional figures of the geometric *method* can no longer provide solutions. The geometric *format*, however, can still be very useful. Three-dimensional animations along the lines of Figure 2 can transform some dryly computed numerical data into lively illustrations. With the reduced cost of computer graphics hardware, three-dimensional animation using the geometric format is becoming more feasible.

REFERENCES

1. BATTEN, R. W. "A Logical Approach to Population Problems," *TSA*, XXI (1969), 49–58.
2. ———. *Mortality Table Constructions*. Englewood Cliffs, N.J.: Prentice-Hall, 1978.
3. CROFTS, G. *Problem Manual in Life Contingencies*. Rev. ed. Boston, Mass.: Northeastern University, 1973.
4. GERSHENSON, H. *Measurement of Mortality*. Chicago: Society of Actuaries, 1961.
5. GRACE, W. L., and NESBITT, C. J. "Actuarial Note: On Average Age at Death Problems," *TSA*, II (1950), 70–74.
6. JORDAN, C. W. *Life Contingencies*. 2d ed. Chicago: Society of Actuaries, 1967.
7. KEYFITZ, N. *Introduction to the Mathematics of Population*. Reading, Mass.: Addison-Wesley, 1968.
8. LEXIS, W. *Einleitung in die Theorie der Bevölkerungs-Statistik*. Strasbourg: Trübner, 1875. (English translation of an excerpt appeared on pp. 39–41 of D. SMITH and N. KEYFITZ, *Mathematical Demography* [Berlin: Springer-Verlag, 1977].)
9. SPEIGELMAN, M. *Introduction to Demography*. Rev. ed. Cambridge, Mass.: Harvard University Press, 1968.
10. TINO, P. Discussion of K. P. VEIT, "Stationary Population Methods," *TSA*, XVI (1965), 253–62.
11. TROWBRIDGE, C. L., and FARR, C. E. *The Theory and Practice of Pension Funding*. Homewood, Ill.: Richard D. Irwin, 1976.
12. VEIT, K. P. "Stationary Population Methods," *TSA*, XVI (1965), 233–43.

DISCUSSION OF PRECEDING PAPER

PAULETTE TINO:

With the publication of Dr. Chan's paper, it is now well established that the solution of any stationary population problem can be derived from the information read on the representation of the population by the area under the l_x curve. This results from the fact that on this representation can be read the past and the future of the $l_x dx$ members now aged x, including the designation of who will die within stipulated ages and/or within a given period. It follows that the latter problem (number of deaths) can be solved on the same representation for a group defined by an age bracket or otherwise.

The calculation of a group's past and future lifetime, keeping the same approach, leads to three-dimensional developments. The choice of the path along which the population ages is arbitrary. What is important is that the surfaces generating the volumes representing future and past lifetime be taken from the l_x graph. As a result, taking the representation of the population at time $t = 0$ to be vertical, it is sufficient to retain the projection on the horizontal plane of the volumes generated by the problem, and to refer to the generating surfaces on the vertical plane for evaluating the weight of the relevant areas on the horizontal plane.

At the risk of being repetitious, I shall illustrate these principles using Example 4, referring to the diagram in Figure 1.

Of the l_{x_1} lives represented by $(x_1 u_1)$, the lives represented by the segment $(v_1 w_1)$ will die between 60 and 80. The number of lives of the population between ages 20 and 70, represented by the area *(20 m c 70)*, who will die between ages 60 and 80 is therefore represented by the area generated by $(v_1 w_1)$ when x_1 varies from 20 to 70, or the area $(a\,b\,f\,c\,d) = 40l_{60} + T_{60} - T_{70} - 50l_{80}$.

All those lives will reach or have reached age 60. For example, for $x = x_1$ the lives represented by $(v_1 w_1)$ will be represented by *(e f)* in $(60 - x_1)$ years, the beginning of the twenty-year period over which they will die. For the sake of problem solving, I will pretend, as explained before, that the plane containing the l_x curve is vertical and that any l_{x_1} lives, aged x_1, will age along a path perpendicular to (20, x) generating an area identical to $(x_1 u_1 \omega)$, the trace of which on the horizontal plane $(0t, 0x)$ is $(x_1 \omega_{x_1})$. In that interpretation, the weight of $(x_1 \omega_{x_1})$ is T_{x_1}. The weight of the area $(\omega_{x_1} x_1 \omega_x)$ is Y_{x_1}. On the line $(N60)$ all lives are aged 60. On the line $(M80)$ all lives are aged 80.

