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# PARAMETRIC MODELS FOR LIFE INSURANCE MORTALITY DATA: GOMPERTZ'S LAW OVER TIME

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ABSTRACT

The use of Gompertz's law to describe mortality between the ages of 30 and 90 is discussed. A planar model is formulated with attained age and experience year as independent variables and the force of mortality as the dependent variable. When Gompertz's law is applied to the ultimate mortality experience from life insurance for 1948–77, the parameter $B$ is shown to have decreased uniformly at a rate of 1 percent per year, and the parameter $c$ is shown to have increased slightly over the first six years of the period and to have remained relatively constant over the last twenty-four years. A further use of Gompertz's law, as a tool in comparing mortality tables, is presented for thirty-three modern tables.

## I. GOMPERTZ'S LAW

It is well established that Gompertz's law holds fairly closely between the ages of 30 and 90 across a wide range of mortality data (see Spiegelman [8: p. 164] and Wetterstrand [12]). In a recent paper Tenenbein and Vanderhoof have reviewed the scope of Gompertz's law and have explored its biophysical implications [10]. Their paper offers excellent insights into the foundations of the law. They have also extended the law to accommodate select mortality data for life insurance purposes.

Gompertz's law dictates the familiar exponential function for the force of mortality:

$$\mu_y = Bc^y = e^{\alpha+\gamma y} , \tag{1}$$

where $B = e^\alpha$ and $c = e^\gamma$. In words, this implies that a person's probability of dying increases at a constant exponential rate as age increases. Thus the law can be said to describe exponential aging.

If we apply the natural logarithmic transformation to equation (1), a linear equation results:

$$\ln \mu_y = \alpha + \gamma y \, . \tag{2}$$

The familiar textbook approximation can be used to estimate the force of mortality from observed mortality rates [2: p. 17]:

$$\mu_{y+1/2} \doteqdot - \ln (1 - q_y) \, . \tag{3}$$

One of the consequences of formula (2) is that a force of mortality satisfying Gompertz's law has a linear graph when plotted on semilog paper. Miller recommends the plotting of observed mortality rates in this way and notes that others have recommended that graduation processes be applied to a logarithm of the observed rates plus a constant. [3: pp. 14, 15, 46, 48]. Ruth showed that the application of linear least squares to formula (2) is an effective way of determining $B$ and $c$ [6]. Tenenbein and Vanderhoof advocate applying weighted linear least squares to formula [2], using as weights the number of deaths making up the numerator in the mortality rate.

One advantage of applying linear least squares to formula (2) is that the result is of the form

$$\ln \mu_y = \hat{\alpha} + \hat{\gamma} y + \hat{f}_y \, , \tag{4}$$

where $\hat{\alpha}$ and $\hat{\gamma}$ are determined in such way that the $\hat{f}_y$'s are minimal in the least-squares sense; that is, the sum of their squares is minimal. When the exponential transformation is applied to formula (4), the result represents the desired fitted model:

$$\mu_y = \hat{\mu}_y e^{\hat{f}_y} , \tag{5}$$

where $\hat{\mu}_y = e^{\hat{\alpha}+\hat{\gamma}y}$ are the fitted values. The point to be made from formula (5) is that if the $\hat{f}_y$'s are minimized in formula (4) the relative (or percentage) error in formula (5) is minimized, the relative error being contained in the term $e^{\hat{f}_y}$. Thus, minimizing the absolute error in formula (4) is equivalent to minimizing the relative error in formula (5). This is important when dealing with data that vary as much in magnitude as mortality data do.

A computer program named GOMPQ has been written to accept mortality rates as input and produce a graph on a video plotter with photostatic copier. The graph contains approximate values of the forces of mortality, along with a regression line fitted to them, between the ages of 30 and 90. The program uses unweighted linear least squares to estimate ln $\mu_{y+1/2}$. (Weighted least squares will be used in future studies, when possible.) The observed, fitted, and residual values of $\mu_{y+1/2}$ are printed out along with the estimates of $B$, $c$, and the multiple regression coefficient between the observed and fitted values of ln $\mu_{y+1/2}$. A program with identical output, called USLIFE, accepts United States census and mortality data ($P_y$ and $D_y$) as input, and forms $q_y$ from the formula in Greville [1]:

$$q_y = \frac{D_y}{3P_y - \frac{1}{2}D_y}. \tag{6}$$

These two programs allow one to process a set of mortality data quite easily in five minutes or less at a graphic terminal.

## II. EXAMPLES

The applicability of Gompertz's law to life insurance mortality data can be explored by considering the results of using GOMPQ to analyze the following sets of data:

1. 1930–40 experience, which was the basis for the Commissioners 1941 Standard Ordinary Mortality Table [9].
2. 1950–54 experience, which was the basis for the Commissioners 1958 Standard Ordinary Mortality Table [9].
3. 1970–75 experience separated by sex, which is the basis for the K-tables proposed as new standards for life insurance valuation [7].

These results are shown graphically in Figures 1–4. The Gompertz constants and corresponding complete expectations of life at age 30 are shown in Table 1.

Good Gompertz fits between ages 30 and 90 can be seen in Figures 1–3. Figures 1 and 2 represent predominantly male mortality, and Figure 3 rep-

TABLE 1

COMBINED, ULTIMATE EXPERIENCE FOR LIFE INSURANCE: COMPARISON OF PARAMETERS OVER FOUR TIME PERIODS

| Period | 1,000$B$ | $c$ | $\mathring{e}_{30}$ |
|---|---|---|---|
| 1930–40 | .1326 | 1.089 | 38.89 |
| 1950–54 | .0569 | 1.098 | 43.65 |
| 1970–75, male | .0416 | 1.100 | 45.55 |
| 1970–75, female | .0312 | 1.097 | 50.29 |

