Risk Capital Decomposition for a Multivariate Dependent Gamma Portfolio

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Abstract

Significant changes in the insurance and financial markets are giving increasing attention to the need for developing a standard framework for risk management. Today's competitive and investment oriented marketplace requires from insurance directors to use all the advantages of investing risk capitals of their enterprises. Recently, there has been growing interest among insurance and investment experts to focus on the use of a tail conditional expectation as a measure of risk, since it shares properties that are considered desirable and applicable in a variety of situations. In particular, such a method allows for a natural allocation of the total risk capital among its various constituents. This paper examines above risk measure in the case of a multivariate gamma portfolio. We demonstrate the explicit formulas for tail conditional expectation and based on it capital allocation when the proposed multivariate model consists of dependent and independent gamma marginals. Financial enterprises are always concerned of fairly allocating the total risk capital to these constituents. Consequently, this work is particularly meaningful in practice in the case of computing capital requirements for an institution who may have several lines of correlated business and whose data is distributed multivariate gamma model considered here.

1 Introduction

Consider a loss random variable X whose distribution density function we denote by $f_X(x)$, distribution function by $F_X(x)$ and then the tail function of X is $\overline{F}_X(x) = 1 - F_X(x)$. This may refer to a total claim for an insurance company or to the total loss in a portfolio of investments for an individual or institution. The *tail conditional expectation* (TCE) is defined to be

$$TCE_X(x_q) = E(X|X > x_q) \tag{1}$$

and it may be interpreted as the mean of worse losses. It gives an average amount of the tail of the distribution. This tail is usually based on the q-th quantile x_q of the loss distribution with the property

$$\overline{F}_X(x_q) = 1 - q,$$

when $0 < q < 1$. It is called value-at-risk, denoted by $VaR_X(q)$ and it is defined as

$$x_q = \inf\{x|F_X(x) \geq q\}. \tag{2}$$

In the case of a continuous random variable the definition above is unique and equaled to $F_X^{-1}(q)$. The formula used to evaluate tail conditional expectation is

$$TCE_X(x_q) = \frac{1}{\overline{F}_X(x_q)} \int_{x_q}^{\infty} x dF_X(x), \tag{3}$$

where $\overline{F}_X(x_q) > 0$.

Tail conditional expectations for the univariate and multivariate Normal family have been well-developed in Panjer (2001). Landsman and Valdez (2003a) extended these results for the essentially larger class of elliptical distributions. Unfortunately, all members of the elliptical family are symmetric. The first step in investigating the tail conditional expectation risk measure in the context of non-symmetric loss distributions was made by Landsman and Valdez (2003b). These authors developed the TCE formulas for the univariate exponential dispersion family (EDF) that includes many well-known distributions like Normal, Gamma and Inverse Gausian, which, except for Normal, are not symmetric, have non-negative support and provide excellent model for fitting insurance losses. It is therefore not surprising to find they are becoming popular among actuaries.

Although the univariate EDF is considerably rich and widely applied, the case is different in the modeling of an n - variate portfolios of financial risks and insurance claims distributed multivariate EDF. For example, it does not include important multivariate distributions whose univariate marginals are inverse Gaussian, gamma or stable (see Bildicar and Potil (1968)). Consequently, the multivariate EDF can not be used to model n - variate portfolios with such claims.

Tail conditional expectation posses a lot of attractive properties. It is proved to be a coherent risk measure (see Artzner, et al. 1999) and it allows for a natural allocation of the total loss among its various constituents. Assume that an insurance company manages n lines of business and the risk managers are interested to know how much risk concealed in line j. The answer to this question is simple. The contribution of the j-th line of business of the insurance company to its total risk capital is

$$TCE_{X_j|S}\left(s_q\right) = E\left(X_j|S > s_q\right), \tag{4}$$

where $S = X_1 + X_2 + \cdots X_n$. Certainly, due to the additive property of conditional expectation, the sum of all marginal risks is equal to the total risk measure for the whole company, i.e.

$$TCE_S\left(s_q\right) = \sum_{j=1}^{n} E\left(X_j|S > s_q\right). \tag{5}$$

In this paper we advance (4) and (5) in the general framework of a multivariate model with univariate gamma marginals and the dependency structure based on the approach in Mathai and Moschopoulos (1991). Multivariate distributions with non-negative support are extremely important in actuarial science and the suggested family may provide a good basis for the modeling of such portfolios.

