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THE MATCHING OF ASSETS AND LIABILITIES

JAMES A. TILLEY

ABSTRACT

In the British actuarial journals most papers on immunization examine the theory as it applies to the valuation of the assets and liabilities of an insurance company or a pension fund. The papers deal primarily with valuation and little with how to determine investment strategy. This paper investigates how the concepts of asset-liability matching can be used to shape investment strategy.

A general model for matching assets and liabilities is developed. Three aspects of the investment problem are discussed: initial investment strategy, reinvestment strategy, and asset liquidation strategy. Reinvestments and disinvestments are handled by an investment-year method. Explicit provision is made for different new-money rates in each future year.

The model is defined by specifying (1) the schedule of interest and principal payments for representative investment instruments comprising the initial portfolio, (2) the expected net cash outflows of the pension fund or other block of business, (3) rollover rates for reinvestments, and (4) a set of patterns of future new-money interest rates. An investment strategy is defined to be a specific allocation of investable funds among the representative instruments. The model solves for a region of strategies that result in a nonnegative total fund value at the end of the investment horizon for each interest rate pattern in the set described in item 4.

Conventional immunization theory is identified as a special case of the general model in which each interest rate pattern represents an immediate and permanent change in the level of interest rates from the current level. The problem of establishing the interest guarantee for a deposit fund is discussed as an example where conventional immunization theory fails but the general model does not.

I. INTRODUCTION

MUCH has been written about the theory of immunization, the matching of assets and liabilities, and the role of the actuary in the investment operations of an insurance company. Contributions range from the original paper by Redington [2] to the records of concurrent sessions of Society of Actuaries meetings. The *Journal of the Institute of Actuaries* and the *Transactions of the Faculty of Actuaries* are rife with articles about these subjects. Perhaps the works most familiar to North American actuaries are those of Vanderhoof [5, 6], who, among actuaries, has almost single-handedly taken up the cause on this side of the Atlantic.

Despite the extensive literature on immunization theory and the fact that the ideas have been in existence for more than twenty-five years, there has been little practical application of the theory to actual investment practice in the United States and Canada. Undoubtedly, one of the reasons for the initial lack of enthusiasm was the fact that interest rates were rather stable from the mid-fifties until the late sixties. Since then, however, we have experienced high inflation, an uncertain economy, and a very different climate for fixed-income instruments. In such an environment the interest rate risks, which immunization is trying to protect against, are apparent. Nevertheless, there still does not appear to be a noticeable move toward applying asset-liability matching theory. There are several reasons why this is so.

1. The backgrounds of actuaries and investment officers are different. In general, neither actuaries nor investment personnel understand the other's professional jargon. This is an impediment to a free exchange of ideas.
2. The organizational structure of most insurance companies segregates investment functions from traditional actuarial functions; usually there are few committees with representation from the two organizational areas.
3. The ideas underlying immunization theory are not well understood by many actuaries and most investment officers, and, when understood, appear to be difficult to apply in practice.
4. Immunization theory in its conventional form places rigid constraints on investment operations, leaving investment officers with too little latitude for making policy decisions.
5. Immunization theory itself is subject to several fundamental criticisms that, if valid, would preclude its use in practical situations.

Before the barriers imposed by the first two items can be broken down, the issues in the last three must be addressed. This is largely a matter of education. We must find better ways of expressing the *concepts* underlying the theory of asset-liability matching and of responding to the explicit criticisms. We then must devise a practical way to apply the theory, a way that involves the interaction of actuarial and investment personnel from the outset.

One of the purposes of this paper is to expose some ideas that should be helpful in eliminating impediments to the successful application of the principles of asset-liability matching. There are many situations today, especially with the proliferation of new-money "accumulation" products, in which these concepts can and should be applied. The formal equations of the theory cannot be used in certain situations, nor should they be applied blindly in any event. With sufficient ingenuity, the ideas can be put to practical use, if not as a guide to determining investment strategy, then as a method of quantifying specific interest rate risks.

II. IMPORTANCE OF INVESTMENT CONSIDERATIONS

All actuaries and investment officers recognize that two primary objectives of investment policy are security of principal and a high rate of return. What is less well recognized is the important connection between investment operations and noninvestment operations. In establishing the benefit structure and contribution level for pension plans and in pricing insurance products, it is essential to take this relationship into account. Doing so involves the concepts of asset-liability matching. The reason for examining the relationship between cash flows from investment operations and cash flows from insurance operations lies in an evaluation of the interest rate risks to which the pension or insurance fund is subject. There are two types of interest rate risk: (*a*) the risk that funds will have to be reinvested when interest rates are lower than those assumed in funding and pricing calculations and (*b*) the risk that fixed-income assets will have to be liquidated at a capital loss when interest rates are high.

Suppose that assets are invested primarily in short-term instruments but the cash-flow requirements of the pension plan or line of business extend many years into the future. The fund will have more asset maturities than it needs to meet its obligations in the short run and may be forced to reinvest funds when interest rates are low, thus running the risk of earning interest at a lower rate than that assumed in the funding or pricing calculations.

When the total fund is a "pooled" fund comprising commingled pension funds or insurance funds for many different blocks or lines of business, the second type of interest rate risk involves issues of equity among the components of the total fund. If the assets underlying a particular line of business are invested primarily in long-term instruments with repayment of principal many years in the future, it can happen that the combination of interest income and premium income is insufficient to meet all the benefit payments, expenses, and taxes in the early years. Assets would have to be liquidated to make up the deficiency, and this would be at a capital loss if interest rates were higher than they were when the assets were purchased. Even if the fund as a whole had a net cash inflow so that asset holdings would not actually have to be liquidated, there could be a serious question of equity. For example, suppose one line of business is experiencing a net cash outflow but another line is experiencing a larger net cash inflow. By diverting the latter to cover the former, the fund will not have to liquidate any assets, but the line with the net cash inflow will have a much smaller amount to invest at the current high interest rates and is deprived of an investment opportunity. To avoid having one line support another, an "accounting" calculation could be instituted to treat the entire situation as if the line with the net outflow had been forced to liquidate assets. As will be seen in Section IV, it is possible to perform a different but substantially equivalent calculation.

III. NATURE OF THE INVESTMENT PROBLEM

The problem of determining investment strategy can be separated into three parts: the initial investment strategy, the reinvestment strategy, and the disinvestment or liquidation strategy. It is necessary to understand the nature of each of these three pieces before attempting to construct a general model of the entire investment strategy problem.

A. *Initial Investment Strategy*

Investment officers and portfolio managers keep abreast of current money and capital market conditions. They have access to many market analyses covering the short-term period (from a few weeks to several months to a year or more). The very nature of their job is to assess the investment opportunities over this period and to make decisions on the allocation of investable funds among various classes of assets and, within a class, among instruments posing different credit risks and having different maturity dates.

Over the very short term, investment officers and portfolio managers are relatively well informed of the key factors necessary for formulating investment strategy: there are reasonably accurate forecasts of the amount and incidence of investable funds, the schedule of commitment takedowns in the near term is known, and there is little uncertainty in yield curves and yield spreads in the public securities markets.

B. *Reinvestment Strategy*

At first sight it would seem that determining reinvestment strategy is no different from determining initial investment strategy. The key point is the time at which such strategy is decided upon. If one is attempting to make decisions now on how to reinvest funds each year in the future, there is considerable uncertainty about all of the factors necessary for making a decision.

Insurance companies channel a significant portion of their funds into instruments whose contractual provisions are directly negotiated with the borrower. (In practice, the insurer may not have much freedom in negotiating these provisions.) When establishing a schedule of commitments to various borrowers, investment officers must estimate the flow of investable funds. These estimates take into account, among other things, scheduled maturities of assets in the existing portfolio and short-term projections of the net cash inflow from insurance operations. Even though funds typically are committed up to a year or two into the future, both competitive considerations and the uncertainty of cash-flow projections prevent the *full* commitment of funds for investment at points several years into the future. Moreover, good borrowers are not willing to tie themselves down to rigid contractual provisions concerning the rate of interest, the schedule of principal repayments, early refunding, and so on, even if they are able to assess their capital needs several years ahead. Thus, apart from publicly traded securities, it is not known with certainty what investment instruments will be in ample supply beyond the period for which a substantial commitment of funds has been made.

The general level of interest rates a year or two into the future is moderately uncertain and beyond that is very uncertain. The more detailed structure of interest rates—the yield curves and yield spreads—is also uncertain. Recent experience has shown that flat and inverted yield curves can arise in situations not expected on the basis of past experience. Yield spreads are determined partly by the investors' overall assessment of the economy and partly by unusual market conditions, and these are completely unpredictable beyond the short term.

The opening sentence of a speech by Dr. Albert M. Wojnilower, managing director and economist of the First Boston Corporation, to the Sixth Institutional Investor Bond Conference in New York City on October 26, 1978, suggests what can be expected for the future: "In less than twenty years the practice of finance, in common with so many aspects of our lifestyle, has been profoundly transformed in structure and habit." The title of his speech, "Suboptimization, or The New Look in Interest Rates," and the main topics discussed—decontrol of interest rates, making all assets marketable, shifting the risk from institutions to the public, interest rates catching up with inflation, changes in the yield curve—point out that it is difficult enough to understand how we arrived at the current situation without trying to predict events that will mold the future.

The best time to make reinvestment decisions is in the future, when the reinvestments are actually made and when the market conditions are known. However, we are trying to protect against interest rate risks brought about by the uncertain future through the choice of an appropriate initial investment strategy. To do this we must have some realistic model for reinvestments, even though it necessarily will be less detailed than the model for the initial investment of funds.

C. *Liquidation Strategy*

Several considerations are important in deciding which assets to liquidate. It may be desirable to improve the quality of the portfolio by eliminating assets that have the greatest chance of default. The duration of the portfolio can be modified by eliminating selectively short-term or long-term assets. However, the dominant consideration is likely to be the capital gains or losses realized on liquidation, and the resulting income tax implications.

When commingled funds or all the assets of the insurer's general account are involved, there probably will be an overall net inflow of funds over a sufficiently long period (three, six, or twelve months) even though certain parts of the fund have a net outflow during this period. In such a situation it is unlikely that the portfolio manager or investment officers would choose deliberately to liquidate assets to meet the demands of those parts of the fund with net outflow. Liquidations probably will occur only for purposes of altering the composition of the portfolio and to optimize the income tax position. It was pointed out in Section II that questions of equity among the separate funds are involved, and it may be necessary to allocate investment income in a manner that recognizes the cash-flow positions of the separate funds. The basis for the allocation does not have to be "as if liquidation of assets actually occurred." This is discussed in the next section.

IV. A GENERAL MODEL FOR MATCHING ASSETS AND LIABILITIES

A general model for matching assets and liabilities is developed in this section. In Section VI it will be seen that conventional immunization is a special case of the general model.