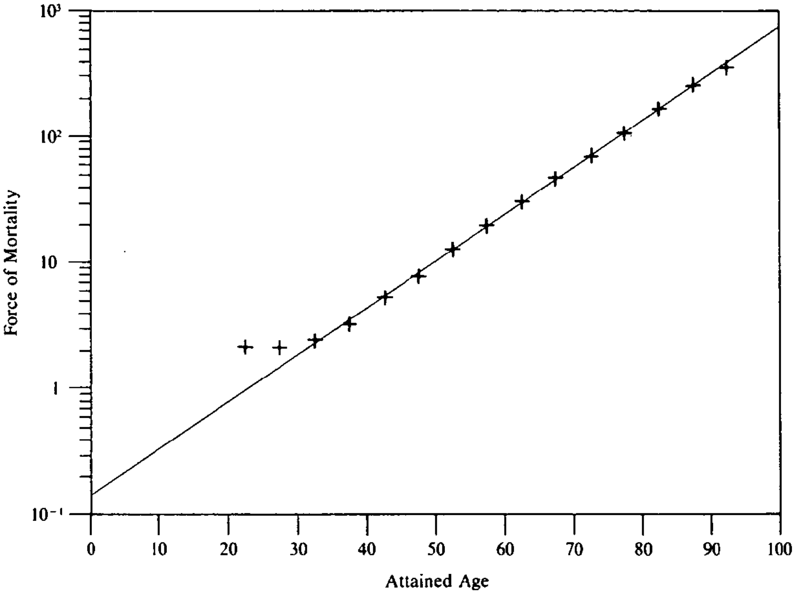


FIG. 1.—Combined, ultimate experience for life insurance: force of mortality by age for 1930–40 anniversaries.

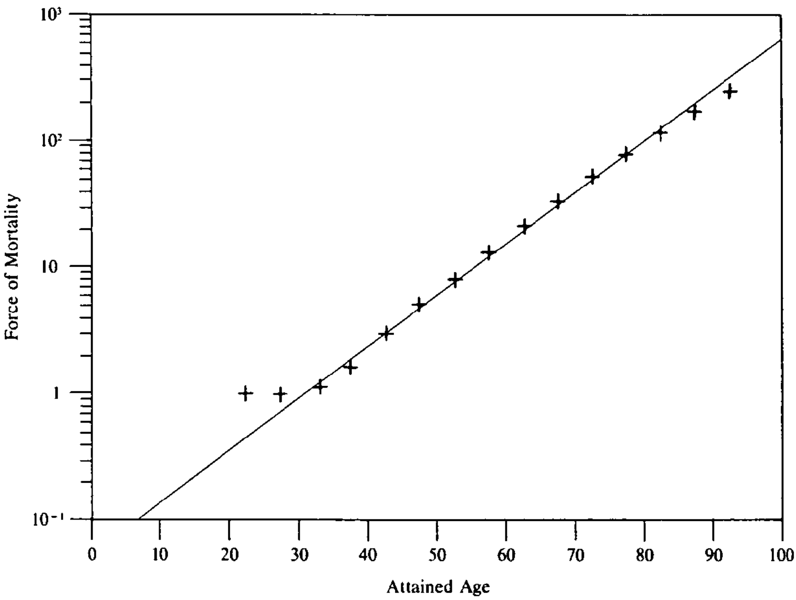


FIG. 2.—Combined, ultimate experience for life insurance: force of mortality by age for 1950–54 anniversaries.

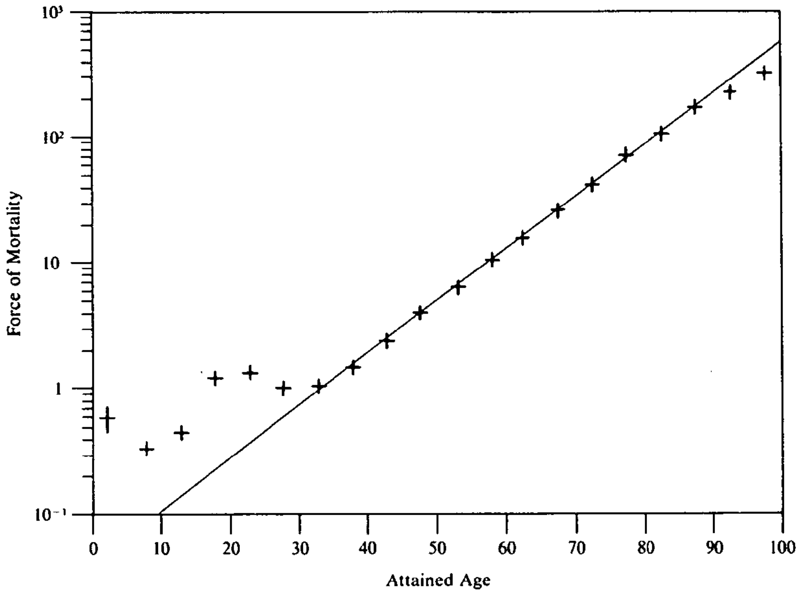


FIG. 3.—Male, ultimate experience for life insurance: force of mortality by age for 1970–75 anniversaries.

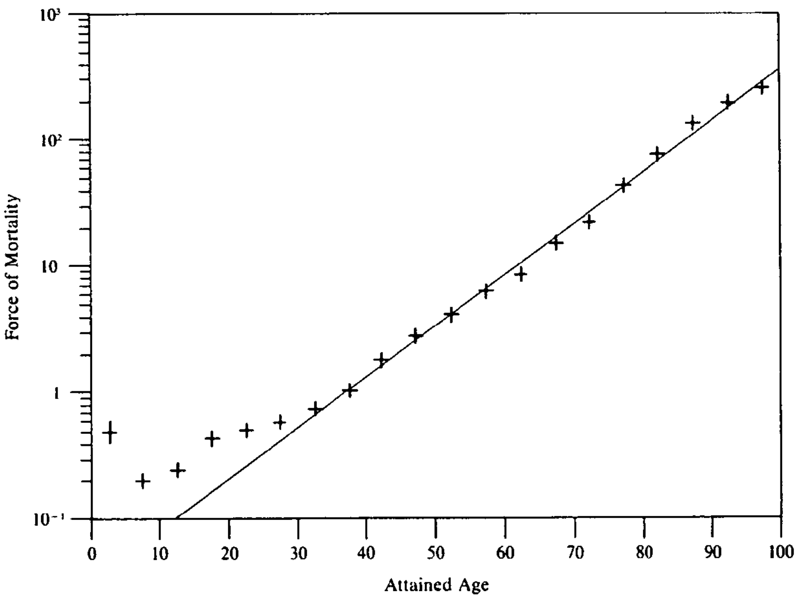


FIG. 4.—Female, ultimate experience for life insurance: force of mortality by age for 1970–75 anniversaries.

resents totally male mortality. There is some convexity of the observed data points above the regression line, centered at age 52. This is particularly evident in the 1950–54 experience and perhaps explains why Gompertz's law was not used in constructing the 1958 CSO Mortality Table. This convexity has been typical of white male mortality in the United States in recent years. Despite the good Gompertz fit for the 1930–40 data, Makeham's law was used in constructing the 1941 CSO Table, giving the familiar asymptotic (rather than linear) shape to the forces of mortality plotted on a logarithmic scale.

