Consistent Pricing for Equity-Linked Products

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I. Objectives and Outline

- **Objectives**:
  - Develop pricing methods for equity-linked products.
  - Reproduce the current market prices of standard insurances and annuities.
  - If insurance products were tradable, their prices would not admit arbitrage (Harrison and Pliska (1981)).

- **Applications** of our model:
  - Variable annuities
  - Segregated funds
  - Unit-linked insurances
  - Universal life
  - Equity-indexed annuities

- **Outline** of the talk:
  - II. Binomial financial and insurance models
  - III. Martingale probability measures:
    - (a) for insurances and annuities
    - (b) for endowment insurances
    - (c) for equity-linked products
  - IV. Equity-indexed annuity valuation
  - V. Numerical examples
  - VI. Concluding remarks
  - VII. References
II. Binomial Financial and Insurance Models

\[ N : \text{number of trading dates per year} \ (\Delta = 1/N) \]

Assume that the short-term rate is deterministic: \( r(0), r(1), r(2), \ldots \) are known at time 0 (can be extended to stochastic interest rates).

Let \( S(k) \) be the “index level” described by a modified Cox, Ross and Rubinstein (1979) model, where \( k = 0, \Delta, 2\Delta, \ldots \).

\( S(k) \) can either increase to \( S(k)u(t^* + 1) \) or decrease to \( S(k)d(t^* + 1) \) at time \( k + \Delta \), where \( t^* = [k] \) and \( u(t) \) and \( d(t) \ (t = 1, 2, \ldots) \) are known at time 0.

Under the martingale measure \( Q \), define
\[
\tilde{\pi}(k) = \tilde{\Pr}[S(k + \Delta) = S(k)u(t^* + 1)|S(k)] = \frac{(1 + r(t^*))^\Delta - d(t^* + 1)}{u(t^* + 1) - d(t^* + 1)}.
\]

Under the CRR model \( \frac{S(t+1)}{S(t)} \) can take one of the following \( N + 1 \) possible values
\[
\gamma(t + 1, i) = u(t + 1)^i d(t + 1)^{N-i},
\]
with corresponding martingale probability
\[
\tilde{\Pr}\left[\frac{S(t + 1)}{S(t)} = \gamma(t + 1, i)\right] = \binom{N}{i} \tilde{\pi}(t)^i (1 - \tilde{\pi}(t))^{N-i},
\]
for \( i = 0, \ldots, N \).

\( K(x) \): curtate-future-lifetime of \( x \).

\( V^{(1)}(x, t, n) \): market price at time \( t \) of an \textbf{n-year term life insurance} issued to \( x \) at time 0.

\( V^{(1)}(x, 0, n) \) are the \textbf{current market prices} (single premium) and are given exogenously \( \forall n \).
II. Binomial Financial and Insurance Models

Let $W^{(1)}(x, t, n)$ be the stochastic processes generated by the $n$-year term life insurance

$$W^{(1)}(x, t, n) = \begin{cases} 
\frac{B(t)}{B(K(x)+1)}, & K(x) < t \\
V^{(1)}(x, t, n), & K(x) \geq t 
\end{cases},$$

where $V^{(1)}(x, n, n) = 0$ and $B(t) = \prod_{i=0}^{t-1} (1 + r(i))$, (the money market account).

**Stochastic Evolution of $W^{(1)}(x, t, 2)$**

\[
\begin{align*}
V^{(1)}(x, 0, 2) & \quad V^{(1)}(x, 1, 2) \\
& \quad \quad 0 \\
& \quad \quad V^{(1)}(x, 1, 2) \\
& \quad \quad \quad 1 \\
& \quad \quad \quad \quad 1 + r(1)
\end{align*}
\]

$V^{(2)}(x, t, n)$: market price at time $t$ of an $n$-year pure endowment insurance issued to $(x)$ at time 0.