All attempts to match assets and liabilities begin with an identification of items of cash inflow and outflow. It is customary to segregate all cash flow into two categories: that arising from investment operations, and all the rest, which, for an insurance company, would be called cash flow from insurance operations. Since only fixed-income investments are considered in this paper, the first category consists of interest and principal payments as inflow items and investment expenses as outflow items. Cash flow from investment operations can be classified further as arising from the initial portfolio of assets or from reinvestments. Cash flow from other than investments consists of premiums or contributions as inflow items, and benefit payments, commissions, expenses, taxes, and policyholder and shareholder dividends as outflow items.

In each of the two major categories, specific items of cash flow will occur throughout the year. To bring the problem to a manageable size, it is customary to lump together all items occurring within a year (plan year, policy year, or calendar year, depending on the circumstances). Funds that cannot be invested long term as soon as they are received are invested instead in very short-term instruments such as commercial paper and Treasury bills until commitments fall due or other investment opportunities open up. When commitment takedowns exceed funds available for investment, the insurer must draw on its lines of credit to borrow funds. In effect, the company invests its cash flow (and borrowed funds) immediately at long-term rates but incurs a short-term borrowing cost. This will be a profitable procedure provided that the borrowing rate is lower than the long-term rate. The net effect of such investment activity can be approximated closely by accumulating the year's cash-flow items to year-end at the prevailing long-term rate.

Let CF_k^{out} denote the *net cash outflow in year k from items other than investments*, with cash flow during year k accumulated to the end of the year at the prevailing long-term interest rate. The cash flows CF_k^{out} are calculated by a fund projection or model-office projection based on actuarial assumptions concerning the amount and incidence of the specific items.

Since the nature of the initial investment problem is different from that of the reinvestment problem, it is customary to treat the cash flow from the initial portfolio of assets separately from the investment cash flow arising from reinvestments. Let CF_k^{in} denote the *net cash inflow in year k from assets associated with the initial portfolio only*, with cash flow during year k accumulated to the end of the year at the prevailing long-term interest rate.

Investment officers can identify representative investment instruments into which currently investable funds can be channeled. These might include public bonds, privately placed bonds, and mortgages. Within each of these broad classes of assets, there may be several instruments, differing by contractual interest rate, maturity date, principal repayment pattern, penalties on early refunding, or other characteristics. Suppose there is a total of n representative investment instruments. Let a_{kj} denote the net interest and principal payments in year k per dollar invested in the jth instrument, with any payments during the year accumulated to year-end at the long-term interest rate. The interest payment is net of investment expenses. Moreover, no credit should be taken for the default premium in this interest rate unless explicit provision has been made in the model for the accumulation of a contingency reserve against default. The initial investment strategy problem consists of specifying how the initial funds are to be allocated among the n representative instruments. Let p_j denote the fraction of initial funds invested in the jth instrument. Then

$$\sum_{j=1}^{n} p_j = 1 ; \tag{1}$$

$$CF_k^{\text{in}} = \sum_{j=1}^{n} a_{kj} p_j . \tag{2}$$

The value of CF_k^{in} in equation (2) is equal to the value previously defined, divided by the value of the initial fund. In the following analysis, CF_k^{out} refers to the value previously defined, divided by the value of the initial fund. The paper "Achieving Consistency between Investment Practice and Investment Assumptions for Single Premium New-Money Products" [4] gives examples of typical investment instruments and the associated matrix $\{a_{kj}\}$.

Before proceeding with the analysis, it is worthwhile to indicate how equation (2) would be modified for immunizing an already existing pension fund or block of business. In that case, CF_k^{in} should include a separate term for cash flow from the already existing portfolio of assets backing the pension fund or block of business. Such a term will be independent of the variables $p_1, p_2, \ldots, p_n$ that apply only to funds currently available for investment. The appropriate modification is

$$CF_k^{\text{in}} = a_{k0} + \sum_{j=1}^{n} a_{kj} p_j . \tag{2'}$$

As each year passes, equation (2′) is the starting point for determining investment strategy to keep the fund in a matched position. The term

a_{k0} applies to the portfolio of assets existing at the start of the first year as a result of investments made in all prior years. For a *new* fund or block of business, $a_{k0} = 0$ for all k. In the remainder of this paper, equation (2) is used in lieu of equation (2′).

From the discussion in the preceding section, we know that it does not make sense to establish reinvestment strategy until the future becomes the present and the investment climate is better known. In particular, no attempt will be made to identify representative investment instruments for the reinvestment of funds in the future. To characterize the impact of reinvestments on the determination of initial investment strategy, it ought to be sufficient to represent the multiplicity of interest rates at any point in the future by a single average rate appropriate to the average term of the reinvestments. This average interest rate is net of investment expenses and excludes an average default premium unless explicit provision is being made for a reserve against defaults.

No mention has been made of how liquidations will be treated in the model. It is of great theoretical convenience to be able to handle liquidations in the same fashion as reinvestments. In the preceding section it was observed that forced liquidation will not actually occur in a commingled fund that has an overall net inflow even though some parts of the fund have net outflow. Equity in allocating investment income to the various parts of the total fund can be achieved by treating each part as investing its net cash flow at the prevailing new-money rate. When one part of the fund has a net outflow, the resulting negative investment it makes can be considered as a loan by the rest of the fund to cover the cash-flow deficiency. This method of allocating investment income is equitable because (*a*) the parts of the total fund experiencing net cash outflows are required to pay interest on their loans at the prevailing new-money rate and (*b*) the parts of the total fund experiencing net cash inflows fare no better and no worse than if their funds had been invested in actual money market or capital market instruments, since the loans bear interest at the prevailing new-money rate.

What is the "loan's" schedule of principal repayment? The "lending" parts of the total fund will be treated fairly if the repayment pattern is the same as that of the typical reinvestment that would have been made if their net inflow had not been diverted to cover the cash-flow deficiencies of the balance of the fund. The "borrowing" parts of the fund will also be treated fairly under this scheme if new-money interest rates in years when repayment is made do not depart significantly from the rate the loan bears.

On the basis of the preceding analysis, the following general model is postulated.

1. Investment operations are simulated by an investment-year method.
2. The model is detailed with respect to the initial investment strategy. The initial fund is assumed to be invested in n representative instruments, with a fraction p_j allocated to instrument j.
3. The model is deliberately less detailed with respect to the reinvestment of funds. Reinvestment or disinvestment in year k ($k \geq 2$) is made at the beginning of the year at new-money rate i_k. The pattern of asset maturities from this reinvestment or disinvestment is characterized by a vector of rollover rates r, component r_j specifying the fraction of the reinvestment repaid j years after the reinvestment is made.

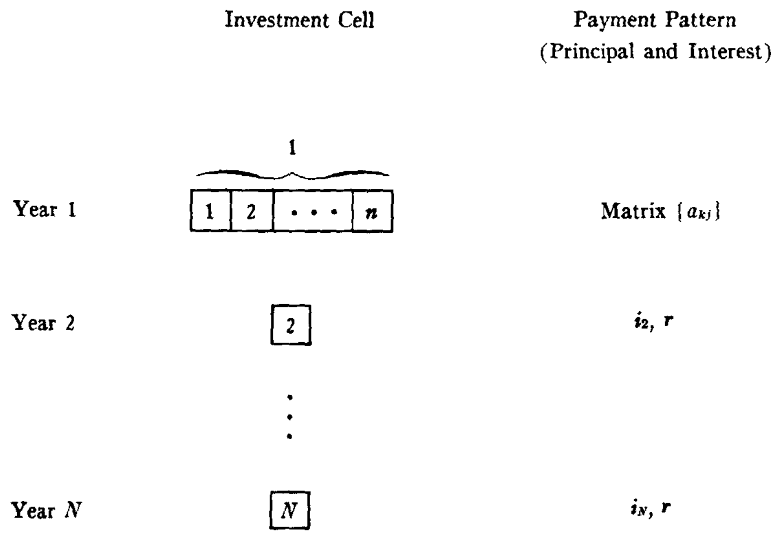

FIG. 1.—Pictorial representation of the investment model

The investment characteristics of the model are pictured in Figure 1. The use of a single vector of rollover rates for all future years is not necessary, but there is no compelling reason to complicate the model further.

Despite what was stated earlier, we now can see how to include the term structure of interest rates in the reinvestment of funds. Reinvestment cell k ($k \geq 2$) can be subdivided into cells according to the fraction of funds that will be rolled over 1, 2, . . . years from the time of reinvestment. These subcells can be characterized by yields, net of investment expenses, applicable to high-quality bonds with terms to maturity of 1, 2, . . . years, respectively. For example, cell k could be divided into subcells $k_1, k_2, \ldots, k_q$ bearing new-money rates (at the beginning of year k) $i_{k1}, i_{k2}, \ldots, i_{kq}$, respectively. Of the total funds invested in cell k at the start of year k, fractions $r_1, r_2, \ldots, r_q$ would be allocated to the respective subcells. The rates $i_{k1}, \ldots, i_{kq}$ describe the yield curve at the beginning of year k. However, the fact that the model can be generalized to account for the term structure of interest rates at times in the future does *not* mean that it should be. There is considerable uncertainty in the shape (rising, flat, inverted) and the slope of the yield curve. Expanding the model in such a fashion adds considerably to the complexity and arbitrariness of the assumptions that must be made, without enhancing the usefulness of the model to any great extent. Accordingly, the form depicted in Figure 1 is used in the remainder of the paper.

The investment horizon (shown as N years in Fig. 1) is the period until the last benefit is paid. This might occur beyond the time of the last scheduled maturity in the initial portfolio of assets. All equations in this paper assume that the assets of the initial portfolio mature fully at or before the investment horizon. It is trivial to modify the equations to cover the situation where the terms of some of the initial investments exceed the term of the contractual obligations of the fund. In such a case, the value of the fund at the investment horizon will include unmatured assets in the initial portfolio.

In circumstances where the contractual obligations of the fund extend far beyond the time of the last scheduled maturity in the initial portfolio of assets, it may be desirable to choose the investment horizon as the time of the last scheduled maturity. All cash flow beyond the investment horizon from other than investment operations can be discounted back to the horizon at a conservative rate of interest. The resulting present value of all obligations beyond the horizon should be included in the net cash flow CF_N^{out}. This approach is analogous to the familiar practice of calculating asset shares for only N years rather than to the end of the mortality table, and then "maturing" the remaining policies for the Nth-year reserve or cash value.

Let $\mathfrak{A}_k$ ($k = 2, 3, \ldots, N$) denote the amount invested in cell k at new-money rate i_k at the beginning of year k. The model permits $\mathfrak{A}_k$ to be negative as well as positive. The vector of rollover rates applicable to reinvestments is $r = (r_1, r_2, \ldots, r_q)$, with $\Sigma_{i=1}^{q} r_i = 1$. (In any expression in the remainder of this paper, it is to be understood that $r_i = 0$ if $i > q$.) Accordingly, the assets remaining in cell k at the start of years $k + 1$,

$k+2, \ldots$ are $\mathcal{A}_k(1-r_1)$, $\mathcal{A}_k(1-r_1-r_2), \ldots$. The funds available for investment at the start of year k ($k \geq 2$) are composed of the following:

1. Interest and maturities during year $k-1$ from the initial portfolio of assets, CF^{in}_{k-1}, *plus*
2. Interest during year $k-1$ on all reinvestments made in years 2, 3, $\ldots, k-1$, $\Sigma_{l=2}^{k-1} i_l \mathcal{A}_l(1-\Sigma_{j=1}^{k-l-1} r_j)$, *plus*
3. Maturities during year $k-1$ from all reinvestments made in years $2, 3, \ldots, k-1$, $\Sigma_{l=2}^{k-1} r_{k-l}\mathcal{A}_l$, *less*
4. Net cash outflow during year $k-1$ from all items not associated with investment operations, CF^{out}_{k-1}.