Table 1 shows that the Gompertz parameter $c$ has increased somewhat over the years, while there has been a steady decrease in the parameter $B$.

Female data are represented in Figure 4. It exhibits a pronounced concavity centered around age 62; this is a characteristic of recent white female mortality in the United States. By comparing the parameters for the 1970–75 Experience Tables in Table 1 with those for the K-tables in Table 2, we see that for the K-tables the parameters $B$ were more than doubled and the parameters $c$ were reduced by 1 percent. The general shapes of the two experience tables were, however, retained in the final loaded, graduated K-tables.

Gompertz's law holds for United States population data as well. Figure 5 contains a graph and least-squares fit for the United States Decennial Life Tables for 1969–71—Total, White [5]. There is the characteristic convexity for the male data and concavity for the female data, as mentioned above for insurance data; however, the two sexes seem to balance each other in combination. As an exercise in the graduation-of-data course at Ball State University, students were asked to construct life tables for a number of selected standard metropolitan areas in the United States for the years 1969–71. They were asked to produce Gompertz fits between ages 30 and 90, and the fits were very good in all cases. There was only one truly discrepant data point (or "outlier"). The parameters $B$ and $c$ varied considerably, however, and would be worthy of study over the whole country, by state and region. Figure 6 displays results for Muncie, Indiana, which are typical.

TABLE 2

COMPARISON OF PARAMETERS FOR PROPOSED VALUATION TABLES (K-TABLES)

| | $1{,}000B$ | $c$ | $\mathring{e}_{30}$ |
|---|---|---|---|
| Male .................... | .0905 | 1.091 | 43.07 |
| Female .................. | .0777 | 1.087 | 47.49 |

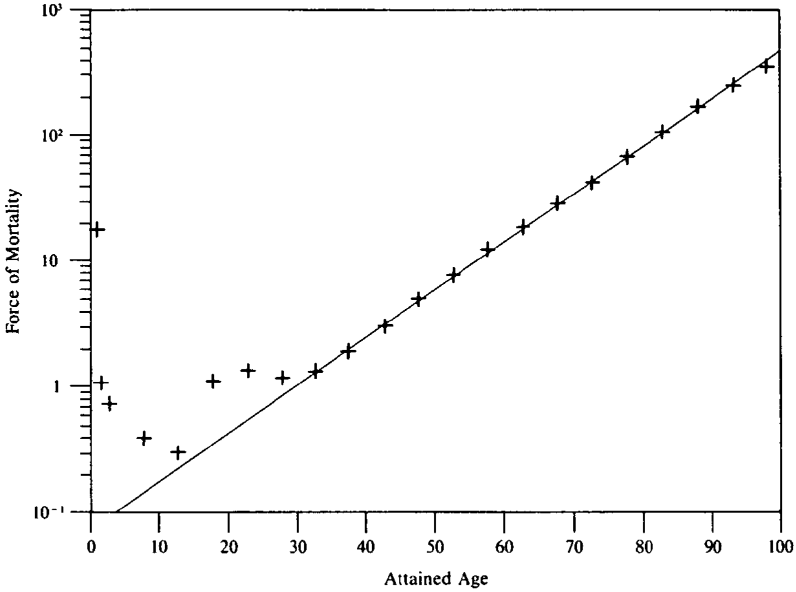


FIG. 5.—United States Life Table for 1969–71—Total, White: force of mortality by age.

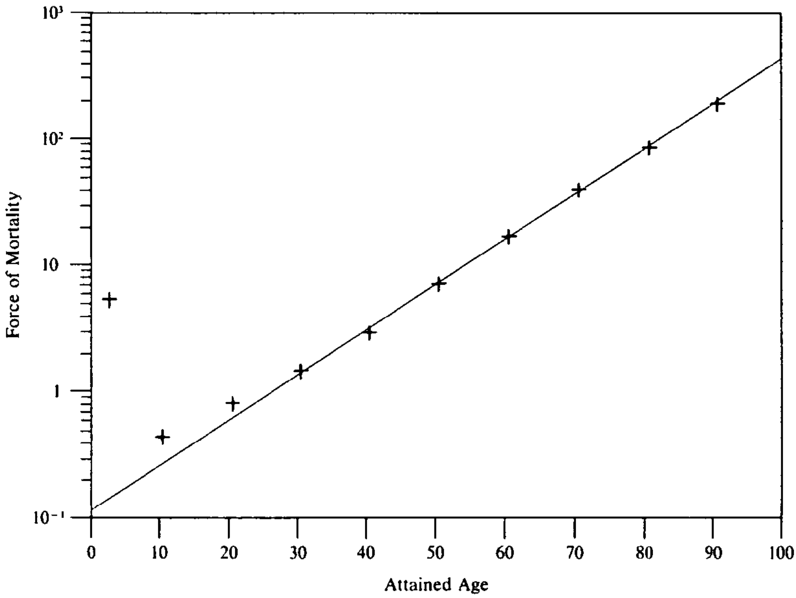


FIG. 6.—1969–71 Life Table for Muncie, Indiana: force of mortality by age

## III. SOCIETY OF ACTUARIES CONTINUING STUDY OF ULTIMATE MORTALITY

The Society of Actuaries, in its annual Reports of Mortality and Morbidity Experience, has published mortality data for standard issues, by attained age, for policy years 16 and over combined, on an annual basis for each experience year beginning with the period between 1947 and 1948 policy anniversaries and continuing to the present. This is the most homogeneous, regular body of mortality data available in which to observe annual changes. For this paper, data for thirty policy years, beginning with the year ending on the 1948 policy anniversary and concluding with the year ending on the 1977 policy anniversary, were studied separately with particular reference to the applicability of Gompertz's law to such data. The experience years are labled 1948–77 in this paper. The label 1948, for example, is used to identify experience between 1947 and 1948 anniversaries, with midpoint January 1, 1948.

