On a Class of Discrete Time Renewal Risk Models

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Abstract

We consider a class of compound renewal (Sparre Andersen) risk process with claim waiting times have a discrete K_m distribution (i.e., the probability generating function (p.g.f.) of the distribution function is a ratio of two polynomials of order at most $m \in \mathbb{N}^+$). The classical compound binomial risk model is a special case when $m = 1$. Both recursive and explicit formulas are derived for the expected discounted penalty function due at ruin, for the surplus before ruin and the deficit at ruin.

Many ruin related quantities can be analyzed through the penalty function, e.g., ruin probability, the p.g.f. of the time of ruin, joint and marginal distributions of the surplus before ruin and the deficit at ruin, claim causing ruin, as well as their moments.

Detailed discussions are given in two special cases: claim sizes are rationally distributed, or the claim sizes distribution have a finite support.

Keywords: Sparre Andersen risk process; K_n family of distributions; Martingale; Generating function; Generalized Lundberg equation; Recursive formula

1 Introduction

Problems associated with the calculation of ultimate ruin probabilities, for the continuous time risk model, have received considerable attention in recent years. These include studies of the distribution of the ruin time (finite-time ruin probabilities), the surplus before ruin and the deficit at ruin, as well as moments of these variables.

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We explore analogue problems, but in the discrete time risk model. A recursive formula for the expected discounted penalty due at ruin is given, using the tool of generating functions, instead of the Laplace transform used for the continuous time model in the recent contributions This discounted penalty depends on the deficit at ruin and the surplus just before ruin. Hence, our recursive formula yields the joint distribution of the three random variables time to ruin, the surplus just before ruin and the deficit at ruin.

Given the discrete nature of our model, probability generating functions (p.g.f.) are used throughout to analyze the time of ruin and its associated random variables. The joint distribution for the compound binomial model is derived in Cheng et al. (2000) using martingale techniques and a duality argument. Li and Garrido (2002) gives a recursive formula for the expected discounted penalty function for the compound binomial risk model. This paper extends the classical compound binomial risk model to a class of discrete time Sparre Andersen risk models.

These results can give a better understanding of their analogues in the continuous time model, but they are also of independent interest. They fill a gap in the scant literature on discrete time risk theory models. Our formulas are readily programmable in practice, while they can still reproduce the continuous versions as limiting cases.

2 Model Description and Notation

Consider the discrete time Sparre Andersen risk process

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i \,, \quad n = 1, 2, \dots \,,$$

where $u \in \mathbb{N}$ is the initial reserve. The X_i are i.i.d. random variables with common probability function (p.f.) $p(x) = P(X = x)$, for $x = 1, 2, \dots$, denoting the i-th claim amount. Denote by $\mu_k = E[X^k]$ the k-th moment of X and by $\hat{p}(s) = \sum_{i=1}^{\infty} s^i p(i)$, $s \in \mathbb{C}$ its p.g.f.. The counting process $\{N(n); n \in \mathbb{N}\}$ denotes the number of claims up to time n and is defined as $N(n) = \max\{k : W_1 + W_2 + \cdots + W_k \leq n\}$, where the claim waiting times W_i are assumed i.i.d. with common probability function $k(x) = P(W = x)$, for $x = 1, 2, \dots$. Denote by $\hat{k}(s) = \sum_{i=1}^{\infty} s^i k(i)$, $s \in \mathbb{C}$ its p.g.f.

We assume that $\{W_i; i \in \mathbb{N}^+\}$ and $\{X_i; i \in \mathbb{N}^+\}$ are independent, and $E(W) = (1 + \theta)E(X) = (1 + \theta)\mu$, in order to have a positive loading factor.

Now define (the possibly defective) random variable $T = \min\{n \in \mathbb{N}^+ : U(n) < 0\}$ to be the ruin time,

$$\Psi(u) = P(T < \infty \,|\, U(0) = u) \ , \quad u \in \mathbb{N} \ ,$$

to be the ultimate ruin probability and

$$\psi(u,n) = P(T = n \,|\, U(0) = u) \ , \quad n = 1, 2, 3, \dots \ ,$$

to be the ruin probability at time t.

Consider $f_3(x, y, t \,|\, u) = P\{U(T-1) = x,\ |U(T)| = y,\ T = t \,|\, U(0) = u\}$, $x \in \mathbb{N}$, $y \in \mathbb{N}^+$, the joint probability function of the surplus just before ruin, deficit at ruin and ruin time. Let $v \in (0,1)$ be the (constant) discount factor over one period and define $f_2(x, y \,|\, u) = \sum_{t=1}^{\infty} v^t f_3(x, y, t \,|\, u)$ as a discounted joint p.d.f. of $U(T-1)$ and $|U(T)|$. Similarly, denote by $f(x \,|\, u) = \sum_{y=0}^{\infty} f_2(x, y \,|\, u)$. The usual conditional probability formulas give the following relation:

$$f_2(x, y \,|\, u) = f(x \,|\, u) \frac{p(x+y+1)}{\bar{P}(x+1)} \ , \quad x \in \mathbb{N}, \ y \in \mathbb{N}^+.$$

Let $w(x, y)$, $x, y = 0, 1, 2, \dots$ be the non-negative values of a penalty function. For $0 < v < 1$, define

$$\phi(u) = E\left[v^T w\big(U(T-1), |U(T)|\big) I(T < \infty) \;\middle|\; U(0) = u\right] \ , \quad u \in \mathbb{N} \ . \tag{1}$$

The quantity $w\big(U(T-1), |U(T)|\big)$ can be interpreted as the penalty at the time of ruin for the surplus $U(T-1)$ and deficit $|U(T)|$. Then $\phi(u)$ is the expected discounted penalty if v is viewed as a discount rate.

The main objective for the rest of the paper is to evaluate the expected discounted penalty function ϕ.

3 An Operator of Discrete Functions

This section gives the definition of an operator to a real valued function with domain in the positive integers (see Dickson and Hipp (2001) for the continuous version of the operator).

Define T_r to be an operator of any real valued function $f(x), x \in \mathbb{N}^+$, by

$$T_r f(y) = \sum_{x=y}^{\infty} r^{x-y} f(x) = \sum_{x=0}^{\infty} r^x \, f(x+y), \quad r \in \mathbb{C}, \ y \in \mathbb{N}^+. \tag{2}$$

Like for the continuous operator T_r in Dickson and Hipp (2001), its discrete restriction has many nice properties, which are helpful to simplify calculations, e.g.,

1. $T_r\, f(1) = \frac{\hat{f}(r)}{r}$, where $\hat{f}(r)$ is the generating function of f.
2. $T_1\, f(y) = \sum_{x=y}^{\infty} f(y)$.
3. If r_1 and r_2 are distinct, then
$$T_{r_2} T_{r_1}\, f(y) = \frac{r_2\, T_{r_2}\, f(y) - r_1\, T_{r_1}\, f(y)}{r_2 - r_1}. \tag{3}$$
4. If r_1 is equal to r_2, then
$$\begin{aligned} T_r^2\, f(y) &= T_r\, T_r f(y) = \lim_{r_1 \to r} T_{r_1} T_r f(y) = \lim_{r_1 \to r} \frac{r_1\, T_{r_1}\, f(y) - r\, T_r\, f(y)}{r_1 - r} \\ &= \frac{d\,[r\, T_r f(y)]}{dr} = \sum_{x=y}^{\infty} (x - y + 1) r^{x-y} f(x). \end{aligned} \tag{4}$$
5. If $r_1, r_2, \ldots, r_k$ are distinct, then
$$T_{r_k} T_{r_{k-1}} \cdots T_{r_1} f(y) = \sum_{j=1}^{k} \frac{r_j^{k-1} T_{r_i} f(y)}{\pi_k'(r_j)}, \tag{5}$$
where $\pi_k(s) = \prod_{i=1}^{k} (s - r_i)$. While its p.g.f. transform is given by
$$s\, T_s T_{r_k} T_{r_{k-1}} \cdots T_{r_1} f(1) = \left[\prod_{i=1}^{k} \frac{s}{s - r_i} \right] \hat{f}(s) - \sum_{j=1}^{k} \left(\frac{s}{s - r_j} \right) \frac{r_j^{k-1}\, \hat{f}(r_j)}{\pi_k'(r_j)}.$$
6. If $r_i = r,$ for $i = 1, 2, \ldots, k,$
$$T_r^k\, f(y) = \underbrace{T_r\, T_r \cdots T_r}_{k} f(y) = \lim_{s \to r} T_s\, T_r^{k-1} = \frac{d[r T_r^{k-1} f(y)]}{dr}. \tag{6}$$

