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MORTALITY COST VALUATION OF UNDERWRITING REQUIREMENTS

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ABSTRACT

The purpose of this paper is to provide a unified theory for the evaluation of underwriting requirements where this evaluation is based on the measurement of the levels of mortality costs associated with such requirements. The two underwriting models presented will be used to develop identities between the observed differentials in the screening potential of the requirements under study and the expected differentials in the mortality costs associated with the various resultant issue classes. The distinguishing feature between the two models is the assumption regarding historical mortality experience data. One model assumes the existence of such data for each of the resultant issue classes and, accordingly, is applicable primarily to the evaluation of a medical examination. The other model assumes the existence of historical mortality experience data only for the combined issue block, and is generally applicable to the evaluation of other types of underwriting requirements. The practical considerations involved in the application of each model are explored in detail.

I. INTRODUCTION

Historically, two approaches to the problem of evaluating an underwriting requirement have evolved, although they are typically applied to different types of requirements. One approach, herein called the actuarial approach, has been utilized primarily for the determination of nonmedical issue limits. It defines the value of a medical examination as the difference between the present value of the mortality costs generated by nonmedical and medical mortality experience, where the calculations are performed on a net-amount-at-risk basis. The other, herein called the underwriting approach, was introduced by Charles A. Ormsby [1] in 1963 and has been applied primarily to such underwriting tools as inspection reports (IRs), attending physician statements (APSs), and electrocardiograms (EKGs). This approach defines the value of such an underwriting tool as the present value of the extra mortality costs (i.e., costs over and above what would be expected of average standard issues) that are saved by removing certain lives from the standard issue class, where, as above, the calculation is on a net-amount-at-risk basis.

Since both methodologies measure the influence of an underwriting tool on mortality costs, they are intimately related. The purpose of this article is to explore such relationships and present a unified theory for the mortality cost valuation of an underwriting requirement.

In Section II, these two methods are surveyed in more detail with particular attention to their respective advantages and disadvantages. In Section III, identities will be developed between several mortality cost differentials, and the theoretical relationship between the above two methods will be made apparent. Section III is separated into two parts, the first developing a model that is more suitable for the evaluation of a medical examination, and the second generalizing the results to other underwriting requirements for which the resultant mortality experience is unknown.

Each of these two subsections will contain a discussion of the validity of the assumptions underlying the given model, as well as applications for its use.

Finally, the appendix contains a mathematical analysis of several properties of a mortality cost function.

II. OVERVIEW OF METHODOLOGIES

A. *The Actuarial Approach*

This is a retrospective approach that attempts to measure the value of a medical examination by the indirect method of comparing medical with nonmedical standard mortality experience. The difference is assumed to be the result of underwriting with a medical examination. The comparison is made with the help of a mortality cost function, M, defined for a given plan, issue age, and sex as the present value of mortality costs per \$1,000 of issue on a net-amount-at-risk basis. Specifically,

$$M = \sum_{t=0}^{n-1} (NAR)_{t+1} v^{t+1/2} \, {}_tp_{[x]} q_{[x]+t} \,, \tag{1}$$

where v, ${}_tp_{[x]}$, and $q_{[x]+t}$ are standard notation, with ${}_tp_{[x]}$ reflecting the appropriate lapse experience, and $(NAR)_{t+1}$ is the approximate net amount at risk during policy year $t+1$ per \$1,000 insured. The net amount at risk could be defined in terms of the policy reserve or the cash value and may provide for the refund of premium in the year of death. The formula for M could be refined to reflect an average-policy-size assumption, or premium modes other than annual, although the effect on M would be small.

Each calculation is performed twice, once utilizing actual standard nonmedical experience to produce M^H (where the superscript H denotes the application medical history questionnaire), and once using actual standard medical experience to produce M^E (where the superscript E denotes the medical examination). The mortality differential, ∇M, is then defined by

$$\nabla M = M^H - M^E . \qquad (2)$$

For a given issue age, ∇M is calculated for several representative plans, and a range is produced. If sufficient issue data are available, a value can be determined that reflects the plan distribution near the current nonmedical limit. In the absence of other factors, the values of ∇M should equal the savings in mortality costs, per \$1,000 of issue, due to the medical examination. Equivalently, these values equal the additional mortality costs that an insurer should incur, per \$1,000 of issue, if the nonmedical limits are increased.

Before setting the ordering limits for a medical examination, it is necessary to discuss those pertinent factors that are not reflected in the calculation of ∇M. In decreasing order of importance, these factors are the socioeconomic factor, the underwriting factor, the omission of values, and the statistical basis of the data.

1. *Socioeconomic factor.* It is well known that mortality experience improves with improved socioeconomic conditions, whether these conditions are measured in terms of education, income, or residence. This is partly attributable to the greater availability and affordability of medical services for the higher socioeconomic classes, and partly to the greater prevalence of occupational and environmental hazards in the lower socioeconomic classes. Since insurance needs are positively correlated to income, socioeconomic status reflects itself in the mortality differential between the medical and nonmedical classes simply because smaller policies are more likely to be applied for nonmedically, and larger policies medically. This factor will also influence mortality results within each of the medical and nonmedical classes.
2. *Underwriting factor.* Since routine underwriting requirements depend on the amount of coverage involved, large-amount policies are subjected to a greater barrage of requirements than smaller policies. For example, larger policies will tend to require more sophisticated inspection reports, as well as such things as X-rays and electrocardiograms. Hence, the difference in mortality levels between the medical and nonmedical groups is affected by the more stringent

underwriting requirements of larger policies, and this will indirectly improve the standard experience of the medical applications proportionately more than that of the nonmedical applications because of the amount correlation noted above. Mortality within each class is also affected by this factor, although more so for the medical class.

3. *Omission of value.* Inherent in the actuarial approach is the assumption that the total value of a medical examination can be determined from an analysis of how the resulting mortality experience affects the mortality costs associated with various base-plan types. This naturally raises concern over the handling of policy riders and benefits when the nonmedical limits refer to base plans alone.

   Policy riders are easy to accommodate, since they can be added to the applicant's total underwriting amount at 100 p percent of the rider amount, where

$$p = \frac{\nabla M'}{\nabla M}; \tag{3}$$

   ∇M is calculated with a net amount at risk valued as if the rider insured the applicant, and $\nabla M'$ with the actual net amount at risk. For example, $\nabla M'$ for a spouse rider should reflect a net amount at risk equal to the cost to pay up the rider. For administrative ease, the value of p could be made independent of age and determined so as to reflect the issue distribution. Of course, $p = 1$ for applicant riders. The "insurance of insurability" benefit can be similarly handled where M' in (3) is calculated to reflect purchase rates and the mortality cost differentials anticipated from these future policies. It is probably safe to ignore the accidental death benefit, since one would not expect any experience differential once the socioeconomic factor is taken into account. However, disability waiver of premium experience would be expected to vary between classes due to antiselection alone. If sufficient experience data could be developed, the value of disability costs could be determined and the resulting differential added to ∇M. This is not practical, however, for most companies.

4. *Statistical basis of data.* Implicit in the use of ∇M as a measure of the value of a medical examination is the assumption that M^E and M^H are calculated with actual standard mortality experience of the medical and nonmedical *applicant* groups, respectively. However, the statistical basis for these experience groups is usually one of *resultant* status, not applicant status. That is, if a nonmedical applicant is required to obtain an examination and is subsequently issued standard, that applicant is categorized with the medical standard issue group. This recategorization would usually be expected to cause some deterioration in the nonmedical experience. Also, because of the socioeconomic and underwriting factors mentioned above, perhaps some deterioration in the medical experience is to be expected as well, although it probably would be slight because of the difference in amount distribution between classes. Hence, if it is deemed necessary to "sweeten" the nonmedical experience, one could offset the recate-

gorization effect simply by assuming that the lives so transferred will experience standard medical mortality, and then calculating the implied applicant experience as the appropriate weighted average of the two groups of resultant experience.

Another small problem within this category is that the mortality experience utilized does not represent the results of the current scale of nonmedical issue limits, or those of any single scale. Rather, it represents a durational cross-sectioning of the results of all scales utilized in the past. For example, tenth-duration experience represents the results of the scale of nonmedical issue limits in effect perhaps a dozen or more years ago. Further, even if it is assumed that this tenth-duration experience is the same as that which would have occurred had the current scales always been in effect, it is likely that it will greatly overstate the mortality experience that the current issue group will produce ten or so years hence. To compensate for this, calculations could be performed that reflect projected mortality improvement trends. For example, $q_{[x]+t}$ could be replaced by $\lambda^t q_{[x]+t}$ in equation (1), reflecting an average mortality improvement of $100(1 - \lambda)$ percent per year. Of course, the ${}_tp_{[x]}$ terms also would be affected, so this calculation is not equivalent to simply increasing the interest rate. As is shown in Theorem 2 in the Appendix, the value of M so calculated will always be smaller than that calculated with $\lambda = 1$, and, more generally, M is an increasing function of λ. Unfortunately, the effect on ∇M is not so easily determined a priori, since one may choose to utilize different values of λ for the M^H and M^E calculations. This would seem necessary on the basis of the historical trends studied in the Reports Number of the *Transactions of the Society of Actuaries*.

The primary advantage of the actuarial approach is that the calculated value of ∇M automatically reflects what will be called the agent-applicant antiselection factor. This is the combination of applicant antiselection, which might be defined as the preference of poorer risks to avoid medical scrutiny, and a hypothesized tendency of agents to submit marginal or questionable risks below the nonmedical limit rather than slightly above it. The proposed motivation for this tendency is that, although the larger, medical policy would pay a proportionately higher commission, the probability of receiving any commission is smaller because of higher average nonissue and not-taken rates on medical applications. Consequently, the agent's expected commission can be smaller than that for a smaller, nonmedical policy. For a specific marginal risk, the above scenario is even more compelling, particularly since an agent has a great deal of influence on what information is reported on the medical history questionnaire.

The effect of this antiselection is that, prior to underwriting, proportionately more individuals below the nonmedical limit than just above that limit would be screened out if examinations were given. In other words, the screening potential of a medical examination is greater on the group of nonmedical applicants than on the group of medical applicants because of the barrier effect that is created by the existence of a nonmedical issue limit. Unlike the underwriting approach, the actuarial approach makes no assumption about this potential; it only measures the *result* of medical as opposed to nonmedical underwriting, and antiselection is part of this result.

Finally, then, once ∇M is calculated from equation (2) and modified to reflect the above remarks (perhaps by using the medical experience of lower-amount policies to diminish the socioeconomic and underwriting factors), ordering limits are set in the following way. Let $\bar{C}^E$ be the average total cost (including clerical and underwriting costs) of the medical examination and all supplemental underwriting tools ordered because of information on the medical, converted to a paid-for issue basis (i.e., equal to the per-application costs, C^E, divided by the probability that a medical application will become a paid-for issue), and let $\bar{C}^H$ be a similarly defined quantity only with regard to the application medical history. The average total mortality and underwriting cost for a medically examined paid-for issue of amount A is then $AM^E + \bar{C}^E$. Similarly, this average total cost for a nonmedical paid-for issue is $AM^H + \bar{C}^H$. Since it is invariably true that $M^E < M^H$ and $\bar{C}^E > \bar{C}^H$, there is a unique amount, A^A, such that below A^A, nonmedical underwriting produces a smaller average total cost than medical underwriting (the reverse being true above A^A), where

$$A^A = \frac{\Delta\bar{C}}{\nabla M}, \tag{4}$$

and

$$\Delta\bar{C} = \bar{C}^E - \bar{C}^H .$$

There is a somewhat dangerous extrapolation here in that it is assumed that the values of $\Delta\bar{C}$ and ∇M, calculated on the basis of the current and past (for ∇M) scales of nonmedical limits, will remain valid for the new limit A^A. That is, it is assumed that underwriting and mortality cost differentials are intrinsic to, and the result of, medical underwriting. In general, the assumption regarding $\Delta\bar{C}$ is probably a safe one, although that for ∇M may be patently false. For example, if a company "piggybacks" another underwriting requirement at the current nonmedical limit, its influence cannot be removed by using lower-amount medical experience for M^E. Consequently, the estimated ∇M will overstate the correct value, thereby artificially suppressing the value of A^A in equation (4). However,

∇M would be appropriate for the purpose of revising the limit for the package, if $\bar{C}^E$ is appropriately redefined.

B. *The Underwriting Approach*

This is a prospective approach that attempts to predict the value of an underwriting requirement by directly measuring its screening potential on the basis of an application-by-application analysis. If two requirements are to be compared, it is the marginal screening potential that is of interest. Hence, the focus of this technique is on the applications that, because of the tool under study, will *not* be issued as applied—that is, the group of applications on which the tool had an effect. Such cases are often labeled "effect" cases. The groups over which the actuarial and underwriting methods measure value are thus distinct and somewhat complementary, since the focus of the actuarial method is on those policies that were issued standard.

The measurement of an underwriting requirement's screening potential is accomplished with the use of an extra mortality cost function, K, defined for a given effect case, per \$1,000 of coverage, by

$$K = \sum_{t=0}^{n-1} (NAR)_{t+1} v^{t+1/2} \, {}_tp'_{[x]} \Delta q_{[x]+t} , \tag{5}$$

where ${}_tp'_{[x]}$ is the probability of survival assuming substandard experience ($\boldsymbol{q}'$) appropriate for the rating and a suitable lapse rate, and

$$\Delta q_{[x]+t} = q'_{[x]+t} - q_{[x]+t} ;$$

$\boldsymbol{q}$ represents standard experience. The value of K, therefore, is an approximation to the extra mortality costs that the company would expect to incur, per \$1,000, had the tool not been ordered and the applicant issued standard. It is an approximation because it would be more appropriate to use K', where

$$K' = M(\boldsymbol{q}') - M(\boldsymbol{q}) , \tag{6}$$

and M is defined in equation (1).

It is straightforward to check that $K' \leq K$ whenever $p'_{[x]+t} \leq p_{[x]+t}$, and this can be expected to be true even if substandard lapse experience is somewhat better than that of standard issue. In the absence of other factors, therefore, K usually will overstate the value of a tool and lead to somewhat conservative underwriting limits.