A program similar to GOMPQ was used to analyze each year's data separately, fitting Gompertz's law to the forces of mortality with unweighted least squares. Data for 1976–77 anniversaries were analyzed separately with GOMPQ. Three typical graphs are presented in Figures 7–9, corresponding

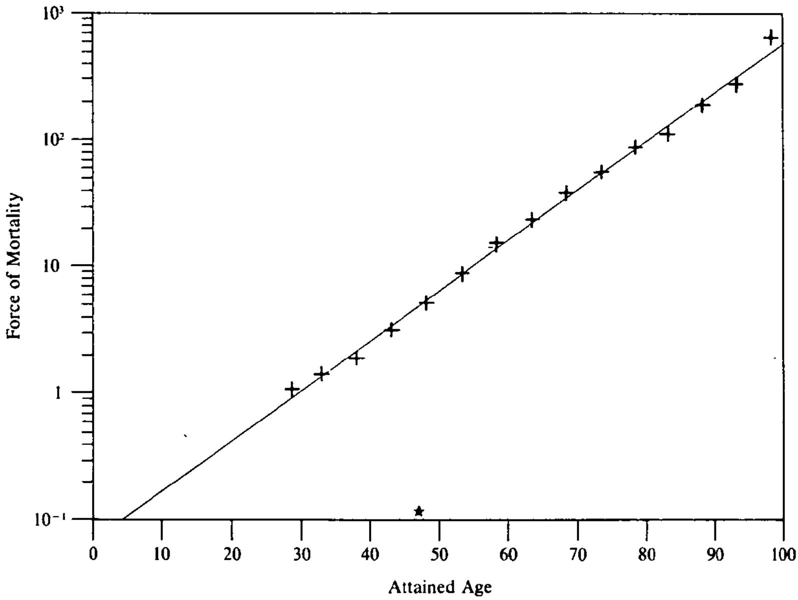


FIG. 7.—Combined, ultimate experience for life insurance: force of mortality by age for 1947–48 anniversaries.

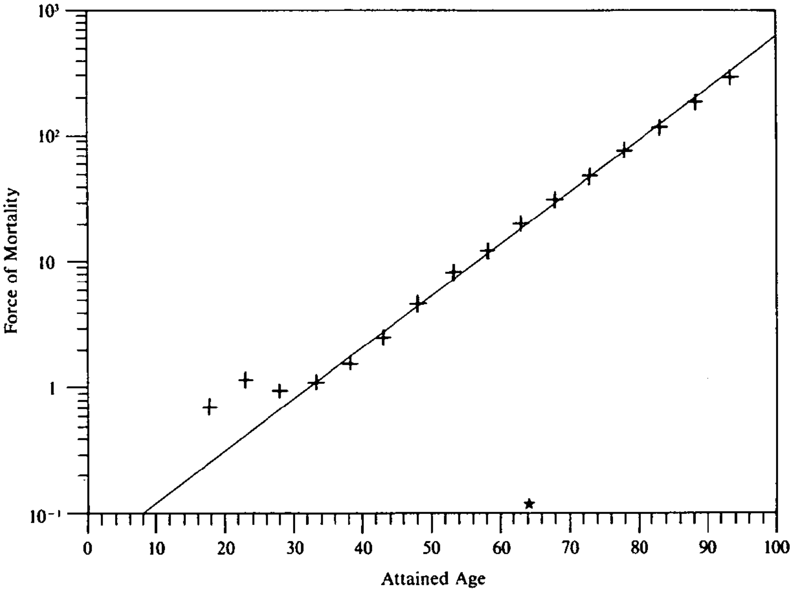


FIG. 8.—Combined, ultimate experience for life insurance: force of mortality by age for 1962–63 anniversaries.

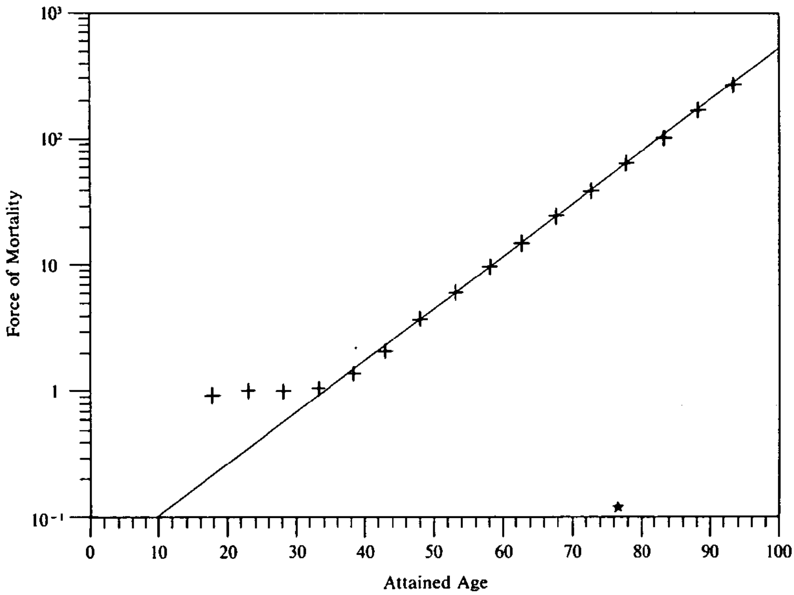


FIG. 9.—Combined, ultimate experience for life insurance: force of mortality by age for 1975–76 anniversaries.