The rest of the paper, is then organized as follows. Section 2 recalls the definition and the most important properties of multivariate gamma distribution in sense of Mathai and Moschopoulos (1991). In this section we also point at the problem of calculating the distribution of the sum of n gamma random variables with any shape and rate parameters and we give a Lemma in order to solve this difficulty. In Section 3 two forms for tail conditional expectation in the case of the univariate gamma distribution are presented. Section 4 provides a general expression for the contribution of a marginal loss X_j to the total risk measure for any non-negative independent random variables with densities $f_{X_j}\left(x\right)$ and finite expectations. In Section 5 we derive the most general representation of TCE in the context of the proposed multivariate gamma distribution. Section 6 presents the formula for risk capital decomposition based on the tail conditional expectation risk measure, and section 7 concludes this paper. At last, we give another, more attractive form of the expression for tail conditional expectation for the sum of n gamma risks in Appendix 1.

2 Multivariate Gamma Distribution

We consider here a multivariate gamma model introduced by Cheriyan (1941) and Ramabhadran (1951), and generalized by Mathai and Moschopoulos (1991). The dependency structure of this distribution is obtained by adding a common random variable to every univariate marginal.

Let $Y_0, Y_1, ..., Y_n$ be mutually independent gamma random variables with shape parameters γ_i and rate parameters α_i, i.e. $Y_i \backsim Ga(\gamma_i, \alpha_i)$. The probability density function of Y_i is then

$$f_{Y_i}(y) = \frac{1}{\Gamma(\gamma_i)} e^{-\alpha_i y} y^{\gamma_i - 1} \alpha_i^{\gamma_i}, y > 0, \alpha_i > 0, \gamma_i > 0, (i = 0, 1, ..., n). \quad (6)$$

Denote

$$X_j = \frac{\alpha_0}{\alpha_j} Y_0 + Y_j, j = 1, 2, ..., n. \quad (7)$$

Definition 1 *The joint distribution of the random vector* $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *is the multivariate gamma distribution in sense of Mathai and Moschopoulos (1991).*

Some important properties of this distribution follow directly from the moment generating function of $\mathbf{X}$, which is easily obtained:

$$\begin{aligned} M_{\mathbf{X}}(t) &= E\left[\exp\left(y_0 \sum_{j=1}^{n} \frac{\alpha_0}{\alpha_j} t_j\right)\right] \prod_{j=1}^{n} E\left[e^{t_j Y_j}\right] \\ &= \left(1 - \sum_{j=1}^{n} \frac{t_j}{\alpha_j}\right)^{-\gamma_0} \prod_{j=1}^{n} \left(1 - \frac{t_j}{\alpha_j}\right)^{-\gamma_j}. \quad (8) \end{aligned}$$

Consequently:

1. $X_j \backsim Ga\left(\gamma_0+\gamma_j, \alpha_j\right)$.

2. $E\left(X_j\right)=\left(\gamma_0+\gamma_j\right)/\alpha_j$.

3. $Var\left(X_j\right)=\left(\gamma_0+\gamma_j\right)/\alpha_j^2$.

4. $Cov\left(X_i, X_j\right)=\gamma_0/\left(\alpha_i\alpha_j\right), i \neq j$.

5. $\rho\left(X_i, X_j\right)=\dfrac{\gamma_0}{\sqrt{\left(\gamma_0+\gamma_i\right)\left(\gamma_0+\gamma_j\right)}}$.

The special case when $\alpha_j=1$ was considered by Cheriyan (1941) and Ramabhadran (1951).

Suppose $\gamma_0 \longrightarrow 0$ then $Y_0 \xrightarrow{a.s.} 0$. We call $Y_0 \backsim Ga\left(0, \alpha_0\right)$ a non-proper gamma distributed random variable. Clearly when this is the case, the random vector $\mathbf{X}$ consists of n independent gamma random variables. Hence, we get a multivariate independent gamma distribution.

Notice, that the distribution of $S=X_1+\cdots+X_n$ is not gamma even if all X_j are mutually independent gamma random variables with different rate parameters (see Mathai and Moschopoulos (1991), Theorem 2.1). The following lemma gives a new interpretation of the Moschopoulos (1985) result concerning the distribution of S.