Thus,

$$\mathcal{A}_k = (CF^{\text{in}}_{k-1} - CF^{\text{out}}_{k-1}) + \sum_{l=2}^{k-1} \mathcal{A}_l\Big[r_{k-l} + i_l\Big(1 - \sum_{j=1}^{k-l-1} r_j\Big)\Big] . \qquad (3)$$

In equation (3) and any of the previous expressions, if the upper limit of summation is less than the lower limit of summation, the value of the sum is to be considered equal to zero. Equation (3) implies that $\mathcal{A}_k$ can be written in the form

$$\mathcal{A}_k = \sum_{l=1}^{k-1} \gamma_{kl}(CF_l^{\text{in}} - CF_l^{\text{out}}) \qquad (k \geq 2) , \qquad (4)$$

where the coefficients γ_{kl} depend only on rollover rates and interest rates. The coefficients γ_{kl} can be determined recursively.

$$\begin{aligned}\mathcal{A}_k &= (CF^{\text{in}}_{k-1} - CF^{\text{out}}_{k-1}) \\ &\quad + \sum_{l=2}^{k-1}\Big[\sum_{m=1}^{l-1} \gamma_{lm}(CF_m^{\text{in}} - CF_m^{\text{out}})\Big]\Big[r_{k-l} + i_l\Big(1 - \sum_{j=1}^{k-l-1} r_j\Big)\Big] \\ &= (CF^{\text{in}}_{k-1} - CF^{\text{out}}_{k-1}) \\ &\quad + \sum_{m=1}^{k-2}\Big\{\sum_{l=m+1}^{k-1}\Big[r_{k-l} + i_l\Big(1 - \sum_{j=1}^{k-l-1} r_j\Big)\Big]\gamma_{lm}\Big\}(CF_m^{\text{in}} - CF_m^{\text{out}}) .\end{aligned} \qquad (5)$$

The latter expression is derived from the former by changing the order of the summations over the dummy indexes l and m. From equation (4) it follows that

$$\begin{aligned}\gamma_{km} &= 0 , & m &> k-1 \\ &= 1 , & m &= k-1 \\ &= \sum_{l=m+1}^{k-1}\Big[r_{k-l} + i_l\Big(1 - \sum_{j=1}^{k-l-1} r_j\Big)\Big]\gamma_{lm} , & k-1 &> m \geq 1 .\end{aligned} \qquad (6)$$

For a given pattern of future new-money interest rates and rollover from reinvestments, the γ matrix can be determined. This matrix is independent of all other variables in the investment strategy problem.

Since the last scheduled maturity in the initial portfolio of assets occurs at or prior to the investment horizon, the amount of assets (valued at cost) at the investment horizon is

$$A_N = \sum_{k=2}^{N+1} \mathfrak{A}_k \left(1 - \sum_{i=1}^{N-k+1} r_i\right) . \tag{7}$$

It should be noted that A_N represents the total amount of assets at the *end* of year N in all reinvestment cells, while $\mathfrak{A}_k$ represents the amount of assets originally invested in cell k at new-money rate i_k at the *start* of year k.

Upon substituting for $\mathfrak{A}_k$ in terms of the γ-coefficients and remembering that $CF_l^{\text{in}} = \Sigma_{j=1}^{n} a_{lj} p_j$, we derive

$$\begin{aligned} A_N = & \sum_{k=2}^{N+1} \sum_{l=1}^{k-1} \gamma_{kl} \left(1 - \sum_{i=1}^{N-k+1} r_i\right) \left(\sum_{j=1}^{n} a_{lj} p_j\right) \\ & - \sum_{k=2}^{N+1} \sum_{l=1}^{k-1} \gamma_{kl} \left(1 - \sum_{i=1}^{N-k+1} r_i\right) CF_l^{\text{out}} , \end{aligned} \tag{8}$$

and, finally,

$$\begin{aligned} A_N = & \sum_{j=1}^{n} \left[\sum_{k=2}^{N+1} \left(1 - \sum_{i=1}^{N-k+1} r_i\right) \sum_{l=1}^{k-1} \gamma_{kl} a_{lj} \right] p_j \\ & - \sum_{k=2}^{N+1} \left(1 - \sum_{i=1}^{N-k+1} r_i\right) \sum_{l=1}^{k-1} \gamma_{kl} CF_l^{\text{out}} . \end{aligned} \tag{9}$$

Although equation (9) appears complicated, it is very neatly structured. The assets at the end of the investment horizon, A_N, are a linear function of the components of p that specify the allocation of initially investable funds among the n subcells of the initial portfolio. The coefficients of the variables p_j and the constant term are determined completely from the rollover rates for reinvestment r, the matrix $\{a_{lj}\}$ (specifying the pattern of interest payments and maturities from the n subcells of the initial portfolio), the matrix $\{\gamma_{kl}\}$ (determined solely from new-money interest rates for future years and rollover rates), and the vector of net liability cash outflows CF^{out}. When programming this model on a computer, one could write several different subroutines to handle the separate pieces of the calculation. A model-office or fund projection would be used to determine the net "liability" outflows. There would be a subroutine to define the classes of assets, representative investment instruments within those classes, and the resulting matrix $\{a_{lj}\}$ of interest and maturity payments. Finally, there would be a subroutine to compute the elements of the matrix $\{\gamma_{kl}\}$. A control program would put the pieces together to arrive at equation (9) for A_N. This fundamental equation, or equations based on it, would serve as input to the linear optimization calculation discussed next.

Let $\boldsymbol{i} = (i_2, i_3, \ldots, i_N)$ denote a pattern of new-money rates applicable to reinvestments at the start of years 2, 3, . . . , N. Let S denote a specified set of interest rate patterns $\boldsymbol{i}$. The generalized problem can be stated as follows:

> *Determine a region R of initial investment strategies $\boldsymbol{p}$, each of which results in nonnegative A_N for each $\boldsymbol{i}$ in S.*

For all $\boldsymbol{i}$ in S, every point $\boldsymbol{p}$ in the region R of feasible investment strategies must satisfy the constraint $A_N \geq 0$. For any given $\boldsymbol{i}$, A_N is a linear function of the components of $\boldsymbol{p}$. In Section VI it will be seen that, in many instances, the set S can be replaced by a finite set of interest rate patterns. Under such circumstances, the region R is defined by the following constraints:

$$p_j \geq 0\,, \qquad j = 1, 2, \ldots, n\,; \tag{10.1}$$

$$\sum_{j=1}^{n} p_j = 1\,; \tag{10.2}$$

$$A_N(\boldsymbol{i}_l, r; \boldsymbol{p}) \geq 0\,, \quad l = 1, 2, \ldots, m\,. \tag{10.3}$$

The set S has been replaced by the finite set of interest rate patterns $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots, \boldsymbol{i}_m\}$. Constraints (10.1) are known as nonnegativity constraints on the variables p_j. Equation (10.2), which states that we can invest only what we have, ensures that region R is bounded. Constraints (10.3) are the generalized asset-liability matching constraints and are linear in the variables p_j.

In practice, there will be further linear constraints on the initial investment strategy. For example, there might be supply constraints on certain investment vehicles, implying upper bounds on some of the p_j's. Also, the investment department may be unwilling to invest less than a certain fraction of funds in a particular asset class. The general model can be used to determine the extent to which such operational constraints limit the competitiveness of the insurer's products or increase its exposure to interest rate risk.

There may be other constraints that should be imposed on the initial investment strategy. Provided that such constraints are linear in the variables p_j, the problem can be handled by the linear optimization calculation described later in this section. For example, it may be desired not only to ensure the solvency of the fund under adverse patterns of future new-money interest rates but also to ensure that expected profit attains a certain minimum level. One way to introduce profit into the model is to include a specific "expense" charge for it in the annual liability outflows. Alternatively, profit can be defined in terms of the value of the fund at the investment horizon. Several equally likely interest rate patterns can be generated from a stochastic model, and the fund value A_N associated with each pattern can be determined. The arithmetic mean of these fund values is an estimate of the expected profit. The associated expected profit constraint is linear in the variables p_j.

The region R defined by constraints (10.1), (10.2), and (10.3) is known as a polytope and is the solution of the asset-liability matching problem. R is difficult to visualize if the number of variables exceeds three. Moreover, if other interest rate patterns are used in (10.3), or if the vector of reinvestment rollover rates r is changed, or if the actuarial assumptions underlying CF^{out} are altered, the region R will change also. It is useful to understand how R changes as the parameters of the problem are modified, but this is not easy as the solution is currently defined. Also, it is essential that the results of the calculation be capable of communication to other actuaries and to investment officers, for, if not, the model will never be implemented.

One resolution of the difficulties outlined above is to use a subset of R as the "solution" of the matching problem in lieu of R. A good candidate for this subset is the largest hypersphere that can be inscribed in R. A sphere is completely characterized by its center point $(\pi_1, \pi_2, \ldots, \pi_n)$ and radius ρ. In a crude sense the center of the sphere is the "center" of the region R and the radius of the sphere measures the "size" of R. Changes in R resulting from changes in the parameters of the problem lead to a new maximal in-sphere, which can be obtained from the former sphere by a translation of the center and an expanding or shrinking of the radius. Hence, the sensitivity of the region of feasible investment strategies to changes in the parameters is much easier to visualize. If R is not very regular in shape, the ratio of the volume of the maximal in-sphere to the volume of R is small and the maximal in-sphere is not a useful substitute for R; in this case, one is probably restricted to dealing with the explicit constraints defining R.

The equation of the maximal in-sphere is

$$\sum_{j=1}^{n} (p_j - \pi_j)^2 = \rho^2 . \tag{11}$$

The parameters π_j ($j = 1, 2, \ldots, n$) and ρ depend on the patterns of interest rates $i_1, i_2, \ldots, i_m$, the reinvestment rollover rates r, the pattern of interest and principal payments $\{a_{kj}\}$ from the initial portfolio of assets, and the actuarial assumptions underlying the liability cash flows CF^{out}. At the expense of eliminating some feasible investment strategies from R, the "solution" to the asset-liability matching problem has been expressed in a single equation. Once the π_j's and ρ are known, it is very easy to check whether a particular initial investment strategy lies within the sphere.

The paper referred to earlier [4] proves that the maximal in-sphere of a region defined by linear equality and inequality constraints is the solution of a standard linear programming problem. The proof is not duplicated here, but the Appendix to this paper includes an APL program that solves both standard linear optimization and associated maximal in-sphere problems.