The aggregate lifetime after age 60 of the lives represented by $(v_1 w_1)$, seen to be equal to area $(efg) = T_{60} - T_{80} - 20l_{80}$ on the vertical plane, is now to be seen as segment (AB), affected with the weight $T_{60} - T_{80} - 20l_{80}$, on

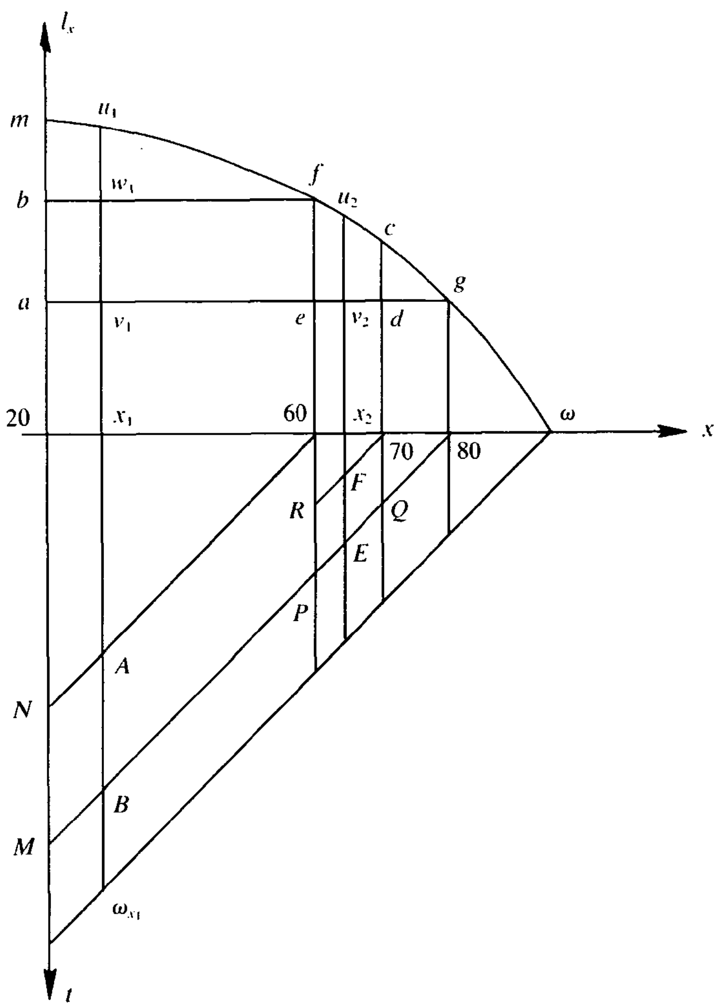

FIG. 1.—Illustration for Example 4

the horizontal plane. When x_1 varies from 20 to 60, (AB) generates the area $(MN60P)$ with weight 40 $(T_{60} - T_{80} - 20l_{80})$.

If we take $x = x_2$, $60 < x_2 < 70$, the lives who will die before age 80 are represented by $(v_2 u_2)$ and their future lifetime on the vertical plane is represented by the area $(v_2 u_2 g) = T_{x_2} - T_{80} - (80 - x_2)l_{80}$. However, the aging will take place along a path perpendicular to $(20, x)$ leaving the trace $(x_2 E)$ on the horizontal plane. The weight of $(x_2 E)$ will be $T_{x_2} - T_{80} - (80 - x_2)l_{80}$. Note that $80 - x_2$ is the length of $(x_2 E)$. The future lifetime of the population represented by the area $(efcd)$ on the vertical plane is equal to the weight of the area $(P6070Q)$ on the horizontal plane. If the weight of $(x_2 E)$ is taken to be $T_{x_2} - T_{80}$, the weight of area $(P6070Q)$ is read to be $Y_{60} - Y_{70} - 10T_{80}$. If the weight of $(x_2 E)$ is taken to be l_{80} at all points, the weight of $(P6070Q)$ is $\frac{1}{2}(20 + 10)l_{30}$. The future lifetime is therefore $Y_{60} - Y_{70} - 10T_{80} - 150l_{80}$.