Denote $\gamma=\sum_{j=1}^{n}\gamma_j$ and $\alpha_{\max}=\max\left(\alpha_1,...,\alpha_n\right)$.

Lemma 1 *The distribution of the sum S is mixed gamma with mixing shape parameter, i.e.*

$$S \backsim Ga\left(\gamma+K, \alpha_{\max}\right),$$

where K is a non negative integer random variable with probabilities

$$p_k=C\delta_k, \quad k \geq 0, \tag{9}$$

where

$$\begin{aligned} C &= \prod_{j=1}^{n}\left(\frac{\alpha_j}{\alpha_{\max}}\right)^{\gamma_j}, \\ \Delta_j &= \left(1-\frac{\alpha_j}{\alpha_{\max}}\right)^{i}, \end{aligned}$$

$$\delta_k = k^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n} \gamma_j \Delta_j \delta_{k-i}, \quad k > 0, \tag{10}$$

and $\delta_0 = 1$.

Proof. Observe that from Moschopoulos (1985), Theorem 1 $\sum_{k \geq 0} p_k = \sum_{k \geq 0} C\delta_k = 1$. ∎

Remark 1 *The case when* $\alpha_1 = \alpha_2 = ... = \alpha_n$ *implies* $\Delta_j = 0$ *and consequently* $P(K = 0) = 1$, *i.e.* $S \backsim Ga(\gamma, \alpha)$.

Finally, we present two graphs, comparing bivariate independent gamma distribution and bivariate dependent one.

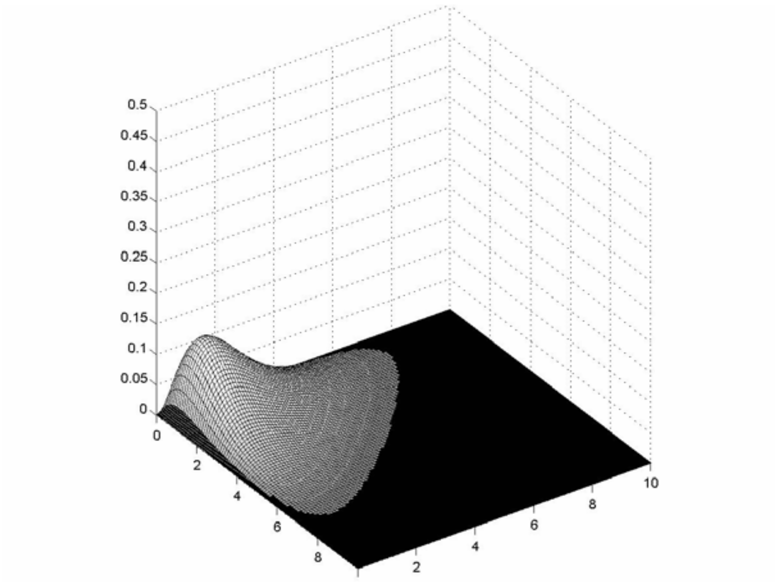

Figure 1: Bivariate Gamma density with independent univariate marginals.

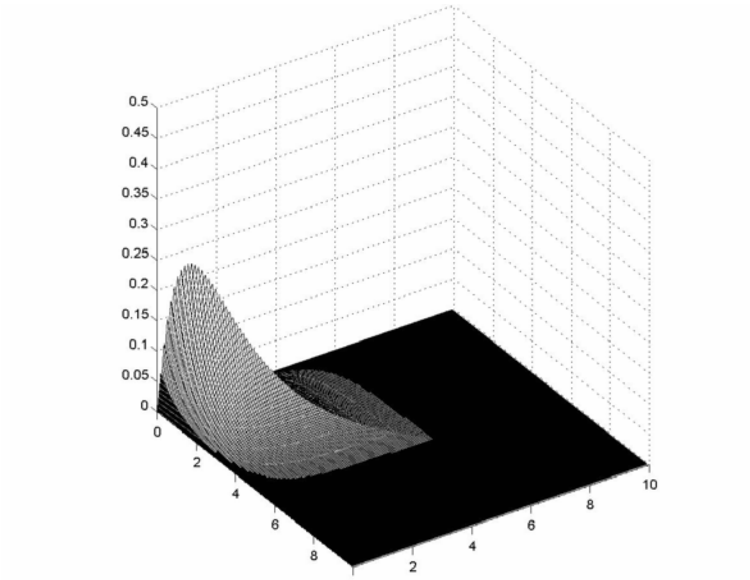

Figure 2: Bivariate Gamma density with dependent univariate marginals.