In the general case where two underwriting requirements are being compared, calculations are performed on both groups of effect cases, and the total value for each group is converted to a per \$1,000 basis. These values then are multiplied by the respective probabilities that an application containing the requirement will become an effect case, thereby producing average values per \$1,000 applied.

For example, if medical examinations are to be valued with this approach, the average values per \$1,000 applied could be denoted by $K^E r^E$ and $K^H r^H$, where r is the probability of becoming an effect case; the superscript H denotes the medical history questionnaire, and the superscript E denotes the medical examination. Hence, since $K^E r^E$ represents the average mortality savings provided by a medical examination, and $K^H r^H$ the corresponding amount for a nonmedical application, $\Delta Kr = K^E r^E - K^H r^H$ will represent, in the absence of other factors, the actual mortality savings attributable to medical underwriting, per \$1,000 applied. Equivalently, this value equals the additional mortality costs that an insurer should incur, per \$1,000, if the nonmedical limits are increased. Although the value of Kr could have been obtained directly by dividing the total value of extra mortality costs by the total amount applied, there are three advantages to determining K and r separately.

First, since the per \$1,000 applied value, Kr, is based on a sample of applications, it is subject to statistical variation. Separating this value into two intuitively appealing quantities, K and r, simplifies the modification of sample data to reflect prior experience and intuition, as well as enabling a more meaningful analysis of trends. That is, a trend in the value of Kr can be better understood in terms of its implications for the underwriting process if it can be linked to trends in K or r separately.

Second, although r theoretically should be calculated on an amount basis, it usually can be defined on an application basis because often there is no statistically significant difference between the average policy size of the sample and that for the effect cases. That being so, r theoretically can be defined on either basis. However, its value on an application basis is preferred, since, as a binomial variable, it is more stable and lends itself more easily to statistical analysis.

Finally, identities developed in the next section will require the value of r.

Before determining the ordering limits for an underwriting tool, it is necessary to discuss the following factors, which affect the calculation and interpretation of the marginal mortality savings.

1. The value of K depends on both the availability of an appropriate mortality standard (q) and the ability to predict mortality levels of effect cases (q')

accurately. The value of q for many application classes will have to be approximated, since it is usually unknown. Predicting the potential mortality experience of applicants whose applications are incomplete or have been declined is also a problem.

2. Fundamental to the conclusion that $\Delta Kr = K^Y r^Y - K^X r^X$ represents the marginal mortality savings of tool Y over tool X is the assumption that the two groups upon which these statistics are based are identical in terms of the screening potential of Y. This is because it is implicit in the above analysis that $K^Y r^Y$ represents the value of Y on the X-class. But this may not be the case, because of the agent-application antiselection factor mentioned earlier.

   As this is a fairly subtle point, consider the following hypothetical example. Assume that tools X and Y produce a similar distribution of effect cases by final underwriting class relative to the appropriate standard, so that a calculation produces $K^X = K^Y$. Assume also that X is only half as likely to discover a ratable impairment as Y, and consequently a "barrier effect" is created, resulting in twice as many impaired risks being submitted below the ordering limit, A, as above it. Hence, it is observed that $r^X = r^Y$ and the marginal mortality savings due to Y is prospectively determined as $\Delta Kr = 0$. Consequently, the limit is increased to $2A$. Of course, twice as many impaired risks will now be submitted below $2A$ as above it—a continuation of what occurred when the limit was A. Retrospectively, then, the return in the amount range A–$2A$ is $K^X r^X$ for X but *would* be $2K^Y r^Y$ for Y. Hence, the real value of Y in this range, which must be calculated retrospectively, was $2K^Y r^Y - K^X r^X = K^Y r^Y$.

   This anomalous situation has nothing to do with the simplifying assumption that $\Delta Kr = 0$, since it will always occur if Y has a greater screening potential on the X-group than on the Y-group. The difference between the retrospective and prospective valuations, $K^Y r^Y$, is a measure of the preventive value of Y. Hence, for certain underwriting requirements, such as a medical examination, that invite antiselection, the calculated value of ΔKr will be an understatement of what is in fact the case.

3. As can be guessed, K is a highly volatile statistic, even under the assumption that the problem of mortality standards could be resolved. This is because its value is influenced by the sample's distribution of effect cases by substandard class, amount applied, sex, and policy type. For a given age group with an expected effect probability of 0.05, for example, one would need to screen about 2,000 applications just to locate 100 effect cases, and this imposes a practical limitation to confidence. For many applications, therefore, it may be advantageous to use the effect cases to construct an explicit model for their distribution by substandard class, as well as the dependence of average policy size on substandard class, if such a dependence is observed. One can then determine the value of K on the basis of this model, assuming the appropriate sex/plan distribution.

4. Since the underwriting method, as well as the actuarial method, attempts to identify the results of the ideal study where the same application group is

subjected to two underwriting scenarios and the results compared, it is important that such characteristics as distribution by sex and plan be similar for the two groups actually studied. This is not a problem, of course, in evaluating the use, as opposed to the nonuse, of a given tool, since only one application group would then be analyzed.

5. The value of r, and perhaps that of K, will depend on how one treats effect cases identified by the tool under study as well as some other tool independently required, that is, multiple-effect cases. In a way, this can be used to advantage by calculating ΔKr, the marginal mortality savings, under the two obvious extreme treatments and producing a range of values.

The primary advantage of the underwriting method is that it provides a method of estimating the value of the various underwriting requirements to which the actuarial method cannot be applied because the values of M^X and M^Y are unknown. This method is also far more sensitive to changes in value caused by changes in the amount or import of the information obtained, such changes not being reflected in the issue experience for several years. Also, this approach allows for the proper handling of the other underwriting requirements that are not under study, although, as will be seen later, ignoring these tools is not the solution.

Finally, then, once the above data have been collected and "massaged," ordering limits can be set, based on one of two philosophies. If underwriting is viewed as an investment of C for a return of Kr, the average underwriting return (or loss, if negative) provided by tool Y for an application of amount A is $AK^Yr^Y - C^Y$. Similarly, the average return provided by tool X is $AK^Xr^X - C^Y$. Similarly, the average return provided by tool X is $AK^Xr^X - C^X$. Under the assumption that Y is the superior tool, one invariably has that $C^Y > C^X$ and $K^Yr^Y > K^Xr^X$. This implies that there is a unique amount, A^U, such that below A^U, tool X maximizes the return (or minimizes the loss) on average, the reverse being true above A^U, where

$$A^U = \frac{\Delta C}{\Delta Kr}, \qquad (7)$$

and $\Delta C = C^Y - C^X$, $\Delta Kr = K^Yr^Y - K^Xr^X$.

Alternatively, if minimizing total mortality and underwriting costs on a paid-for issue basis is the criterion, as was the case for the actuarial method, this total cost for a policy of amount A on which tool X is used can only be improved by the use of tool Y if

$$\bar{C}^Y - \bar{C}^X \leq A(K^Yr^Y - K^Xr^X) .$$

Consequently, the ordering limit under this criterion, $\bar{A}^U$, satisfies

$$\bar{A}^U = \frac{\Delta\bar{C}}{\Delta Kr}. \tag{8}$$

Since a superior underwriting tool usually carries with it a smaller probability that the application will become a paid-for issue, and since $\bar{C}$ is C adjusted for not-takens and nonissues, it is clear that $\Delta\bar{C} > \Delta C$, and, consequently, that $\bar{A}^U > A^U$.

As was the case with the actuarial method, the validity of the above analysis depends on the assumption that the values of ΔC ($\Delta\bar{C}$) and ΔKr, calculated on the basis of the current ordering limit, accurately reflects what their values will be based on the new limit A^U ($\bar{A}^U$). Intuitively, one may wonder how this could be the case without a more careful analysis of the other tools used on each class. As a specific example, it should be expected that if a medical examination is eliminated in the amount range A–$2A$, one would lose proportionately more effect cases than if these examinations were eliminated in the amount range $2A$–$3A$. This is a result of the fact that the other tools utilized within $2A$–$3A$ should be more sophisticated than those used within A–$2A$. Consequently, there will be proportionately more multiple-effect cases there and, hence, fewer lost if examinations are forgone. In other words, with all else equal, the higher one sets nonmedical limits, the less valuable these examinations become. Clearly, a similar statement would be true for other tools as well.

The underwriting method does not accurately reflect this redundancy in underwriting information. The actuarial method would reflect this redundancy to some extent, although its influence on issue mortality will be slow to emerge and will be masked to a great degree because of the underwriting factor discussed above. The role of these other underwriting tools will become apparent in the identities that emerge in the next section.

III. THEORETICAL UNDERWRITING MODELS

In this section, the relationship between extra mortality cost differentials (ΔKr) and standard mortality cost differentials (∇M) will be explored under a number of different hypotheses. Of course, since this is a theoretical investigation, various factors must be idealized, and this will become apparent in the definitions and assumptions.

In Section A below, the "dual mortality standard" approach will be considered, where it is assumed that actual historical mortality experience of each of the two underwriting groups is known. Since the primary use of this approach will be to determine nonmedical issue limits, notation will be used that reflects this specific application.

In Section B below, the "single mortality standard" approach is developed, which will generalize the results of Section A, to the situation where two or more arbitrary underwriting tools are to be evaluated, but where the historical mortality experience of each underwriting group separately is assumed to be unknown. There it will be shown that the use of a composite of this experience as a mortality standard is sufficient if it is properly chosen or constructed.

The application of these models to the real-world situation of determining ordering limits for specific underwriting requirements is also discussed in each section.

A. *The Dual Mortality Standard Approach*

Throughout this section, the following notation will be used, where by "mortality cost" or "extra mortality cost" is meant the present value of such cost per $1,000 applied. Let

M^E (M^H) = Mortality cost for a group of applicants deemed standard according to all underwriting tools required for the medical (nonmedical) class;

$\bar{M}^E$ ($\bar{M}^H$) = Mortality cost for a group of applicants deemed standard on the basis of the medical examination (history) alone;

$\bar{\bar{M}}^H$ = Mortality cost for a group of applicants deemed standard on the basis of the medical history questionnaire and all other underwriting tools usually required for the *medical* class;

K^E (K^H) = Extra mortality cost for the medical (nonmedical) effect cases calculated according to equation (6), where $M(q)$ is taken as M^E (M^H);

L^E (L^H) = Extra mortality cost for those effect cases within the medical (nonmedical) class attributable to some other underwriting tool but *not* simultaneously attributable to the medical examination (history), calculated, as is K, relative to the appropriate standard;

r^E (r^H) = Probability that a medical (nonmedical) application will become a medical (nonmedical) effect case;

s^E (s^H) = Probability that a medical (nonmedical) application will become an effect case for the other tools utilized within the medical (nonmedical) class, calculated according to the convention underlying L^E (L^H) that all multiple-effect cases are attributed to r^E (r^H) alone.

Also, let such symbols as $\tilde{K}^E$ and $\tilde{s}^H$ be defined as above, only under the alternative convention that all multiple-effect cases are credited to the other tools alone and not to the examination or history. Since insurance companies often have more conservative "special" medical requirements for certain groups of applicants, such as relatives of agents or those applying for "preferred" class policies, it is natural to categorize the effect cases resulting from such examinations with $\bar{s}^E$. This is consistent with the intent of $\bar{s}^E$ to reflect all effect cases that would be produced independent of the existence of the regular nonmedical issue limit.

In anticipation of Section A, 3, below, dealing with practical considerations, it will be assumed that substandard mortality experience is known only for those applicants who were rated subsequent to medical examination. Consequently, all extra mortality costs defined above must be calculated with this experience for q'. Since nonmedical underwriting is generous in terms of its categorization of standard lives (i.e., $M^E < M^H$), it seems reasonable to expect that it will also be generous in categorizing substandard lives. Hence, the above assumption will understate the mortality experience of nonmedical substandard issue. To accommodate this let X^H be the mortality cost of an effect case within the nonmedical class in excess of that predicted by the medical issue q'. Although the value of X^H will be discussed further in Section A, 3, below, it will be assumed here that

$$X^H = \lambda(M^H - M^E) , \quad 0 \leq \lambda \leq 1 . \tag{9}$$

When no other underwriting tools are utilized, the value of a medical examination is well defined and simply equals the mortality cost differential that it creates in the standard issue class. In the more realistic setting, however, there are a number of mortality differentials of interest that reflect some aspect of the "value" of medical underwriting. Three such differentials are the following:

1. The total issue differential—the difference between the mortality costs of the medical and nonmedical standard issue classes $(M^H - M^E)$, denoted by ∇M.
2. The absolute differential—the difference between the mortality costs of the medical and nonmedical standard issue classes on the assumption that no other underwriting tools were utilized $(\bar{M}^H - \bar{M}^E)$, denoted by $\nabla\bar{M}$.
3. The marginal differential—the difference between the mortality costs of the medical standard issue group and the standard issue group that would exist if the examination had been replaced by a medical history, but all other tools utilized on this group were left unchanged $(\bar{\bar{M}}^H - M^E)$, denoted by $\nabla\bar{\bar{M}}$.

It should be noted that the only difference between definitions 1 and 3 is the assumption regarding the other tools utilized on the nonmedical group. Of course, it is the marginal differential that should be utilized for the purpose of establishing the nonmedical issue limit.

In the following, identities between these differentials and the extra mortality cost differentials will be derived under the ideal setting that the same applicant group could be subjected to two underwriting scenarios and the results compared. It will also be assumed that the mortality experience of the standard issue groups and all effect cases can be accurately predicted and that all calculations of the various M, K, and L values are performed with regard to the same lapse assumption.

1. A SPECIAL CASE: NO OTHER TOOLS UTILIZED

Given an applicant group of amount A, medical underwriting will partition A into E^M (the standard group) and E^K (the effect cases). Consequently, the total mortality cost of the applicant group is predicted to be

$$E^M M^E + E^K(M^E + K^E) .$$

Had this group been underwritten nonmedically, the total mortality cost would have been predicted to be

$$H^M M^H + H^K(M^H + K^H) ,$$

but because of the understatement of nonmedical experience, it is actually this quantity plus $H^K X^H$.