The aggregate past lifetime since age 60 of the population represented by area $(efcd)$ on the vertical plane is equal to its future lifetime to age 70. It is therefore to be seen on the horizontal plane as the weight of the area $(6070R)$, with generator $(x_2 F)$, having itself a weight of $T_{x_2} - T_{70} - (70 - x_2)l_{80}$. Breaking the evaluation into two steps as above, the aggregate past lifetime since age 60 is equal to $Y_{60} - Y_{70} - 10T_{70} - \frac{1}{2}(10 \times 10)l_{80}$.

The average age at death, [60 + (Lifetime after 60/Number of deaths)], can now be found.

E. S. SHIU:

The three-dimensional diagrams in this paper are beautiful. The most interesting feature of the model is Figure 8, which illustrates the two interpretations of the quantity $Y_x - Y_{x+n} - nT_{x+n}$. The corresponding Tino geometric model is given in Figure 13, where it is not immediately obvious that the volume of *ABEFGH* is equal to the volume of *ABEFCD*, since they have different shapes. Each horizontal slice of Figure 13, however, is a rhombus partitioned by the plane *ABEF* into two triangles of equal area; thus *ABEFGH* and *ABEFCD* have equal volumes.

The easiest way to tackle an "average age at death" problem is to apply Veit's "in-and-out" method to determine the total number of deaths, and then apply the Grace-Nesbitt transformations to determine their total lifetime. The two-dimensional Lexis diagrams are very easy to sketch, and Mrs. Tino's geometric model (reference [10] of the paper) elegantly explains that the areas above the vertical and horizontal lines in a Lexis diagram are T_x and l_x, respectively. To find the number of deaths with Dr. Chan's method, we need to draw a three-dimensional diagram, determine the two-dimensional region corresponding to the deaths, and then project the region northward to get rid of the factor $\sqrt{2}$. To find the lifetime in terms of the traditional actuarial symbols,we need to recognize various shapes and be able to add and subtract volumes mentally.

Many three-dimensional diagrams are awkward to sketch on paper. For instance, consider the following problem (reference [3] of the paper, p. 90, No. 4b): "How many years do the people who have any birthday from 20th to 29th inclusive during 1966 live from that birthday until December 31, 1975?" It is somewhat difficult to draw a three-dimensional diagram for a problem of this type, since such a figure has many faces. However, the problem can be solved readily by integration. Let $\lceil y \rceil$ denote the least integer greater than or equal to y; then the number of years is

$$\int_{y=19}^{y=29}\left(\int_{y+t=\lceil y \rceil}^{t=10} l_{y+t}dt\right)dy = \int_{19}^{29} (T_{\lceil y \rceil} - T_{y+10})dy$$

$$= \sum_{j=20}^{29} T_j - Y_{29} + Y_{39} .$$

Despite its many interesting features, Dr. Chan's model might not be an effective tool for actuarial students writing the Part 5 examination. Since solid geometry is usually absent from the curricula of high schools and universities, many students will have difficulty visualizing the diagrams in this paper. In the mid-1960s, my colleague H. J. Boom introduced diagrams similar to those in Figures 5–8 to his life contingencies classes, but he did not pursue the approach because most of his students could not interpret the diagrams in three dimensions. We are hopeful, however, that future generations of students will find three-dimensional problems easier because of their familiarity with Rubik's cube! Students who are good at geometry might find it more advantageous to master the Tino geometric model, since Lexis diagrams are used in Dr. Chan's reference [2], which is a textbook for the Part 5B examination. Both geometric models can be used to solve stationary population problems by inspection of appropriate figures. Students who prefer calculus to geometry should follow J. Maynard's advice in reference [12] of the paper (p. 263): "A student who can use the [Lexis] diagram to write correct expressions, and has mastered double and single integration using actuarial symbols, should have a sure-fire approach to this kind of problem."