Whenever asset-liability matching is discussed, the question arises as to whether a deterministic or stochastic model should be used. Undoubtedly both provide insight into the problem. Academicians seem to favor stochastic models, since these are representative of the manner in which the uncertain world actually works, while practitioners whose experience is based largely on evolved history tend to think in terms of specific scenarios and favor deterministic models. The academicians indicate that historical data give but one "snapshot" from the statistical ensemble of the ways things could have turned out and, at best, merely serve to eliminate certain stochastic models as incapable of explaining actual results. Since the future is uncertain, they argue, it is better to use a stochastic model and to make probability statements about eventualities. For the investment problem studied in this paper, the risks are associated with certain "adverse" patterns of future interest rates. It is important to determine how various investment strategies fare under these critical patterns so that strategies that eliminate or reduce the risk can be chosen. There may be less concern about how the strategies fare under fluctuating interest rate scenarios that lie between the critical patterns. A deterministic model is adequate in such circumstances, and it is this type of model that is used in this paper. Stochastic effects can be explored by scrutinizing particular strategies under several different interest rate patterns generated by a stochastic time series model, an approach used by Ziock in a paper investigating the effects of an inflationary economy on profits from nonparticipating insurance policies [7].

V. ABSOLUTE MATCHING

Before analyzing particular interest rate scenarios in the next section, it makes sense to mention briefly a model that is independent of the future interest rate environment (except for the possibility of prepayment or default of assets, or significant dependence of cash withdrawals on prevailing interest rates). This model is known in the literature as "absolute matching."

Absolute matching can be expressed mathematically as $CF_k^{\text{in}} = CF_k^{\text{out}}$ for $k = 1, 2, \ldots, N$. From equation (4) this implies that $\mathcal{A}_k = 0$ for $k = 2, 3, \ldots, N+1$, and from equation (7) this in turn implies that $A_N = 0$. These equalities are independent of the pattern of future new-money interest rates. Since interest and principal from the initial portfolio of assets match exactly the net cash outflow requirements at every duration up to the investment horizon, there are never any funds to invest during that period. It does not matter how interest rates behave.

The absolute matching conditions are very restrictive on investment policy. In fact, there may be no investment strategies that permit cash inflows and outflows to be matched exactly. The absolute matching constraints can be relaxed to the inequalities $CF_k^{\text{in}} \geq CF_k^{\text{out}}$ $(k = 1, 2, \ldots, N)$. A practical application of this more general "matching" model to the pricing of single premium immediate annuities is discussed in [4].

VI. INTEREST RATE SCENARIOS

A. *Level Reinvestment Rates*

Let the set S of interest rate patterns defined earlier consist only of level reinvestment rates $\boldsymbol{i} = (i_2, i_3, \ldots, i_N)$, with $i_2 = i_3 = \ldots = i_N = i$ and $i_L \leq i \leq i_U$. Since cells 2, 3, ..., N all bear new-money interest rate i, there is no distinction among them and they can all be replaced by a single reinvestment cell. Moreover, the exact pattern of rollover rates is inconsequential; the amount of assets at the end of the investment horizon is independent of the vector of rollover rates. Hence,

$$A_N = \sum_{k=1}^{N} (CF_k^{\text{in}} - CF_k^{\text{out}})(1 + i)^{N-k} . \qquad (12)$$

If A_N is divided by $(1+i)^N$, the resulting function of i, denoted by $S(i)$, is the gross premium valuation surplus.

$$S(i) = \sum_{k=1}^{N} v^k (CF_k^{\text{in}} - CF_k^{\text{out}}) . \tag{13}$$

Immunization theory attempts to mold the behavior of $S(i)$ as a function of i by choosing the initial portfolio of assets appropriately. Let i_0 be the point in the interval $[i_L, i_U]$ that is the best estimate of the average future reinvestment rate. Expand $S(i)$ in a Taylor series about i_0:

$$S(i) = S(i_0) + \left(\frac{dS}{di}\right)_{i_0}(i - i_0) + \frac{1}{2!}\left(\frac{d^2S}{di^2}\right)_{i_0}(i - i_0)^2 + \ldots . \tag{14}$$

By choosing the initial investment strategy so that

$$S(i_0) = \left(\frac{dS}{di}\right)_{i_0} = 0 \quad \text{and} \quad \left(\frac{d^2S}{di^2}\right)_{i_0} > 0 , \tag{15}$$

one may ensure that the surplus function will have a local minimum at i_0. Therefore, for small changes in i, either above i_0 or below i_0, $S(i)$ will be larger than $S(i_0)$. This is depicted in Figure 2. The question marks indicate that higher-order terms of the Taylor series are important for sufficiently large deviations from i_0, and there is no guarantee that $S(i)$ will not bend over and cross the i-axis. Conventional immunization theory gives no a priori information concerning the size of the interval about i_0 for which $S(i) \geq 0$. Of course, for a given investment strategy satisfying the immunization equations, the surplus function can be plotted. The

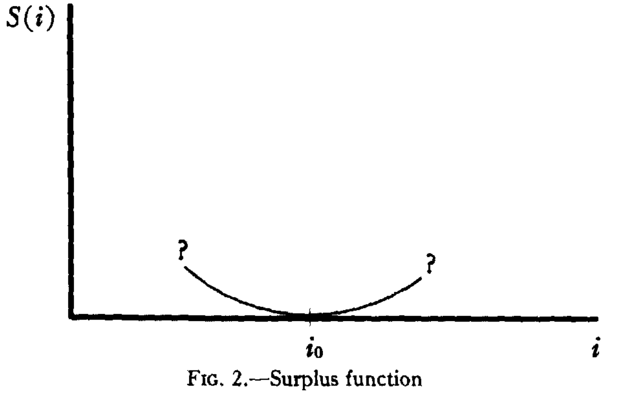

Fig. 2.—Surplus function

extent of the "upside" and "downside" protection can then be determined from the graph.

Constraints (15) can be expressed in terms of the cash inflows and cash outflows by using equation (13).

$$\sum_{k=1}^{N} v_0^k CF_k^{\text{in}} = \sum_{k=1}^{N} v_0^k CF_k^{\text{out}}\,; \tag{16.1}$$

$$\sum_{k=1}^{N} k v_0^k CF_k^{\text{in}} = \sum_{k=1}^{N} k v_0^k CF_k^{\text{out}}\,; \tag{16.2}$$

$$\sum_{k=1}^{N} k^2 v_0^k CF_k^{\text{in}} > \sum_{k=1}^{N} k^2 v_0^k CF_k^{\text{out}}\,. \tag{16.3}$$

Expressions (16) are the familiar forms of the conventional immunization equations.

The derivation of expressions (16) has assumed that the interest rate dependence of the surplus function lies only in the discount factor v. More generally, the cash inflows and outflows also depend on the interest rate. This can arise in the following ways: First, bonds with call provisions and mortgages with prepayment clauses may be refunded prior to their full term if interest rates drop sufficiently to make it attractive for the issuer to refinance his debt. This possibility means that CF_k^{in} will depend on the new-money interest rates $i_2, \ldots, i_{k+1}$ if the initial portfolio contains assets that permit early refunding. Second, the rate of cash withdrawals on investment contracts and the rate of policy loans on permanent insurance policies depend on the level of prevailing interest rates. This should be reflected in CF_k^{out}. If an appropriate functional dependence of CF_k^{in} and CF_k^{out} on i can be postulated, the immunization equations can be rederived taking explicit account of the derivatives of the cash flows with respect to the interest rate. These derivatives may dominate those arising from the discount factor.

There are two criticisms of conventional immunization theory that do not necessarily afflict the general model developed in this paper.

1. The immunization expressions (15) constraining the initial investment strategy are unduly restrictive because each *equality* constraint reduces the dimensionality of the region of feasible investment strategies. This can be alleviated somewhat by using the constraint $S(i_0) \geq 0$ instead of $S(i_0) = 0$.
2. The impact of reinvestment of funds in uncertain market conditions can be treated more accurately by using a model that permits different new-money rates in each future year and recognizes the pattern of rollover from reinvestments. If a nonlevel pattern of reinvestment rates is used, the repayment pattern of negative reinvestments (disinvestments) may be important.

The second criticism was addressed directly in the formulation of the general asset-liability matching model. The following discussion shows how alternatives to the immunization constraints can be used to remove the first criticism without sacrificing protection against interest rate risks.

Expanding $S(i)$ in a Taylor series about i_0 and ensuring that a local minimum in $S(i)$ occurs at i_0 is merely a convenient way to force $S(i)$ to be positive in a neighborhood of i_0. What is important is that $S(i)$ be positive in a neighborhood of i_0 whether or not a minimum occurs at i_0. Surplus function S_2 in Figure 3 protects against immediate and permanent changes in interest rates better than does S_1, even though S_2 has a relative maximum (not a minimum) at i_0.

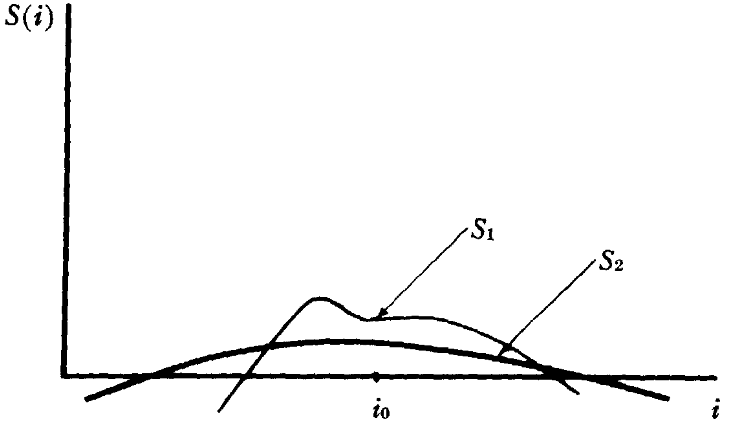

FIG. 3.—Alternative surplus functions

It is suggested that the pair of constraints

$$\begin{aligned} &S(i_L) \geq 0\,, \quad i_L < i_0 \quad \text{(downside risk)}\,, \\ &S(i_U) \geq 0\,, \quad i_U > i_0 \quad \text{(upside risk)} \end{aligned} \tag{17}$$

be used in place of the conventional immunization constraints. How should i_L and i_U be chosen? Provided that $S(i)$ is continuous in some neighborhood of i_0, and provided that there exists an investment strategy for which $S(i_0) > 0$, it is possible to find an $i_L < i_0$ and an $i_U > i_0$ such that $S(i) > 0$ for all i in the interval (i_L, i_U). If downside protection of Δi_L and upside protection of Δi_U are desired, then $i_L = i_0 - \Delta i_L$ and $i_U = i_0 + \Delta i_U$. The linear programming problem for the maximal insphere would then be solved to see whether there are any strategies satisfying constraints (17). If not, i_L will have to be increased and/or i_U decreased until a solution exists. For a given strategy satisfying constraints (17), it is necessary to plot the function $S(i)$ on the interval $[i_L, i_U]$ to verify that $S(i)$ has no roots on the interior of the interval. The example in Section VII illustrates these points.