Since the total mortality cost is independent of the partitioning, the above predictions can be equated, producing

$$\nabla M = \Delta Kr - X^H r^H , \tag{10}$$

where $\Delta Kr = K^E r^E - K^H r^H$. Hence, if nonmedical substandard experience were known and used to calculate K^H, or if all nonmedical effect cases were required to be medically examined, equation (10) would reduce to

$$\nabla M = \Delta Kr . \tag{11}$$

In this best of all worlds the above identity certainly is expected, since each term is a measure of the mortality cost of the group of poorer risks that slipped through the nonmedical sieve. The term ∇M isolates this group as the difference between the groups that were issued standard, whereas ΔKr identifies this group as the difference between the groups that were screened out. Utilizing equation (9) in equation (10), we would expect in general that

$$\frac{\Delta Kr}{1 + r^H} \leqslant \nabla M \leqslant \Delta Kr . \tag{12}$$

2. THE GENERAL CASE

Subjected to complete medical underwriting, an applicant group of amount A will be partitioned into the groups E^M, E^K, and E^L, and the total mortality cost predicted to be

$$E^M M^E + E^K(M^E + K^E) + E^L(M^E + L^E) . \tag{13}$$

Similarly, a prediction based on complete nonmedical underwriting can be made and augmented by $(H^K + H^L)X^H$. Equating these values, we get an identity for the total issue differential,

$$\nabla M = \Delta Kr + \Delta Ls - X^H(r^H + s^H) . \tag{14}$$

Utilizing equation (9), we expect the following bounds:

$$\frac{\Delta Kr + \Delta Ls}{1 + r^H + s^H} \leq \nabla M \leq \Delta Kr + \Delta Ls . \tag{15}$$

For the absolute differential, it is clear that $\tilde{M}^E \geq M^E$, since the "standard" issue class under this scenario is $E^M + E^L$. Similarly, $\tilde{M}^H \geq M^H$. However, the relationship between ∇M and $\nabla \tilde{M}$ is less apparent.

Given this scenario, the medical applicant group will be partitioned into $E^M + E^L$ and E^K, and the total mortality cost will be predicted to be

$$(E^M + E^L)\tilde{M}^E + E^K(M^E + K^E) . \tag{16}$$

Since this expression must equal expression (13), we obtain

$$\tilde{M}^E = M^E + \frac{s^E}{1 - r^E} L^E .$$

Similarly,

$$\tilde{M}^H = M^H + \frac{s^H}{1 - r^H} (L^H + X^H) .$$

Hence,

$$\nabla \tilde{M} = \nabla M + \frac{s^H}{1 - r^H} (L^H + X^H) - \frac{s^E}{1 - r^E} L^E , \tag{17}$$

and bounds for $\nabla \tilde{M}$ can be produced by using (9) and (15). Depending on the relative value of the other tools, therefore, the difference between

$\nabla\bar{M}$ and ∇M can be positive or negative. It should be noted that one cannot conclude, a priori, that $s^H \leqslant s^E$, even though the other tools used on the medical group are assumed to be more effective than those used on the nonmedical group. This is because we certainly have $r^E > r^H$, and, by convention, multiple-effect cases are not attributed to these other tools.

In order to develop an expression for $\bar{\bar{M}}^H$, it is first necessary to consider more carefully the impact of the underlying underwriting scenario on the effect cases, compared with the usual medical underwriting underlying the development of M^E.

Since the same package of other tools is assumed to be used in each scenario, none of the effect cases that make up s^E will be lost. Also, none of the effect cases attributable to both the medical examination and some other tool will be lost, since the examination was redundant for these cases anyway. Hence, the only effect cases lost under the new scenario will be those cases identified by the medical examination alone that cannot be discovered by use of the medical history.

By definition, $\bar{r}^E$ reflects effect cases due to the examination alone. Unfortunately, although $\bar{r}^H$ (like $\bar{r}^E$) can be calculated moderately easily, it is not exactly what is needed for this scenario. What is needed is the medical history probability of producing an effect calculated so as to exclude all multiple effects that would be attributable to the other tools ordinarily utilized on the *medical* class, say $\hat{r}^H$, and this is unknown in general. What can be noted, however, is that $\hat{r}^H \leqslant \bar{r}^H$, since the other tools used on the medical class are at least as efficient as those used on the nonmedical class. Moreover, since the group of nonmedical effect cases can be assumed to be a subset of the group of medical effect cases, it cannot overlap the group of medical class other tools' effect cases any more than the group of medical effect cases overlaps that group. Consequently,

$$r^H - (r^E - \bar{r}^E) \leqslant \hat{r}^H \leqslant \bar{r}^H \; . \tag{18}$$

In the following, $\hat{r}^H$ will be assumed to be equal to $\bar{r}^H$ in order to simplify notation. Also, the unit extra mortality costs of this group, say $\hat{K}^H$, will be taken to be equal to $\bar{K}^H$, which is readily calculated. This assumption may be considered reasonable even if $\hat{r}^H \neq \bar{r}^H$, and, if so, the sensitivity of the resulting identity for $\nabla\bar{\bar{M}}$ to (18) can be observed by simply re-evaluating this identity by substituting $r^H - (r^E - \bar{r}^E)$ everywhere for $\bar{r}^H$, thereby producing a range for $\nabla\bar{\bar{M}}$.

Now an applicant group submitted to complete medical underwriting will have a predicted total mortality cost of

$$E^M M^E + \tilde{E}^K(M^E + \tilde{K}^E) + \tilde{E}^L(M^E + \tilde{L}^E) \,, \tag{19}$$

which equals expression (13) with multiple-effect cases categorized differently.

Subjected to the hypothesized nonmedical underwriting scenario, this total cost, augmented by the value of X^H, is given by

$$\tilde{H}^M \bar{\bar{M}}^H + \tilde{H}^K(M^H + \tilde{K}^H + \tilde{X}^H) + \tilde{H}^L(M^H + \tilde{L}^H + \tilde{X}^H) \,. \tag{20}$$

It should be noted that the value of effect cases can be calculated with respect to any mortality standard, and, as will be seen in a moment, it is worthwhile to calculate those in expression (20) with respect to M^H rather than $\bar{\bar{M}}^H$.

As was noted earlier, $\tilde{E}^L = \tilde{H}^L$ because of the convention regarding multiple-effect cases. It is also clear, therefore, that $M^H + \tilde{L}^H + \tilde{X}^H = M^E + \tilde{L}^E$, since each represents the total mortality cost of this common group, symbolically expressed with respect to two different standards. Finally, according to earlier observations, we have

$$\tilde{H}^M = E^M + \tilde{E}^K - \tilde{H}^K \,. \tag{21}$$

Equating (19) to (20), and utilizing (21) and the above remarks, we have, sequentially,

$$(E^M + \tilde{E}^K)M^E + \tilde{E}^K\tilde{K}^E = (E^M + \tilde{E}^K - \tilde{H}^K)\bar{\bar{M}}^H + \tilde{H}^K(M^H + \tilde{K}^H + \tilde{X}^H) \,,$$

$$(E^M + \tilde{E}^K - \tilde{H}^K)(\bar{\bar{M}}^H - M^E) = \tilde{E}^K\tilde{K}^E - \tilde{H}^K\tilde{K}^H - \tilde{H}^K(M^H - M^E + \tilde{X}^H) \,,$$

$$\nabla\bar{\bar{M}} = \frac{\Delta\tilde{K}\tilde{r} - \tilde{r}^H(\nabla M + \tilde{X}^H)}{1 - \tilde{s}^E - \tilde{r}^H} \,, \tag{22}$$

$$\frac{\Delta\tilde{K}\tilde{r} - 2r^H\nabla M}{1 - \tilde{s}^E - \tilde{r}^H} \leqslant \nabla\bar{\bar{M}} \leqslant \frac{\Delta\tilde{K}\tilde{r} - \tilde{r}^H\nabla M}{1 - \tilde{s}^E - \tilde{r}^H} \,. \tag{23}$$

3. PRACTICAL CONSIDERATIONS

Each of the identities derived above shows the necessary relationship between differences in standard issue mortality costs (∇M) and differences in extra mortality costs saved by underwriting (ΔK and ΔL), under a certain set of assumptions. As part of the derivations, it was assumed that the future mortality levels of the current applicant group were known, which is never the case. However, on the basis of current mortality experience, it is possible to determine $M(q)$ for the standard issue classes, and $M(q')$ for the substandard classes (see below), with which current values of K and L can be determined. Although neither $M(q)$ nor $M(q')$ will accurately predict the future experience of the current applicant group, it is reasonable to assume that K and L calculated with these values *are* fairly accurate measures of the mortality savings due to the various tools. This is based on the assumption that both $M(q)$ and $M(q')$ are in error in the same way, namely, they both overstate probable future results. Hence, some of this error is offset in the calculation of K and L.

Therefore, on the basis of an application-by-application study, estimates of the parameters r, s, K, L, and λ are made (see below) and can be substituted in formulas (14) and (15) to predict a level of ∇M that is consistent with these data. The new value of ∇M as well as the other estimated parameters can then be used to predict $\nabla \bar{M}$ and $\nabla \bar{\bar{M}}$ from (17), (22), and (23). As was noted earlier, $\nabla \bar{\bar{M}}$ should also be evaluated to reflect (18), and it is one's final estimate of $\nabla \bar{\bar{M}}$ that would be used in (4), for example, to determine nonmedical issue limits.

Hence, the utility of these identities depends solely on the credibility of the assumptions upon which they were based, and this is discussed next.

First of all, it is necessary to be able to calculate or estimate all of the parameters mentioned above. For the probability r^E (r^H), one should reflect *all* adverse underwriting actions taken, or expected to be taken, as a result of information on the medical examination (history). This includes ratings based directly on this initial information as well as those resulting from any information obtained from another tool, the ordering of which was, or would have been, motivated by this initial information. The corresponding value of s is then defined as the overall probability of becoming an effect case (which is well defined), *minus* the value of r.

To recategorize multiple-effect cases, the probability $\bar{s}^E$ should reflect all adverse actions taken, or expected to be taken, as a result of information on examinations ordered because of special requirements, or information from any other underwriting tool utilized *except* those tools ordered exclusively because of information noted on the medical examination. The value of r is then defined as the total probability minus the appropriate s.

Of course, the above process will require professional judgment, but this is unavoidable. One particularly problematic group is the group of applications incompleted before all necessary tools are received and a mortality class determined. One needs to estimate both the likelihood that the case would have become an effect case, and the most probable mortality class given that it did become an effect case. For such cases, the value of this probability could be used in the estimation of r or s, rather than the more usual procedure for finalized cases of counting such a case as a zero or 1.

The value of λ might be estimated as follows, where $X^H = \lambda\,(\nabla M)$. If a substandard nonmedical applicant is required to undergo an examination, let $X^H = 0$. Otherwise, it seems reasonable that the "error" in the rating will decrease as the rating class increases. Using ∇M as the error in the standard class, zero for the declined cases, and some actuarial archery, λ could be estimated by

$$\lambda = \sum_{j=1}^{n} \left(\frac{n-j}{n}\right) I_j \, , \tag{24}$$

where I_j, $j = 1, \ldots, n-1$, equals the proportion of nonmedical effect cases in the jth mortality class that were not subsequently examined, and I_n is the proportion of those cases either declined or class-rated subsequent to examination. Obviously, the term in (24) corresponding to $j = n$ adds nothing, but was notationally included as a reminder that

$$\sum_{j=1}^{n} I_j = 1 \, .$$

In order to calculate K and L as they were defined above, it is necessary to have standard and substandard mortality experience for the respective *issue* groups. Unfortunately, one can expect to develop such experience only for the *paid-for issue* groups, and this causes one to make an assumption regarding the selective potential of not-taken policies. For standard issue, it is fairly easy to believe that the not-taken policies are randomly distributed throughout the issue class, since these applicants were issued exactly as applied. Hence, they have no *new* reason for comparison shopping and selecting against the company by searching for a more favorable underwriting decision elsewhere. Applicants who receive a rated policy do have such a reason. Depending on perspective, they have been issued either a more costly policy than originally quoted or a policy with a smaller face amount for the same premium. In either case, this perceived "bait-and-switch" probably motivates many applicants to reassess their need for insurance and, if it still exists, to consider shopping for a better deal. One may speculate, therefore, that those applicants who refuse substandard policies because they disagree with the need for the rating given have, as a group, better mortality than those who subsequently pay for them. The reliability of a person's perception of his health is best appreciated by considering individual annuitant mortality.

In any event, actual substandard experience probably overstates the experience of the associated issue group. One way to modify this experience would be to set

$$q' = q_1' - (q_1' - q_2')q^{nt} , \tag{25}$$

where q_1' is the actual experience, q_2' is one's guess as to the appropriate level for the not-taken policies, and q^{nt} is the probability than an issue in the given substandard premium class will become a not-taken policy.

Declined policies are another problem, since by definition *no* experience exists. As is the case for incompleted applications, mortality estimates should be made with the assistance of the underwriting and medical staff. For declined applicants, estimates should at least distinguish between those declined because of predictable yet exceedingly high mortality (e.g., obese hypertensive diabetics) and those whose mortality is entirely unpredictable for several years, yet will be more stable after this period (e.g., those anticipating or recovering from major surgery).

The appropriateness of the lapse assumption may be questioned, although it is often the case that substandard experience is quite similar to that of standard issue. It should be noted that the above identities depend on this assumption, since this justified the equating of total mortality costs, independent of the underwriting scenario. However, if this assumption is not considered appropriate, it is possible to determine the sign of the "error" in $\nabla\bar{\bar{M}}$ by using the results in the Appendix. This is because M is a decreasing function of the lapse assumption, that is, M decreases as the probabilities of lapse increase. For example, if the lapse assumption utilized is appropriate for $\underline{M}^E$, yet is considered overstated for $\bar{\bar{M}}^H$, the calculated value of $\nabla\bar{\bar{M}}$ will understate the value that would have been produced if $\bar{\bar{M}}^H$ could have been calculated with the appropriate lower lapse probabilities.