ROBERT L. BROWN AND BEN W. LUTEK:*

Dr. Chan is to be congratulated on his well-written paper, which documents a powerful new geometric approach to population problems in the

* Mr. Lutek, not a member of the Society, is a student of actuarial science at the University of Waterloo.

flavor of Mrs. Tino's geometrical model (discussion of Veit's paper, *TSA*, Vol. XVI). Dr. Chan's paper allows a whole new population (nonstationary) of actuarial philosophers to put pen to paper and support their favorite method of solving these difficult problems.

Most students of life contingencies have not read the methods proposed by Maynard, Grace and Nesbitt, Veit, Tino, or Batten, so they depend exclusively on the integration approach outlined in chapter viii of Jordan.

Dr. Chan presents a method that does not require multiple integration, and claims that "it appears to be more concrete and comprehensible than conventional methods." But, as he points out in his conclusion, "Considering the time and effort incurred in the preparation of the figures . . . , the geometric method seems unlikely to be the most efficient method." We fully agree with these comments, and, in the spirit of refining (rather than reinventing) the wheel, we wish to document a convenient and quick technique for solving a wide class of stationary population problems in which a group of lives is specified by an initial age range and a time-age range for deaths. This class includes any problem for which a "death region" can be sketched on a Veit diagram, and furthermore includes all seven examples given in Dr. Chan's paper.

The method incorporates Grace and Nesbitt's substitutions for total lifetime, along with equally basic substitutions for past and future lifetimes. Dr. Chan is to be commended on providing a visible model, which co-discussant B. Lutek used to derive these new substitutions.

Consider first a Chan diagram (Fig. 1) on which the present group of lives is described by a region (*A*) on the diagonal surface. As Dr. Chan states, the projection (*B*) of this region onto the north l_x wall describes the number of lives in the group. An eastward projection specifies a "death region" (*C*) on the overhead l_x roof (lives passing through points in this region die). The vertical projection of this death region onto the Chan diagram base yields a death region (*D*) that (as Dr. Chan observes) is related directly by a linear transformation to the death region one draws on a Veit diagram. The substitutions of Grace and Nesbitt that link projected lives to volumes on Chan's diagrams consequently link lives on a Veit diagram to volumes in an implicit three-dimensional structure above the Veit diagram. The reader is invited to convince himself of the substitutions in Table 1 by considering the direct relationship between the Chan, Veit, and Grace and Nesbitt models. In the table, k is a scalar multiplier, and α is the distance through water (Veit's analogy) immediately north of a boundary's east end (or simply due north of a north-south boundary). *Alternatively,* α is the distance from a boundary's east end to the north-south Veit diagram axis. Two of Dr. Chan's examples serve as illustrations and can be used to explain k and α.

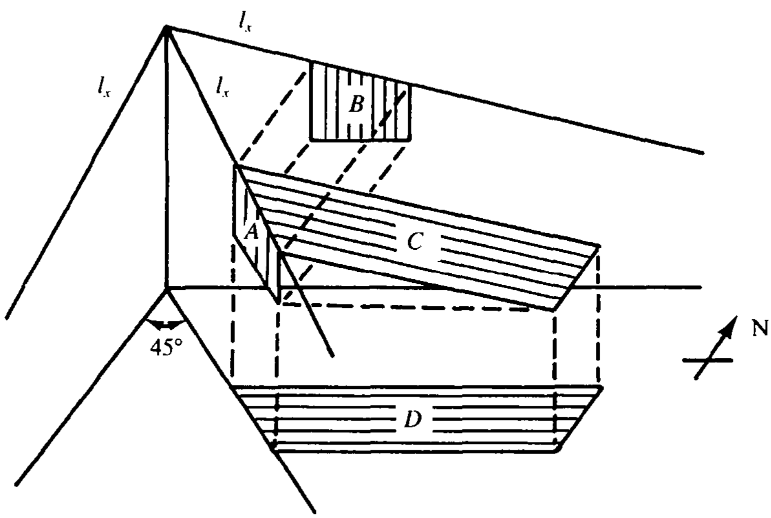

FIG. 1.—The generic Chan diagram

TABLE 1

SUBSTITUTIONS

	kl_x	T_x
Past lifetime	$kl_x(x-\alpha+k/2)$	$(x-\alpha)T_x+Y_x$
Future lifetime	$k[(\alpha-k/2)l_x+T_x]$	αT_x+Y_x
Total lifetime (Grace and Nesbitt)	$kF_x=k(xl_x+T_x)$	$G_x=xT_x+2Y_x$

"EXAMPLE 5. *Find an expression for the average attained age of those persons in the stationary population now between ages 25 and 40 who will die between the ages of 30 and 50 within the next twenty years.*"

Solution: Using Veit's approach (Fig. 2), we can quickly ascertain that the number of members in the group is $5l_{30} + (T_{30} - T_{40}) - [10l_{50} + (T_{45} - T_{50})]$. To determine average attained age, one must find total past lifetime. This can be done using our proposed formulas.