3 TCE Formula for Univariate Gamma Distribution

Consider gamma distributed loss random variable X with shape parameter γ and rate parameter α. Let q be such that $0 < q < 1$ and let x_q denote the q-th quantile of the distribution of X. We denote by $g(x|\gamma, \alpha)$ and $G(x|\gamma, \alpha)$ the density and the cumulative distribution functions respectively and let the tail probability function of X be $\overline{G}(x|\gamma, \alpha) = 1 - G(x|\gamma, \alpha)$. We call X a standardized gamma random variable if $\alpha = 1$ and then we write simply $g(x|\gamma)$, $G(x|\gamma)$ and $\overline{G}(x|\gamma)$.

Theorem 1 *Let $X \backsim Ga(\gamma, \alpha)$ then the tail conditional expectation of X is given by*

$$TCE_X(x_q) = E(X)\frac{\overline{G}(\alpha x_q|\gamma+1)}{\overline{G}(\alpha x_q|\gamma)} \quad (11)$$

$$= E(X) + x_q\lambda, \quad (12)$$

where $\lambda = \dfrac{g(\alpha x_q|\gamma)}{\overline{G}(\alpha x_q|\gamma)}$ is a hazard function.

We would like to notice here, that representation (11) coincides with that given in Landsman and Valdez (2003b) . In the same time the form of (12) was first obtained in the context of multivariate Normal family (see Panjer (2001)) However, for the case of non-negative losses, representation (11) is more convenient.

4 Portfolio Risk Decomposition with TCE for Nonnegative Independent Losses

When uncertainty is due to different sources, it is often natural to ask how to decompose the total level of uncertainty to these sources. Frees (1998) suggested methods for quantifying the degree of importance of various sources of uncertainty for insurance systems.

For our purposes, suppose that the total loss or claim is equaled to $S = \sum_{j=1}^{n} X_j$, where one can think of each X_j as the claim arising from a particular line of business or product line in the case of insurance, or the loss resulting from a financial instrument or a portfolio of instruments in any other case. As it was noticed by Panjer (2001), from the additivity of expectation, the tail conditional expectation allows for a natural decomposition of the total loss:

$$TCE_S(s_q) = \sum_{j=1}^{n} E(X_j | S > s_q). \tag{13}$$

Note that this is not in general equivalent to the sum of the tail conditional expectations of the individual components. This is because

$$TCE_{X_j}(s_q) \neq E(X_j | S > s_q).$$

Instead we denote this as

$$TCE_{X_j|S}(s_q) = E(X_j | S > s_q) \tag{14}$$

the contribution to the total risk attributable to risk j. It can be interpreted as follows: that in the case of a disaster as measured by an amount at least as large as the quantile of the total loss distribution, this refers to the average amount that would be due to the presence of risk j.

TCE based allocation formulas for symmetric distributions, having elliptical dependence structure, were studied in Landsman and Valdez (2003a). Notice, that TCE decomposition technics for portfolios with non-negative risks essentially differ from the methods applied to the situations when the risks are symmetric. For example, the distribution function of the convolution of two independent non-negative random variables X and Y may be rewritten as

$$F_{X+Y}(x) = \int_0^x F_X(x-t)\, dF_Y(t). \tag{15}$$

The following Lemma brings out the contribution of the marginal loss X_j given that the aggregate risk S is bigger than any shortfall x_q.We consider this Lemma an important tool for the allocation technics concerning non-negative risks in the most general form.

Lemma 2 *Let* $\mathbf{X} = (X_1, X_2, ..., X_n)^{\mathbb{T}}$ *be a multivariate portfolio, constructed by independent non-negative risks* $X_1, X_2, ..., X_n$ *with densities* $f_{X_j}(x)$, $j = 1, 2, ..., n$ *and finite expectations. Then*

$$E(X_j|S > s_q) = \frac{E(X_j)\left(1 - \frac{1}{E(X_j)}\int_0^{s_q} x f_{X_j}(x)\, F_{S-X_j}(s_q - x)\, dx\right)}{\overline{F}_S(s_q)}. \tag{16}$$