Finally, in the real world, the same group cannot be underwritten twice. Therefore, it is important that the study be designed to normalize such observable characteristics as the sex distribution of the two classes under study.

In addition, as was noted in Section II, A, it is not really expected that the preunderwriting mortality profiles of the medical and nonmedical applicant groups are equal, yet the above model assumes this. Although there does not appear to be an accurate method of quantifying the effect of the socioeconomic and antiselection factors, it is straightforward to represent symbolically what their effect would be. Let D be the unit mortality cost of the total nonmedical applicant group in excess of the cost for the medical applicant group. Then the development preceding (14) could be modified to produce

$$\nabla M = \Delta K r + \Delta L s - X^H(r^H + s^H) + D\,. \tag{26}$$

The formula for $\nabla\tilde{M}$ would not change in appearance, but would change in reality, because it depends on the value of ∇M.

Finally, only the antiselection component of D, D^A, would play a role in the development of $\nabla\bar{\bar{M}}$ because of the assumed underwriting scenario. The derivation preceding (22) would have produced, in this case,

$$\nabla\bar{\bar{M}} = \frac{\Delta\tilde{\bar{K}}\tilde{\bar{r}} - \tilde{\bar{r}}^H(\Delta M + \tilde{\bar{X}}^H) + cD^A}{1 - \tilde{\bar{s}}^E - \tilde{\bar{r}}^H}\,, \tag{27}$$

where $c \leq 1$ reflects the fact that some antiselection would be prevented by the improved package of other tools. For example, one might propose that c be proportional (or equal) to $\tilde{\bar{r}}^E/r^E$.

B. *The Single Mortality Standard Approach*

In this section, the results of Section A above will be generalized in order to allow the evaluation of underwriting tools other than the medical examination. The primary difference here is that mortality experience usually is not partitioned on the basis of the underwriting tools utilized, except for the medical/nonmedical partition.

The difficulty that this lack of experience data introduces is one of choosing an appropriate standard or standards against which the extra mortality costs of the effect cases can be evaluated. It will be shown that the use of one standard, judiciously chosen and dependent on the tools under study, is sufficient to enable the evaluation of the counterparts to the three mortality differentials defined above.

In the most general setting, there will be n primary tools under study, T_i, $i = 1, \ldots, n$, along with which n secondary groups of other tools, O_i, are utilized. As a convention, it will be assumed that the sophistication of the primary tool or secondary group increases as i increases. In particular, T_1 is the crudest or least valuable, and will often represent the use of no primary tool. The associated secondary groups of tools, O_i, are simply defined as everything else utilized during the underwriting of the T_i group *except* those tools that were ordered exclusively because of T_i. In general, these secondary groups do not represent well-defined packages of tools applied uniformly throughout the respective T_i classes, since the ordering criteria for each of the other tools will not usually coincide with those for T_i.

Throughout this section, the following notation will be used, where, as before, "cost" means the present value of such future costs per $1,000 applied. Let

M_i = Mortality cost for the group of applicants deemed standard according to T_i and O_i ($i = 1, \ldots, n$);

$\bar{M}_i$ = Mortality cost for the group of applicants deemed standard according to T_i ($i = 1, \ldots, n$);

$\bar{\bar{M}}_i$ = Mortality cost for the group of applicants deemed standard according to T_i and O_{i+1} ($i = 1, \ldots, n - 1$);

M = Mortality cost for some group of standard issue to be determined later;

K_i = Extra mortality cost for the T_i effect cases calculated according to (6) with $M(q)$ taken as M;

L_i = Extra mortality cost for those effect cases within the T_i class attributable to O_i but *not* simultaneously attributable to T_i, calculated with $M(q)$ taken as M;

r_i = Probability that an application in the T_i class will become a T_i effect case;

s_i = Probability that an application in the T_i class will become an O_i effect case calculated according to the convention underlying L_i; and

$\tilde{K}_i$, $\tilde{L}_i$, $\tilde{r}_i$, $\tilde{s}_i$ = Counterparts of K_i, L_i, r_i, and s_i, respectively, defined so that O_i has priority for multiple-effect cases. As was discussed in Section A above, effect cases produced by T_i that were ordered because of some special requirement, independent of the regular age/amount requirement, should be categorized with $\tilde{s}_i$. This is consistent with the intent of $\tilde{s}_i$ to reflect all effect cases that would be produced independent of the regular ordering limit.

As an example of a multiple-effect case, consider the following. Assume that information on an inspection report (T_i) motivates the need for a medical examination, because of an indication of, say, hypertension. Assume also that the information obtained on the examination results in a class rating. Then, if the inspection report was the only reason for ordering this examination, the examination would not be considered part of the O_i group, and credit for the effect case would be counted in r_i and $\tilde{r}_i$. However, if the examination was also required because of, say, age/amount, it would be considered part of the O_i group, and credit for the effect case counted in r_i and $\tilde{s}_i$.

As before, three sets of mortality differentials will be considered:

1. Total issue differentials: $\nabla M_i = M_i - M_{i-1}$ $(i = 1, \ldots, n-1)$;
2. Absolute differentials: $\nabla \tilde{M}_i = \tilde{M}_i - \tilde{M}_{i-1}$ $(i = 1, \ldots, n-1)$; and
3. Marginal differentials: $\nabla \bar{\tilde{M}}_i = \bar{\tilde{M}}_i - M_{i-1}$ $(i = 1, \ldots, n-1)$.

In the following sections, identities between these differentials and the extra mortality cost differentials will be derived under the ideal setting that the same applicant group could be subjected to two underwriting scenarios and the results compared. It will also be assumed that the mortality experience of the standard issue groups and all effect cases can be accurately predicted and that all calculations of the various M, K, and L values are performed with regard to the same lapse assumption.

In order to motivate the methodology used, the special case $n = 2$ will be treated before the case for general n.

1. A SPECIAL CASE: $n = 2$

Subjected to the underwriting requirements appropriate for the T_1 group, an applicant group of amount A will be partitioned into A_1^M, A_1^K, and A_1^L, where these symbols have the obvious meaning. On the basis of this partitioning, the total mortality cost is predicted to be

$$A_1^M M_1 + A_1^K(M + K_1) + A_1^L(M + L_1) ,$$

which can also be written as

$$AM_1 + A_1^K K_1 + A_1^L L_1 + (A_1^K + A_1^L)(M - M_1) .$$

Subjected to the requirements appropriate for the T_2 group, the cost is predicted to be

$$AM_2 + A_2^K K_2 + A_2^L L_2 + (A_2^K + A_2^L)(M - M_2) .$$

Equating and simplifying, we obtain the following identity:

$$\nabla M_1 = \Delta K_1 r_1 + \Delta L_1 s_1 + (r_2 + s_2)(M - M_2) - (r_1 + s_1)(M - M_1) , \tag{28}$$

where Δ is the usual forward-difference operator.

In its present form, equation (28) is not too useful because of the presence of M_1 and M_2 in the expression on the right. Although the applications will be discussed in Section B, 3, below, let us assume for the moment that it is known that

$$M = a_1 M_1 + a_2 M_2 , \quad a_1 + a_2 = 1 . \tag{29}$$

Then it is easy to verify that

$$M - M_1 = -a_2 \nabla M_1 , \quad M - M_2 = a_1 \nabla M_1 . \tag{30}$$

Substituting expression (30) in equation (28) and simplifying, we obtain

$$\nabla M_1 = \frac{\Delta K_1 r_1 + \Delta L_1 s_1}{1 - a_1(r_2 + s_2) - a_2(r_1 + s_1)} . \tag{31}$$

Had the applicant group been subjected only to T_1, the predicted costs would have been

$$(A_1^M + A_1^L)\bar{M}_1 + A_1^K(M + K_1) ,$$

which, when equated to the cost above determined by T_1 and O_1, results in

$$\bar{M}_1 = M_1 + \frac{s_1}{1 - r_1}(L_1 - a_2 \nabla M_1) .$$

Similarly,

$$\bar{M}_2 = M_2 + \frac{s_2}{1 - r_2}(L_2 + a_1 \nabla M_1) .$$

Hence,

$$\nabla \bar{M}_1 = \nabla M_1 \left(1 - \frac{a_2 s_1}{1 - r_1} - \frac{a_1 s_2}{1 - r_2}\right) + \frac{s_1}{1 - r_1} L_1 - \frac{s_2}{1 - r_2} L_2 . \tag{32}$$

As was the case before, the evaluation of the hypothetical underwriting scenario of T_1 and O_2 involves the small problem of estimating $\hat{r}_1$, which is the counterpart to $\bar{r}_1$ when O_2 is the collection of other tools rather than O_1. One can justify the following expression, corresponding to expression (18):

$$r_1 - (r_2 - \bar{r}_2) \leq \hat{r}_1 \leq \bar{r}_1 . \tag{33}$$

As before, in order to simplify notation, the identity for $\nabla\bar{\bar{M}}_1$ will be derived assuming that $\hat{r}_1 = \tilde{r}_1$ and $\hat{K}_1 = \tilde{K}_1$. The resulting expression can then be reevaluated reflecting the potential range of $\hat{r}_1$ given in (33). Of course, if T_1 represents the use of no tool, then $r_1 = \tilde{r}_1 = \hat{r}_1 = 0$.

Subjected to T_2 and O_2, the predicted cost, allocating multiple-effect cases to O_2, becomes

$$A_2^M M_2 + \tilde{A}_2^K(M + \tilde{K}_2) + \tilde{A}_2^L(M + \tilde{L}_2) .$$

When the scenario of T_1 and O_2 is utilized, this predicted cost becomes

$$\tilde{A}_1^M \bar{\bar{M}}_1 + \tilde{A}_1^K(M + \tilde{K}_1) + \tilde{A}_1^L(M + \tilde{L}_1) .$$

However, on the basis of the observations made prior to and including equation (21),

$$\tilde{A}_1^M = A_2^M + \tilde{A}_2^K - \tilde{A}_1^K , \quad \text{and} \quad \tilde{A}_1^L(M + \tilde{L}_1) = \tilde{A}_2^L(M + \tilde{L}_2) . \tag{34}$$

Equating the above predicted costs and using equations (30) and (34), we obtain, sequentially,

$$A_2^M M_2 + \tilde{A}_2^K(M + \tilde{K}_2) = (A_2^M + \tilde{A}_2^K - \tilde{A}_1^K)\bar{\bar{M}}_1 + \tilde{A}_1^K(M + \tilde{K}_1) ,$$

$$(A_2^M + \tilde{A}_2^K - \tilde{A}_1^K)(\bar{\bar{M}}_1 - M_2) = \tilde{A}_2^K(\tilde{K}_2 + M - M_2) - \tilde{A}_1^K(\tilde{K}_1 + M - M_2) ,$$

and

$$\nabla\bar{\bar{M}}_1 = \frac{\Delta\tilde{K}_1\,\tilde{r}_1 + a_1\Delta\tilde{r}_1\nabla M_1}{1 - \tilde{s}_2 - \tilde{r}_1} . \tag{35}$$

2. THE GENERAL CASE

By comparing the predicted cost based on T_i and O_i with that based on T_{i+1} and O_{i+1}, we produce the following identity, which is entirely analogous to (28):

$$M_i - M_{i+1} = \Delta K_i r_i + \Delta L_i s_i + (r_{i+1} + s_{i+1})(M - M_{i+1}) - (r_i + s_i)(M - M_i) . \tag{36}$$

Rewriting $M_i - M_{i+1}$ in (36) as $(M - M_{i+1}) - (M - M_i)$ and rearranging, we produce the following $n - 1$ equations in the n unknowns $M - M_i$, where $i = 1, \ldots, n$:

$$\begin{aligned}(1 - r_{i+1} - s_{i+1})(M - M_{i+1}) - (1 - r_i - s_i)(M - M_i) \\ = \Delta K_i r_i + \Delta L_i s_i \,, \qquad i = 1, \ldots, n - 1.\end{aligned} \tag{37}$$

Assuming, as before in (29), that it is known that

$$M = \sum_{i=1}^{n} a_i M_i \,, \qquad \sum_{i=1}^{n} a_i = 1 \,, \tag{38}$$

the underdetermined system in (37) can be augmented with the necessary nth equation, producing the following system:

$$\begin{aligned}(1 - r_{i+1} - s_{i+1})Y_{i+1} - (1 - r_i - s_i)Y_i = \Delta K_i r_i + \Delta L_i s_i \,, \\ i = 1, \ldots, n - 1 \,, \\ \sum_{i=1}^{n} a_i Y_i = 0 \,,\end{aligned} \tag{39}$$

where $Y_i = M - M_i$.

Of course, if (39) is solvable, the total issue differentials, ∇M_i, are easily obtained from the solution of this system by noting as above that

$$\nabla M_i = \Delta Y_i \,, \qquad i = 1, \ldots, n - 1 \,. \tag{40}$$

As it turns out, this system is always uniquely solvable for the Y_i's, since if A denotes the matrix of coefficients in (39), the determinant of A is given by

$$\text{Det (A)} = (-1)^{n-1} \sum_{i=1}^{n} a_i \prod_{j \neq i} (1 - r_j - s_j) \,, \tag{41}$$

and this can never be equal to zero given the natural restrictions that for all i

$$a_i \geqslant 0 \,, \qquad r_i + s_i < 1 \,. \tag{42}$$

Since (41) is obvious for $n = 2$, let us assume that it is true for $n =$

$m - 1$ and apply induction to the case $n = m$. If A_m is the corresponding matrix of coefficients, Det (A_m) can be expressed in terms of cofactors based on the last column of A_m, producing

$$\text{Det}\,(A_m) = -(1 - r_m - s_m)\,\text{Det}\,(A_{m-1}) + a_m\,\text{Det}\,(U)\,, \tag{43}$$

where U is an upper triangular matrix with diagonal elements equal to $-(1 - r_i - s_i)$ for $i = 1, \ldots, m - 1$. Hence, since (41) is assumed to be true for A_{m-1}, (43) can be written as

$$\begin{aligned}\text{Det}\,(A_m) &= (-1)^{m-1}(1 - r_m - s_m) \\ &\times \sum_{i=1}^{m-1} a_i \prod_{j \neq i} (1 - r_j - s_j) + (-1)^{m-1} a_m \prod_{j=1}^{m-1} (1 - r_j - s_j)\,,\end{aligned} \tag{44}$$

which equals expression (41) for $r = m$, so the proof is complete.