For the term $5l_{30}$, $k = 5$ and $\alpha = 5$; therefore, past lifetime $= 5l_{30}(30 - 5 + 5/2) = 137.5l_{30}$.

For the term $(T_{30} - T_{40})$, $\alpha = 0$, since these lives enter on the extreme "west frontier"; therefore, past lifetime $= (30T_{30} + Y_{30}) - (40T_{40} + Y_{40})$.

For the term $10l_{50}$, $k = 10$ and $\alpha = 20$; therefore, past lifetime $= 10l_{50}(50 - 20) + (100/2)l_{50} = 350l_{50}$.

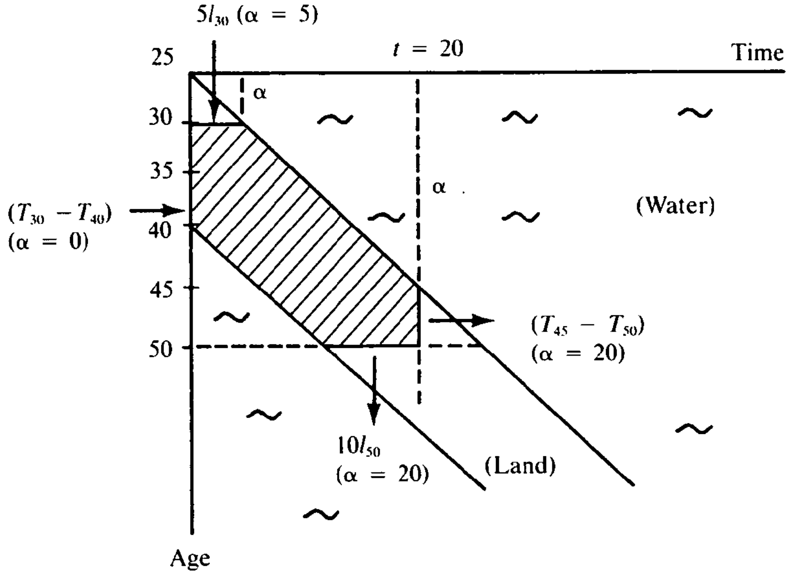

FIG. 2.—Determination of the number of members in the group for Example 5

For the term $(T_{45} - T_{50})$, $\alpha = 20$; therefore, past lifetime $= [(45 - 20)T_{45} + Y_{45}] - [(50 - 20)T_{50} + Y_{50}] = 25T_{45} + Y_{45} - 30T_{50} - Y_{50}$.

Putting these four pieces together, one quickly arrives at the correct answer.

Example 7 of Dr. Chan's paper provides a representative illustration of the total lifetime substitutions.

> "EXAMPLE 7. *Find the number of deaths in 1966–70 for those between ages 50 and 60 on January 1, 1960, who die between ages 65 and 75 in the given five-year span; and find the total of their ages at death.*"

Solution: We determine the number of members by using a Veit diagram (Fig. 3). In this case, the age 75 restriction is redundant, since, of the original group, no one can exceed age 71 by January 1, 1971. From the Veit diagram we see that the number of members is $5l_{65} + (T_{65} - T_{66}) - (T_{65} - T_{71})$, or $5l_{65} + T_{71} - T_{66}$. This is identically the number of deaths, since the group is defined according to when its members die. The sum of their ages at death equals total lifetime, so, by the corresponding substitutions (Grace and Nesbitt), we have