$V^{(2)}(x, 0, n)$ are the current market prices and are given exogenously $\forall n$. 
II. Binomial Financial and Insurance Models

Let $W^{(2)}(x, t, n)$ be the stochastic processes generated by the $n$-year pure endowment insurance

$$W^{(2)}(x, t, n) = \begin{cases} 
0, & K(x) < t \\
V^{(2)}(x, t, n), & K(x) \geq t 
\end{cases},$$

where $V^{(2)}(x, n, n) = 1$.

**Stochastic Evolution of $W^{(2)}(x, t, 2)$**

![Diagram](attachment:image.png)
III.(a) Martingale Probability Measures for Insurances and Annuities

Method A:
- We suppose a separation of the insurance and the annuity markets.
- The price of any product containing death and accumulation benefits is then the sum of the two insurances priced separately.

Martingale measures $Q^j_x$ ($j = 1, 2$) are any probability measure, equivalent to the insurance market measure, such that $\frac{W^j(x,t,n)}{B(t)}$ are martingales.

Let $\tilde{q}^j_x(t)$ ($j = 1, 2$) be the probability under $Q^j_x$ that $(x)$ dies before age $x + t + 1$ given that the insured is alive at age $x + t$

$$\tilde{q}^j_x(t) = \Pr^j_x [K(x) = t|K(x) \geq t].$$

Define the probability (under $Q^j_x$) that $(x)$ survives at least 1 year given that the insured is alive at age $x + t$ by

$$\tilde{p}^j_x(t) = \Pr^j_x [K(x) > t|K(x) \geq t] = 1 - \tilde{q}^j_x(t).$$

The goal is to define $\tilde{q}^j_x$’s and $\tilde{p}^j_x$’s for $j = 1, 2$ using the martingale properties.

For given $V^j(x,0,n)$ ($j = 1, 2$ and $n = 1, 2, ...$) the age-dependent, mortality risk-adjusted probabilities are

$$\tilde{q}^{(1)}_x(n-1) = \frac{(V^{(1)}(x,0,n) - V^{(1)}(x,0,n-1)) \Pi_{i=0}^{n-1} (1 + r(i))}{\Pi_{i=0}^{n-2} \tilde{p}^{(1)}_x(i)},$$

and

$$\tilde{p}^{(2)}_x(n) = \frac{V^{(2)}(x,0,n+1)}{V^{(2)}(x,0,n)} (1 + r(n)).$$
III.(b) Martingale Probability Measures for Endowment
Insurances

Method B:

- Any product containing death and accumulation benefits is now priced by unifying the underlying contingent claims.

\( V^{(3)}(x, t, n) \): market price at time \( t \) of an \( n \)-year endowment insurance issued to \( (x) \) at time 0.

\( V^{(3)}(x, 0, n) \) are the current market prices and are given exogenously \( \forall n \).

For given \( V^{(3)}(x, 0, n) \) \((n = 1, 2, ...\) the age-dependent, mortality risk-adjusted probabilities are

\[
\prod_{i=0}^{n-1} p^{(3)}_x(i) = \frac{V^{(3)}(x, 0, n + 1) - V^{(3)}(x, 0, n)}{\prod_{i=0}^{n} (1 + r(i))^{-1} - \prod_{i=0}^{n-1} (1 + r(i))^{-1} }.
\]

III.(c) Martingale Probability Measures for Equity-Linked
Products

- Equity-linked products provide death and accumulation benefits.
- Those benefits are based on the performance of the underlying index.
- The equity-linked products market is composed by the financial and the insurance markets.
- The goal is to determine the martingale measure for the combined market.
- The combined martingale measure reproduces the insurance and the financial martingale measures.
Stochastic Evolution of the Combined Insurance and Financial Markets Between $t$ and $t+1$

For a fixed term $n$, define

$$
\tilde{e}_{x,n}^{(1)+}(t, i) = \tilde{\Pr}^{(1)+}\left[ \frac{S(t + 1)}{S(t)} = \gamma(t + 1, i), W^{(1)}(x, t + 1, n) = V^{(1)}(x, t + 1, n) S(t), K(x) \geq t | S(t), K(x) \geq t \right], i = 0, \ldots, N
$$

and

$$
\tilde{e}_{x,n}^{(1)+}(t, i) = \tilde{\Pr}^{(1)+}\left[ \frac{S(t + 1)}{S(t)} = \gamma(t + 1, i - N - 1), W^{(1)}(x, t + 1, n) = 1 | S(t), K(x) \geq t \right], i = N + 1, \ldots, 2N + 1
$$

where $\tilde{\Pr}^{(1)+}[\cdot]$ represents the probability under $Q_{x,n}^{(1)+}$. 