B. *Increasing or Decreasing Reinvestment Rates*

Consider the patterns of interest rates shown in Figure 4. For each pattern in this family of patterns, the amount of assets at the end of the

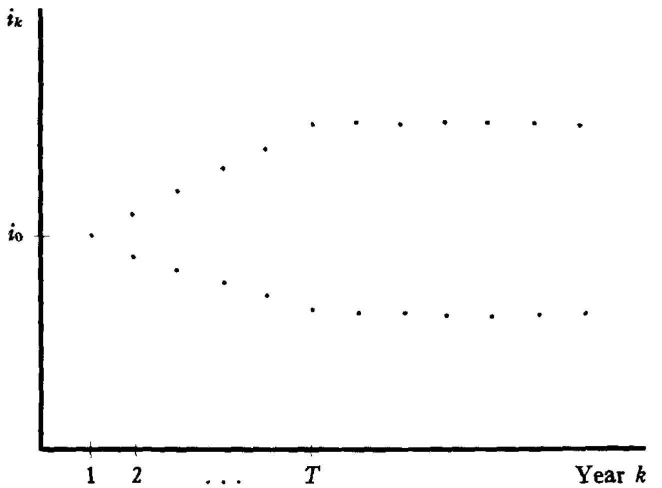

FIG. 4.—The family of new-money interest rate patterns is characterized by the equation $i_k = i_0 + [\min(k, T) - 1]\Delta i$, with $k = 1, 2, \ldots$. Increasing and decreasing patterns occur for $\Delta i > 0$ and $\Delta i < 0$, respectively.

investment horizon is a function of i_0, Δi, and T. These parameters control where the interest rates start, how steeply they rise or fall, and the point at which they level off, respectively. The investment problem might be posed as finding strategies that satisfy $A_N(i_0, \Delta i, T; p) \geq 0$ for a given i_0 and all Δi and T, subject to $\Delta i_L \leq \Delta i \leq \Delta i_U$ and $T = 1, 2, \ldots, T_U$, respectively, where $\Delta i_L < 0$ and $\Delta i_U > 0$.

For a fixed value of T, say T_0, $A_N(i_0, \Delta i, T_0; p)$ can be expanded in a Taylor series:

$$A_N(\Delta i) = A_N(0) + \left(\frac{dA_N}{d\Delta i}\right)_0 \Delta i + \frac{1}{2!}\left(\frac{d^2A_N}{d\Delta i^2}\right)_0 (\Delta i)^2 + \ldots \quad (18)$$

The explicit dependence of A_N on i_0, T_0, and p has been suppressed. The "immunization" conditions are

$$A_N(0) \geq 0\,, \quad \left(\frac{dA_N}{d\Delta i}\right)_0 = 0\,, \quad \left(\frac{d^2A_N}{d\Delta i^2}\right)_0 > 0\,. \tag{19}$$

In many situations the two critical patterns are the steepest rise and fall over the longest period; that is, $\Delta i = \Delta i_U$ and $\Delta i = \Delta i_L$ with $T = T_U$. The corresponding critical constraints are

$$\begin{aligned} A_N(i_0, \Delta i_U, T_U; p) &\geq 0 \quad \text{(increasing rates)}\,, \\ A_N(i_0, \Delta i_L, T_U; p) &\geq 0 \quad \text{(decreasing rates)}\,. \end{aligned} \tag{20}$$

When the problem has been solved, the function $A_N(i_0, \Delta i, T; p)$ can be computed on the two-dimensional region $\Delta i_L < \Delta i < \Delta i_U$, $T = 1, 2, \ldots, T_U$ to determine whether A_N has any roots in the region. This approach is bound to be more fruitful than that based on the Taylor series expansion of A_N. First, both "matching" constraints in equation (20) are inequalities rather than equalities, so the dimensionality of the region of feasible solutions is not reduced unnecessarily. Second, under conditions analogous to those described for level interest rate patterns, it is certain that values of Δi_L and Δi_U can be found for which $A_N(\Delta i, T)$ has no roots in the region $\Delta i_L < \Delta i < \Delta i_U$, $T = 1, 2, \ldots, T_U$. However, under those same conditions, there is no guarantee that "immunization" strategies exist.

C. *Other Patterns*

It is useful to solve the initial strategy problem for a set of interest rate scenarios that includes a few patterns with level reinvestment rates at high and low levels, a few patterns with increasing and decreasing rates that ultimately level off, and adverse patterns of fluctuating rates derived from a stochastic time series model. Each pattern gives rise to a linear constraint $A_N \geq 0$. Some of these constraints may be redundant, but that poses no theoretical difficulty in solving the linear programming problem for the maximal in-sphere of initial investment strategies. However, computer core limitations, execution time, and round-off error will be important if the number of constraints is large.

VII. AN EXAMPLE

The ideas presented in this paper can be given much more meaning through an example. The following example involves the determination of a level interest guarantee for a three-year period on a deposit fund similar to a flexible premium annuity. The purpose of the example is to illustrate the investment aspects of the problem, so most other features are glossed over or ignored.

The contract holder can deposit funds at any time. A certain percentage (called the load) is assessed against each deposit and provides for commissions to the sales representative and for some expenses of handling the contract. In addition, there may be periodic service fees charged to the contract, but they are not considered here. The net deposit—the original deposit less the load—accumulates at a guaranteed rate of interest (compounded annually) for three years, at which point the net deposit with accumulated interest may be rolled over into a new three-year interest guarantee to be established at that time. The product allows partial or total withdrawal of funds at the end of contract years *without* any asset-liquidation or surrender charges, regardless of the prevailing level of new-money interest rates. The market for this product is expected to consist of investors having large amounts to invest and desiring complete security of principal at a rate of return that is competitive for the one- to three-year portion of the yield curve.

This product carries two obvious investment risks:

1. If interest rates rise significantly above the guaranteed rate, it is expected that there will be massive cash withdrawals as investors look elsewhere to reinvest their funds in low-load or no-load vehicles bearing the current high yield. The fund would be faced with the possibility of liquidating assets at a capital loss. Since the contract provides for security of principal and a guaranteed rate of return, the company would bear the entire loss.
2. The risk described in item 1 suggests that funds be invested short to provide sufficient early maturities to pay off withdrawals if interest rates rise sharply. However, if such a strategy is followed and interest rates drop, there will be few withdrawals and the fund will be forced to reinvest substantial amounts when interest rates are low. The fund may then be unable to earn interest at a rate equal to that guaranteed.

When interest rates are rising, there is likely to be a large inflow of new funds coupled with extensive withdrawals on in-force contracts. Therefore, the situation discussed in item 1 above may not be so much a matter of actual asset liquidation as a matter of recognizing the cash-flow positions of the several parts of the total fund so that an equitable allocation of investment income can be made. Since the rate of interest is guaranteed, it might seem that a method for allocating investment income is unnecessary, but without such an approach the company has no way of quantifying its gain and loss positions on the several parts of the fund.

The investment officers suggest that net deposits be invested in one-, two-, and three-year government notes. The notes pay semiannual coupons, but for purposes of this example it will be assumed that interest payments are made annually. Principal is fully repaid at the maturity date. The initial portfolio consists of three cells.

Cell	Coupon Rate
1-year notes..............	g_1
2-year notes..............	g_2
3-year notes..............	g_3

Each dollar of net deposit is allocated among the three cells according to the fractions p_1, p_2, and p_3, respectively, with $p_1 + p_2 + p_3 = 1$. The

TABLE 1

Cell	Pattern of Principal and Interest Payments		
	End of Year 1	End of Year 2	End of Year 3
1-year notes........	$1+g_1$	0	0
2-year notes........	g_2	$1+g_2$	0
3-year notes........	g_3	g_3	$1+g_3$

pattern of cash inflow is shown in Table 1. Thus, the components of CF^{in} are given by

$$CF_1^{\text{in}} = (1 + g_1)p_1 + g_2p_2 + g_3p_3 ,$$

$$CF_2^{\text{in}} = (1 + g_2)p_2 + g_3p_3 ,$$

$$CF_3^{\text{in}} = (1 + g_3)p_3 .$$

Since service fees and taxes are being ignored, the only cash outflows are fund withdrawals. Let w_1 and w_2 denote the withdrawal rates by amount (including both partial and full withdrawals) at the ends of years 1 and 2, respectively. Because all funds are withdrawn or rolled over at the end of three years, $w_3 = 1$. Let i_g denote the guaranteed interest rate. The cash outflows per dollar of net deposit are

$$CF_1^{\text{out}} = w_1(1 + i_g) ,$$

$$CF_2^{\text{out}} = w_2(1 - w_1)(1 + i_g)^2 ,$$

$$CF_3^{\text{out}} = (1 - w_2)(1 - w_1)(1 + i_g)^3 .$$

A crucial assumption in the determination of the guaranteed interest rate is the dependence of the rates of cash withdrawal on the prevailing new-money interest rate. It is assumed that cash withdrawals will amount to 10 percent per year as long as interest rates remain at the level of the guarantee or drop below that level. To be conservative, it is estimated that withdrawals will reach 40 percent if interest rates rise 2 percent above the guarantee and will level off at 70 percent if interest rates soar 5 percent or more above the guarantee. The qualitative aspects of this behavior are sketched in Figure 5. The graph resembles a cumula-

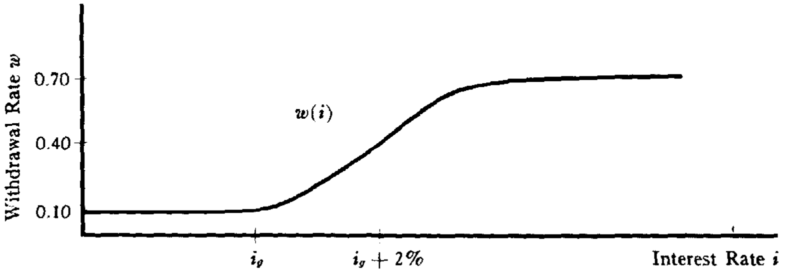

FIG. 5.—Dependence of cash withdrawal rate on new-money interest rate

tive probability distribution function. Let $F(i)$ denote such a function. Then

$$w(i) = 0.10 + 0.60F(i) ,$$

where

$$F(i) = \int_{-\infty}^{i} dF(i') .$$

Suppose $F(i)$ is chosen to be the cdf of the normal distribution with mean at $i_g + 0.02$ and standard deviation 0.01. Then

$$w(i) = 0.10 + \frac{0.60}{0.01\sqrt{(2\pi)}} \int_{-\infty}^{i} \exp\left[-\frac{1}{2}\left(\frac{i' - i_g - 0.02}{0.01}\right)^2\right] di' .$$

The first and second derivatives of $w(i)$ with respect to i can be computed easily for use in the conventional immunization constraints.