$$5(65l_{65} + T_{65}) + (71T_{71} + 2Y_{71}) - (66T_{66} + 2Y_{66}) .$$

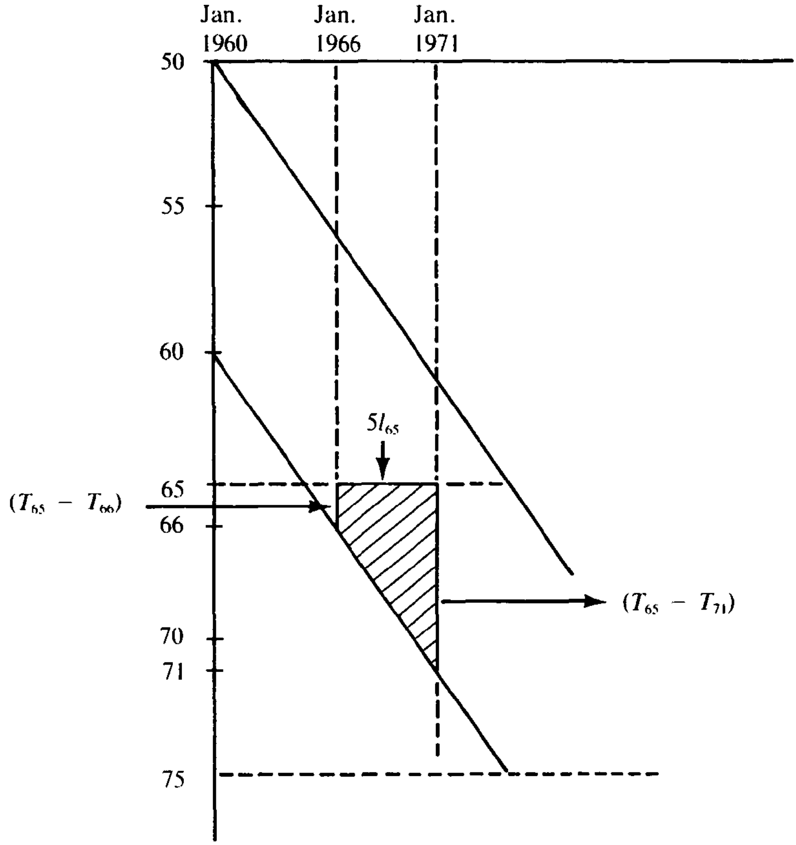

FIG. 3.—Determination of the number of members in the group for Example 7

This solution takes less than a minute.

Note that in Example 7 we have

$$\begin{aligned}\text{Past lifetime} &= 5l_{65}(65 - 11 + 5/2) + (65 - 6)T_{65} + Y_{65} \\ &\quad - (66 - 6)T_{66} - Y_{66} - (65 - 11)T_{65} - Y_{65} \\ &\quad - (71 - 11)T_{71} + Y_{71} \\ &= 282.5l_{65} + 5T_{65} - 60T_{66} - Y_{66} + 60T_{71} + Y_{71} ;\end{aligned}$$

and

$$\begin{aligned}\text{Future lifetime} &= 5[(11 - 5/2)l_{65} + T_{65}] + 6T_{65} + Y_{65} \\ &\quad - 6T_{66} - Y_{66} - 11T_{65} - Y_{65} + 11T_{71} + Y_{71} \\ &= 42.5l_{65} - 6T_{66} - Y_{66} + 11T_{71} + Y_{71} .\end{aligned}$$

We offer this methodology as a fast, efficient technique for solving stationary population problems. At the same time, we repeat Batten's caveat (reference [2] of the paper, p. 49): "In some instances, however, it is the feeling that certain short-cut devices have worked against the actuarial student's best interests. If a student finds himself able to solve certain problems with an entirely mechanical process, the natural result often is unwillingness to investigate the theoretical aspects of the situation, thus defeating the purposes of the Society examinations."

HUNG-PING TSAO:

Beda Chan's paper is excellent for those who need to reinforce their understanding of an integral. Although his intention is to avoid integration, he is, in fact, explaining what an integral is. Nevertheless, this masterpiece of mathematics is worth reviewing at times.

As stated in the conclusion of the paper, geometric solutions to stationary population problems are by no means efficient. I would like to present here a rather simple-minded and yet efficient method that uses a concept of flow.

A stationary population is like a special kind of water flow whose cross-sections become smaller and smaller and which runs out at the end. In this analogy, l_x corresponds to the amount of water running through cross-section x at any moment. The discussion that follows, however, applies to any flow.