It is interesting to compare the method used here to that used in the special case of $n = 2$. Clearly, the starting points, (28) and (36), are identical. The difference was that, for (28), the $M = M_i$ terms were expressed as multiples of ∇M_1 given in (30), whereas for (36), the ∇M_i were expressed in terms of $M - M_i$ and $M - M_{i+1}$.

To imitate the method used in the special case above, it is only necessary to be able to express $M - M_i$ as a linear combination of ∇M_j, and this can be done with the following generalization of (30):

$$\begin{aligned} M - M_i &= \sum_{j=1}^{n-1} b_j^i \nabla M_j\,, \\ b_j^i &= \sum_{k=1}^{j} a_k\,, \qquad 1 \leq j \leq i - 1 \\ &= \sum_{k=1}^{j} a_k - 1, \quad i \leq j \leq n - 1\,. \end{aligned} \tag{45}$$

Had (45) been used in (36), the following $n - 1$ equations in the $n - 1$ unknowns ∇M_j would have emerged:

$$\sum_{j=1}^{n} [b_j^i(r_i + s_i) - b_j^{i+1}(r_{i+1} + s_{i+1}) + \delta_{ij}]\nabla M_j = \Delta K_i r_i + \Delta L_i s_i\,, \qquad i = 1, \ldots, n - 1\,, \tag{46}$$

where δ_{ij} is the Kronecker delta, defined as 1 if $i = j$, and zero otherwise.

It is not difficult to show that (39) and (46) are equivalent systems in the sense that (1) if $\{Y_i\}_{i=1}^{n}$ solves (39), then $\{\Delta Y_j\}_{j=1}^{n-1}$ solves (46), and (2) if $\{\nabla M_j\}_{j=1}^{n-1}$ solves (46), then $\{\Sigma_{j=1}^{n-1} b_j^i \nabla M_j\}_{i=1}^{n}$ solves (39).

For absolute differentials, the same method used for the case $n = 2$ produces

$$\bar{M}_i = M_i + \frac{s_i}{1 - r_i}(L_i + Y_i)\ , \quad i = 1, \ldots, n\ , \tag{47}$$

where $\{Y_i\}$ is the unique solution to (39). Hence,

$$\nabla\bar{M}_i = \nabla M_i + \nabla\left[\frac{s_i}{1 - r_i}(L_i + Y_i)\right]\ , \quad i = 1, \ldots, n - 1\ . \tag{48}$$

If Y_i is expressed in terms of the ∇M_j by using (45), then (48) will be analogous to (32) in form.

Finally, letting $\hat{r}_i = \tilde{r}_i$ for notational convenience, where $\hat{r}_i$ is defined as before, the following identity for the $n - 1$ marginal differentials, $\nabla\bar{\bar{M}}_i$, can be derived as for the case $n = 2$:

$$\nabla\bar{\bar{M}}_i = \frac{\Delta\tilde{K}_i\tilde{r}_i + Y_{i+1}\Delta\tilde{r}_i}{1 - \tilde{s}_{i+1} - \tilde{r}_i}\ , \quad i = 1, \ldots, n - 1\ . \tag{49}$$

Expressing Y_{i+1} in terms of the ∇M_j's by using (45), then (49) will be analogous to (35) in form.

As always, (49) should be reevaluated to reflect the range of $\hat{r}_i$, which can be expressed as

$$r_i - (r_{i+1} - \tilde{r}_{i+1}) \leq \hat{r}_i \leq \tilde{r}_i\ , \quad i = 1, \ldots, n - 1\ . \tag{50}$$

3. PRACTICAL CONSIDERATIONS

The considerations noted in Section A, 3, above, with two additional considerations, are sufficient here as well. These two additional considerations are (1) the determination of a mortality standard by means of choosing a_i in (29) or (38) and (2) the determination of K_i and L_i for the various T_i underwriting classes.

In general, the choice of the standard on which to calculate M depends on both the tool under study and the level of refinement available from past mortality experience data. However, most requirements can be valued with the following list of experience classes: (1) the nonmedical mortality class, (2) a "lower-amount" medical mortality class, and (3) one to three "large-amount" medical mortality classes.

In the best of all worlds, $n = 2$ and one might have access to experience studies upon which M_2, the mortality cost for the group on which T_2 is utilized, can be determined. If this is the case, then (35) reduces to the equivalent of (22) with $\tilde{X}^H = 0$:

$$\nabla \bar{\bar{M}}_1 = \frac{\Delta \tilde{K}_1 \tilde{r}_1}{1 - \tilde{s}_2 - \tilde{r}_1}, \tag{51}$$

where $\tilde{r}_1$ is to be varied as $\hat{r}_1$ to reflect (33). If the alternative tool, T_1, is the use of no tool, then $\hat{r}_1 = \tilde{r}_1 = r_1 = 0$, and all the remaining parameters in (51) can be estimated on the basis only of an analysis of applications containing T_2. Otherwise, both applications containing T_2 and those containing T_1 must be sampled.

For example, many insurers could develop mortality estimates for their large-amount experience (over \$100,000, over \$250,000, etc.). Such experience data could be used to calculate M_2, where T_2 corresponded to certain specialty requirements such as urinalysis, X-ray, or EKG. All of these studies would then require only a sampling of applications above the respective ordering limits.

As another example, the evaluation of what is known as a "checkup" APS can be handled in a similar manner. This underwriting group of APSs excludes those ordered because of specific impairments identified on some information source such as the application medical history, a medical examination, or an inspection report. Rather, a checkup APS is a request for information from a physician that is motivated solely by the applicant's acknowledgment of a recent physical examination or routine checkup. Age/amount ordering limits usually depend on the relative timing of the checkup, with more liberal limits suitable as the duration increases.

To evaluate this requirement, it may seem natural to assume that the mortality of those applicants on whom such an APS was ordered and who were subsequently issued standard is similar to lower-amount medical mortality, whether the application was submitted medically or nonmedically (an alternative "standard" is discussed below). With such an assumption, the value of this requirement can be determined by a study of only those applications containing such an APS by using (51) with $\hat{r}_1 = 0$, and calculating extra mortality costs based on the assumed mortality standard.

As an example of a nonzero r_1, consider the evaluation of an appropriate limit for the so-called special service inspection report. This report may be of particular interest because of its relatively high average cost, which is usually determined relative to an hourly rate rather than a fixed rate per report. Of course, this average cost should be defined so as to include the cost of all necessary geographical transfers that may be billed at a much later time. Here, one of the large-amount mortality classes usually would be suitable for use as a mortality standard. In contrast to the above examples, however, it would also be necessary here to sample applications that contained the next lower report in the IR hierarchy and vary the value of $\hat{r}_1$ to reflect the range for $\hat{r}_1$ defined in (33).

For the evaluation of some requirements, however, it will happen that either $n = 2$ and historical data on which to base M_2 are unavailable, or $n > 2$. In these cases, one of two approaches should suffice.

First, it might happen that, although the experience of no individual T_i group is available, the experience of the group of all T_i's is known. If this occurs, the combined experience group would be a suitable mortality standard. The value of the a_i's in (29) or (38) could then be defined relative to the proportion of applications (by amount) in a random sample that contained T_i (T_1 could be "no tool"). More specifically, if c_i equals this proportion, then a_i is taken as proportional to $c_i(1 - r_i - s_i - q_i^{ni})$, where q_i^{ni} is the probability that an application containing T_i will not become an effect case and yet will not be issued, because of the applicant's request. The proportionality constant is defined so that $\Sigma a_i = 1$. If it is believed that the not-taken probabilities for the standard issue T_i subgroups are appreciably different, this could be reflected in these proportions in the obvious way.

Of course, the actual study need not be conducted on a random sample of the group of applications containing some T_i, since such a sample is likely to be heavily weighted with the T_i's with smaller i. The random sample is really needed only for the determination of a_i, and the actual study can then be performed on samples of some T_i subgroups, and supplemented versions of others, in order to have comparable group sizes for each T_i.

As an example for $n > 2$, consider the evaluation of the hierarchy of "medical" limits above the nonmedical limit that usually separate the medical class into paramedicals, exams given by several "grades" of physicians, and/or "double examinations." Here, combined medical experience would be the appropriate standard.

This approach could also be used for the simultaneous evaluation of the several scales of ordering limits utilized for the various cost inspection reports. Here, the appropriate standard would be an "aggregate" mortality table based on all issue experience. This application, however, involves a great deal of inefficiency because a random sample will be so heavily weighted with applications containing no inspection report, or the lowest cost report.

To remedy this, another approach could be utilized that is workable under the condition that, although the experience of the group of all T_i's is unavailable (or impractical to use as above), this group can be partitioned into several groups for which the experience is known. This partitioning need not be with respect to parameters involving the T_i's, since any partitioning will do. Unlike the above two general approaches, this method utilizes a value of M that is constructed.

For example, it may be assumed as part of the checkup APS study discussed above that the group of medical (nonmedical) standard applicants with a recent checkup is a random sample, from a mortality point of view, of the group of all medical (nonmedical) standard applicants. If this is the case, the appropriate standard would be constructed as

$$M = b_1M^H + b_2M^E + b_3M^{E+} , \qquad (52)$$

where M^H, M^E, and M^{E+} are the mortality costs based on the nonmedical and two medical issue classes (partitioned by amount), and b_j is defined relative to the proportion c_j, where c_j is the probability that a randomly sampled application containing such an APS will be in the jth experience group. As before, this value of M would be used as M_2, and (51) applied with $r_1 = 0$.

As a final example, consider the IR study noted above, where the group of all T_i's is the entire applicant class. Assume that nonmedical experience and two or three classes of medical experience have been defined and quantified and the respective values of M, denoted as in (52), calculated. The initial sample is then drawn as three or four random samples, one from each of these subgroups of applicants. The proportion drawn can be chosen in such a way as to offset to a great extent the skewness by number of reports that would occur with a single random sample. On the basis of this combined group of applications, the standard would be defined by

$$M = b_1M^H + b_2M^E + b_3M^{E+} + b_4M^{E++} , \qquad (53)$$

with

$$b_j \approx \frac{c_j}{k_j}[1 - r(j) - s(j) - q^{n(j)}_{(j)}] , \qquad (54)$$

where c_j is the proportion of the total sample that was drawn from the respective experience class, k_j the sample proportion used, and $r(j)$, $s(j)$, and $q^{ni}_{(j)}$ have their usual meaning, but generalized to the M^H class for $j = 1$, the M^E class for $j = 2$, and so on. For example, $r(1)$ is the all-T_i probability of producing an effect case in the M^H class. Again, the proportionality constant in (54) is determined so that $\Sigma b_j = 1$. Of course, M in (53) will theoretically be equal to the "aggregate" mortality cost for a single random sample and can be defined that way directly, based on actual experience, if it is believed that the parameters in (54) have been reasonably stable for several years.

Finally, let c^i_j be the proportion of applications from the jth experience class that contains T_i, the ith report in the IR hierarchy (T_1 = no report). Then $M = \Sigma a_i M_i$, where the a_i's are defined by

$$a_i \approx \left(\sum_j \frac{c_j c^i_j}{k_j} \right) (1 - r_i - s_i - q^{ni}_i) \ . \tag{55}$$

As before, not-taken experience could be reflected in the definition of a_i and/or b_j if desired.

Hence, fairly detailed analyses of many requirements can be made on the basis of only four or so mortality standards if the sample for the study is carefully drawn.

One final consideration, as noted above, is the determination of K_i and L_i for the various T_i underwriting classes. It was assumed in the above development that the mortality experience of each substandard class for each T_i applicant group could be produced. In reality, however, historic substandard mortality experience usually will be available for analysis only separated by substandard class, and the vast majority of this experience will be from the medically underwritten applicant class. Hence, mortality predictions for effect cases in an underwriting class must be based on experience that will not in general be of that class.

In Section A above, it was assumed that mortality predictions for all effect cases were based on historic substandard experience of medically examined lives, and an adjustment X^H was proposed to correct the understatement that this caused in the predictions for the nonmedical effect cases. A similar procedure will now be investigated in this more general setting.

Let

M^j_i = Mortality cost of the effect cases from the T_i applicant group that are in the jth substandard class;

M^j = Mortality cost of the jth substandard experience class, modified as in (25) for not-taken policies if deemed necessary;
d_i^j = Proportion of effect cases from the T_i applicant group in the jth substandard class, $\Sigma_j d_i^j = 1$ for all i.

Hence, per unit of effect case, the mortality cost of all effect cases from the T_i applicant group is predicted to be $\Sigma_j d_i^j M^j$, but is really equal to $\Sigma_j d_i^j M_i^j$. Hence, the prediction error per unit, E_i, which must be *added* to the cost predicted, is given by

$$E_i = \Sigma_j d_i^j (M_i^j - M^j) . \tag{56}$$

Assume that, corresponding to (38), for each j there is $\{b_k^j\}$ such that

$$M^j = \Sigma_k b_k^j M_k^j , \quad \Sigma_k b_k^j = 1 . \tag{57}$$

This was the case in Section A above, where $b_1^j = 0$ and $b_2^j = 1$ for all j, $E_1 = X^H$, and $E_2 = 0$. Also, if M in (38) is based on the experience of medical issues, then, since the experience underlying M^j is usually from this same group, it is clear that b_k^j could be estimated from a_k in (38) by taking into consideration the various probabilities of becoming an effect case, the d_i^j's above, and the probabilities of becoming a noneffect, nonissue case, a not-taken standard issue, and a not-taken substandard issue. Hence, in a number of instances, the parameters b_k^j can be estimated.