TABLE 3

COMBINED, ULTIMATE EXPERIENCE FOR LIFE INSURANCE:
COMPARISON OF PARAMETERS FROM 1947–48 TO 1976–77

| Year | 1,000$B$ | $c$ | $\mathring{e}_{30}$ | Year | 1,000$B$ | $c$ | $\mathring{e}_{30}$ |
|---|---|---|---|---|---|---|---|
| 1948 .... | .074 | 1.095 | 42.7 | 1963 .... | .049 | 1.099 | 44.5* |
| 1949 .... | .072 | 1.095 | 43.0 | 1964 .... | .055 | 1.097 | 44.5 |
| 1950 .... | .070 | 1.095 | 43.5 | 1965 .... | .054 | 1.097 | 44.7 |
| 1951 .... | .069 | 1.095 | 43.3* | 1966 .... | .052 | 1.098 | 44.8 |
| 1952 .... | .073 | 1.094 | 43.7 | 1967 .... | .050 | 1.098 | 44.8 |
| 1953 .... | .068 | 1.095 | 43.7 | 1968 .... | .048 | 1.099 | 44.8 |
| 1954 .... | .061 | 1.097 | 43.8 | 1969 .... | .053 | 1.097 | 45.0 |
| 1955 .... | .057 | 1.097 | 44.0 | 1970 .... | .054 | 1.096 | 45.3 |
| 1956 .... | .056 | 1.097 | 44.2 | 1971 .... | .050 | 1.097 | 45.5 |
| 1957 .... | .050 | 1.099 | 44.2 | 1972 .... | .048 | 1.098 | 45.6 |
| 1958 .... | .055 | 1.098 | 44.2 | 1973 .... | .044 | 1.099 | 45.6 |
| 1959 .... | .050 | 1.099 | 44.4 | 1974 .... | .046 | 1.098 | 45.8 |
| 1960 .... | .046 | 1.100 | 44.4 | 1975 .... | .044 | 1.098 | 46.3 |
| 1961 .... | .051 | 1.098 | 44.6 | 1976 .... | .041 | 1.099 | 46.7 |
| 1962 .... | .049 | 1.099 | 44.6 | 1977 .... | .043 | 1.098 | 46.6* |

*Decreased from previous year.

to the experience years 1948, 1963, and 1976, respectively. Characteristic convexity can be observed for 1948 and 1963, but not for 1976. The values of the parameters 1,000$B$ and $c$, along with $\mathring{e}_{30}$, are shown in Table 3 for all years. There is a fairly uniform decrease in the parameter 1,000$B$ over the years, from 0.074 in 1948 to 0.043 in 1977. There is one sharp break in the values of $c$ between 1953 and 1954. After 1954, $c$ stays between 1.096 and 1.100. From these data we see that Gompertz's law holds quite closely in annual cross sections of the mortality surface. We will now investigate that surface in its entirety.

## IV. PLANAR MODELS INCORPORATING TIME TRENDS

Two versions of a three-dimensional model have been formulated, each with two independent variables:

1. Attained age $y$, which is time from birth.
2. Two alternative time measures:
   a) Experience year $s$, which is time from the base year (1900).
   b) Birth year, $u = s - y$, which is constant for any individual, measured from the base year (1900).

Parameters in the model include the following:

1. $B_0$, the Gompertz location parameter for the base year (1900).

2. $c$, the Gompertz aging parameter.
3. $d$, a new parameter to allow discounting of $B$ with respect to time.

Two equivalent models are considered here. They are both extensions of Gompertz's law and correspond to using the two time measures described above. The first model is

$$\mu_{y,s} = B_0 d^s c^y = B_s c^y , \tag{7}$$

where $B_s$ $(= B_0 d^s)$ varies by experience year but $c$ does not. The second model results from a simple reparameterization of the first:

$$\mu_{y,u} = B_0 d^u \tilde{c}^y = B_u \tilde{c}^y , \tag{8}$$

where $B_u$ $(= B_0 d^u)$ varies by birth year (or cohort) and $\tilde{c} = cd$. Since the two are equivalent, we will concentrate on the experience year in fitting a planar model to the data considered in the previous section.

We first transform formula (7) into exponential form:

$$\mu_{y,s} = e^{\alpha + \beta s + \gamma y} , \tag{9}$$

where $B_0 = e^{\alpha}$, $d = e^{\beta}$ , and $c = e^{\gamma}$. Taking the natural logarithm of formula (9) yields

$$\ln \mu_{y,s} = \alpha + \beta s + \gamma y . \tag{10}$$

This expression represents a plane for $\mu_{y,s}$.

Figure 10 shows the surface of ln $\mu_{y,s}$ for the data considered previously covering the years 1948–76. At first glance it seems quite planar, almost dull. There is some jaggedness at the higher ages, and a slight hump at the middle ages for the earlier years.

Unweighted least-squares analyses were performed on the data, using the model in formula (10) for two time periods: 1948–76 and 1954–76. Because of the break in the value of $c$ around 1954, noted in the univariate analyses, the fit of the model for the second time period was found to be slightly better than for the first time period.

The values of the parameters for both time periods are shown in Table 4. The values of $c$ are in the range established previously by the univariate analyses. The parameter $d$ corresponds to a rate of discount of about 1 percent per year. The parameter $B_0$ is arbitrary, since it depends on the choice of the base year.

The fit of the planar model can be judged from the residuals, or deviations of the observed values of ln $\mu_{y,s}$ from their fitted values. These are exhibited in Table 5. If we consider a residual value of zero to be the level of water in a topological map, we can speak of "islands," "peninsulas," and "bays" representing curvature above and below the regression plane. There is a good deal of curvature above the plane between the ages 47.5 and 67.5. Data for 1975 and 1976 are under water and fairly level. These characteristics

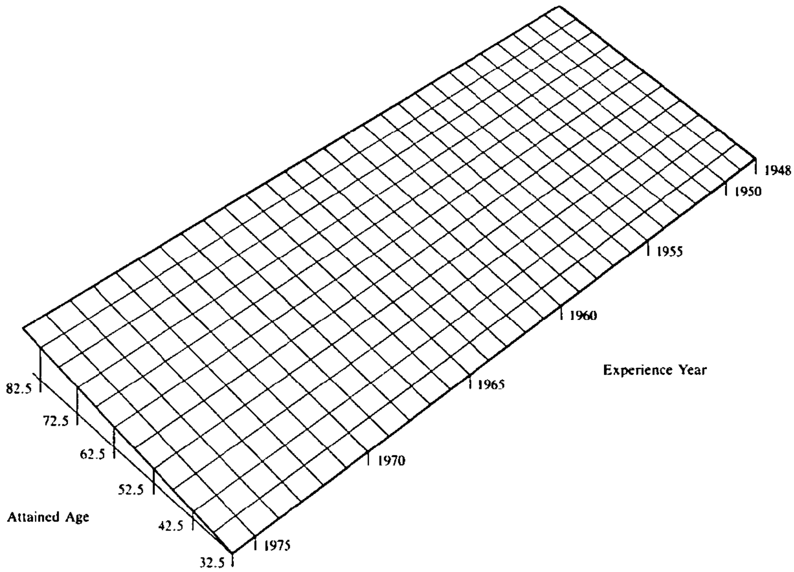


FIG. 10.—Combined, ultimate experience for life insurance: logarithm of force of mortality by age and year.