We have already pointed out that there is some complexity with the convolution of gamma losses. In fact, even the distribution of the sum of n independent gamma random variables with different rates is rather complicated. We treated this problem by introducing some integer random variables. One of them K, defined earlier in Lemma 1, has the probability distribution function

$$p_k = C\delta_k, \quad K \geq 0.$$

Another integer random variable V appears after summarizing two gamma risks $\widetilde{Y} \backsim Ga(\widetilde{\gamma} + k + 1, \widetilde{\alpha}_{\max})$ and $Y_0' \backsim Ga(\gamma_0, \alpha_0/\eta)$, where $\widetilde{\alpha}_{\max} = \max(\alpha_1, \alpha_2, ..., \alpha_n)$ and $\widetilde{\gamma} = \sum_{j=1}^n \gamma_j$. In fact, V is also determined by Lemma 1 in the sense that the distribution of $\widetilde{S} = \widetilde{Y} + Y_0'$ may be considered a mixture gamma $Ga(\gamma + k + V + 1, \widetilde{\alpha}_{\max})$, where the random shape $\gamma + k + V + 1$ is the mixing parameter.

Let us define $\alpha_{\max} = \max(\alpha_0, \widetilde{\alpha}_{\max})$ and $\gamma = \widetilde{\gamma} + \gamma_0$. We denote Z_t to be a gamma random variable with shape parameter equaled 1 and rate parameter α_t, $t = 0, 1, ..., n$ and for the case when $\alpha_t = \alpha_{\max}$, we write $Z_{\max}$. Let $E_K(\cdot)$ and $E_V(\cdot)$ be the expectations with respect to K and V respectively.

We now state a theorem for calculating the TCE of the sum S.

Theorem 2 *The tail conditional expectation can be expressed*

$$\begin{aligned} & TCE_S(s_q) \qquad (17) \\ = \ & \eta \frac{\gamma_0}{\alpha_0} \frac{\overline{F}_{S+\eta Z_0}(s_q)}{\overline{F}_S(s_q)} + \frac{\widetilde{\gamma}}{\widetilde{\alpha}_{\max}} \frac{\overline{F}_{S+Z_{\max}}(s_q)}{\overline{F}_S(s_q)} + \frac{E_K\left(K E_V \overline{G}(s_q | \gamma + K + V + 1, \widetilde{\alpha}_{\max})\right)}{\widetilde{\alpha}_{\max} \overline{F}_S(s_q)}. \end{aligned}$$

Corollary 1 *Suppose* $\gamma_0 \longrightarrow 0$ *then* $Y_0 \overset{a.s.}{\longrightarrow} 0$, $V \overset{a.s.}{\longrightarrow} 0$. *In the limit case, the random vector* $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *becomes a vector of* n *independent univariate gamma random variables with any rate and shape parameters. Consequently* $V \overset{a.s.}{=} 0$, *and the formula for tail conditional expectation simplifies to*

$$TCE_S(s_q) = \frac{\widetilde{\gamma}}{\widetilde{\alpha}_{\max}} \frac{\overline{F}_{S+Z_{\max}}(s_q)}{\overline{F}_S(s_q)} + \frac{E_K\left(K \overline{G}(s_q | \widetilde{\gamma} + K + 1, \widetilde{\alpha}_{\max})\right)}{\widetilde{\alpha}_{\max} \overline{F}_S(s_q)}. \qquad (18)$$

Furthermore, if in addition to independency the equality of all rate parameters $a_1, \alpha_2, ..., a_n$ *to some* a *is implied, then the formula for TCE becomes even simpler, because in this case* $P(K = 0) = 1$

$$TCE_S(s_q) = E(S) \frac{\overline{G}(s_q | \widetilde{\gamma} + 1)}{\overline{G}(s_q | \widetilde{\gamma})}. \qquad (19)$$

5 TCE based capital allocation

In Section 4 we have already emphasized that TCE based allocation technics for portfolios with non-negative risks essentially differ from the methods applied to elliptical portfolios. Additional distinction appears because of the fact that elliptical marginals are closed under convolutions. In the same time, the distribution of the sum of gamma random variables with different rate parameters is not gamma and therefore is much more complicated then the distributions of the constituents of such a sum.

In the following Theorem we show that the contribution of each marginal risk to the shortfall is stipulated by its expectation, risk's rate parameter and additional gamma random variable added to the aggregate sum.