4 On Martingales and a Generalized Lundberg Equation

Let $\tau_k = \sum_{j=1}^{k} W_j$ be the arrival time of the k-th claim and $U_k = U(\tau_k)$ be the surplus immediately after k-th claim. Defining $\tau_0 = 0$ gives $U_0 = u$, and for $k = 1, 2, \ldots,$

$$U_k = U(\tau_k) = u + \tau_k - \sum_{j=1}^{k} X_j = u + \sum_{j=1}^{k} [W_j - X_j].$$

We seek a number $s \in \mathbb{C}$ such that the process:

$$\left\{ v^{\tau_k}\, s^{-U_k};\ k \in \mathbb{N} \right\} \tag{7}$$

will form a martingale. Here the martingale condition is equivalent to

$$E[v^{W_1}\, s^{X_1 - W_1}] = E\left[(v/s)^{W_1}\, s^{X_1}\right] = E\left[(v/s)^{W_1}\right] E\left[s^{X_1}\right] = 1,$$

which is

$$\hat{k}(v/s)\, \hat{p}(s) = 1. \tag{8}$$

Equation (8) is a generalized version of Lundberg equation.

In the rest of this paper, we assume that the claim inter-arrival times have a discrete K_m distribution, i.e., the p.g.f. of $k(x)$, $x \in \mathbb{N}^+$ can be expressed as

$$\hat{k}(s) = \frac{s[\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1} \beta_j (s-1)^j]}{\prod_{i=1}^{m}(1 - s\, q_i)}, \tag{9}$$

where $0 < q_i < 1$, for $i = 1, 2, \ldots, m$, and the coefficients $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are such that $\hat{k}'(s) > 0$, $s \in (0, 1)$, to guarantee that $k(x)$, $x \in \mathbb{N}^+$ is a p.f.. The mean and second factorial moment of the claim inter-arrival times r.v.'s are thus given by

$$E(W) = \hat{k}'(1) = 1 + \sum_{i=1}^{m} \frac{q_i}{(1-q_i)} + \frac{\beta_1}{\prod_{i=1}^{m}(1-q_i)}. \tag{10}$$

$$\begin{aligned} E[W^{(2)}] \quad &= \quad \hat{k}''(1) = \frac{2\beta_2 + \beta_1 \sum_{i=1}^{m} \frac{q_i}{(1-q_i)}}{\prod_{i=1}^{m}(1-q_i)} \\ &\quad + E(W) \sum_{i=1}^{m} \frac{q_i}{(1-q_i)} + \sum_{i=1}^{m} \left(\frac{q_i}{1-q_i} \right)^2, \end{aligned} \tag{11}$$

where $x^{(2)} = x(x-1)$ is the second factorial power of x.

This class of distributions includes, as special cases, the shifted geometric, shifted or truncated negative binomial, as well as linear combinations (including mixture) of these.

In particular, if $q_1, q_2, \ldots, q_m$ are distinct, by partial fractions, k can be expressed as a linear combination of m geometric distributions with parameters q_i :

$$k(x) = \sum_{i=1}^{m} \theta_i \, (1-q_i) \, q_i^{x-1}, \qquad x = 1, 2, \ldots, \tag{12}$$

where θ_i are such that $\sum_{i=1}^{m} \theta_i = 1$ and given explicitly by

$$\theta_i = \frac{\sum_{k=1}^{m-1} \beta_k (1/q_i - 1)^k + \prod_{j=1}^{m} (1-q_j)}{(1-q_i) \left[\prod_{j=1,\, j \neq i}^{m} (1 - q_j/q_i) \right]}. \tag{13}$$

Under the assumption that $\hat{k}(s)$ is given by (9), the generalized Lundberg equation $\hat{k}(v/s)\, \hat{p}(s) = 1$ simplifies to

$$\begin{aligned} \gamma(s) : \; &= \frac{1}{\hat{k}(v/s)} \qquad (14) \\ &= \frac{\prod_{i=1}^{m} (s - v\, q_i)}{v \left[s^{m-1} \prod_{i=1}^{m} (1-q_i) + \sum_{j=1}^{m-1} \beta_j s^{m-1-j} \, (v-s)^j \right]} = \hat{p}(s), \qquad s \in \mathbb{C}. \end{aligned}$$

The roots of the equation above play a key role in this paper, and are discussed in the following theorem.

Theorem 1 For $0 < v < 1$, and $m \in \mathbb{N}^+$, equation (14) has exactly m roots, say $\rho_i(v)$, $i = 1, 2, \ldots, m$ with $0 < |\rho_i| < 1$.

Remarks:

1. Define $l(s) := \hat{p}(s) - \frac{1}{\hat{k}(v/s)}$. Since $l(1) < 0$ and $\lim_{s \to \infty} l(s) = +\infty$, then if $p(x)$ is sufficiently regular, $l(s) = 0$ has one root greater than 1. Hence denote by $R(v)$, which can be called a generalized *adjustment coefficient.*

2. $R(v) \to R(1)$, as $v \to 1^-$, and $\rho_j(v) \to \rho_j(1)$, for $1 \leq j \leq m$, where $R(1)$ and $\rho_j(1)$ are roots to $\frac{1}{\hat{k}(1/s)} = \hat{p}(s)$.

3. For simplicity, $R(v)$ and $\rho_j(v)$ are denoted by R and ρ_j, for $1 \leq j \leq m$ and $0 < v < 1$.

5 Probability Generating Function

Conditioning the time and amount of the first claim, one obtains that for $u \in \mathbb{N}$:

$$\phi(u) = E[v^{W_1}\phi(U_1)] = E[v^{W_1}\, \phi(u + W_1 - X_1)] = \sum_{t=1}^{\infty} v^t\, k(t) E[\phi(u + t - X_1)].$$

Now define $\hat{\phi}(s) = \sum_{u=0}^{\infty} s^u \phi(u)$ to be the p.g.f. transform of ϕ, then by (15),

$$\begin{aligned}
\hat{\phi}(s) &= \sum_{u=0}^{\infty} s^u \phi(u) = \sum_{u=0}^{\infty} s^u \sum_{t=1}^{\infty} v^t\, k(t) E[\phi(u + t - X_1)] \\
&= \sum_{u=0}^{\infty} s^u \sum_{y=u+1}^{\infty} v^{y-u}\, k(y-u)\, E[\phi(y - X_1)] \\
&= \sum_{y=1}^{\infty} v^y\, E[\phi(y - X_1)] \sum_{u=0}^{y-1} (s/v)^u\, k(y-u) \\
&= \sum_{y=1}^{\infty} s^y\, E[\phi(y - X_1)] \sum_{t=1}^{y} (v/s)^t\, k(t). \qquad (15)
\end{aligned}$$

If $q_1, q_2 \ldots, q_m$ are distinct, then $k(t)$ has the form in (12). Substituting it into (15) yields

$$\begin{aligned}
\hat{\phi}(s) &= \sum_{i=1}^{m} \frac{\theta_i\,(1-q_i)}{q_i} \sum_{y=1}^{\infty} s^y\, E[\phi(y - X_1)] \sum_{t=1}^{y} (v/s)^t\, q_i^t \\
&= \sum_{i=1}^{m} \frac{\theta_i\,(1-q_i)(v/s)}{[1-(v/s)\, q_i]} \left\{ \sum_{y=1}^{\infty} s^y\, E[\phi(y - X_1)] - \sum_{y=1}^{\infty} (v\, q_i)^y\, E[\phi(y - X_1)] \right\} \\
&= \hat{k}(v/s) \sum_{y=1}^{\infty} s^y\, E[\phi(y - X_1)] - \sum_{i=1}^{m} \frac{\theta_i(1-q_i)\, v\, b_i}{(s - v\, q_i)}, \qquad (16)
\end{aligned}$$

where $b_i = \sum_{y=1}^{\infty} (v\, q_i)^y\, E[\phi(y - X_1)]$. By definition of $\phi(u)$,

$$E[\phi(y - X_1)] = \sum_{x=1}^{y} \phi(y-x)\, p(x) + \sum_{x=y+1}^{\infty} w(y-1, x-y)\, p(x). \qquad (17)$$