TABLE 4

COMBINED, ULTIMATE EXPERIENCE FOR LIFE INSURANCE: PARAMETERS FOR PLANAR MODEL OVER TWO TERM PERIODS

| Parameter | 1948–76 | 1954–76 |
|---|---|---|
| $\alpha$ | −2.255 | −2.340 |
| $B_0$ | .10487 | .09630 |
| $\beta$ | − .010667 | − .009949 |
| $d$ | .98939 | .99070 |
| $\gamma$ | .09278 | .09339 |
| $c$ | 1.09722 | 1.09789 |

TABLE 5

COMBINED ULTIMATE EXPERIENCE FOR LIFE INSURANCE: RELATIVE RESIDUALS FOR PLANAR MODEL

| YEAR | AGE | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 32.5 | 37.5 | 42.5 | 47.5 | 52.5 | 57.5 | 62.5 | 67.5 | 72.5 | 77.5 | 82.5 | 87.5 |
| 1948 .... | .07 | -.04 | .03 | .05 | .12 | .18 | .14 | .15 | .04 | .04 | -.17 | -.09 |
| 1949 .... | .06 | -.03 | -.06 | .05 | .10 | .14 | .15 | .09 | -.02 | -.07 | -.09 | -.11 |
| 1950 .... | -.09 | -.11 | -.03 | .04 | .15 | .12 | .14 | .06 | .02 | .02 | -.17 | -.35 |
| 1951 .... | .08 | -.18 | .10 | .06 | .09 | .10 | .12 | .05 | .05 | .09 | -.08 | -.11 |
| 1952 .... | .05 | -.09 | .01 | .02 | .09 | .11 | .08 | .04 | .07 | -.11 | -.17 | -.25 |
| 1953 .... | -.02 | -.13 | .06 | .01 | .05 | .10 | .15 | .10 | .04 | -.05 | -.14 | -.26 |
| 1954 .... | .06 | -.14 | -.10 | .03 | .06 | .07 | .08 | .03 | -.03 | -.05 | -.03 | -.11 |
| 1955 .... | .02 | -.11 | -.11 | -.03 | .00 | .05 | .08 | .05 | -.04 | -.02 | -.07 | -.05 |
| 1956 .... | .03 | -.24 | -.12 | -.02 | .03 | .06 | .09 | .02 | .05 | -.06 | -.09 | -.18 |
| 1957 .... | -.08 | -.19 | -.11 | -.04 | .05 | .06 | .05 | .06 | .01 | .00 | -.03 | -.09 |
| 1958 .... | -.04 | -.16 | -.09 | .07 | .05 | .08 | .13 | .06 | .02 | -.06 | -.04 | -.09 |
| 1959 .... | -.08 | -.16 | -.14 | .02 | .06 | .04 | .08 | .08 | -.02 | -.01 | -.05 | -.05 |
| 1960 .... | -.15 | -.20 | -.11 | -.01 | .08 | .08 | .05 | .07 | .05 | -.05 | -.08 | .03 |
| 1961 .... | -.08 | -.17 | -.06 | .04 | .04 | .06 | .07 | .08 | .03 | -.03 | -.05 | -.08 |
| 1962 .... | -.09 | -.18 | -.08 | .01 | .08 | .07 | .08 | .04 | .04 | -.00 | -.02 | -.07 |
| 1963 .... | -.05 | -.20 | -.10 | .03 | .14 | .05 | .11 | .08 | .04 | .02 | -.02 | -.05 |
| 1964 .... | .09 | -.10 | -.09 | .10 | .09 | .10 | .11 | .10 | .03 | .01 | -.02 | -.03 |
| 1965 .... | .05 | -.04 | -.06 | .05 | .01 | .10 | .10 | .04 | .04 | -.00 | -.01 | -.05 |
| 1966 .... | .10 | -.12 | -.06 | .02 | .04 | .09 | .11 | .07 | -.00 | .01 | .00 | -.02 |
| 1967 .... | .10 | -.13 | -.06 | -.02 | .05 | .06 | .09 | .10 | .04 | .00 | -.00 | .00 |
| 1968 .... | .01 | -.07 | -.05 | .02 | .04 | .10 | .08 | .12 | .04 | .09 | .00 | .02 |
| 1969 .... | .06 | -.05 | .02 | .02 | .09 | .11 | .08 | .07 | .04 | .05 | .01 | -.08 |
| 1970 .... | .14 | -.06 | -.01 | .00 | .06 | .07 | .07 | .05 | .04 | -.01 | -.01 | -.07 |
| 1971 .... | .13 | -.15 | -.02 | -.03 | .00 | .06 | .07 | .04 | .00 | .03 | -.04 | -.03 |
| 1972 .... | .09 | -.09 | -.12 | .02 | .03 | .04 | .06 | .06 | .04 | .00 | .01 | -.04 |
| 1973 .... | -.03 | -.12 | -.05 | .02 | -.01 | .05 | .06 | .06 | .04 | .02 | -.00 | .02 |
| 1974 .... | .10 | -.06 | -.10 | -.04 | -.02 | .08 | .03 | .07 | .03 | .06 | .02 | -.04 |
| 1975 .... | .12 | -.19 | -.12 | -.05 | -.06 | -.01 | -.02 | -.00 | -.04 | -.03 | -.03 | -.04 |
| 1976 .... | .05 | -.15 | -.19 | -.09 | -.08 | -.09 | -.03 | -.03 | -.04 | -.03 | -.04 | -.05 |

were noted previously in the cross sections by experience year. Also, there is an underwater trench for all years at ages 37.5 and 42.5. Finally, there is an underwater low spot at the higher ages and earlier years.

## V. POLYNOMIAL MODELS

Polynomial models involving up to fourth-degree powers of the variables $s$ and $y$ were fitted to the data in an attempt to describe better the curvature noted above. Although the fit was improved by expanding the planar model in this way, the improvement was not very noticeable. The residuals for the polynomial models still displayed systematic patterns. Because of the simplicity of the planar models and the only marginal improvement achieved by using the polynomial models, the planar models are to be preferred and the results for the polynomial models are not given here.