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III.(c) Martingale Probability Measures for Equity-Linked Products

Martingale measures $Q^{(1)+}_{x,n}$ are any probability measure, equivalent to the combined market physical measure, such that $\frac{W^{(1)}(x,t,n)}{B(t)}$ and $\frac{S(t)}{B(t)}$ are martingales.

Define the joint c.d.f. of $S(t+1)$ and $W^{(1)}(x,t+1,n)$ between $t$ and $t+1$ by $G^{(1)+}(y_1,y_2) = \Pr^{(1)+} [S(t+1) \leq y_1, W^{(1)}(x,t+1,n) \leq y_2 | S(t), K(x) \geq t]$.

From the martingale property of $\frac{W^{(1)}(x,t,n)}{B(t)}$ $G^{(1)+}(\infty,y_2) = \Pr \left[ W^{(1)}(x,t+1,n) \leq y_2 | K(x) \geq t \right]$.

From the martingale property of $\frac{S(t)}{B(t)}$ $G^{(1)+}(y_1,\infty) = \Pr [S(t+1) \leq y_1 | S(t)]$.

Based on the copulas approach, the joint c.d.f. is defined by $G^{(1)+}(y_1,y_2) = C \left( G^{(1)+}(y_1,\infty), G^{(1)+}(\infty,y_2); \kappa(t) \right)$, where $C : [0,1]^2 \rightarrow [0,1]$ and $\kappa(t)$ is the copula’s free parameter indicating the level of dependence between the index and the insured’s life.

The following three non-parametric copulas are used to determine the dependence relation between the index and the insured’s life.

Independent copula: $G^{(1)+}(y_1,y_2) = G^{(1)+}(y_1,\infty)G^{(1)+}(\infty,y_2)$

Upper Frechet bound: $G^{(1)+}(y_1,y_2) = \min \left( G^{(1)+}(y_1,\infty), G^{(1)+}(\infty,y_2) \right)$

Lower Frechet bound: $G^{(1)+}(y_1,y_2) = \max \left( G^{(1)+}(y_1,\infty) + G^{(1)+}(\infty,y_2) - 1, 0 \right)$

Using Cossette, Gaillardetz, Marceau and Rioux (2002) the $\hat{e}_{x,n}^{(1)+}$’s can be extracted from $G^{(1)+}$. 

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IV. Equity-Indexed Annuity Valuation

The payoff of the **Total Return** EIA class can be expressed by

\[ D(t) = \max \left[ 1 + \alpha R(t), \beta (1 + g)^t \right] . \]

\( \alpha \): participation level.
\( \beta \) and \( g \): minimum guarantee.

**Point-to-Point:** \( R(t) = \frac{S(t)}{S(0)} - 1 \)

**High-Water Mark:** \( R(t) = \max_{k \in \{0, \Delta, 2\Delta, ..., t\}} \frac{S(k)}{S(0)} - 1 \)

\( C_{x,n} \): price of the EIA.

**Method A:** \( C_{x,n} = C_{x,n}^{(1)} + C_{x,n}^{(2)} \)

**Method B:** \( C_{x,n} = C_{x,n}^{(3)} \)

Leading to

\[ C_{x,n}^{(1)} = \tilde{E}_{+}^{(1)} + \left[ \sum_{k=1}^{n} \frac{D(k) I(K(x) = k-1)}{B(k)} \right] , \quad C_{x,n}^{(2)} = \tilde{E}_{+}^{(2)} + \left[ \frac{D(n) I(K(x) \geq n)}{B(n)} \right] , \]

and

\[ C_{x,n}^{(3)} = \tilde{E}_{+}^{(3)} + \left[ \sum_{k=1}^{n-1} \frac{D(k) I(K(x) = k-1)}{B(k)} + \frac{D(n) I(K(x) \geq n-1)}{B(n)} \right] , \]

where \( \tilde{E}_{+}^{(j)}[.] \) represents the expected value under \( Q_{x,n}^{(j)} \).
V. Numerical Examples