For simplicity, let $\omega(y) = \sum_{x=y+1}^{\infty} w(y-1, x-y)\, p(x)$. Substituting (17) into (16) yields

$$\hat{\phi}(s) = \frac{\hat{k}(v/s)\, \hat{\omega}(s) - \sum_{i=1}^{m} \frac{\theta_i(1-q_i)\, v\, b_i}{(s - v\, q_i)}}{[1 - \hat{k}(v/s)\, \hat{p}(s)]}, \qquad (18)$$

where $\hat{\omega}(s) = \sum_{y=1}^{\infty} s^y \, \omega(y)$. Multiplying both denominator and numerator by $\gamma(s) = \frac{1}{\hat{k}(v/s)}$, (18) can be rewritten as

$$\hat{\phi}(s) = \frac{\hat{\omega}(s) - \frac{Q_{m-1}(s)}{v\left[s^{m-1}\prod_{i=1}^{m}(1-q_i)+\sum_{j=1}^{m-1}\beta_j s^{m-1-j}\,(v-s)^j\right]}}{[\gamma(s) - \hat{p}(s)]}, \tag{19}$$

where $Q_{m-1}(s) = [\prod_{i=1}^{m}(s - v\,q_i)]\left[\sum_{i=1}^{m} \frac{\theta_i(1-q_i)\,v\,b_i}{(s-v\,q_i)}\right]$ is a polynomial of degree $m-1$ or less. Since $\hat{\phi}(s)$ is finite for all s such that $0 < |\Re(s)| < 1$, the numerator on the right hand of (19) must be zero whenever the denominator is zero. Then $Q_{m-1}(s)$ can be determined by the linear system for $j = 1, 2, \ldots, m$,

$$Q_{m-1}(\rho_j) = \hat{\omega}(\rho_j)\left\{v\left[{\rho_j}^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{t=1}^{m-1}\beta_t\,{\rho_j}^{m-1-t}\,(v-\rho_j)^t\right]\right\}.$$

Further, if $\rho_1, \rho_2, \ldots, \rho_m$ are distinct, by the *Lagrange interpolation formula,* one obtains

$$Q_{m-1}(s) = \sum_{j=1}^{m} c_j\,\hat{\omega}(\rho_j)\left[\prod_{k=1,k\neq j}^{m}\frac{(s-\rho_k)}{(\rho_j-\rho_k)}\right], \tag{20}$$

where $c_j = v\left[{\rho_j}^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{t=1}^{m-1}\beta_t\,{\rho_j}^{m-1-t}\,(v-\rho_j)^t\right]$, for $j = 1, 2, \ldots, m$.

We remark that if some q_i are equal, formula (19) still holds, and (20) still holds for the case where $\rho_1, \rho_2, \ldots, \rho_m$ are distinct, by the continuity property.

6 Analysis when $u = 0$

We now turn to finding ruin related quantities when $u = 0$. For simplicity, we assume that the $\rho_1, \rho_2, \ldots, \rho_m$ are distinct. First

$$\begin{aligned}
\phi(0) &= \lim_{s\to 0}\hat{\phi}(s) = \lim_{s\to 0}\frac{\hat{\omega}(s) - \frac{Q_{m-1}(s)}{v\left[s^{m-1}\prod_{i=1}^{m}(1-q_i)+\sum_{j=1}^{m-1}\beta_j s^{m-1-j}\,(v-s)^j\right]}}{[\gamma(s)-\hat{p}(s)]} \\
&= \frac{\sum_{j=1}^{m} c_j\,\hat{\omega}(\rho_j)\,\left[\prod_{k=1,k\neq j}^{m}\frac{\rho_k}{\rho_j-\rho_k}\right]}{v^m\,\prod_{i=1}^{m} q_i} \\
&= \left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\sum_{j=1}^{m}\frac{c_j\,\hat{\omega}(\rho_j)}{\rho_j\prod_{k=1,k\neq j}^{m}(\rho_j-\rho_k)}.
\end{aligned} \tag{21}$$

Since $\omega(y) = \sum_{x=y+1}^{\infty} w(y-1, x-y)\, p(x) = \sum_{t=1}^{\infty} w(y-1,t)\, p(y+t)$, and then

$$\hat{\omega}(s) = \sum_{y=1}^{\infty} s^y \omega(y) = \sum_{y=1}^{\infty}\sum_{t=1}^{\infty} s^y\, w(y-1,t) p(y+t) = \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} s^{x+1} w(x,y)\, p(x+y+1),$$

therefore, (21) can be rewritten as

$$\phi(0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} {\rho_j}^{x+1} w(x,y)\, p(x+y+1)}{\rho_j \prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}. \tag{22}$$

On the other hand,

$$\begin{aligned} \phi(0) &= E[v^T w(U(T-1), |U(T)|) I(T<\infty) | U(0) = 0] \\ &= \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} w(x,y) f_2(x,y\,|\,0). \end{aligned} \tag{23}$$

Comparing these two formulas yields

$$f_2(x,y\,|\,0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, {\rho_j}^x\, p(x+y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \qquad x \in \mathbb{N},\, y \in \mathbb{N}^+, \tag{24}$$

so

$$f_1(x\,|0) = \sum_{y=1}^{\infty} f_2(x,y|0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, {\rho_j}^x\, \bar{P}(x+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \qquad x \in \mathbb{N}, \tag{25}$$

where $\bar{P}(x+1) = P(X > x+1) = \sum_{y=x+2} p(y)$, finally,

$$\begin{aligned} g(y) := g(y\,|\,0) &= \sum_{x=0}^{\infty} f_2(x,y\,|\,0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j \sum_{x=0}^{\infty} {\rho_j}^x\, p(x+y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \\ &= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, T_{\rho_j} p(y+1)}{\prod_{k=1,k\neq j}^{m} (\rho_j - \rho_k)}, \quad y \in \mathbb{N}^+, \end{aligned} \tag{26}$$

where T_r is an operator defined in section 3.

The function g is a defective distribution function. It plays a very important role in this paper. Define $\hat{g}(s) = \sum_{y=1}^{\infty} s^y\, g(y)$ to be the generating function of g, then we have the following Lemma.

Lemma 1 The generating function of g is given by

$$\hat{g}(s) = 1 - \frac{\prod_{i=1}^m (s - v\,q_i) - v\,\hat{p}(s)[s^{m-1}\prod_{i=1}^m (1-q_i) + \sum_{j=1}^{m-1} \beta_j s^{m-1-j}\,(v-s)^j]}{\left(\prod_{i=1}^m \frac{v\,q_i}{\rho_i}\right)\prod_{i=1}^m (s-\rho_i)}. \tag{27}$$

Using Lemma 1, one obtains

$$\begin{aligned}
\phi_T(0) &= E[v^T I(T<\infty)|U(0)=0] = \sum_{y=1}^{\infty} g(y) = \lim_{s\to 1}\hat{g}(s) \\
&= 1 - \left[\prod_{i=1}^m \frac{\rho_i}{v\,q_i}\right]\left[\frac{\prod_{i=1}^m (1-v\,q_i) - v\left[\prod_{i=1}^m (1-q_i) + \sum_{t=1}^{m-1}\beta_t\,(v-1)^t\right]}{\prod_{i=1}(1-\rho_i)}\right] \\
&= 1 - \left[\prod_{i=1}^m \frac{\rho_i}{v\,q_i}\right]\left[\frac{\prod_{i=1}^m (1-v\,q_i)[1-\hat{k}(v)]}{\prod_{i=1}^m (1-\rho_i)}\right] < 1, \qquad (28)
\end{aligned}$$

where the last step follows from the definition of $\hat{k}(s)$.

Finally,

$$\begin{aligned}
\Psi(0) &= \lim_{v\to 1^-} E[v^T I(T<\infty)|U(0)=0] \\
&= 1 - \lim_{v\to 1^-}\left[\prod_{i=1}^m \frac{\rho_i}{v\,q_i}\right]\left[\frac{\prod_{i=1}^m (1-v\,q_i)[1-\hat{k}(v)]}{\prod_{i=1}(1-\rho_i)}\right] \\
&= 1 - \left(\prod_{i=1}^m \frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1}\frac{\rho_i(1)}{1-\rho_i(1)}\right]\lim_{v\to 1^-}\frac{\left[\frac{1-\hat{k}(v)}{1-v}\right]}{\left[\frac{1-\rho_m(v)}{1-v}\right]} \\
&= 1 - \left(\prod_{i=1}^m \frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1}\frac{\rho_i(1)}{1-\rho_i(1)}\right]\left[\frac{\hat{k}'(1)}{\rho_m'(1)}\right] \\
&= 1 - \left(\prod_{i=1}^m \frac{1-q_i}{q_i}\right)\left[\prod_{i=1}^{m-1}\frac{\rho_i(1)}{1-\rho_i(1)}\right][E(W)-E(X)], \qquad (29)
\end{aligned}$$

where the last step follows from $\hat{k}'(1) = E(W)$ and $\rho_m'(1) = \frac{E(W)}{[E(W)-E(X)]}$, which is obtained by taking derivatives w.r.t. v on both sides of Lundberg's equation $\hat{k}(v/\rho_m(v))\,\hat{p}(\rho_m(v)) = 1$, letting $v \to 1^-$, and noting that $\lim_{v\to 1^-}\rho_m(v) = 1$.