Theorem 3 *Let* $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *be an* n *variate dependent gamma distributed random vector, where* $X_j \backsim Ga\left(\gamma_0 + \gamma_j, \alpha_j\right)$, *then*

$$TCE_{X_j|S}\left(s_q\right) = E\left(Y_0\right) \frac{\alpha_0 \overline{F}_{S+\eta Z_0}\left(s_q\right)}{\alpha_j \overline{F}_S\left(s_q\right)} + \frac{\gamma_j}{\alpha_j} \frac{\overline{F}_{S+Z_j}\left(s_q\right)}{\overline{F}_S\left(s_q\right)}. \tag{20}$$

Corollary 2 *In the case when the random vector* $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *consists of* n *independent univariate gammas with arbitrary rate and shape parameters, i.e.* $Y_0 \overset{a.s.}{=} 0$ *(see Corollary 1) the formula for the allocation reduces to*

$$TCE_{X_j|S}\left(s_q\right) = \frac{\gamma_j}{\alpha_j} \frac{\overline{F}_{S+Z_j}\left(s_q\right)}{\overline{F}_S\left(s_q\right)}.$$

In the situation when $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ *consists of* n *independent gamma random variables with the same rate* a *and any shapes the formula takes the following form*

$$TCE_{X_j|S}\left(s_q\right) = E\left(Y_j\right) \frac{\overline{G}\left(s_q|\widetilde{\gamma}+1\right)}{\overline{G}\left(s_q|\widetilde{\gamma}\right)}.$$

Let us notice, that due the full allocation principle, one can get the expression for tail conditional expectation risk measure (17) from the formula for TCE based allocation (20), i.e.

$$TCE_S\left(s_q\right) = E\left(\eta Y_0\right) \frac{\overline{F}_{S+\eta Z_0}\left(s_q\right)}{\overline{F}_S\left(s_q\right)} + \sum_{j=1}^{n} \frac{\gamma_j}{\alpha_j} \frac{\overline{F}_{S+Z_j}\left(s_q\right)}{\overline{F}_S\left(s_q\right)} \tag{21}$$

We prove the correctness of this result in Appendix 1.

6 Conclusions

In this paper we examined tail conditional expectation and based on it capital allocation for loss random variables that belong to multivariate Gamma distribution. The discussed distribution posses some features that are considered desirable for actuarial professionals. For instance it is non-symmetric, it has positive support and it is relatively tolerant to big losses. We studied the multivariate Gamma model presented by Mathai and Moschopoulos (1991).

The dependence structure of this distribution is obtained by adding a common random variable to every univariate marginal. We found an appealing way to express the tail conditional expectation formula for Gamma random variables. Furthermore, we evaluated the formula for the contribution of j-th loss to the whole insurance company risk capital and we showed that the full allocation principle holds in our context. Anybody believing his data is distributed multivariate Gamma model considered in here may find this work self-contained.

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7 Appendix

Define $K^{(j)}$ be an integer non-negative random variable generated by the convolution $\sum_{i=1}^{n} Y_i + Z_j$, thus $K^{(j)}$ has the probability distribution function

$$p_k^{(j)} = C^{(j)} \delta_k^{(j)}, k \geq 0, \tag{22}$$

where $C^{(j)}$ and $\delta_k^{(j)}$ follow directly from Lemma 1.

Theorem 4 *1. The expression for tail conditional expectation may be written as follows*

$$TCE_S(s_q) = E(\eta Y_0) \frac{\overline{F}_{S+\eta Z_0}(s_q)}{\overline{F}_S(s_q)} + \sum_{j=1}^{n} \frac{\gamma_j}{\alpha_j} \frac{\overline{F}_{S+Z_j}(s_q)}{\overline{F}_S(s_q)}. \tag{23}$$

2. Two useful relations between p_k and $p_k^{(j)}$ are

$$p_k^{(j)} = p_k \frac{\alpha_j}{\alpha_{\max}} \left(1 + \frac{k}{\gamma_j} \left(\frac{\gamma_j \Delta_j \delta_{k-1}^{(j)}}{k \delta_k}\right)\right), \tag{24}$$

$$\delta_k = \frac{1}{k} \sum_{j=1}^{n} \gamma_j \Delta_j \delta_{k-1}^{(j)}, k > 0 \tag{25}$$

and $p_0^{(j)} = p_0 \frac{\alpha_j}{\alpha_{\max}}, \delta_0^{(j)} = \delta_0 = 1.$