For a life age (55) and term of 5 years, the following data are observed (Bowers et al. (1997)):

\[
\begin{array}{cccccc}
 t & r(t-1) & q_{55+t-1} & V^{(1)}(55,0,t) & V^{(2)}(55,0,t) & V^{(3)}(55,0,t) \\
1 & 4.50\% & 8.9605 & 13.1464 & 957.4531 & 961.5385 \\
2 & 5.00\% & 9.7538 & 23.8093 & 904.8839 & 916.3769 \\
3 & 5.50\% & 10.6230 & 34.1888 & 850.0821 & 869.8067 \\
4 & 5.75\% & 11.5230 & 44.4799 & 795.8125 & 824.3219 \\
5 & 6.00\% & 12.6181 & 54.7425 & 742.4034 & 780.1222 \\
\end{array}
\]

Probabilities and prices are multiplied by 1,000.

For illustration purposes, assume that the market prices are determined using the standard deviation premium principle with a factor of 5.00%.

For the financial model, assume \( N = 3 (\Delta = 1/3) \), \( u(t) = e^{0.15\sqrt{\Delta}} = 1.0905 \) and \( d(t) = e^{-0.1\sqrt{\Delta}} = 0.9439 \) for \( t = 1, ..., 5 \).

The martingale probabilities are:

\[
\begin{array}{ccccccc}
 t & \tilde{q}_{55}^{(1)}(t) & \text{Loading} & \tilde{q}_{55}^{(2)}(t) & \text{Loading} & \tilde{q}_{55}^{(3)}(t) & \text{Loading} & \tilde{\pi}(t) \\
1 & 11.8053 & 2.0515 & 7.6505 & -2.1033 & 10.9922 & 1.2384 & 494.6 \\
2 & 12.2683 & 1.6453 & 8.8932 & -1.7298 & 12.0553 & 1.4322 & 505.6 \\
3 & 13.0232 & 1.4480 & 10.0113 & -1.5639 & 12.8658 & 1.2906 & 511.1 \\
\end{array}
\]

Probabilities and loadings are multiplied by 1,000.

Determine \( \alpha \) by numerical methods after setting \( g = 3.00\% \) and \( \beta \).
Method A: Point-to-Point Design

![Graph showing participation rate vs. percentage of minimum guarantee. The graph includes lines for Independence, Upper Frechet Bound, and Lower Frechet Bound. The lines are labeled in the legend.]
Method A: High-Water Mark Design

![Graph showing participation rates and bounds for different percentage of minimum guarantee.]

- **Independence**
- **Upper Frechet Bound**
- **Lower Frechet Bound**

### Table

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<th>Percentage of Minimum Guarantee</th>
<th>Participation Rate</th>
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Method B: Point-to-Point Design

![Graph showing the relationship between Participation Rate and Percentage of Minimum Guarantee for Method B: Point-to-Point Design. The graph includes lines for Independence, Upper Frechet Bound, and Lower Frechet Bound.]
Method B: High-Water Mark Design

Independence
Upper Frechet Bound
Lower Frechet Bound

Percentage of Minimum Guarantee
Participation Rate
VI. Concluding Remarks

- We derived an age-dependent, mortality risk-adjusted martingale probability measure for each market.
- We combined the information from the insurance and the financial markets and derived martingale measures.
- We introduced two pricing methods for equity-linked products:
  Method A: split the benefits and use the insurance and annuity markets;
  Method B: unify the contingent claims and use the endowment market.
- Difficulty to find current market prices.
- Use other parametric copulas (free-parameters can fit the equity-linked product prices).
- Extend to surrender charges and stochastic interest rates.

VII. References