Combining (56) with (57), we have, since $\Sigma_k b_k^j = 1$,

$$E_i = \Sigma_j d_i^j [\Sigma_k b_k^j (M_i^j - M_k^j)] . \tag{58}$$

In Section A it was effectively assumed that $M_1^j - M_2^j = [(n - j)/n]$ $(M_1 - M_2)$, where M_1 (M_2) corresponds to M^H (M^E), $j = 1, \ldots, n - 1$, enumerates the substandard classes, and $j = n$ the declined class. Generalizing this, assume that

$$M_i^j - M_k^j = x_j(M_i - M_k) , \quad 0 \leq x_n < x_{n-1} < \ldots < x_1 \leq 1 . \tag{59}$$

Then,

$$E_i = \Sigma_j d_i^j x_j (M_i - \Sigma_k b_k^j M_k) . \tag{60}$$

Adjusting the predictions made in (36) with those values of E_i, for example, the system that is analogous to that in (39) becomes

$$(1 - r_{i+1} - s_{i+1})Y_{i+1} - (1 - r_i - s_i)Y_i$$

$$= \Delta K_i r_i + \Delta L_i s_i + \Delta[(r_i + s_i)E_i]\,, \quad i = 1\,, \ldots, n - 1\,; \quad (61)$$

$$\sum_{i=1}^{n} a_i Y_i = 0\,.$$

In practice, the solution $\{Y_i\}$ to the unmodified system (39) can be utilized to produce E_i in (60) (recalling that $Y_i = M - M_i$), and then the modified system in (61) can be solved. Of course, the iterative procedure

$$E_i = 0 \longrightarrow Y_i \longrightarrow M_i \longrightarrow E_i \longrightarrow Y_i \longrightarrow M_i \longrightarrow \ldots \quad (62)$$

could be repeated again and again, although it is an open question as to when such a procedure "converges."

In a similar fashion, (31) and (46) can be modified. The identity for the absolute mortality differentials in (32) or (48) then becomes

$$\nabla \tilde{M}_i = \nabla M_i + \nabla \left[\frac{s_i}{1 - r_i} (L_i + Y_i + E_i) \right]\,, \quad i = 1, \ldots, n - 1\,, \quad (63)$$

and that of the marginal mortality differentials in (35) or (49) becomes

$$\nabla \tilde{M}_i = \frac{\Delta \tilde{K}_i \tilde{r}_i + \Delta E_i \tilde{r}_i + Y_{i+1} \Delta \tilde{r}_i}{1 - \tilde{s}_{i+1} - \tilde{r}_i}\,, \quad i = 1, \ldots, n - 1\,. \quad (64)$$

IV. OTHER MODELS

Although the models presented in this paper can be considered fairly complete in their ability to accommodate the evaluation of various underwriting tools, it should be kept in mind that it was assumed throughout this paper that the present value of mortality costs was the correct measure with which to evaluate an underwriting requirement. More specifically, differentials in the mortality costs associated with various resultant classes of standard issue were used to estimate value. Although this measure has intuitive appeal and reflects the majority of the value of underwriting, it is basically a "local" measure in scope and does not necessarily reflect the "corporate" impact of underwriting in full.

For example, this measure is not sensitive to the effect of underwriting on not-taken experience and implicitly assumes that the discovery of *all* ratable information "saves" the insurer money, even though the likelihood that a substandard issue will generate premium income is far less than that for a standard issue. Also, the measure utilized here does not lend itself easily to the analysis of various issue-volume models and the impact of underwriting liberalizations on these models.

A more general measure for evaluating underwriting requirements is one based on asset shares. It allows a comparison of the total assets generated by the entire issue blocks produced by different underwriting scenarios and is, therefore, quite sensitive to the insurer's standard and substandard gross premium margins and the effect of underwriting on nonissue and not-taken experience. The effect of underwriting requirements on the applicant distribution can also be modeled and measured and would be of interest for tools thought to have a "barrier effect" on application amounts (e.g., medical examinations).

In addition, such a method allows for a more consistent treatment among issue blocks in various markets because the components of a gross premium are more accurately reflected. Finally, the concepts of "cash flow" underwriting can be quantified and incorporated into the technique by considering various patterns of future interest rates. This measure, therefore, is more appropriate as the basis of what might be called the "corporate approach."

These considerations will be more fully explored in a forthcoming paper entitled "Asset Share Valuation of Underwriting Requirements." The relationship of that approach to the mortality cost method investigated here will also be developed.

APPENDIX

In this appendix, several of the properties of a mortality cost function will be analyzed. To this end, let

$$\begin{aligned} M(\boldsymbol{a}, v, \boldsymbol{q}, \boldsymbol{w}) &= \sum_{k=1}^{n} a_k v^k q_k \prod_{j=1}^{k-1} (1 - q_j - w_j) \\ &= \sum_{k=1}^{n} a_k v^k q_k p_{k-1} \, , \end{aligned} \tag{A.1}$$

where

v = Discount factor, $0 < v \leq 1$, $v = (1 + i)^{-1}$;

$\boldsymbol{a}$ = $(a_1, \ldots, a_n)$ = Net-amount-at-risk vector per unit insured, a_k being the value for duration k, where $0 \leq a_j \leq a_k \leq 1$ for $j > k$;

$\boldsymbol{q} = (q_1, \ldots, q_n)$ = Mortality vector, q_k being the probability that an insured life entering the kth duration will die during that duration, $0 \leq q_k < 1$, $k = 1, \ldots, n-1$, $0 \leq q_n \leq 1$;

$\boldsymbol{w} = (w_1, \ldots, w_{n-1})$ is the lapse vector, w_k being the probability that an insured life entering the kth duration will lapse during that duration, $0 \leq w_k \leq 1 - q_k$; and

$$p_{k-1} = \prod_{j=1}^{k-1} (1 - q_j - w_j) .$$

LEMMA 1. *Let $\boldsymbol{q}$ be a mortality vector. Then for any lapse vector $\boldsymbol{w}$,*

$$\sum_{k=j}^{n} q_k p_{k-1} \leq p_{j-1} , \quad j = 1, \ldots, n . \tag{A.2}$$

That is, at every duration, the total probability of a future death cannot exceed 1.

Proof. Let

$$m_j = \sum_{k=j}^{n} q_k \prod_{i=j}^{k-1} (1 - q_i - w_i) .$$

To prove that $m_j \leq 1$, it is sufficient to consider the special case when $\boldsymbol{w} = 0$, since m_j will be smaller for any other $\boldsymbol{w}$.

Further, for $\boldsymbol{w} = 0$,

$$m_{j+1} = \frac{m_j - q_j}{1 - q_j} ,$$

and if $m_j \leq 1$, this expression is bounded above by m_j. Hence, if $m_j \leq 1$, we have $m_{j+1} \leq 1$. To prove that $m_1 \leq 1$, note that

$$m_1 = 1 - \prod_{k=1}^{n} (1 - q_k) . \quad \square$$

It is apparent from the proof of Lemma 1 that if $m_j = 1$, then $w_i = 0$ and $m_i = 1$ for $i \geq j$. The implication regarding m_i is also clear when we make the observation that $m_j = 1$ if and only if $q_n = 1$ and $w_i = 0$ for $i \geq j$.

For notational convenience, M will always be expressed as a function of the variable(s) of interest. For example, $M(\boldsymbol{a}_0, v_0, \boldsymbol{q}, \boldsymbol{w}_0)$, where $\boldsymbol{a}_0$, v_0, and w_0 are assumed fixed, will be denoted by $M(q)$. Also, $q < q'$ will be used to denote that $q_j \leq q'_j$ for all j and $q_j < q'_j$ for at least one j.

M is not a very interesting function of v or a, since it is clear that $M(v) \leq M(v')$ if $v < v'$, and $M(a) \leq M(a')$ if $a < a'$. That is, M is an increasing function of v (decreasing function of i) and a. If it could be assumed that q and w are independent, then it would also be clear that $M(w') \leq M(w)$ for $w < w'$, that is, M is a decreasing function of w. However, on the basis of the theory that lapsation is selective, it might be expected that q will increase if w increases. Hence, the ultimate impact of increased lapsation on the value of M is closely related to the behavior of M as a function of q. The following results identify some of this behavior.

THEOREM 2. *Let $M(q)$ be a mortality cost function, and let q, q' be mortality vectors such that $0 < q < q'$. Then*

$$0 \leq M(q') - M(q) < M(q' - q) , \tag{A.3}$$

with equality if and only if either

(a) $a_{k_0} = 0$, *where k_0 is the first duration for which $q'_k > q_k$, or*
(b) $a_k = a_{k_0}$ *for* $k \geq k_0$, $v = 1$, *and*

$$\sum_{k=k_0+1}^{n} q_k p_{k-1} = p_{k_0} \quad (so\ w_k = 0\ for\ k \geq k_0 + 1) .$$

Proof. To prove that $M(q) \leq M(q')$, that is, that M is an increasing function of q, it is sufficient to consider the special case where q and q' differ only in one component. To see this, note that the general case can be reduced to at most n special cases by the sequence q^k, $k = 0, 1, \ldots, n$, defined by $q^k = (q_1^k, \ldots, q_n^k)$, where

$$\begin{aligned} q_j^k &= q'_j , \quad j \leq k \\ &= q_j , \quad j > k . \end{aligned} \tag{A.4}$$

Since $q^0 = q$, $q^n = q'$, we have

$$M(q') - M(q) = \sum_{k=1}^{n} [M(q^k) - M(q^{k-1})] . \tag{A.5}$$

Now,

$$M(q^k) = \sum_{j=1}^{n} a_j v^j q_j^k q_{j-1}^k ,$$

and this can be rewritten as

$$M(\boldsymbol{q}^k) = \sum_{j=1}^{k} a_j v^j q_j^{k-1} p_{j-1}^{k-1} + (q'_k - q_k) a_k v^k p_{k-1}^{k-1} + \sum_{j=k+1}^{n} a_j v^j q_j^{k-1} p_{j-1}^{k} , \quad \text{(A.6)}$$

since $q_j^k = q_j^{k-1}$ for $j \neq k$. For $j \geq k + 1$,

$$p_{j-1}^k = \left(1 - \frac{q'_k - q_k}{1 - q_k - w_k}\right) p_{j-1}^{k-1} ,$$

which, when substituted in (A.6), produces

$$M(\boldsymbol{q}^k) = M(\boldsymbol{q}^{k-1}) + (q'_k - q_k) \left(a_k v^k p_{k-1}^{k-1} - \frac{1}{1 - q_k w_k} \sum_{j=k+1}^{n} a_j v^j q_j^{k-1} p_{j-1}^{k-1}\right) . \quad \text{(A.7)}$$

Now if condition (a) is satisfied, clearly $M(\boldsymbol{q}^k) = M(\boldsymbol{q}^{k-1})$ for all k, so $M(\boldsymbol{q}) = M(\boldsymbol{q}')$. Otherwise $a_k > 0$ for $k = k_0$, and the last expression in (A.8) can be written as

$$a_k v^k p_{k-1}^{k-1} \left(1 - \sum_{j=k+1}^{n} \frac{a_j}{a_k} v^{j-k} q_j^{k-1} \frac{p_{j-1}^{k-1}}{p_k^{k-1}}\right) .$$

Now if condition (b) is satisfied, $M(\boldsymbol{q}^k) = M(\boldsymbol{q}^{k-1})$ for $k = k_0$ clearly, and on the basis of the remarks following Lemma 1, equality holds also for $k > k_0$, so $M(\boldsymbol{q}') = M(\boldsymbol{q})$. Otherwise, this expression is strictly greater than zero, because of Lemma 1, and the first inequality in (A.3) is verified.

For the second inequality, note that because of (A.7) we have

$$M(\boldsymbol{q}^k) - M(\boldsymbol{q}^{k-1}) \leq (q'_k - q_k) a_k v^k p_{k-1}^{k-1} . \quad \text{(A.8)}$$

Now,

$$p_{k-1}^{k-1} = \prod_{j=1}^{k-1} (1 - q'_j - w_j)$$

$$< \prod_{j=1}^{k-1} [1 - (q'_j - q_j) - w_j] .$$

Hence, combining this with (A.5) and (A.8), we have

$$M(\boldsymbol{q}') - M(\boldsymbol{q}) < \sum_{k=1}^{n} (q'_k - q_k)a_k v^k \prod_{j=1}^{k-1} [1 - (q'_j - q_j) - w_j]$$

$$= M(\boldsymbol{q}' - \boldsymbol{q}) . \quad \square$$

COROLLARY. *Let $\boldsymbol{q}$, $\boldsymbol{q}'$ be mortality vectors. Then*

$$|M(\boldsymbol{q}') - M(\boldsymbol{q})| \leq M(|\boldsymbol{q}' - \boldsymbol{q}|) , \tag{A.9}$$

where $|\boldsymbol{q}' - \boldsymbol{q}|$ is the vector whose jth component is $|q'_j - q_j|$.

Proof. Let $\boldsymbol{q}^U$ and $\boldsymbol{q}^L$ be vectors defined by

$$q_j^U = \max (q'_j, q_j) , \quad q_j^L = \min (q'_j, q_j) .$$

Then, clearly, $\boldsymbol{q}^L < \boldsymbol{q}, \boldsymbol{q}' < \boldsymbol{q}^U$, unless $\boldsymbol{q} = \boldsymbol{q}'$ and (A.9) is immediate. According to Theorem 2, then,

$$|M(\boldsymbol{q}') - M(\boldsymbol{q})| = c[M(\boldsymbol{q}') - M(\boldsymbol{q})]$$
$$\leq M(\boldsymbol{q}^U) - M(\boldsymbol{q}^L) ,$$

where $c = 1$ or -1. Applying Theorem 2 to this last term and observing that $\boldsymbol{q}^U - \boldsymbol{q}^L = |\boldsymbol{q}' - \boldsymbol{q}|$ completes the proof. $\square$

Next, the directional derivatives of $M(\boldsymbol{q})$ will be considered. Recall that if $\boldsymbol{u}$ is a unit vector, that is, $|\boldsymbol{u}| \equiv (\Sigma u_i^2)^{1/2} = 1$, then the directional derivative of $M(\boldsymbol{q})$ at $\boldsymbol{q}_0$ in the direction of $\boldsymbol{u}$, denoted $(\partial M/\partial \boldsymbol{u})|_{\boldsymbol{q}_0}$, is defined by the following limit when it exists:

$$\left.\frac{\partial M}{\partial \boldsymbol{u}}\right|_{\boldsymbol{q}_0} = \lim_{\lambda \to 0} \frac{M(\boldsymbol{q}_0 + \lambda \boldsymbol{u}) - M(\boldsymbol{q}_0)}{\lambda} . \tag{A.10}$$

Of course, if $\boldsymbol{u}$ equals one of the canonical unit vectors, $\boldsymbol{u} = (0, \ldots, 1, \ldots, 0)$, then (A.10) is the definition of the jth partial derivative, where $u_j = 1$. For notational convenience, let $M'_u(\boldsymbol{q}_0)$ denote the directional derivative, and $M'_j(\boldsymbol{q}_0)$, $j = 1, \ldots, n$, the n partial derivatives.