For this example it is assumed that the current yields of one-, two-, and three-year government notes, net of investment expenses, are 7.50, 7.75, and 8.00 percent, respectively. The interest guarantee will lie somewhere between 7.50 and 8.00 percent. In attempting to find conventional immunization strategies, we discovered that the interest rate dependence of the withdrawal rate completely overwhelms the interest rate dependence of the discount factor. It is *not* possible to select an investment strategy that results in a local minimum of $S(i)$ at $i = i_g$. Attempts to match cash inflows with cash outflows at each duration are also thwarted by the interest rate dependence of the withdrawal rates. For fixed withdrawal rates, a guaranteed rate can be chosen so that there is a nonempty region of investment strategies for which cash inflows equal or exceed cash outflows at the ends of years 1, 2, and 3, but these cash flows are not matched if interest rates rise or fall significantly, because the withdrawal rates also change.

It is sometimes claimed that conventional immunization theory cannot be used when contracts contain guaranteed cash values. This criticism is not strictly valid, but it does apply to the example presented here. By appealing to the general asset-liability matching model developed in this paper, the investment strategy problem for the deposit fund can be solved.

Let A_3 denote the total assets at the end of the third year. The general problem involves finding strategies for which $A_3 \geq 0$ under specified patterns of future new-money interest rates. Suppose these patterns are restricted to level reinvestment rates, corresponding to immediate and permanent (three years) changes in the level of interest rates. Let i denote such a level rate. The problem is to determine investment strategies for which $A_3(i) \geq 0$ for all i in some specified interval about the guaranteed rate i_g. To be specific, suppose that most forecasts call for increasing interest rates over the next year. It is desired to protect the fund against immediate increases of 2 percent or less and decreases of 1 percent or less. The "solution" is the maximal in-sphere of the region defined by the following linear constraints.

$$p_1 \geq 0\,, \quad p_2 \geq 0\,, \quad p_3 \geq 0\,;$$

$$p_1 + p_2 + p_3 = 1\,;$$

$$A_3(i_g - 0.01; p_1, p_2, p_3) \geq 0 \quad \text{(downside risk)}\,,$$

$$A_3(i_g + 0.02; p_1, p_2, p_3) \geq 0 \quad \text{(upside risk)}\,.$$

The problem was solved for i_g = 7.50, 7.55, 7.60, 7.65, and 7.70 percent. For $i_g \geq 7.71$ percent there are no investment strategies satisfying both the downside and the upside constraint. Table 2 and Figure 6 summarize the results. In Figure 6, the investment strategy associated with a particular surplus function is the center of the maximal in-sphere corresponding to the guaranteed interest rate that identifies its graph.

As the interest rate guarantee is increased, it becomes increasingly more difficult to find investment strategies that cover the interest rate risks on both the downside and the upside. This can be seen in the shrinking of the radius of the maximal in-sphere and the graphs of the surplus function. It is helpful to view the results pictorially. Figure 7 is applicable to an interest guarantee of 7.50 percent. The triangle shown with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1) is the part of the plane $p_1 + p_2 + p_3 = 1$ lying in the positive octant $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$. The shaded part of the triangle is the region of investment strategies satisfying the downside constraint when $i_g = 7.50$ percent. The downside constraint forces us to invest long—that is, toward vertex (0, 0, 1). Similarly, as shown in Figure 8, the upside constraint forces us to invest short—toward vertex (1, 0, 0). The upside and downside constraints fight each other.

TABLE 2

Guaranteed Interest Rate	Center of Maximal In-Sphere			Radius
	π_1	π_2	π_3	
7.50%.....	0.209	0.179	0.612	0.219
7.55.......	0.242	0.133	0.625	0.163
7.60.......	0.271	0.089	0.640	0.109
7.65.......	0.298	0.045	0.657	0.055
7.70.......	0.322	0.002	0.676	0.002

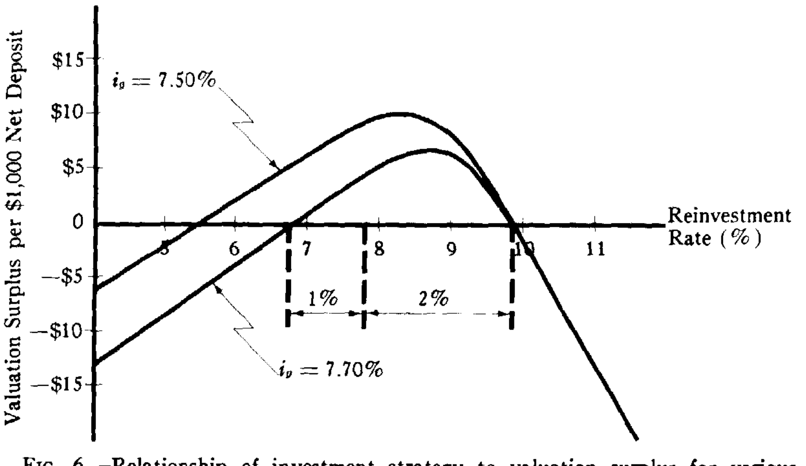

FIG. 6.—Relationship of investment strategy to valuation surplus for various guaranteed interest rates.

The region of feasible investment strategies is the compromise solution of this fight. That region and its maximal in-sphere are shown in Figure 9.

As the interest guarantee is increased, the fight between the downside and the upside constraints intensifies until, for $i_g > 7.70$ percent, there are no feasible solutions. Upside protection of 2 percent and downside protection of 1 percent cannot be provided simultaneously if the guarantee exceeds 7.70 percent. Figure 10 illustrates how the region of feasible investment strategies shrinks as the interest guarantee increases toward 7.70 percent. The locus of the center points is shown as a dotted line.

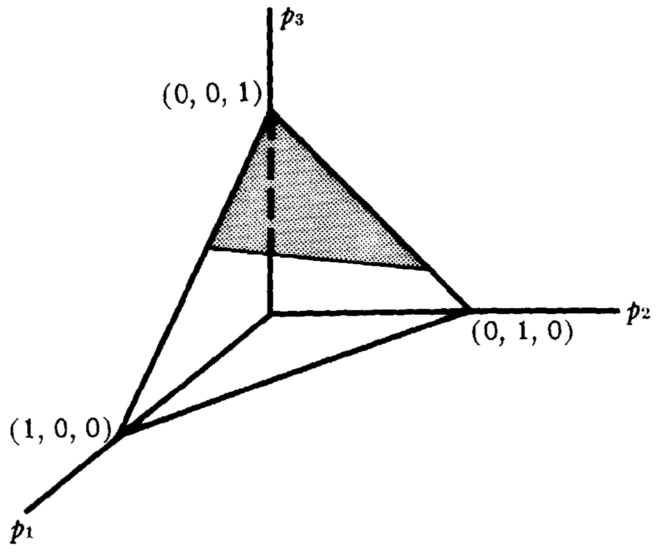

FIG. 7.—Shaded area represents region of investment strategies satisfying downside constraint.

The analysis of this example would not be complete without consideration of other interest rate patterns. Rollover from reinvestments affects the constraints of the general model only when future interest rates are not level. Rollover rates used in the calculations presented in the remainder of this section were determined iteratively to be consistent with the center point of the maximal in-sphere for the interest guarantee at the maximum value for which feasible solutions exist. For this limiting situation, roll-off from reinvestments is approximately the same as that from the initial investment. It should be pointed out that A_3 depends only on the first component of the rollover vector.

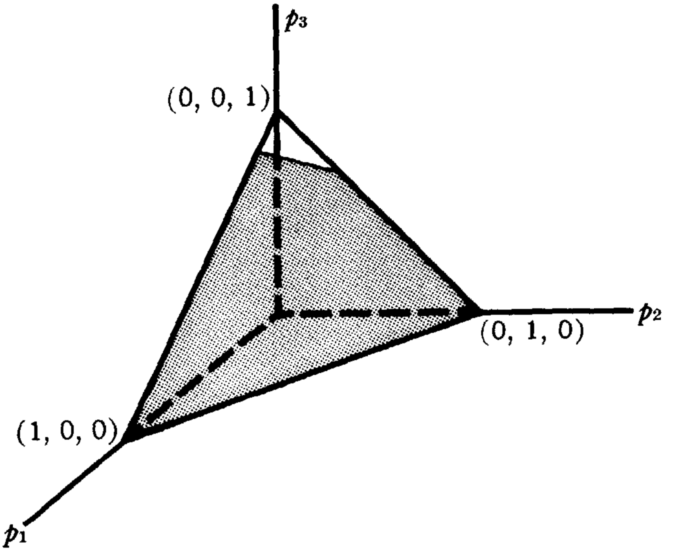

FIG. 8.—Shaded area represents region of investment strategies satisfying upside constraint.

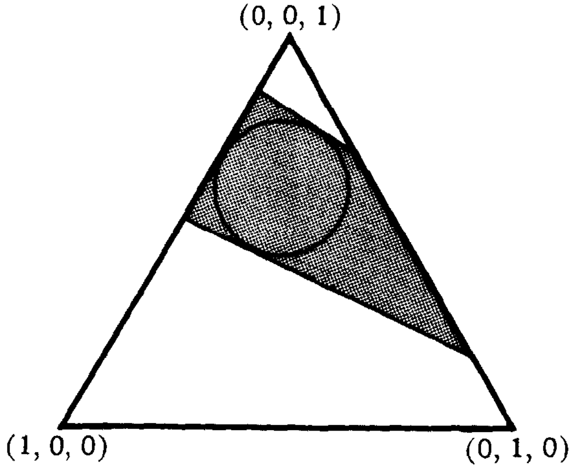

FIG. 9.—Shaded area shows region of feasible investment strategies and its maximal in-sphere.

Table 3 gives the solution of the investment strategy problem when $r = (0, \frac{1}{2}, \frac{1}{2})$ and there are two interest rate patterns—an increasing one with $i_2 = i_g + 1\frac{1}{2}$ percent and $i_3 = i_g + 3$ percent, and a decreasing one with $i_2 = i_g - \frac{3}{4}$ percent and $i_3 = i_g - 1\frac{1}{2}$ percent. For $i_g \geq 7.68$ percent there are no feasible solutions.

Although the limiting values of the interest guarantees under the two different sets of interest rate patterns are very close, the limiting investment strategies are quite different. Both strategies lie on the boundary of the triangle formed by the intersection of the coordinate planes with the plane $p_1 + p_2 + p_3 = 1$. For the level patterns, the limiting strategy

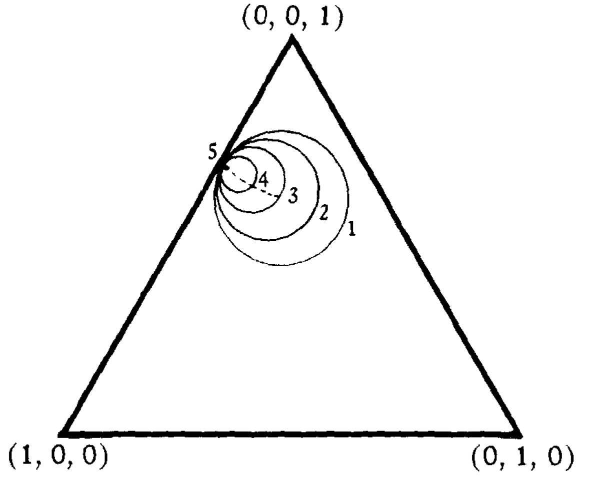

FIG. 10.—Maximal in-spheres 1, 2, 3, 4, and 5 correspond to guaranteed interest rates of 7.50, 7.55, 7.60, 7.65, and 7.70 percent, respectively.