7 Recursive Formula for $\phi(u)$

In this section, a recursive formula for $\phi(u)$ is given by renewal argument, which can be used to analyze other ruin related problems. The starting point of the recursion $\phi(0)$ is given in (21) by

$$\phi(0) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} {\rho_j}^x w(x,y)\, p(x+y+1)}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)}.$$

For $u \geq 1$, by similar arguments as in Gerber and Shiu (1998) for the continuous case, we condition on the first time when the surplus process drops below the initial surplus u :

$$\begin{aligned}
\phi(u) &= \sum_{y=1}^{u} \sum_{x=0}^{\infty} \sum_{t=1}^{\infty} v^t\, \phi(u-y)\, f_3(x,y,t|0) \\
&\quad + \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} \sum_{t=1}^{\infty} v^t\, w(x+u, y-u) f_3(x,y,t|0) \\
&= \sum_{y=1}^{u} \sum_{x=0}^{\infty} \phi(u-y) f_2(x,y|0) + \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w(x+u, y-u) f_2(x,y|0) \\
&= \sum_{y=1}^{u} \phi(u-y) g(y) + H(u), \qquad u \in \mathbb{N}^+, \qquad (30)
\end{aligned}$$

where

$$\begin{aligned}
H(u) &= \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} w(x+u, y-u) f_2(x,y|0) = \sum_{y=1}^{\infty} \sum_{x=u}^{\infty} w(x,y) f_2(x-u, y+u|0) \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} \sum_{x=u}^{\infty} \rho_j^{x-u} \sum_{y=1}^{\infty} w(x,y) p(x+y+1) \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} \sum_{x=u}^{\infty} \rho_j^{x-u} \omega(x+1) \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} T_{\rho_j} \omega(u+1), \qquad u \in \mathbb{N}^+. \qquad (31)
\end{aligned}$$

Eq. (30) is a recursive formula for $\phi(u)$ with the starting point $\phi(0)$. In particular, if $w(x,y) = 1$, then $\phi(u)$ simplifies to the p.g.f. transform of ruin time T w.r.t. discount factor v, which is now defined as

$$\phi_T(u) := E[v^T\, I(T < \infty) \,|\, U(0) = u], \qquad u \in \mathbb{N}.$$

In this case, ω simplifies to $\omega(u) = \sum_{x=u+1}^{\infty} p(x) = \bar{P}(u) = T_1 p(u)$, and $H(u)$ simplifies to

$$
\begin{aligned}
H(u) &= \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} T_{\rho_j} T_1 p(u+1) \\
&= T_1 g(u) = \sum_{y=u+1}^{\infty} g(u), \qquad (32)
\end{aligned}
$$

then $\phi_T(u)$ has the following recursive formula,

$$
\phi_T(u) = \sum_{y=1}^{u} \phi_T(u-y)\, g(y) + \sum_{y=u+1}^{\infty} g(y), \qquad u \in \mathbb{N}^+. \qquad (33)
$$

The ruin probability $\Psi(u)$ can thus be obtained by taking limit for $\phi_T(u)$ when $v \to 1^-$, i.e.,

$$
\begin{aligned}
\Psi(u) &= \lim_{v\to 1^-} E[v^T I(T<\infty)|U(0)=u] \\
&= \sum_{y=1}^{u} \Psi(u-y)\, g_1(y) + \sum_{y=u+1}^{\infty} g_1(y), \qquad u \in \mathbb{N}^+, \qquad (34)
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(y) &= \lim_{v\to 1^-} g(y) = \lim_{v\to 1^-} \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i}\right] \sum_{j=1}^{m} \frac{c_j\, T_{\rho_j} p(y+1)}{\prod_{k=1,k\neq j}^{m}(\rho_j - \rho_k)} \\
&= \left[\prod_{i=1}^{m} \frac{\rho_i(1)}{q_i}\right] \left[\sum_{j=1}^{m-1} \frac{c_j\, T_{\rho_j(1)} p(y+1)}{\prod_{k=1,k\neq j}^{m}[\rho_j(1) - \rho_k(1)]} + \frac{\prod_{i=1}^{m}(1-q_i) T_1 p(y+1)}{\prod_{i=1}^{m-1}[1-\rho_i(1)]}\right],
\end{aligned}
$$

the last step follows from the fact of $\lim_{v\to 1^-} \rho_m(v) = 1$.

8 Explicit Expression for $\phi(u)$

In this section, we show that the discounted penalty function $\phi(u)$ can be expressed explicitly in terms of a compound geometric d.f.'s. First rewrite (30) as

$$
\phi(u) = \frac{1}{1+\xi_v} \sum_{y=1}^{u} \phi(u-y)\, l(y) + \frac{1}{1+\xi_v} M(u), \qquad u \geq 1, \qquad (35)
$$

where ξ_v is such that $\frac{1}{1+\xi_v} = \phi_T(0)$, $l(y) = (1+\xi_v)g(y)$ is a proper d.f., $M(u) = (1+\xi_v)\, H(u)$ and $\phi(0) = \frac{1}{1+\xi_v} K(0) = H(0)$, specially, if $w(x,y) = 1$,

$$\phi_T(u) = \frac{1}{1+\xi_v} \sum_{y=1}^{u} \phi_T(u-y)\, l(y) + \frac{1}{1+\xi_v} \bar{L}(u), \qquad u \geq 1, \tag{36}$$

where $\bar{L}(u) = \sum_{y=u+1}^{\infty} l(y)$ is the tail of l.

Define a compound geometric d.f. by $z(u) = \frac{\xi_v}{1+\xi_v} \sum_{n=0}^{\infty} \left(\frac{1}{1+\xi_v}\right)^n l^{*n}(u)$, for $u \in \mathbb{N}$, with $z(0) = \frac{\xi_v}{1+\xi_v}$, where $*$ denotes the convolution. Then it is easy to show, using generating functions, that $\phi_T(u)$ can be expressed as the tail of the compound geometric d.f. z as follows:

$$\phi_T(u) = \bar{Z}(u) = \sum_{y=u+1}^{\infty} z(y) = \frac{\xi_v}{1+\xi_v} \sum_{n=1}^{\infty} \left(\frac{1}{1+\xi_v}\right)^n \bar{L}^{*n}(u), \quad u \geq 0. \tag{37}$$

Remarks:

1. Since the support of $l(y)$ is $\mathbb{N}^+$, then $l^{*n}(u) = 0$, if $n > u$, therefore $z(u)$ can be expressed as a finite sum by $z(u) = \frac{\xi_v}{1+\xi_v} \sum_{n=0}^{u} \left(\frac{1}{1+\xi_v}\right)^n l^{*n}(u)$.

2. $L^{*n}(u) = 0$, if $n > u$, then $\phi_T(u)$ can be expressed as the sum of finite terms by

$$\phi_T(u) = 1 - \frac{\xi_v}{1+\xi_v} \sum_{n=0}^{u} \left(\frac{1}{1+\xi_v}\right)^n L^{*n}(u), \qquad u \in \mathbb{N}. \tag{38}$$

The following theorem shows that, for general $w(x,y)$, the expected discounted penalty function $\phi(u)$ can be expressed explicitly in terms of the compound geometric d.f. $z(u)$.

Theorem 2

$$\phi(u) = \frac{1}{\xi_v} \sum_{y=0}^{u} M(u-y)\, z(y), \qquad u \geq 0. \tag{39}$$

9 Distribution of the Surplus Before Ruin and the Deficit at Ruin

In this section, we study the discounted joint and marginal distributions of surplus before ruin $U(T-1)$ and deficit at ruin $|U(T)|$.