## VI. VALIDITY OF THE PLANAR MODEL

It is interesting to compare the results for the planar model applied to insurance data with results using population data. Myers presents a table of ratios of the mortality rates in the United States Life Tables from 1900 onward to corresponding rates in the 1969–71 table [4]. The ratios for the 1949–51 and 1959–61 tables are fairly constant for ages 30–90, indicating that the slope of a fitted Gompertz curve might be similar to that for the 1969–71 table. Thus, a planar fit might be appropriate for population data in the same time period as covered under the insurance study. Earlier tables in the series show a definite downward trend of ratios with advancing age, indicating a decrease in $c$ (the slope) as time recedes. Further study of the United States Life Tables is warranted, taking into account the factors of sex and race as well as age.

The conclusions of the study of planar models presented here are fairly straightforward. Gompertz's law describes fairly accurately the ultimate mortality of insured lives over the last thirty years. The parameter $B$ has decreased uniformly over that time period at a rate of about 1 percent per year. The parameter $c$ increased slightly over the first six years of the period and has remained relatively constant over the last twenty-four years. One might be tempted to expect, for prediction purposes, that these relationships will continue to hold in the future. Gompertz's law is useful, if not as a formal graduation process, then at least as a descriptive tool in mortality study.

## VII. COMPARISON OF THIRTY-THREE MORTALITY TABLES

An actuary is often called upon to make quick comparisons of mortality tables. One way to do this is to compare expectations of life from one table to another. The expectation of life at age 30 might be relevant for many insurance purposes, whereas the expectation of life at age 65 might be relevant to pension costs. Ultimately, for more careful comparisons, actuaries would consider annuity values at particular ages for relevant interest rates.

In addition to complete expectations of life, it is useful to compare the Gompertz parameters $B$ and $c$. Table 6 contains values of $1{,}000B$, $c$, $\mathring{e}_{30}$, and $\mathring{e}_{65}$ for thirty-three common mortality tables. The Gompertz parameters were obtained by using GOMPQ to process mortality rates from Tillinghast, Nelson and Warren's widely distributed compendium of mortality tables [5]. The expectations of life were obtained from that source also, except for

TABLE 6

COMPARISON OF MORTALITY TABLES—AGES 30–90

| Mortality Table | $1{,}000B$ | $c$ | $\mathring{e}_{30}$ | $\mathring{e}_{65}$ |
|---|---|---|---|---|
| American Experience | .561 | 1.071 | 35.3 | 11.1 |
| Standard Industrial | 1.050 | 1.066 | 30.6 | 9.4 |
| 1941 Standard Industrial | .475 | 1.075 | 34.2 | 10.1 |
| American Men Ultimate (5) | .269 | 1.081 | 37.7 | 11.3 |
| 1930–40 Experience | .133 | 1.089 | 38.9* | 12.3* |
| 1941 CSO | .272 | 1.080 | 37.7 | 11.6 |
| 1950–54 Experience | .057 | 1.098 | 43.7* | 13.9* |
| 1958 CSO | .120 | 1.089 | 41.3 | 12.9 |
| 1960 CSG | .147 | 1.087 | 40.3 | 12.5 |
| 1961 CSI | .193 | 1.083 | 39.7 | 12.6 |
| 1965–70 Ultimate Basic—Combined | .051 | 1.098 | 44.3 | 14.5 |
| 1965–70 Ultimate Basic—Male | .055 | 1.098 | 43.9 | 14.2 |
| 1965–70 Ultimate Basic—Female | .032 | 1.098 | 49.2 | 17.8 |
| 1970–75 Experience—Male | .042 | 1.100 | 45.6* | 15.1* |
| 1970–75 Experience—Female | .031 | 1.097 | 50.3* | 19.1* |
| K—Male | .091 | 1.091 | 43.1* | 14.0* |
| K—Female | .078 | 1.087 | 47.1* | 17.4* |
| Combined Annuity | .182 | 1.084 | 40.0 | 12.7 |
| 1937 Standard Annuity | .201 | 1.079 | 41.9 | 14.4 |
| $a$-1949–Male | .057 | 1.096 | 44.6 | 15.0 |
| G$a$-1951—Male | .049 | 1.100 | 44.0 | 14.2 |
| G$a$-1951—Male Projected to 1965 | .037 | 1.101 | 45.5 | 15.1 |
| 1955 American Annuity Male | .046 | 1.097 | 46.5 | 15.9 |
| 1971 Individual Annuity Male | .049 | 1.094 | 47.4 | 17.2 |
| 1971 Individual Annuity Female | .018 | 1.102 | 52.4 | 20.1 |
| 1971 GAM—Male | .037 | 1.102 | 45.6 | 15.1 |
| 1971 GAM—Female | .015 | 1.105 | 51.8 | 19.2 |
| 1969—71 U.S. Life—Male, White | .112 | 1.090 | 41.1 | 13.0 |
| 1969—71 U.S. Life—Female, White | .048 | 1.095 | 47.6 | 16.9 |
| 1969—71 U.S. Life—Total, White | .081 | 1.091 | 44.3 | 15.1 |
| 1969—71 U.S. Life—Male, Nonwhite | .839 | 1.061 | 36.2 | 12.9 |
| 1969—71 U.S. Life—Female, Nonwhite | .337 | 1.068 | 42.6 | 16.0 |
| 1969—71 U.S. Life—Total, Nonwhite | .563 | 1.064 | 39.4 | 14.5 |

*Computed from Gompertz parameters.

those marked with an asterisk, which were computed from the Gompertz parameters, as discussed previously.

Two tables can be compared if their values for $c$ are identical or very close. The logarithmic slopes of the forces of mortality are equal (or nearly so), and the values of $B$ can be compared directly as representing the levels of mortality in the two tables. The 1965–70 Ultimate Basic Tables have the same value of $c$ for male and female. The ratio of male to female values of $B$ is 1.719, and one can therefore conclude that male mortality is 172 percent of female mortality, uniformly, from age 30 to age 90. The ratio can be translated to an age setback for females of $(\log 1.719)/(\log 1.089) = 6.35$ years, confirming the six-year setback in the 1976 amendments to the standard nonforfeiture law. The 1971 Group Annuity Mortality Male Table and the 1970–75 Experience Male Table have very close values of $B$ and $c$, from which one can conclude that they have very similar levels of mortality from age 30 to age 90.