TABLE 3

Guaranteed Interest Rate	Center of Maximal In-Sphere			Radius
	π_1	π_2	π_3	
7.50%.....	0.182	0.236	0.582	0.223
7.55.......	0.133	0.304	0.563	0.163
7.60.......	0.083	0.370	0.547	0.102
7.65.......	0.031	0.436	0.533	0.038
7.67.......	0.010	0.461	0.529	0.012

is approximately $\pi = (\frac{1}{3}, 0, \frac{2}{3})$, while for the increasing and decreasing patterns the limiting strategy is approximately $\pi = (0, \frac{1}{2}, \frac{1}{2})$. How should the initial investment strategy be chosen?

The obvious way to proceed is to analyze the problem under a set of patterns that includes both immediate and permanent changes in the level of interest rates, and also increasing and decreasing patterns. Table 4 gives solutions that satisfy $A_3 \geq 0$ for each of the four patterns examined previously, but this time considered simultaneously. The vector of roll-over rates is (0.167, 0.298, 0.535). There are no feasible solutions for $i_g \geq 7.66$ percent.

The center of the maximal in-sphere does not move much as the interest guarantee is raised from 7.50 to 7.65 percent. The limiting point

TABLE 4

Guaranteed Interest Rate	Center of Maximal In-Sphere			Radius
	π_1	π_2	π_3	
7.50%.....	0.177	0.243	0.580	0.208
7.55.......	0.177	0.262	0.561	0.139
7.60.......	0.173	0.281	0.546	0.071
7.65.......	0.167	0.298	0.535	0.005

does not lie on the boundary of the previously mentioned triangle. Increasing the interest guarantee merely decreases the size of the region of feasible investment strategies; it does not change appreciably the investment strategy on which the region is "centered." The model has pointed to a relatively unambiguous way to invest the net deposits regardless of the actual interest guarantee.

Weighting the net yields of the one-, two-, and three-year government notes by the corresponding allocation of investable funds shown in the last line of the immediately preceding table results in

$$(0.167 \times 7.50\%) + (0.298 \times 7.75\%) + (0.535 \times 8.00\%) = 7.84\% .$$

To cover the interest rate risks associated with guaranteeing 7.65 percent for three years and providing annual withdrawal privileges without asset-liquidation (surrender) charges, it is necessary to strip 19 basis points from the initial portfolio's weighted net rate of return.

VIII. SUMMARY

A recent paper by Shedden [3] has examined immunization theory under a deterministic model of the yield curve, and another by Boyle [1] has used a stochastic model for the term structure of interest rates. It is important to indicate why the general model developed in this paper appears to attach less significance to the term structure. The above papers are concerned with the relationship of immunization to the valuation of the company's assets and liabilities. If an asset is to be valued on a liquidation basis, the prevailing yield appropriate to its term must be used to discount future interest and principal payments in order to arrive at the proper market value. Thus, the structure of the prevailing yield curve is essential to valuing assets at market. This paper has focused on the investment strategy problem, not the valuation problem. Currently prevailing yield curves and yield spreads are recognized explicitly in the matrix $\{a_{kj}\}$ describing the n representative instruments of the initial portfolio. For reasons stated in Sections III and IV, no attempt is made in the reinvestment problem to identify specific investment instruments or to recognize the term structure of interest rates. The effects of reinvestments are characterized by a single interest rate in each future year, together with a vector of rollover rates.

This paper has been concerned with the extent to which the requirements of asset-liability matching constrain the setting of investment policy. The accompanying example illustrated the connection between product design and investment strategy. In the context of determining investment strategy, the following two criticisms can be made of conventional immunization theory:

1. Conventional immunization theory restricts investment strategy unnecessarily.
2. Only immediate and permanent changes in the level of interest rates are considered in the conventional theory.

The solution of the investment problem is a region of feasible investment strategies. Regarding the first criticism, the general model produces regions of higher dimensionality than does immunization theory. Moreover, there are situations in which no conventional immunization strategies exist but for which there are other strategies that provide adequate protection against interest rate risks. To assist in visualizing the region of feasible strategies and to facilitate communication of the results between investment officers and actuaries, the idea of a "maximal sphere" of initial investment strategies was introduced. The example proved the value of this construct.

The general model presented in this paper is not subject to the second limitation above. It is based on an investment-year method, and, theoretically, any number of patterns of future new-money interest rates can be included in the model, although the size of the resulting linear programming problem may pose computational difficulties.

The coordination of investment operations and insurance operations is an area that is ripe for research and creative thinking. There are many ways in which the general model can be expanded. In particular, it would be interesting to explore the stochastic version. From a practical viewpoint, however, it is more important to adapt the model as it stands to real-life situations.

IX. ACKNOWLEDGMENTS

The author would like to express his gratitude to Robert P. Clancy, Judith Markland, and Joseph A. Tomlinson for their constructive criticism of early drafts of this paper. Incorporating their suggestions has led to a clearer and more accurate exposition.

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APPENDIX

Consider the general linear programming problem with n variables and m constraints.

Maximize (Minimize) the function

$$y = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to the constraints

$$(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n)\ r_1\ (b_1)\ ,$$

$$(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n)\ r_2\ (b_2)\ ,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$(a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n)\ r_m\ (b_m)\ ,$$

where each r_i is one of $\leq$, $=$, or $\geq$.

The APL function *SIMPLEX* included in Exhibit I solves the above problem using the simplex method. The left argument A of the function is the "character" vector $(r_1, r_2, \ldots, r_m)$. The right argument B of the function is the following numeric matrix:

$$\begin{bmatrix} c_1 & c_2 & \ldots & c_n & 0 \\ a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ . & . & . & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

The function *SIMPLEX* has been designed to determine the radius and center point of the largest sphere that can be inscribed in the closed convex region defined by the constraints. To indicate that the maximal in-sphere is desired, all elements of the first row of the right-hand argument of the function should be set equal to zero. The function *SIMPLEX* invokes the functions *SPHERE* and *UNIQUE*.

The programs as listed were written for an MCM-800. Minor modifications are required before they can be used on any of the APL time-sharing systems. In particular, the function SPHERE invokes the function *INV* in line 10. Apart from returning the explicit result 0 (scalar) if its argument is a singular matrix, the function *INV* is identical with the monadic domino primitive ⌹.

EXHIBIT I

```
     ∇ A SIMPLEX B ;M;N;NN;MIN;MAX;REG;ART;BASIS;IN;OUT;CTR;R;S
[1]    ⍝
[2]    ⍝THE FIRST PART OF THE PROGRAM VALIDATES THE INPUT AGAINST CONFORMABILITY CONDITIONS.
[3]    ⍝
[4]    →(2=⍴⍴B)/⎕LC+2
[5]    →0 ∘ ⎕←'RIGHT ARGUMENT MUST BE RANK 2'
[6]    N←¯1+(⍴B)[2] ∘ M←¯1+(⍴B)[1]
[7]    →(0=B[1;N+1])/⎕LC+2
[8]    →0 ∘ ⎕←'LAST ELEMENT OF FIRST ROW OF RIGHT ARGUMENT MUST BE ZERO'
[9]    →(1=⍴⍴A)/⎕LC+2
[10]   →0 ∘ ⎕←'LEFT ARGUMENT MUST BE RANK 1'
[11]   →(∧/A∊'≥=≤')/⎕LC+2
[12]   →0 ∘ ⎕←'LEFT ARGUMENT MUST BE LITERAL AND CONTAIN ONLY ≥, =, AND ≤'
[13]   →(M=⍴A)/⎕LC+2
[14]   →0 ∘ ⎕←'LENGTH OF LEFT ARGUMENT MUST BE ONE LESS THAN NO. OF ROWS IN RIGHT ARGUMENT'
[15]   ⍝
[16]   ⍝THE FOLLOWING PART OF THE PROGRAM SETS UP THE INITIAL SIMPLEX TABLEAU,
[17]   ⍝THE FIRST ROW OF WHICH IS ADJUSTED ACCORDING TO THE 'BIG M' METHOD.
[18]   ⍝
[19]   MIN←~MAX←1
[20]   →L1×⍳∧/B[1;]=0
[21]   →L1×⍳(MAX←∧/'MAX'=R)∨MIN←∧/'MIN'=R←¯3↑⍞'MAX OR MIN PROBLEM? '
[22]   'INVALID RESPONSE'
[23]   →⎕LC-2
[24] L1:B←B×⍉((N+1),M+1)⍴R←(MIN-MAX),1+(B[;N+1]≥0)-B[;N+1]<0
[25]   A←(1↓R)×('≤'=A)-'≥'=A
[26]   →L2×⍳REG←∨/B[1;]≠0
[27]   →0×⍳~SPHERE
[28] L2:B←1⌽(¯1⌽B),((⍳M+1)∘.=1+⍳M),(0,R)⍀-(⍳+/R)∘.=⍳+/R←A=¯1
[29]   R←(~(⍳NN)∊N+ART←(A≠1)/⍳M)/⍳NN←(⍴B)[2]
[30]   B[1;R]←B[1;R]-+⌿B[ART+1;R]×1000×⌈/|B[1;]
```

EXHIBIT I—*Continued*

```
 ⍝
 ⍝THE NEXT PART OF THE PROGRAM USES THE SIMPLEX ALGORITHM TO
 ⍝DETERMINE A FINITE OPTIMAL BASIC FEASIBLE SOLUTION, IF ONE EXISTS.
 ⍝
 BASIS←N+⍳M ⋄ CTR←1
LOOP:→END×⍳¯1E¯20≤B[1;IN←R⍳⌊/R←¯1↓B[1;]]
 →END×⍳0=⍴R←(B[;IN]>1E¯20)/⍳M+1
 BASIS[¯1+OUT←R[S⍳⌊/S←B[R;NN]÷B[R;IN]]]←IN
 B←B-(B[;IN]×OUT≠⍳M+1)∘.×B[OUT;]←B[OUT;]÷B[OUT;IN]
 →LOOP ⋄ CTR←CTR+1
END:→CHECK×⍳~∨/1E¯20<|0,B[1+BASIS⍳(BASIS∊N+ART)/BASIS;NN]
 →0 ⋄ ⎕←'NO FEASIBLE SOLUTION'
CHECK:→L3×⍳0<⍴R
 →0 ⋄ ⎕←'UNBOUNDED SOLUTION'
 ⍝
 ⍝THE LAST PART OF THE PROGRAM PRINTS THE OUTPUT.
 ⍝
L3:→L4×⍳REG
 ⎕←'RADIUS OF MAXIMAL IN-SPHERE: ',⍕B[1;NN]
 ⎕←'COORDINATES OF CENTER POINT'
 ⎕←'      VARIABLE      VALUE'
 ⎕←(2 3⍴10 0 0 14 0 ¯1)⍕⍉(2,N-1)⍴(⍳N-1),(N-1)↑((⍳NN-1)∊BASIS)\B[1+⍋BASIS;NN]
 →L5
L4:⎕←'BASIS VARIABLE      VALUE'
 ⎕←(2 3⍴9 0 0 16 0 ¯1)⍕⍉(2,M)⍴BASIS[R],B[1+R←⍋BASIS;NN]
 ⎕←((6⍴MAX,MIN)/'MMAIXN'),'IMUM VALUE OF OBJECTIVE FUNCTION: ',⍕B[1;NN]×MAX-MIN
L5:→L6×⍳∧/1E¯10<|(~(⍳NN-1)∊BASIS)/¯1↓B[1;]
 UNIQUE
L6:⎕←'NUMBER OF ITERATIONS: ',⍕CTR
∇
```