Theorem 3 For $x \geq 0$, $y \geq 1$, and $u \geq 1$:

$$\begin{aligned} f_2(x, y \,|\, u) &= \sum_{z=1}^{u} f_2(x,\, y \,|\, u-z) g(z) + I(u \leq x) f_2(x-u, y+u \,|\, 0), & (40) \\ f_1(x \,|\, u) &= \sum_{z=1}^{u} f_1(x \,|\, u-z) g(z) + I(u \leq x) \sum_{l=u+1}^{\infty} f_2(x-u,\, l \,|\, 0), & (41) \\ g(y \,|\, u) &= \sum_{z=1}^{u} g(y \,|\, u-z) g(z) + g(y+u \,|\, 0), & (42) \end{aligned}$$

where the starting points $f_2(x,\, y \,|\, 0)$, $f_1(x \,|\, 0)$ and $g(y \,|\, 0)$ are given by (24), (25) and (26), respectively.

Define $Z = U(T-1) + |U(T)| + 1$ to be the claim causing ruin and let $h(z \,|\, u), z \geq 2$ be its probability distribution, then

Theorem 4 For $u \geq 1$ and $z \geq 2$,

$$h(z \,|\, u) = \sum_{y=1}^{u} h(z \,|\, u-y)\, g(y) + I(z \geq u+2)\, p(z)\, A(u), \tag{43}$$

where

$$A(u) = \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i} \right] \sum_{j=1}^{m} \frac{c_j\, (1 - \rho_j^{z-u-1})}{(1-\rho_j) \prod_{k=1, k \neq j}^{m} (\rho_j - \rho_k)}. \tag{44}$$

While the starting point is given by

$$h(z \,|\, 0) = p(z)\, A(0) = p(z) \left[\prod_{i=1}^{m} \frac{\rho_i}{v\, q_i} \right] \sum_{j=1}^{m} \frac{c_j\, (1 - \rho_j^{z-1})}{(1-\rho_j) \prod_{k=1, k \neq j}^{m} (\rho_j - \rho_k)}. \tag{45}$$

As an application of Theorem 2, we show that $f_2(x,\, y| \, u)$, $f_1(x \,|\, u)$, $g(y \,|\, u)$ and $h(z \,|\, u)$ all find explicit expressions in terms of the compound geometric p.f. z.

Theorem 5 For $x \in \mathbb{N}$, and $y \in \mathbb{N}^{+}$,

$$f_2(x,\, y \,|\, u) = \begin{cases} \left(\frac{1+\xi_v}{\xi_v}\right) \left(\frac{\prod_{i=1}^{m} \rho_i}{\prod_{i=1}^{m} v\, q_i}\right) p(x+y+1) \sum_{j=1}^{m} \frac{c_j\, \rho_j^{x-u} \sum_{n=0}^{u} \rho_j^{n}\, z(n)}{\prod_{k=1, k\neq j}^{m} (\rho_j - \rho_k)}, & 0 \leq u \leq x, \\ \left(\frac{1+\xi_v}{\xi_v}\right) \left(\frac{\prod_{i=1}^{m} \rho_i}{\prod_{i=1}^{m} v\, q_i}\right) p(x+y+1) \sum_{j=1}^{m} \frac{c_j\, \rho_j^{x-u} \sum_{n=u-x}^{u} \rho_j^{n}\, z(n)}{\prod_{k=1, k\neq j}^{m} (\rho_j - \rho_k)}, & u > x. \end{cases} \tag{46}$$

Corollary 1 For $x \in \mathbb{N}$,

$$f_1(x \mid u) = \begin{cases} \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) \bar{P}(x) \sum_{j=1}^m \frac{c_j\, \rho_j^{x-u} \sum_{n=0}^u \rho_j^n\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j - \rho_k)}, & 0 \le u \le x, \\ \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) \bar{P}(x) \sum_{j=1}^m \frac{c_j\, \rho_j^{x-u} \sum_{n=u-x}^u \rho_j^n\, z(n)}{\prod_{k=1,k\neq j}^m (\rho_j - \rho_k)}, & u > x. \end{cases} \tag{47}$$

Remark: If $m = 1$, then (46) simplifies to

$$f_2(x, y \mid u) = \begin{cases} f_2(x, y \mid 0) \left(\frac{1+\xi_v}{\xi_v}\right) \sum_{n=0}^{u} \rho^{n-u}\, z(n), & 0 \le u \le x \\ f_2(x, y \mid 0) \left(\frac{1+\xi_v}{\xi_v}\right) \sum_{n=u-x}^{u} \rho^{n-u}\, z(n), & u > x \end{cases}, \tag{48}$$

which can be found in Li and Garrido (2002). Specially, if setting $v = 1$, then the joint distribution of $U(T-1)$ and $|U(T)|$ is given by

$$f_2(x, y \mid u) = \begin{cases} f_2(x, y \mid 0) \frac{1-\Psi(u)}{1-\Psi(0)}, & 0 \le u \le x \\ f_2(x, y \mid 0) \frac{\Psi(u-x)-\Psi(u)}{1-\Psi(0)}, & u > x \end{cases}, \tag{49}$$

which is Dickson's classical formula in the discrete model.

For the distribution of the claim causing ruin $Z = U(T-1) + |U(T)| + 1$, we have the following result.

Theorem 6 For $u \in \mathbb{N}$, and $u + 2 \le z$,

$$h(z \mid u) = \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) p(z) \sum_{j=1}^m \frac{c_j \sum_{n=0}^u (1 - \rho_j^{z-u+n-1})\, z(n)}{(1-\rho_j) \prod_{k=1,k\neq j}^m (\rho_j - \rho_k)}, \tag{50}$$

for $2 \le z < u + 2$,

$$h(z \mid u) = \left(\frac{1+\xi_v}{\xi_v}\right)\left(\frac{\prod_{i=1}^m \rho_i}{\prod_{i=1}^m v\, q_i}\right) p(z) \sum_{j=1}^m \frac{c_j \sum_{n=u+2-z}^u (1 - \rho_j^{z-u+n-1})\, z(n)}{(1-\rho_j) \prod_{k=1,k\neq j}^m (\rho_j - \rho_k)}. \tag{51}$$

In particular, if $m = 1$, we have the following Corollary.

Corollary 2 If the claim waiting times are geometrically distributed with $k(x) = (1-q)\, q^{x-1} I(x \ge 1)$, then

$$h(z \mid u) = \begin{cases} \frac{v(1-\rho)(1-q)}{1-v} p(z) \sum_{n=0}^u \frac{1-\rho^{z-u+n-1}}{1-\rho} z(n), & u + 2 \le z \\ \frac{v(1-\rho)(1-q)}{1-v} p(z) \sum_{n=u+2-z}^u \frac{1-\rho^{z-u+n-1}}{1-\rho} z(n), & 2 \le z < u + 2 \end{cases}. \tag{52}$$

Further, if $v = 1$, $\rho = 1$, and $\rho'(1) = \frac{1+\theta}{\theta}$, then

$$h(z \,|\, u) = \begin{cases} \frac{(1-q)(1+\theta)}{\theta} p(z) \sum_{n=0}^{u} (z - u + n - 1)\, z(n), & u + 2 \leq z, \\ \frac{(1-q)(1+\theta)}{\theta} p(z) \sum_{n=u+2-z}^{u} (z - u + n - 1)\, z(n), & 2 \leq z < u + 2, \end{cases} \tag{53}$$

where $0 < \theta$ is the security loading factor.

10 Explicit Results for Two Classes of Claim Size Distributions

Theorem 2 shows that the expected discounted penalty function $\phi(u)$ can be expressed explicitly in terms of the compound geometric p.f. $z(u)$, with $z(0) = \phi_T(0)$, and $z(u) = \phi_T(u-1) - \phi_T(u)$, for $u \geq 1$, that is to say, if $\phi_T(u)$ can be obtained explicitly, then so can $\phi(u)$. One such case where $\phi_T(u)$ finds an explicit expression is when it admits a rational generating function. It follows from (33) that $\hat{\phi}_T(s)$ is a rational function if and only if $\hat{g}(s)$ is a rational function, while $\hat{g}(s)$ is rational function if and only if $\hat{p}(s)$ is a rational function. Another case for which $\phi_T(u)$ has an explicit expression is when $\hat{p}(s)$ is a polynomial (or $p(x)$ has a finite support), since, in this case, $\hat{\phi}_T(s)$ also has a rational generating function.