More generally, a multivariate function $M(\boldsymbol{q})$ is said to be differentiable at $\boldsymbol{q}_0$ if there is a linear function L, $L(\boldsymbol{w}) = \Sigma a_i w_i$, such that

$$\frac{M(\boldsymbol{q}_0 + \boldsymbol{w}) - M(\boldsymbol{q}_0)}{|\boldsymbol{w}|} - L(\boldsymbol{w}) = O(|\boldsymbol{w}|) .$$

That is, this difference tends to zero as $|\boldsymbol{w}| \longrightarrow 0$.

It is not too difficult to prove that if $M(\boldsymbol{q})$ is differentiable, then $a_i = M_i'(\boldsymbol{q}_0)$, and, hence,

$$M_u'(\boldsymbol{q}_0) = \Sigma u_i M_i'(\boldsymbol{q}_0) . \tag{A.11}$$

The existence of partial derivatives does not necessarily imply differentiability, but the existence of continuous partials does (although this is not a necessary condition). For our purposes, $M(\boldsymbol{q})$ as defined in (A.1) is certainly differentiable, since it is a polynomial function.

THEOREM 3. *Let $M(\boldsymbol{q})$ be defined as in (A.1), and let $\boldsymbol{u} > 0$ be a unit vector. Then $M_u'(\boldsymbol{q})$ is a decreasing function. That is,*

$$M_u'(\boldsymbol{q}') \leqslant M_u'(\boldsymbol{q}) \quad \text{if } \boldsymbol{q} < \boldsymbol{q}' . \tag{A.12}$$

In particular, if $\boldsymbol{q}^k$ is defined as in (A.4),

$$\begin{aligned} M_j'(\boldsymbol{q}^k) - M_j'(\boldsymbol{q}^{k-1}) &= \frac{-(q_k' - q_k)}{1 - q_k - w_k} M_j'(\boldsymbol{q}^{k-1}) , & k < j \\ &= 0 , & k = j \\ &= \frac{-(q_k' - q_k)}{1 - q_j' - w_j} M_k'(\boldsymbol{q}^{k-1}) , & k > j . \end{aligned} \tag{A.13}$$

Proof. Since (A.5) is an identity, it holds equally well for $M_u'(\boldsymbol{q})$. Hence, it is clear that (A.12) is implied by (A.13) and (A.11), since $\boldsymbol{u} > 0$, and

$$M_u'(\boldsymbol{q}') - M_u'(\boldsymbol{q}) = \sum_{k=1}^{n} \sum_{j=1}^{n} u_j [M_j'(\boldsymbol{q}^k) - M_j'(\boldsymbol{q}^{k-1})] . \tag{A.14}$$

To prove (A.13), note that

$$M_j'(\boldsymbol{q}) = a_j v^j p_{j-1} - \frac{1}{1 - q_j - w_j} \sum_{i=j+1}^{n} a_i v^i q_i p_{i-1} . \tag{A.15}$$

This can be verified directly or by noting that because of (A.7), the quotient in (A.10) is constant, and hence the limit equals this constant. Since $M_j'(\boldsymbol{q})$ is independent of q_j, it is clear that $M_j'(\boldsymbol{q}^j) - M_j'(\boldsymbol{q}^{j-1}) = 0$.

Now if $k < j$, all p terms in (A.15) contain q'_k, so

$$M'_j(\boldsymbol{q}^k) = a_j v^j p^k_{j-1} - \frac{1}{1 - q^k_j - w_j} \sum_{i=j+1}^{n} a_i v^i q^k_i p^k_{i-1}$$

$$= \frac{1 - q'_k - w_k}{1 - q_k - w_k} \left(a_j v^j p^{k-1}_{j-1} - \frac{1}{1 - q^{k-1}_j - w_j} \sum_{i=j+1}^{n} a_i v^i q^{k-1}_i p^{k-1}_{i-1} \right)$$

$$= \left(1 - \frac{q'_k - q_k}{1 - q_k - w_k} \right) M'_j(\boldsymbol{q}^{k-1}) \; .$$

Alternatively, if $k > j$,

$$M'_j(\boldsymbol{q}^k) = a_j v^j p^k_{j-1} - \frac{1}{1 - q^k_j - w_j} \left(\sum_{i=j+1}^{k} a_i v^i q^k_i p^k_{i-1} + \sum_{i=k+1}^{n} a_i v^i q^k_i p^k_{i-1} \right) . \tag{A.16}$$

Now for $i \leqslant k$, $p^k_{i-1} = p^{k-1}_{i-1}$. Otherwise, as above,

$$p^k_{i-1} = \frac{1 - q'_k - w_k}{1 - q_k - w_k} p^{k-1}_{i-1} \; .$$

Also, $q^k_k = q^{k-1}_k + (q'_k - q_k)$ and $q^k_i = q^{k-1}_i$ for $i \neq k$.
Rewriting (A.16), we obtain

$$M'_j(\boldsymbol{q}^k) = a_j v^j p^{k-1}_{j-1} - \frac{1}{1 - q^{k-1}_j - w_j} \times \left[\sum_{i=j+1}^{k} a_i v^i q^{k-1}_i p^{k-1}_{i-1} + (q'_k - q_k) a_k v^k p^{k-1}_{k-1} \right]$$
$$- \frac{1}{1 - q^{k-1}_j - w_j} \times \left(1 - \frac{q'_k - q_k}{1 - q_k - w_k} \right) \sum_{i=k+1}^{n} a_i v^i q^{k-1}_i p^{k-1}_{i-1}$$

$$= M_j'(\boldsymbol{q}^{k-1}) - \frac{q_k' - q_k}{1 - q_j' - w_j}$$

$$\times \left(a_k v^k p_{k-1}^{k-1} - \frac{1}{1 - q_k - w_k} \sum_{i=k+1}^{n} a_i v^i q_i^{k-1} p_i^{k-1} \right)$$

$$= M_j'(\boldsymbol{q}^{k-1}) - \frac{q_k' - q_k}{1 - q_j' - w_j} M_k'(\boldsymbol{q}^{k-1}) . \quad \square$$

COROLLARY. *Let* $\boldsymbol{q}, \boldsymbol{q}'$ *be mortality vectors satisfying* $\boldsymbol{q} < \boldsymbol{q}'$. *Then*

$$M(\boldsymbol{q}') = M(\boldsymbol{q}) + \sum_{k=1}^{n} (q_k' - q_k) \prod_{j=1}^{k-1} \left(1 - \frac{q_j' - q_j}{1 - q_j - w_j}\right) M_k'(\boldsymbol{q}) . \quad \text{(A.17)}$$

Proof. Rewriting (A.7), we have

$$M(\boldsymbol{q}^k) - M(\boldsymbol{q}^{k-1}) = (q_k' - q_k) M_k'(\boldsymbol{q}^{k-1}) . \quad \text{(A.18)}$$

Applying (A.13) $k - 1$ times to $M_k'(\boldsymbol{q}^{k-1})$ yields

$$M_k'(\boldsymbol{q}^{k-1}) = \prod_{j=1}^{k-1} \left(1 - \frac{q_j' - q_j}{1 - q_j - w_j}\right) M_k'(\boldsymbol{q}) . \quad \text{(A.19)}$$

Combining (A.5), (A.18), and (A.19) completes the proof. $\square$

COROLLARY. *Let* $\boldsymbol{q}$, $\boldsymbol{q}'$, *and* $\boldsymbol{q}''$ *be mortality vectors and* $\boldsymbol{w}$ *a lapse vector satisfying* $\boldsymbol{q} < \boldsymbol{q}'$, $q_j'' \leq 1 - q_j' - w_j$ *for* $j = 1, 2, \ldots, n - 1$. *Then*

$$M(\boldsymbol{q}' + \boldsymbol{q}'') - M(\boldsymbol{q}') \leq M(\boldsymbol{q} + \boldsymbol{q}'') - M(\boldsymbol{q}) . \quad \text{(A.20)}$$

Proof. Applying (A.17) to each difference and utilizing Theorem 3 complete the proof. $\square$

As was noted before, if it is assumed that $\boldsymbol{q}$ and $\boldsymbol{w}$ are independent, it becomes clear that M is a decreasing function of $\boldsymbol{w}$. That is, $M(\boldsymbol{w}') \leq M(\boldsymbol{w})$ for $\boldsymbol{w} < \boldsymbol{w}'$. However, it might be assumed that as $\boldsymbol{w}$ increases, $\boldsymbol{q}$ increases as well, reflecting the opinion that healthy lives are more likely to lapse. It could be possible, therefore, that the value of M remains stable as $\boldsymbol{w}$ increases. Indeed, given $\boldsymbol{q}$, $\boldsymbol{w}$, $\boldsymbol{w}'$ with $\boldsymbol{w} < \boldsymbol{w}'$, there is a "surface" of solutions $\boldsymbol{q}'$ in n-dimensional space that satisfies

$$M(\boldsymbol{q}', \boldsymbol{w}') = M(\boldsymbol{q}, \boldsymbol{w}) . \quad \text{(A.21)}$$

It is natural to wonder whether a lapse vector $\boldsymbol{w}'$, where $\boldsymbol{w} < \boldsymbol{w}'$, could be created by a simple forward "redistribution" of the $\boldsymbol{w}$ lapsed group, without the need for "extra" lapses. This would leave the actual yearly claim amount experience, as well as the value of M, stable.

The following lemma indicates that this is not possible.

LEMMA 4. *Let d_j equal the number of claim units and s_j the number of surrender units for duration j, $j = 1, \ldots, n-1$. Let s'_j be any nonnegative sequence satisfying*

$$\sum_{j=1}^{n-1} s'_j \leq \sum_{j=1}^{n-1} s_j \; ; \quad s'_i \neq s_i \text{ for some } i \; . \tag{A.22}$$

Then there exists k such that $w'_k < w_k$, where $\boldsymbol{w}$ and $\boldsymbol{w}'$ are the lapse vectors implied by s_j, d_j and s'_j, d_j, respectively.

Proof. Let $\{s'_j\}$ be given, and let x_j be defined by $s'_j = s_j + x_j$. Hence, $\Sigma x_j \leq 0$; letting $l_{j+1} = l_j - d_j - s_j$, we have

$$w'_j = (s_j + x_j) \Big/ \left(l_j - \sum_{i=1}^{j-1} x_i \right) , \quad w_j = \frac{s_j}{l_j} \, . \tag{A.23}$$

Now assume that $w'_j \geq w_j$ for all j. From (A.23), this implies that there is a solution, $\boldsymbol{x} = (x_1, \ldots, x_{n-1})$, to the following system of equations:

$$x_j + w_j \sum_{i=1}^{j-1} x_i \geq 0 \, , \quad j = 1, \ldots, n-1 \; ; \quad \sum_{i=1}^{n-1} x_i \leq 0 \, , \tag{A.24}$$

where $w_j < 1$ for all j.

Now

$$\sum_{i=1}^{n-1} x_i \leq 0 \, , \quad x_{n-1} + w_{n-1} \sum_{i=1}^{n-2} x_i \geq 0$$

imply that either

$$\text{(a)} \quad x_{n-1} = \sum_{i=1}^{n-2} x_i = 0$$

or

$$\text{(b)} \quad x_{n-1} > 0 \, , \quad \sum_{i=1}^{n-2} x_i < 0 \, .$$

Assuming (a) and (A.24) for $j = n - 2$ implies that either

$$\text{(aa)} \quad x_{n-2} = \sum_{i=1}^{n-3} x_i = 0 ,$$

or

$$\text{(ab)} \quad x_{n-2} > 0 , \quad \sum_{i=1}^{n-3} x_i < 0 .$$

Assuming (b) and (A.24) for $j = n - 2$ implies (ab) above. Continuing in this way, one of two cases finally emerges: (A) $x_j = 0$ for all j; (B) $x_2 > 0$ and $x_1 < 0$. Case (A) contradicts the assumption that $s_i' \neq s_i$ for some i, and Case (B) contradicts the assumption that $w_j' \geqslant w_j$ for all j. Therefore, it must be that $w_k' < w_k$ for some k, completing the proof. []

Hence, if $\boldsymbol{q}$, $\boldsymbol{w}$, $\boldsymbol{w}'$ are given, $\boldsymbol{w} < \boldsymbol{w}'$, and $\boldsymbol{q}'$ can be defined to produce the same annual claim amounts as $\boldsymbol{q}$, it must be true that there are "extra" lapses given by $\boldsymbol{w}'$ in the sense that $\Sigma s_i' > \Sigma s_i$.

If we assume that all the additional lapsed policies came from the group that would have survived n policy durations given the decrements $\boldsymbol{q}$ and $\boldsymbol{w}$, it is clear that the actual amount of claims each year would be the same, and the "implied" $\boldsymbol{q}'$ would satisfy (A.21). This is truly a worst-case scenario of selective lapsation, and the resultant $\boldsymbol{q}'$ represents a realistic upper bound to the mortality vectors that could be "caused" by the increased $\boldsymbol{w}'$. Specifically, $\boldsymbol{q}'$ is defined recursively by

$$q_k' = q_k \frac{p_{k-1}}{p_{k-1}'} , \quad k = 1, \ldots , n , \tag{A.25}$$

where $\boldsymbol{p}'$ is defined with $\boldsymbol{q}'$ and $\boldsymbol{w}'$.