EXHIBIT I—*Continued*

```
    ∇ Z←SPHERE ;C;D;J;K;V
[1]    ⍝
[2]    ⍝SUBPROGRAM OF 'SIMPLEX' MODIFYING THE CONSTRAINT VECTOR 'A' AND
[3]    ⍝COEFFICIENT MATRIX 'B' DEFINED IN PROGRAM 'SIMPLEX' SO THAT
[4]    ⍝THE MAXIMAL IN-SPHERE CAN BE FOUND.  THE EXPLICIT RESULT
[5]    ⍝IS 1 IF NO DIFFICULTIES ARE ENCOUNTERED, OTHERWISE 0.
[6]    ⍝
[7]    B←((1,A≠0)⌿B)⍪(-(⍳N)∘.=⍳N+1)⍪C←(0,A=0)⌿B
[8]    K←+/0≠A←((A≠0)/A),(N⍴1),(A=0)/A
[9]    V←-J←,1
[10] L1:→L2×⍳1E¯6<Z←|1⍴INV D+.×⍉D←0 ¯1↓B[J+1;]⍪C
[11]   →Z←0 ⋄ ⎕←'NO FEASIBLE SOLUTION'
[12] L2:V←V,A[J]÷Z*.5
[13]   →L1×⍳K≥J←J+1
[14]   B←1⌽(¯1⌽B),((M+N+1),1)⍴V,(M+N-K)⍴0
[15]   N←N+Z←1 ⋄ M←M+N
    ∇
```

EXHIBIT I—*Continued*

```
    ∇ UNIQUE ;R;T;X
[1]    ⍝
[2]    ⍝
[3]    ⍝LET 'X' BE THE MATRIX FORMED BY KEEPING ONLY THOSE COLUMNS OF THE FINAL
[4]    ⍝SIMPLEX TABLEAU ASSOCIATED WITH NON-BASIC VARIABLES AND FOR WHICH THE
[5]    ⍝FIRST ELEMENT IS ZERO.  THEN, THE OPTIMAL SOLUTION IS NOT UNIQUE IF AND
[6]    ⍝ONLY IF FOR SOME COLUMN OF 'X', EITHER
[7]    ⍝
[8]    ⍝    (1) THERE ARE ONLY NON-POSITIVE ELEMENTS, OR
[9]    ⍝    (2) THE MINIMUM OF THE RATIOS OF THE ELEMENTS OF THE LAST COLUMN
[10]   ⍝        OF THE FINAL SIMPLEX TABLEAU TO THE CORRESPONDING ELEMENTS OF
[11]   ⍝        THE COLUMN OF 'X' (IGNORING NON-POSITIVE DENOMINATORS) IS
[12]   ⍝        POSITIVE.
[13]   ⍝
[14]   X←(1E¯10≥X[T←1;])/X←(~(⍳NN-1)∊BASIS)/0 ¯1↓B
[15]  TEST:→LOOP×⍳1E¯10≥⌊/B[R;NN]÷X[R←(X[;T]>1E¯10)/⍳M+1;T]
[16]   →0 ◊ ⎕←'SOLUTION IS NOT UNIQUE'
[17]  LOOP:→TEST×⍳(¯1↑⍴X)≥T←T+1
    ∇
```

DISCUSSION OF PRECEDING PAPER

BENTTI O. HOISKA:

Mr. Tilley's paper provides a very clear and useful explanation of his extension of immunization theory. This discussion reformulates his theory for the special case of absolute matching (Sec. V) and shows that this reformulation allows the solution set for the absolute-matching problem to be narrowed greatly. Furthermore, the solution set for the reformulated problem has a minimum-cost (economic) interpretation instead of a maximal-sphere (geometric) interpretation. Finally, we show that the maximal in-sphere may fail to contain the minimum-cost absolute-matching solution.

Absolute matching occurs when the cash flow from initial investments (typically fixed income) covers the benefit payments due in each future year. If the final benefit payment is due before the latest maturity date of the available assets, then absolute matching can be achieved simply by investing a large amount in each available asset. Obviously, such a crude approach can be very expensive. The best approach is to find the mix of initial assets that allows benefits to be matched at minimum cost. This is the approach described below.

We start by introducing some additional notation. Recalling that k denotes years and j denotes asset classes, let

a_{kj} = Cash flow (interest plus principal) in year k, per unit of asset j;
$A = (a_{kj})$ = Cash-flow matrix (N rows by n columns);
b_k = Projected benefit payments due in year k;
x_j = Units of asset j initially purchased;
$x = (x_1, \ldots, x_n)$ = Asset-mix vector;
p_j = Current market price of a unit of asset j;
$p = (p_1, \ldots, p_n)$ = Market price vector;
F = Total market value of initial assets
$= p_1x_1 + \ldots + p_nx_n$.

The absolute-matching problem now can be formulated as the following linear program.

Minimize

$$F = p_1x_1 + \ldots + p_nx_n$$

subject to

$$a_{k1}x_1 + \ldots + a_{kn}x_n \geq b_k \quad (k = 1, \ldots, N),$$

$$x_j \geq 0 \quad (j = 1, \ldots, n).$$

Using vectors and matrices, we can rewrite this linear program more concisely as

Minimize

$$F = p \cdot x$$

subject to

$$Ax \geq b, \quad x \geq 0 .$$

The objective is to minimize the initial dollar investment F while satisfying the cash-flow constraints. The first N inequalities require that cash flow cover benefits in each year k. The n nonnegativity constraints prohibit negative investments (i.e., "short" positions). These constraints are similar to Tilley's equations (1) and (2).

Tilley's approach is to determine the sphere with maximal radius that can be inscribed in the polytope of feasible solutions to the linear program. The idea is to give the investment department some discretion in picking the asset-mix vector x. But, as we have argued, there is no need to give the investment department discretion in the absolute-matching case, because the minimum-cost solution is best. (However, see remark 3 below.)

It is easy to show that minimizing the initial investment is equivalent to maximizing the yield on the initial fund. In effect, the linear program examines the shape of the current yield curve to find the highest-yielding assets. If the yield curve is positively sloped, the maturity structure described by the solution has maximum duration. If the yield curve is inverted, the opposite is true. If the yield curve is flat, all feasible solutions are optimal. These statements can be proved using the duality theory for linear programming.

Four final remarks:

1. Since Tilley's in-sphere does not completely fill the feasible set (polytope), it is possible for the minimum-cost asset-mix vector x to fall outside this sphere. In that case, absolute matching is not achieved at minimum cost if the asset-mix vector is restricted to the maximal in-sphere.
2. By minimizing the required initial investment, our approach allows the company to charge the lowest annuity premiums or, alternatively, to realize the largest contribution to surplus for a given premium schedule.
3. Although the set of immunized strategies has been narrowed greatly (perhaps to a single point), the above formulation still leaves the investment department with an important role to play. Indeed, the investment department must specify the list of investment candidates j ($j = 1, \ldots, n$) among which funds are to be allocated. Presumably, this list will be determined on the basis of such considerations as quality rating, call protection, and diversification. Diversification requirements determine the number of

assets n to be considered. Furthermore, if no more than a specified amount can be allocated to a particular asset, constraints of the form $x_j \leq c_j$ ($j = 1, \ldots, n$) must be added to the linear program.

Since the linear programming solution depends on the level and shape of the yield curve, the investment department also can exercise its market-timing judgment to indicate whether investments should be made now or later. However, anticipating yield-curve shifts runs counter to the spirit of immunization.

4. Absolute matching is usually associated with single premium immediate annuities. However, a noninsured pension fund can use the above approach to immunize the benefits for retired lives. The solution specifies the minimum size and composition of the segregated fund that, when set aside, will meet the retired life liability given current investment opportunities.

(AUTHOR'S REVIEW OF DISCUSSION)

JAMES A. TILLEY:

I would like to thank Mr. Hoiska for his discussion of my paper. The absolute-matching conditions he describes are identical with those developed in my paper "Achieving Consistency between Investment Practice and Investment Assumptions for Single Premium New-Money Products" (reference 4 of the paper). My principal motivation for constructing a more general asset-liability matching model was a dissatisfaction with the impracticalities of absolute matching:

1. In many circumstances, the liability cash flows extend beyond the latest maturity of the securities under consideration and absolute matching cannot be achieved.
2. Implicit in the formulation of absolute matching is the assumption that both asset and liability cash flows are independent of prevailing interest rates. This certainly is not true of assets having call or prepayment provisions, nor is it true of products where cash withdrawals or loans against policy values are permitted. Even the liabilities of a closed group of pensioners may be linked indirectly to interest rate movements through cost-of-living adjustments.
3. Absolute matching will be upset if there are substantial deviations from the *expected* liability cash flow, as might occur in a pension fund if retirees live longer than originally assumed.
4. Absolute matching is such a restrictive investment strategy that feasible solutions do not exist in many situations. Often a more flexible immunization strategy can be used whether or not absolute matching can be used.

The paper mentioned previously highlights some difficulties in achieving absolute matching for single premium immediate annuities.

Mr. Hoiska states that it is possible for his minimum-cost asset-mix vector x to fall outside the maximal in-sphere of investment strategies. If the sphere has a positive radius, this is not only possible but certain! Unless there are multiple optimal solutions, his minimum-cost solution must lie at a corner point of the polytope of feasible strategies, and no inscribed sphere can touch a corner point. As was pointed out in my paper, the sphere is useful only if there is no natural objective function to optimize and if the polytope is sufficiently regular that the sphere contains a large proportion of all feasible investment strategies. However, in many practical applications, these considerations are moot because the actuary is forced to price his product so competitively that only a single feasible strategy (if any) exists.

In general, it is not true that the best of all investment strategies satisfying given asset-liability matching constraints is the one with maximum yield to maturity. For example, often one must sacrifice going-in yield to be able to offer the highest interest guarantee for a guaranteed investment contract while meeting all the matching constraints. A better goal than maximum initial yield might be maximum expected surplus at the end of a specified period.