10.1 K_n Claim Size Distribution

From (33), the generating function of $\phi_T(s)$ is given by

$$\begin{aligned} \hat{\phi}_T(s) &= \frac{\phi_T(0) - \hat{g}(s)}{(1-s)[1-\hat{g}(s)]} \\ &= \frac{\prod_{i=1}^{m}(s - v\,q_i) - \hat{p}(s)B_{m-1}(s) - \left(\prod_{i=1}^{m} \frac{v\,q_i}{\rho_i}\right)[1-\phi_T(0)]\prod_{i=1}^{m}(s-\rho_i)}{(1-s)\{\prod_{i=1}^{m}(s - v\,q_i) - \hat{p}(s)B_{m-1}(s)\}}, \end{aligned} \tag{54}$$

where $B_{m-1}(s) = v[s^{m-1}\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1}\beta_j s^{m-1-j}\,(v-s)^j]$ is a polynomial of degree $m-1$, with leading coefficient $B_{m-1} = v[\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1}(-1)^j\beta_j$.

In this section, we assume that $p(x)$ is K_n distributed for x, $n \in \mathbb{N}^+$, i.e., its generating function is given by

$$\hat{p}(s) = \frac{E_n(s)}{\prod_{i=1}^{n}(1 - s\,\alpha_i)}, \qquad \Re(s) < \min\left\{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \cdots \frac{1}{\alpha_n}\right\}, \tag{55}$$

where $E_n(s)$ is a polynomial of degree n with $E_n(0) = 0$, $E_n(1) = \prod_{i=1}^n (1 - \alpha_i)$, and $0 < \alpha_i < 1$, for $i = 1, 2, \ldots, n$. In this case, $\hat{\phi}_T(s)$ can be transformed to a rational function, which is given in the following theorem.

Define $E_{m,n}(s) = [\prod_{i=1}^m (s - v\, q_i)][\prod_{i=1}^n (1 - s\, \alpha_i)] - E_n(s) B_{m-1}(s)$ to be a polynomial of degree $n + m$ with leading coefficient $(-1)^n (\prod_{i=1}^n \alpha_i)$. Then it is easily verified that the roots to the generalized Lundberg equation (8), $\rho_1, \rho_2, \ldots, \rho_m$ with $0 < |\rho_i| < 1$, are m roots to the equation $E_{m,n}(s) = 0$. Let $R_1, R_2, \ldots, R_n$ with $|R_i| \geq 1$ be the remaining n roots of $E_{m,n}(s) = 0$. We remark that there is a relation among the roots $\rho_1, \rho_2, \ldots, \rho_m$ and $R_1, R_2, \ldots, R_n$, i.e.,

$$\left[\prod_{i=1}^m \rho_i\right] \left[\prod_{i=1}^n R_i\right] = \frac{\prod_{i=1}^n v\, q_i}{\prod_{i=1}^n \alpha_i}. \tag{56}$$

Theorem 7 For above defined $\hat{p}(s)$, the generating function of $\phi_T(u)$ is given by

$$\hat{\phi}_T(s) = \frac{\varpi_{n-1}(s)}{(R_1 - s)(R_2 - s) \cdots (R_n - s)}, \tag{57}$$

where $\varpi_{n-1}(s) = \{\prod_{i=1}^n (R_i - s) - (\prod_{i=1}^m \frac{v\, q_i}{\rho_i})[1 - \phi_T(0)] \prod_{i=1}^n (1/\alpha_i - s)\}/(1 - s)$ is a polynomial of degree $n - 1$.

Further, if R_i are distinct, then by partial fractions,

$$\hat{\phi}_T(s) = \sum_{i=1}^n \frac{r_i}{(R_i - s)}. \tag{58}$$

Accordingly,

$$\phi_T(u) = \sum_{i=1}^n \left(\frac{r_i}{R_i}\right) R_i^{-u}, \qquad u \in \mathbb{N}, \tag{59}$$

in particular,

$$\phi_T(0) = 1 - \left(\prod_{i=1}^m \frac{\rho_i}{v\, q_i}\right) \frac{\prod_{i=1}^n (R_i - 1)}{\prod_{i=1}^n (1/\alpha_i - 1)}, \tag{60}$$

where $r_i = \left(\prod_{k=1}^n \frac{1 - R_i\, \alpha_k}{1 - \alpha_k}\right) \left(\prod_{j=1, j \neq i}^n \frac{R_j - 1}{R_j - R_i}\right)$, for $i = 1, 2, \ldots, n$.

Remark: If $\hat{p}(s)$ is given by (55), the $\hat{g}(s)$ can be simplified to

$$\hat{g}(s) = 1 - \left(\prod_{i=1}^m \frac{\rho_i}{v\, q_i}\right) \left(\prod_{i=1}^n \frac{R_i - s}{1/\alpha_i - s}\right) = 1 - \frac{\prod_{i=1}^n (1 - s/R_i)}{\prod_{i=1}^n (1 - \alpha_i\, s)}. \tag{61}$$

Example 1 In this example, we assume that the claim waiting times are shifted negative binomial distributed with $k(x) = x\,(1-q)^2\,q^{x-1} I(x \geq 1)$, and $\hat{k}(s) = \frac{s(1-q)^2}{(1-s\,q)^2}$. Claim amounts have a mixture of two geometric distributions with $p(x) = \vartheta\,(1-\alpha_1)\alpha_1^{x-1} + (1-\vartheta)(1-\alpha_2)\alpha_2^{x-1}$, for $x \geq 1$ and $0 < \vartheta,\ \alpha_1,\ \alpha_2 < 1$. Then $\hat{p}(s) = \frac{s[(1-\alpha_1)(1-\alpha_2)+\beta(1-s)]}{(1-s\,\alpha_1)(1-s\,\alpha_2)}$, with $\beta = \vartheta\,\alpha_2(1-\alpha_1) + (1-\vartheta)\,\alpha_1\,(1-\alpha_2)$.

Since the equation

$$E_{2,2}(s) = (s-v\,q)^2(1-s\,\alpha_1)(1-s\,\alpha_2) - v\,(1-q)^2\,s^2[(1-\alpha_1)(1-\alpha_2)+\beta(1-s)] = 0,$$

has two roots, say ρ_1, ρ_2 with $|\rho_i| < 1$, and two roots, say $R_1,\ R_2$ with $|R_i| > 1$. It is easy to check that the relation $\rho_1\,\rho_2\,R_1\,R_2 = \frac{v^2\,q^2}{\alpha_1\,\alpha_2}$ holds.

By (60) and the above relation

$$\phi_T(0) = 1 - \frac{1}{R_1\,R_2}\frac{(R_1-1)(R_2-1)}{(1-\alpha_1)(1-\alpha_2)}, \tag{62}$$

and for $u \geq 0$,

$$\begin{aligned} \phi_T(u) \quad &= \quad \frac{(1-R_1\,\alpha_1)(1-R_2\,\alpha_2)(R_2-1)R_1^{-(u+1)}}{(1-\alpha_1)(1-\alpha_2)(R_2-R_1)} \\ &\quad - \frac{(1-R_2\,\alpha_1)(1-R_2\,\alpha_2)(R_1-1)R_2^{-(u+1)}}{(1-\alpha_1)(1-\alpha_2)(R_2-R_1)} \end{aligned} \tag{63}$$

(61) gives

$$\hat{g}(s) = \frac{s\left[(\alpha_1\,\alpha_2 - \frac{1}{R_1 R_2})s + \frac{R_1+R_2}{R_1\,R_2} - (\alpha_1+\alpha_2)\right]}{(1-s\,\alpha_1)(1-s\,\alpha_2)},$$

inverting yields

$$g(y) = \varsigma_1\,\alpha_1^{y-1} + \varsigma_2\,\alpha_2^{y-1}, \qquad y \geq 1, \tag{64}$$

where

$$\begin{aligned} \varsigma_1 \quad &= \quad \frac{(R_1 R_2\,\alpha_1\,\alpha_2 - 1) + \alpha_1[R_1 + R_2 - R_1 R_2(\alpha_1+\alpha_2)]}{R_1 R_2(\alpha_1 - \alpha_2)}, \\ \varsigma_2 \quad &= \quad \frac{(R_1 R_2\,\alpha_1\,\alpha_2 - 1) + \alpha_2[R_1 + R_2 - R_1 R_2(\alpha_1+\alpha_2)]}{R_1 R_2(\alpha_2 - \alpha_1)}. \end{aligned}$$

If setting $v = 1$, and $w(x,y)$ to be $x\,y$, x and y, respectively, then $\phi(u)$ simplifies to the joint and marginal moments of $U(T-1)$ and $|U(T)|$, which can be obtained by the recursive formula $\phi(u) = \sum_{y=1}^{u} \phi(u-y)\,g(y) + H(u)$.