It is interesting to note that tables intended for use in the valuation of life insurance have higher $B$-values and lower $c$-values than the experience tables upon which they are based. This is because the margins added to the basic experience have been decreasing percentages of the mortality rates as age increases. When $B$ and $c$ move in opposite directions, nothing definitive can be said concerning the relative levels of mortality in two tables.

In population data, the values of $B$ and $c$ are markedly different for whites and nonwhites. For nonwhites, $c$ is much lower and $B$ is much higher. Gompertz's law applies to both, nonetheless.

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# DISCUSSION OF PRECEDING PAPER

ROGER S. LUMSDEN:*

Mr. Wetterstrand's comment in Section VI that "one might be tempted to expect, for prediction purposes, that these relationships will continue to hold in the future" was interesting. In effect, he has created a facility for projecting insurance mortality tables, for the important insurance ages, similar in some ways to the projection scales associated with the $a$-1949 and 1971 Individual Annuity Mortality Tables. The difference is that his method has a much more solid base than the rather arbitrary factors associated with the many annuity projection scales.

It would certainly be pleasant if data had been available to permit a study similar to Tables 3 and 4 for the individual annuity experience. Less use is available for projected insurance mortality than for projected annuitant mortality, since in the insurance case projection is not conservative. However, a member attempting to value by using currently appropriate assumptions may benefit from knowing the probable expected improvement in mortality to date in such a precise and compact form.

It is also possible to use the data in Table 4 to create the insurance analogue of a progressive annuity table, where the improvement is assumed to continue indefinitely. Using the Table 4 data for the years 1954–76, I estimate that a static table for 1980 would have an age 30 complete expectation of life of about 45.9 years, but a progressive table would project at about 49.8 years, an increase in life expectancy of 8½ percent if mortality continues to improve.

(AUTHOR'S REVIEW OF DISCUSSION)

W. H. WETTERSTRAND:

Mr. Lumsden's comment concerning the application of the planar model to annuitant mortality, for the purpose of developing projection factors for the rate of improvement of mortality, is well taken.

Since the first appearance of the paper in preprint form, I have been able to convert the computer programs to include weighted least-squares analysis, on another computer, and have run several interesting analyses, including the following:

1. Mortality for Individual Immediate Annuities, Male and Female, Refund and Non-

* Mr. Lumsden, not a member of the Society, is assistant superintendent, actuarial valuation, Crown Life Insurance Company.

TABLE 1

SUMMARY OF RESULTS FOR PLANAR MODEL

| Combined, Insurance Data | $1{,}000B_0$ | $d$ | $c$ | $1-R^2$ | RMSE |
|---|---|---|---|---|---|
| 1948–76: unweighted | .1049 | .9894 | 1.0972 | .27% | ...... |
| 1948–78: weighted | .1163 | .9884 | 1.0972 | .28 | 7% |
| 1948–53: weighted | .1995 | .9806 | 1.0947 | .43 | 10 |
| 1954–76: unweighted | .0963 | .9901 | 1.0979 | .21 | ...... |
| 1954–73: weighted | .0897 | .9921 | 1.0979 | .20 | 7 |
| 1974–78: weighted | .0623 | .9660 | 1.0978 | .17 | 2 |
| Female data: | | | | | |
| Annuity; nonrefund* | .0209 | 1.0210 | 1.0888 | 8.66 | 29 |
| Annuity; refund* | .0589 | .9974 | 1.0942 | 2.93 | 17 |
| Medicare: 1968–78 | .1081 | .9764 | 1.1068 | .11 | 2.5 |
| Medicare: 1968–73 | .0661 | .9824 | 1.1078 | .07 | 1.9 |
| Medicare: 1974–78 | .1094 | .9772 | 1.1057 | .12 | 2.5 |
| Male data: | | | | | |
| Annuity; non-refund* | .1434 | 1.0041 | 1.0805 | 9.80 | 28 |
| Annuity; refund* | .1909 | .9985 | 1.0821 | 3.02 | 15 |
| Medicare: 1968–78 | .6080 | .9847 | 1.0812 | .08 | 1.6 |
| Medicare: 1968–73 | .3706 | .9916 | 1.0813 | .06 | 1.5 |
| Medicare: 1974–78 | .5851 | .9853 | 1.0810 | .04 | 1.1 |

* "By number."

refund, for policy anniversaries in 1948–76, grouped into four- or five-year periods, as reported in the *TSA Reports* for 1954, 1959, 1964, 1969, 1973, and 1979.

2. Mortality for Medicare Recipients, Male and Female, for the years 1968–78, reported by John C. Wilkin elsewhere in this volume.
3. Combined, Ultimate Mortality Experience for life insurance, as analyzed in the paper, but now broken down into three time periods: 1948–53, 1954–73, and 1974–78.

The results for these analyses are presented in Table 1 of this discussion. From the Gompertz model in the paper, $\mu_{y,s} = B_0 d^s c^y$, where $B$ is the location parameter, $d$ is the projection factor, and $c$ is the aging factor. The rate of improvement of mortality is $1 - d$ expressed as a percentage. The quantities labeled $1 - R^2$ and RMSE are measures of fit; $R$ is the multiple correlation coefficient, and RMSE stands for root mean square error, which is the square root of the average squared residual and thus is comparable to an average residual for the model. (The smallness of either indicates a good fit.)

The annuitant data are highly irregular and display a severe lack of fit to the model. In fact, an average mortality increase is exhibited in several cases. Unfortunately, one must look elsewhere for a basis for estimating projection factors.

The medicare data provide such a basis, for ages 65–90, since they are homogeneous, complete, and accurate. For 1968–78, uniform, average, annual improvement factors are exhibited as 1.5 percent for males and 2.4 percent for females. The fits for the medicare data are the best found. A similar figure for 1948–78 using the insurance data exhibits 1.2 percent, which is predominantly male mortality.

Splitting the data into before and after 1974 shows an increase of 2.2 percent, to a 3.4 percent rate of improvement for the insurance data; no change for males receiving medicare; and an increase of 0.1 percent for like females. With essentially no change in the rate of improvement in recent years for medicare, the significant change from a 1.2 percent to a 3.4 percent rate of improvement in the insurance might be questioned. Perhaps it is due to some factor other than pure mortality.

It is interesting to note that the values for $c$ differ among the three groups but have not changed within the groups since 1954.

Photocopies of the tables of residuals for the analyses cited above are available from the author.