Solution to Example 4

Thanks to Grace and Nesbitt, we need only obtain the amount of water in question (the number of lives in the case of stationary population problems). The same will apply to all other examples.

First we construct a simple linear diagram (Fig. 1) depicting the flow. We want to obtain the amount, D, of water lost while the water in the part of

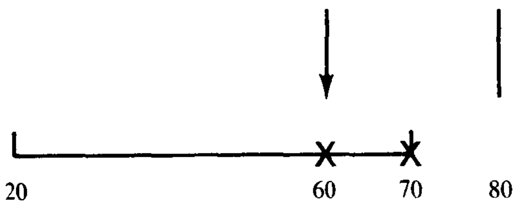

FIG. 1.—Flow diagram for Example 4

the flow between cross-sections 20 and 70 (⌞———⌟) runs from point 60 with the arrow (↓) to point 80 with the bar (|). The amount, B, of water between cross-sections 60 and 70 (x———x) is

$$T_{60} - T_{70} .$$

Then

$$D = B + S - E ,$$

where S is the amount of water in the part of the flow between cross-sections 20 and 60 (⌞———x) run through point 60, and E is the amount of water in the part of the flow between cross-sections 20 and 70 (⌞———⌟) run through point 80. From Figure 1 we see that

$$S = (60 - 20)l_{60}$$

and

$$E = (70 - 20)l_{80} .$$

Therefore,

$$D = T_{60} - T_{70} + 40l_{60} - 50l_{80} .$$

Solution to Example 3

First we construct a diagram (Fig. 2) depicting the flow. We can readily see from Figure 2 that

$$\begin{aligned} D &= T_x - T_{x+n} + (x - x)l_x - [(x + n) - x]l_{x+n} \\ &= T_x - T_{x+n} - nl_{x+n} . \end{aligned}$$

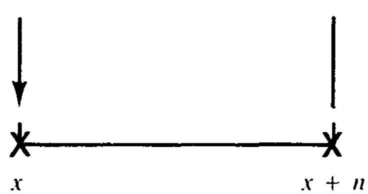

FIG. 2.—Flow diagram for Example 3

Solution to Example 5

In this case, we construct a somewhat more elaborate diagram (Fig. 3) to depict the flow. Since the loss of water accounts for only the next 20

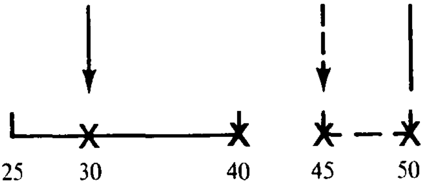

FIG. 3.—Flow diagram for Example 5

points of time, D should exclude the amount, $T_{45} - T_{50}$, of water between cross-sections 45 and 50 (x— — —x), less the amount, $(50 - 45)l_{50}$, of water in the part of the flow between cross-sections 45 and 50 run through point 50 with the dotted arrow (¦). Therefore,

$$D = (T_{30} - T_{40}) + (30 - 25)l_{30} - (40 - 25)l_{50}$$
$$- [(T_{45} - T_{50}) - (50 - 45)l_{50}]$$
$$= T_{30} - T_{40} - T_{45} + T_{50} + 5l_{30} - 10l_{50} .$$

Solution to Example 6

First we construct the flow diagram (Fig. 4). From the figure we see that

$$D = T_{60} - T_{70} + 40l_{60} - 50l_{80} - (T_{70} - T_{80} - 10l_{80})$$
$$= T_{60} - 2T_{70} + T_{80} + 40l_{60} - 40l_{80} .$$

Solution to Example 7

The water in the part of the flow between cross-sections 55 and 60 (└──┘) has to run through point 65 and has to be lost completely before