Now let $q = \frac{1}{3}$, $\vartheta = 0.6$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $v = 1$, then $E(W) = 2 > E(X) = 1.8$ means a positive loading. The equation $E_{2,2}(s) = 0$ gives four roots, $\rho_1 = 1$, $\rho_2 = 0.2183$, $R_1 = 1.1344$ and $R_2 = 2.6917$. This gives

$$\begin{aligned} \Psi(u) &= 0.7731 \times 1.1344^{-u} + 0.00342 \times 2.6917^{-u}, \qquad u \geq 0, \\ g(y) &= 0.2941\, \alpha_1^{y-1} + 0.1256\, \alpha_2^{y-1}, \qquad y \geq 1. \end{aligned}$$

Table 1 gives the joint and marginal moments of the surplus before ruin and deficit at ruin, together with the mean of the claim amount causing ruin. It shows that: (i) the joint moment given ruin occurs is increasing in u; (ii) the first two moments of $U(T-1)$ and $|U(T)|$ are increasing in u, while the effect of u on the first two moments of $U(T-1)$ is greater than that of $|U(T)|$; (iii) the mean of the claim causing ruin is increasing in u, and greater than the mean of claim amount r.v.'s; (iv) finally, the effect of u on all these quantities is greater for small u, and smaller for big initial surplus values u.

Table 2 gives the covariance and correlation coefficient of the surplus before ruin and the deficit at ruin, given that ruin occurs. It can be seen that the covariance is increasing in u, and the two random variables are positively correlated, while the smaller correlation coefficient mean that they are weakly correlated.

10.2 Claim Amounts Distributions with Finite Support

In this section, we assume that the claim amount distribution has a finite support, i.e., for $N \geq 2$:

$$p(n) = P(X = n) = p_n, \qquad n = 1, 2, \ldots N. \tag{65}$$

Then

$$\hat{p}(s) = D_N(s) := p_1\, s + p_2\, s^2 + \cdots + p_N\, s^N, \qquad -1 < \Re(s) < 1, \tag{66}$$

is a polynomial of degree N. For example, the binomial distribution, the discrete uniform, and the hyper-geometric distribution all have a finite support.

Define $V(s) := D_N(s) B_{m-1}(s) - \prod_{i=1}^{m}(s - v\, q_i)$ to be a polynomial of degree $N+m-1$, with leading coefficient $V_{N+m-1} = p_N\, B_{m-1}$, where $B_{m-1} = v[\prod_{i=1}^{m}(1-q_i) + \sum_{j=1}^{m-1}(-1)^j \beta_j]$ is the leading coefficient of polynomial $B_{m-1}(s)$. Of all the $N+m-1$ roots to the equation $V(s) = 0$, $\rho_1, \rho_2, \ldots, \rho_m$ are m roots with $0 < |\rho_i| < 1$. Let R_1, $R_2, \ldots R_{N-1}$ with $|R_i| > 1$ be the remaining $N-1$ roots. Therefore,

Table 1: Moments for the surplus before ruin and deficit at ruin for different u

u	A	B	C	D	E	F
0	1.9107	0.9904	1.8784	2.8557	5.2716	3.8688
1	2.95803	1.53196	1.89591	4.53027	5.37623	4.4279
2	3.53798	1.82529	1.90329	6.02392	5.42065	4.7286
3	3.86556	1.98875	1.90645	7.17367	5.43939	4.8952
4	4.05238	2.08156	1.90785	8.00108	5.44744	4.9894
5	4.15964	2.13462	1.90838	8.57300	5.45077	5.0430
6	4.22144	2.16502	1.90862	8.95754	5.45238	5.0736
7	4.25669	2.18245	1.90879	9.21084	5.45301	5.0912
8	4.27691	2.19274	1.90889	9.37486	5.45347	5.1016
9	4.28879	2.19854	1.90913	9.48004	5.45368	5.1077
10	4.29552	2.20187	1.90917	9.54631	5.45399	5.1110
11	4.29957	2.20366	1.90942	9.58805	5.45424	5.1131
12	4.30171	2.20491	1.90932	9.61360	5.45411	5.1142
13	4.30269	2.20535	1.90935	9.62939	5.45403	5.1147
14	4.30409	2.20612	1.90966	9.63979	5.45443	5.1158
15	4.30447	2.20586	1.90899	9.64538	5.45502	5.1149

A: joint moments of $U(T-1)$ and $U(T)$, given that ruin occurs
B: mean of $U(T-1)$, given that ruin occurs
C: mean of $|U(T)|$, given that ruin occurs
D: second moment of $U(T-1)$ about the origin, given that ruin occurs
E: second moment of $|U(T)|$ about the origin, given that ruin occurs
F: mean of the claim amount causing ruin, given that ruin occurs

Table 2: Covariance and coefficient of correlation between the surplus before ruin and the deficit at ruin for different u

u	0	1	2	3	4	5	6	7
G	0.05036	0.05356	0.06391	0.07411	0.08109	0.08599	0.08921	0.09085
H	0.02785	0.02716	0.02905	0.03075	0.03149	0.03189	0.03209	0.03202

u	8	9	10	11	12	13	14	15
G	0.09121	0.09151	0.09173	0.09185	0.09189	0.09192	0.09216	0.09349
H	0.03173	0.03156	0.03148	0.03139	0.03132	0.03131	0.03104	0.03178

G stands for the covariance, H stands for the coefficient of correlation.

$V(s)$ can be factored as $V(s) = V_{N+m-1} \left[\prod_{i=1}^{m}(s-\rho_i)\right] \left[\prod_{i=1}^{N-1}(s-R_i)\right]$. Setting $s=0$, it is easily shown that

$$(-1)^N V_{N+m-1}\Big(\prod_{i=1}^{m}\rho_i\Big)\Big(\prod_{i=1}^{N-1}R_i\Big) = \prod_{i=1}^{m}(v\,q_i). \tag{67}$$

Then (54) can be rewritten as

$$\hat{\phi}_T(s) = \frac{V_{N+m-1}\prod_{i=1}^{N-1}(s-R_i) + \prod_{i=1}^{m}(\frac{v\,q_i}{\rho_i})[1-\phi_T(0)]}{V_{N+m-1}(1-s)\prod_{i=1}^{N-1}(s-R_i)}, \tag{68}$$

Since $s=1$ is a removable singularity of $\hat{\phi}_T(s)$, then we have

$$1-\phi_T(0) = -V_{N+m-1}\left[\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right]\Big[\prod_{i=1}^{N-1}(1-R_i)\Big] = \prod_{i=1}^{N-1}\frac{R_i-1}{R_i} \tag{69}$$

and (68) simplifies to

$$\hat{\phi}_T(s) = \frac{F_{N-2}(s)}{\prod_{i=1}^{N-1}(R_i-s)}, \qquad -1 < \Re(s) < 1, \tag{70}$$

where $F_{N-2}(s) := \frac{\prod_{i=1}^{N-1}(R_i-s)-\prod_{i=1}^{N-1}(R_i-1)}{(1-s)}$ is a polynomial of degree $N-2$. By partial fractions,

$$\hat{\phi}_T(s) = \sum_{i=1}^{N-1}\frac{r_i}{(R_i-s)} = \sum_{i=1}^{N-1}(\frac{r_i}{R_i})\frac{1}{(1-\frac{s}{R_i})}, \tag{71}$$

where $r_i = \prod_{j=1,j\neq i}^{N-1}\left(\frac{R_j-1}{R_j-R_i}\right)$. Inverting yields

$$\phi_T(u) = \sum_{i=1}^{N-1}\left(\frac{r_i}{R_i}\right)R_i^{-u}, \qquad u \in \mathbb{N}^+. \tag{72}$$

Finally, if $\hat{p}(s)$ is given by (66), then $\hat{g}(s)$ simplifies to

$$\hat{g}(s) = 1 + V_{N+m-1}\left(\prod_{i=1}^{m}\frac{\rho_i}{v\,q_i}\right)\prod_{i=1}^{N-1}(s-R_i) = 1 - \prod_{i=1}^{N-1}\frac{R_i-s}{R_i}. \tag{73}$$

Isolating the coefficient of s^n gives $g(n)$, for $n = 1, 2, \ldots, N-1$, e.g.,

$$\begin{aligned}
g(1) &= \left[\sum_{i=1}^{N-1} \frac{1}{R_i}\right], && (74)\\
g(2) &= -\sum_{1\le i<j\le N-1} \frac{1}{R_i R_j}, && (75)\\
\vdots \quad & \quad \vdots \\
g(N-2) &= (-1)^{N-3}\left[\prod_{i=1}^{N-1} \frac{1}{R_i}\right]\sum_{i=1}^{N-1} R_i, && (76)\\
g(N-1) &= (-1)^{N-2}\prod_{i=1}^{N-1} \frac{1}{R_i}. && (77)
\end{aligned}$$