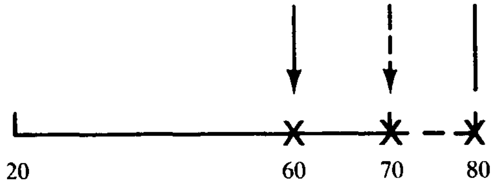

FIG. 4.—Flow diagram for Example 6

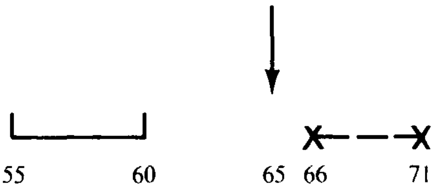

FIG. 5.—Flow diagram for Example 7

the running section matches the part of the flow between cross-sections 66 and 71 (x— — —x). Hence the flow diagram appears as in Figure 5. From the figure we see that

$$D = (60 - 55)l_{65} - (T_{66} - T_{71})$$
$$= 5l_{65} - T_{66} + T_{71} .$$

(AUTHOR'S REVIEW OF DISCUSSION)

BEDA CHAN:

I wish to thank the discussants for adding valuable materials to the solution of stationary population problems.

It is trivial to verify that

$$T_x = [l_0/2\omega](\omega - x)^2$$

under de Moivre's hypothesis, but it will take more work to show that

$$T_x = [l_0 e^{B/\ln c}/\ln c]E_1(\alpha)$$

under Gompertz's hypothesis and to show that

$$T_x = [l_0 e^{B/\ln c} e^{-Ax}/\alpha^z \ln c]\Gamma(z, \alpha)$$

under Makeham's hypothesis. Here,

$$\alpha = Bc^x/\ln c , \quad z = -A/\ln c ;$$

$$E_1(\alpha) = \int_\alpha^\infty e^{-t}t^{-1}dt$$

is the exponential integral function; and

$$\Gamma(z, \alpha) = \int_{\alpha}^{\infty} e^{-t} t^{z-1} dt$$

is the incomplete gamma function.

It probably is because of elaborate computations such as those encountered above that questions on stationary populations where l_x assumes specific analytic forms are seldom considered. Instead, l_x is assumed axiomatically to be some nonincreasing, nonnegative function defined on the nonnegative real axis. Integration is used as a *concept* to represent quantities but not as a tool for computation. Thus, as the paper has shown, the *concept* of integration alone will be sufficient to carry out all necessary analysis.

This approach of avoiding integration is again illustrated by Mrs. Tino's discussion. In her earlier discussion (*TSA*, XXI [1969], 289–95) of Batten's paper (reference [1] of my paper), she introduced a method that begins with a diagram constructed within the l_x curve and arrives at the solution by integration. In her current discussion, she expands her diagram and makes integration unnecessary.

The purpose of this paper is to present the theory of the stationary population at the most elementary level. An intuitive illustration of a concept is frequently more comprehensible than, and historically can appear before, its analytic formulation. An example is Cavalieri's principle (1629), which preceded the integral calculus: Solids with the same height and with cross-sections of equal area have the same volume. This principle explains the equality of the two volumes *AGEI* and *AECI* in Figure 13, establishes the equivalence of Tino's method (reference [10]) and the geometric method in this paper, and delivers the solution to the following problem (Brian Bambrough, *Problem Solving in Life Contingencies,* p. 12.5): "What is the aggregate time lived in the next year by those dying between ages x and $x + n$ in the next year?" The solid that represents the aggregate time has height $l_x - l_{x+n}$ and cross-sections of congruent triangles with area ½. Its volume, by Cavalieri's principle, is $(l_x - l_{x+n})/2$. I thank Dr. Shiu for bringing my attention to this problem.

I agree with Messrs. Brown and Lutek and with Dr. Shiu that Veit's "in-and-out" method followed by the Grace-Nesbitt principle is the most efficient, at least for most people on most problems. The effectiveness of this method is further enhanced by the expressions for past and future lifetime given in Messrs. Brown and Lutek's discussion. In place of Veit's method, Mr. Tsao suggests the analogy of fluid flow to determine the number of lives and presents a shorthand notation for remembering the three-dimensional information suppressed in his diagrams.

I agree also that the classical method of integration is powerful. The problem solved so compactly by integration in Dr. Shiu's discussion will take up more space by the geometric method: for those who have their twentieth birthday during 1966, mental reference to the geometric method will show that they live $T_{20} - Y_{29} + Y_{30}$ person-years until December 31, 1975. Thus, for the twentieth through the twenty-ninth birthday, the answer is

$$\sum_{k=0}^{9} T_{20+k} - Y_{29+k} + Y_{30+k} .$$

I thank the discussants again for their stimulating discussions.