Example 2 In this example, we assume that the claim waiting times have a negative binomial$(2, q)$ distribution, with $k(x) = x\,(1-q)^2\, q^{x-1} I(x \geq 1)$ and $\hat{k}(s) = \frac{s(1-q)^2}{(1-s\,q)^2}$. The claim sizes are uniformly distributed with $P(X=1) = P(X=2) = P(X=3) = \frac{1}{3}$ and with $\hat{p}(s) = \frac{(s+s^2+s^3)}{3}$. Then $V(s) = \hat{p}(s) B_1(s) - (s - v\,q)^2 = \frac{v(1-q)^2}{3}(s^2+s^3+s^4) - (s-v\,q)^2$ is a polynomial of degree 4 with leading coefficient $\frac{v(1-q)^2}{3}$. It can be factored as $V(s) = \frac{v(1-q)^2}{3}(s-\rho_1)(s-\rho_2)(s-R_1)(s-R_2)$. We remark that the relation $\rho_1\,\rho_2\,R_1 R_2 = -\frac{3\,v\,q^2}{(1-q)^2}$ holds by setting $s=0$ in the above factorization.

(69) together with above relation gives

$$\phi_T(0) = 1 + \frac{(1-q)^2\,\rho_1\,\rho_2}{3\,v\,q^2}(R_1-1)(R_2-1) = \frac{R_1+R_2-1}{R_1\,R_2},$$

and for $u \in \mathbb{N}^+$,

$$\phi_T(u) = \frac{R_2-1}{R_2-R_1} R_1^{-(u+1)} + \frac{R_1-1}{R_1-R_2} R_2^{-(u+1)}. \qquad (78)$$

Together (74) and (77) give that

$$g(1) = \frac{R_1+R_2}{R_1\,R_2}, \qquad g(2) = -\frac{1}{R_1\,R_2} \quad \text{and} \quad g(i) = 0, \quad \text{for } i \geq 3. \qquad (79)$$

Thus the recursive formula for $\phi(u)$ simplifies to

$$\begin{aligned}
\phi(0) &= H(0),\\
\phi(1) &= \phi(0)\,g(1) + H(1),\\
\phi(u) &= \phi(u-1)\,g(1) + \phi(u-2)\,g(2) + H(u), \qquad u \geq 2,
\end{aligned}$$

where in this example, $H(0) = \frac{\rho_1\,\rho_2(1-q)^2}{3\,v\,q^2}[w(0,1) + w(0,2) + (\rho_1 + \rho_2)w(1,\,1)]$, $H(1) = \frac{\rho_1\,\rho_2(1-q)^2}{3\,v\,q^2}w(1,1)$, and $H(u) = 0$, for $u \geq 2$.

$\phi(u)$ can also be evaluated explicitly by

$$\phi(u) = \frac{1}{1-\phi_T(0)} \sum_{y=1}^{u} H(u-y)\,z(y) + H(u), \qquad u \geq 1, \tag{80}$$

where $\frac{1}{1-\phi_T(0)} = \frac{R_1\,R_2}{(R_1-1)(R_2-1)}$, and

$$z(u) = \phi_T(u-1) - \phi_T(u) = \frac{(R_1-1)(R_2-1)}{R_2 - R_1}[R_1^{-(u+1} - R_2^{-(u+1)}].$$

Thus (80) simplifies to

$$\phi(u) = \frac{R_2}{R_2 - R_1} \sum_{y=1}^{u} H(u-y)\,R_1^{-y} + \frac{R_1}{R_1 - R_2} \sum_{y=1}^{u} H(u-y)\,R_2^{-y} + H(u). \tag{81}$$

Since $H(u)$ is not zero only at $u = 0$ and $u = 1$, the above formula is equivalent to

$$\begin{aligned} \phi(0) &= H(0), \qquad \phi(1) = H(1) + \frac{R_1 + R_2}{R_1 R_2} H(0), \\ \phi(u) &= \frac{R_1 R_2}{R_2 - R_1}\Big\{H(0)\big[R_1^{-(u+1)} - R_2^{-(u+1)}\big] + H(1)\big[R_1^{-u} - R_2^{-u}\big]\Big\}, \quad u \geq 2. \end{aligned}$$

Setting $w(x,y) = I(x+y+1 = z)$, for $z = 2, 3, \ldots$, and $v = 1$, implies that $\phi(u)$ simplifies to the distribution function $h(z\,|\,u)$ of $Z = U(T-1) + |U(T)| + 1$. In particular, $z = 2$, $H(0) = \frac{(1-q)^2 \rho_1\,\rho_2(\rho_1+\rho_2)}{3\,q^2} = -\frac{1}{R_1 R_2}$ and $H(i) = 0$, for $i = 1, 2, \ldots$. Then

$$h(2\,|\,u) = \frac{R_1^{-(u+1)} - R_2^{-(u+1)}}{R_1 - R_2}, \qquad u \geq 0.$$

Similarly, $z = 3$, $H(0) = -\frac{1+\rho_1+\rho_2}{R_1\,R_2}$, $H(1) = -\frac{1}{R_1\,R_2}$ and $H(i) = 0$, for $i \geq 2$. Then

$$h(3\,|\,u) = \frac{1+\rho_1+\rho_2}{R_1 - R_2}\big[R_1^{-(u+1)} - R_2^{-(u+1)}\big] + \frac{1}{R_1 - R_2}\big[R_1^{-u} - R_2^{-u}\big], \qquad u \geq 0.$$

Finally, for $z \geq 4$, $h(z\,|\,u) = 0$, for all $u \geq 0$.

If instead, we set $v = 1$ and $w(x, y) = xy$ (alternatively x, or y), then $\phi(u)$ simplifies to $E[U(T-1)|U(T)|I(T<\infty)\,\big|U(0)=u]$ ($E[U(T-1)I(T<\infty)\,\big|U(0)=u]$, or $E[|U(T)|I(T<\infty)\,\big|U(0)=u]$), and

$$
\begin{aligned}
&E[U(T-1)|U(T)|I(T<\infty)\,\big|U(0)=u] \\
&\quad = E[U(T-1)I(T<\infty)\,\big|U(0)=u] \\
&\quad = \frac{(\rho_1+\rho_2)[R_1^{-(u+1)} - R_2^{-(u+1)}] + (R_1^{-u} - R_2^{-u})}{R_1 - R_2}, \\
&E[|U(T)|I(T<\infty)\,\big|U(0)=u] \\
&\quad = \frac{(3+\rho_1+\rho_2)[R_1^{-(u+1)} - R_2^{-(u+1)}] + (R_1^{-u} - R_2^{-u})}{R_1 - R_2}.
\end{aligned}
$$

Now setting $q = 0.35$, implies that $E(W) = \frac{1+q}{1-q} = 2.077 > E(X) = 2$ and equation

$$
V(s) = \frac{(1-q)^2}{3}(s^2 + s^3 + s^4) - (s-q)^2 = 0
$$

has four roots, say, $\rho_1 = 1$, $\rho_2 = 0.2449$, $R_1 = 1.0708$ and $R_2 = -3.3158$. The following table gives the moments of $U(T-1)$ and $|U(T)|$, as well as the covariance, for $u = 0, 1, 2, \ldots, 10$.

Table 3: Moments and covariance of the surplus before ruin and the deficit at ruin for different u

u	Joint Moment	$E[U(T-1)\|T<\infty]$	$E[\|U(T)\|\big\|T<\infty]$	Cov
0	0.3836	0.3836	1.3081	-0.1182
1	0.5856	0.5856	1.2072	-0.1213
2	0.5207	0.5207	1.2396	-0.1248
3	0.5417	0.5417	1.2291	-0.1241
4	0.5349	0.5349	1.2325	-0.1244
5	0.5371	0.5371	1.2314	-0.1243
6	0.5364	0.5364	1.23176	-0.12432
7	0.5366	0.5366	1.23165	-0.12430
8	0.53656	0.53656	1.23169	-0.12432
9	0.53657	0.53657	1.23168	-0.124312
10	0.53656	0.53656	1.23168	-0.124310

Joint Moment stands for $E[U(T-1)|U(T)|\big|T<\infty]$.

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