Otherwise, if any of the additional lapsed policies came from the group of future mortality claims given $\boldsymbol{q}$ and $\boldsymbol{w}$, the construction in (A.25) would eventually fail, since at least one of the necessary q_k''s would require claims from the survivor group. In this case, one would always have $M(\boldsymbol{q}', \boldsymbol{w}') \leq M(\boldsymbol{q}, \boldsymbol{w})$ for the implied $\boldsymbol{q}'$.

For all practical purposes, therefore, M is a decreasing function of $\boldsymbol{w}$ even when selective lapsation is taken into account.

Our final investigation concerns the behavior of $M(\boldsymbol{q}', \boldsymbol{w}) - M(\boldsymbol{q}, \boldsymbol{w})$ as a function of $\boldsymbol{w}$ when $\boldsymbol{q}' > \boldsymbol{q}$ and each is assumed independent of $\boldsymbol{w}$.

As it turns out, $M(\boldsymbol{q}', \boldsymbol{w}) - M(\boldsymbol{q}, \boldsymbol{w})$ is not a monotonic function of $\boldsymbol{w}$ without further restrictions. To see this, let k_1, k_2 be such that $q_i' = q_i$ for $i < k_1$ and $i > k_2$. Then since $q_i' = q_i$ for $i < k_1$,

$$
\begin{aligned}
M(\boldsymbol{q}', \boldsymbol{w}) &- M(\boldsymbol{q}, \boldsymbol{w}) \\
&= \sum_{k=k_1}^{n} a_k v^k \left[q_k' \prod_{j=1}^{k-1} (1 - q_j' - w_j) - q_k \prod_{j=1}^{k-1} (1 - q_j - w_j) \right] \\
&= \prod_{j=1}^{k_1 - 1} (1 - q_j - w_j) \sum_{k=k_1}^{n} a_k v^k \\
&\quad \times \left[q_k' \prod_{j=k_1}^{k-1} (1 - q_j' - w_j) - q_k \prod_{j=k_1}^{k-1} (1 - q_j - w_j) \right] .
\end{aligned}
\tag{A.26}
$$

Now since the last sum in (A.26) is nonnegative by Theorem 2, and apparently independent of w_j for $j = 1, \ldots, k_1 - 1$, it is clear that $M(\boldsymbol{q}', \boldsymbol{w}) - M(\boldsymbol{q}, \boldsymbol{w})$ is a decreasing function of these w_j's, since this is true of the product in (A.26).

On the other hand, since $q_i' = q_i$ for $i > k_2$,

$$
\begin{aligned}
M(\boldsymbol{q}', \boldsymbol{w}) &- M(\boldsymbol{q}, \boldsymbol{w}) \\
&= \sum_{k=1}^{k_2} a_k v^k (q_k' p_{k-1}' - q_k p_{k-1}) + \sum_{k=k_2+1}^{n} a_k v^k (p_{k-1}' - p_{k-1}) q_k .
\end{aligned}
\tag{A.27}
$$

Now the first sum in (A.27) is independent of w_j for $j = k_2, \ldots, n - 1$. Also, the second sum can be rewritten by noting that

$$
\begin{aligned}
p_{k-1}' - p_{k-1} &= \prod_{j=k_2+1}^{k-1} (1 - q_j - w_j) \\
&\quad \times \left[\prod_{j=1}^{k_2} (1 - q_j' - w_j) - \prod_{j=1}^{k_2} (1 - q_j - w_j) \right] ,
\end{aligned}
\tag{A.28}
$$

for $k \geqslant k_2 + 1$, and this is clearly an increasing function of w_j for $j = k_2, \ldots, k - 1$, since $p_j' < p_j$ for $k_1 \leqslant j \leqslant k_2$ implies that all partial derivatives with respect to these w_j's are positive. Hence, $M(\boldsymbol{q}', \boldsymbol{w}) - M(\boldsymbol{q}, \boldsymbol{w})$ is an increasing function of w_j, for $j = k_2, \ldots, n - 1$.

The following theorem provides a condition under which $M(\boldsymbol{q}', \boldsymbol{w}) - M(\boldsymbol{q}, \boldsymbol{w})$ is a decreasing function of $\boldsymbol{w}$.

THEOREM 5. *Let* $\boldsymbol{q}$, $\boldsymbol{q}'$, *and* $\boldsymbol{w}_0$ *be given, such that* $\boldsymbol{q} < \boldsymbol{q}'$ *and*

$$
q_k' p_{k-1}' \geqslant q_k p_{k-1} , \quad k = 1, \ldots, n , \tag{A.29}
$$

where p'_{k-1} and p_{k-1} are defined with $\mathbf{w}_0$. Let H_0 be the hypercube in $\Re^{n-1}$ determined by $\mathbf{w}_0$, that is,

$$H_0 = \{\mathbf{w} \in \Re^{n-1} \mid \mathbf{w} < \mathbf{w}_0\} . \quad (A.30)$$

Then, $M(\mathbf{q}', \mathbf{w}) - M(\mathbf{q}, \mathbf{w})$ is a decreasing function of $\mathbf{w}$ for $\mathbf{w} \in H_0$.

Proof. First of all, note that

$$\frac{p'_{k-1}}{p_{k-1}} = \prod_{j=1}^{k-1} \left(1 - \frac{q'_j - q_j}{1 - q_j - w_j}\right) , \quad (A.31)$$

and this is a decreasing function of w_j, $j = 1, \ldots, k - 1$, since $\mathbf{q} < \mathbf{q}'$. Hence, if (A.29) is true for $\mathbf{w}_0$, it is also true for all $\mathbf{w} < \mathbf{w}_0$, that is, everywhere in H_0.

Now let $\mathbf{q}, \mathbf{q}', \mathbf{q} < \mathbf{q}', \mathbf{w} \in H_0$ be given. Then

$$M(\mathbf{q}', \mathbf{w}) - M(\mathbf{q}, \mathbf{w}) = \sum_{k=1}^{n} a_k v^k \left(\frac{p'_{k-1}}{p_{k-1}} q'_k - q_k\right) p_{k-1} . \quad (A.32)$$

By the above remarks, the bracketed expressions in (A.32) are nonnegative, decreasing functions of the w_j's, and since the p_{k-1}'s clearly have the same property, their products are decreasing functions, which completes the proof. []

It should be noted that the inequality in (A.29) cannot be productively reversed, since $\mathbf{q} < \mathbf{q}'$ and $q'_k p'_{k-1} \leq q_k p_{k-1}$ would imply that $\mathbf{q} = \mathbf{q}'$. Also, this inequality is somewhat restrictive, since it requires that if $q'_j > q_j$ for $j = k$, then $q'_j > q_j$ all $j > k$. Hence, this theorem has applicability only to such mortality vectors.

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DISCUSSION OF PRECEDING PAPER

E. S. SHIU:

Mr. Reitano is to be complimented for this comprehensive study on evaluating underwriting requirements. I wish to make the following two points.

1. It is formulated in Section II, B, that the value of an underwriting requirement for an application is given by

$$M(\boldsymbol{q}') - M(\boldsymbol{q}) , \tag{6}$$

where $\boldsymbol{q}$ is the standard mortality vector and $\boldsymbol{q}'$ is the mortality vector determined by the underwriting tool. I find this puzzling.

Consider the very simple case where the application is for a single premium whole life insurance policy with the death benefit of $1 payable at the end of the year of death. Since we are dealing with a single premium policy, we need not consider lapse rates; thus

$$\begin{aligned} M(\boldsymbol{q}) &= \Sigma_{t\geq 0} (1 - {}_{t+1}V)v^{t+1} {}_{t}p_x q_{x+t} \\ &= \Sigma_{t\geq 0} (1 - A_{x+t+1})v^{t+1} {}_{t|}q_x \\ &= A_x - \Sigma_{t\geq 0} A_{x+t+1} v^{t+1} {}_{t|}q_x . \end{aligned}$$

Hence

$$M(\boldsymbol{q}') - M(\boldsymbol{q}) - (A'_x - A_x) = \Sigma_{t\geq 0} v^{t+1}(A_{x+t+1}\, {}_{t|}q_x - A'_{x+t+1}\, {}_{t|}q'_x) . \tag{†}$$

On the other hand, if the underwriting tool had not been ordered and the single premium policy issued standard, the extra costs the company would expect to incur would be the extra premium $A'_x - A_x$. However, the right-hand side of (†) is not necessarily zero. If by extra mortality costs we mean net extra premiums, then the proof given by W. Shur on page 102 of his paper "A General Method of Calculating Experience Net Extra Premiums Based on the Standard Net Amount at Risk" (*TSA*, Vol. VI [1954]) shows that equation (5) is the correct formula when lapse rates are zero.

It is interesting to note that A_x can be expressed in a form like the definition of the mortality cost function given in equation (1), that is,

$$A_x = \Sigma_{t\geq 0} (1 - {}_{t+1}V)v^{t+1}q_{x+t} .$$

2. The results in the Appendix are quite interesting. Some alternative proofs are outlined below.

a) Lemma 1 is a consequence of the equation

$$\sum_{k=j}^{n} {}_{k-1|}q_x = {}_{j-1}p_x - {}_{n}p_x \,.$$

b) Several of the inequalities can be proved by the mean value theorem: If f is a real-valued and continuously differentiable function defined on a Euclidean space, then for each pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, there exists a number t, $t \in (0, 1)$, such that

$$f(\boldsymbol{x}) = f(\boldsymbol{y}) + (\boldsymbol{x} - \boldsymbol{y}) \cdot \nabla f(t\boldsymbol{x} + (1 - t)\boldsymbol{y}) \,.$$

Thus, to prove the first inequality in Theorem 2, we simply check that the gradient vector of m is nonnegative, that is,

$$\frac{\partial}{\partial q_j} M(\boldsymbol{q}) \geqslant 0, \qquad j = 1, 2, \ldots, n \,.$$

Multiplying $(1 - q_j - w_j)$ to both sides of equation (A.15) and applying Lemma 1 yield the result.

Similarly, as we verify the inequalities

$$\frac{\partial^2}{\partial q_i \, \partial q_j} M(\boldsymbol{q}) \leqslant 0 \,, \qquad i, j = 1, 2, \ldots, n \,,$$

and because the vector $\boldsymbol{u}$ is nonnegative, we prove the first half of Theorem 3.

c) The second inequality in (A.3) is a "triangle inequality." Put $\boldsymbol{q}^* = \boldsymbol{q}' - \boldsymbol{q}$. To prove

$$M(\boldsymbol{q}') = M(\boldsymbol{q} + \boldsymbol{q}^*) \leqslant M(\boldsymbol{q}) + M(\boldsymbol{q}^*) \,,$$

it is sufficient to show,for each k,

$$p'_k q'_{k+1} \leqslant p_k q_{k+1} + p^*_k q^*_{k+1} \,.$$

The last inequality holds because

$$p'_k \leqslant \text{minimum } \{p_k, p^*_k\} \,.$$

(AUTHOR'S REVIEW OF DISCUSSION)

ROBERT R. REITANO:

I would like to thank Dr. Shiu for his discussion of my paper. Before commenting on his first remark, it is valuable to point out that the definition of M given in equation (1) and formalized as the function $M(\boldsymbol{a}, v, \boldsymbol{q}, \boldsymbol{w})$ in equation (A.1) did not imply that $(NAR)_{t+1}$ or a_k was a function of v, $\boldsymbol{q}$, or $\boldsymbol{w}$. This term was intended to reflect the policy values under consideration. Although it could be defined in terms of policy reserves, it might also be defined to reflect policy cash values, gross premiums, and the insurer's practices regarding annual and settlement dividends, or simply set equal to 1 in some cases. In addition, the parameters v, $\boldsymbol{q}$, $\boldsymbol{q}'$, and $\boldsymbol{w}$ were intended to represent actual experience, and even if $(NAR)_{t+1}$ is defined in terms of policy reserves, the v and $\boldsymbol{q}$ underlying those reserves rarely would be utilized for the respective variables in M. Finally, once $(NAR)_{t+1}$ is defined for a given policy, it is to be used in the calculation of both $M(\boldsymbol{q})$ and $M(\boldsymbol{q}')$.

If we assume, then, as did Dr. Shiu, that $(NAR)_{t+1}$ is equal to $1 - {}_{t+1}V_x$, his equation (†) becomes

$$M(\boldsymbol{q}') - M(\boldsymbol{q}) = A'_x - A_x - \Sigma_{t\geq 0}\, v^{t+1}\, {}_{t+1}V_x({}_{t|}q'_x - {}_{t|}q_x)\ , \qquad \text{(D.1)}$$

where A_x (A'_x) is standard notation for the net single premium defined with respect to $\boldsymbol{q}$ ($\boldsymbol{q}'$) and v, and for the moment can be considered independent of the factors underlying ${}_{t+1}V_x$. However, his question is still material, since it is clear from equation (D.1) above that, in general,

$$M(\boldsymbol{q}') - M(\boldsymbol{q}) \neq A'_x - A_x\ , \qquad \text{(D.2)}$$

where perhaps equality was expected. He also points out the following related result from Mr. Shur's paper (eq. [7] on p. 101 of *TSA*, Vol. VI): If ${}_{t+1}V_x$ and A_x are defined in terms of v and $\boldsymbol{q}$, A'_x defined in terms of v and $\boldsymbol{q}'$, then, with $\boldsymbol{w} = 0$,

$$A'_x - A_x = \Sigma_{t\geq 0} v^{t+1}\, {}_tp'_x(q'_{x+t} - q_{x+t})(1 - {}_{t+1}V_x)\ , \qquad \text{(D.3)}$$

where the right-hand side of (D.3) is essentially the same as K defined in (5), with $(NAR)_{t+1}$ set equal to $1 - {}_{t+1}V_x$.

As noted above, the parameters v, $\boldsymbol{q}$, $\boldsymbol{q}'$, and $\boldsymbol{w}$ are intended to represent actual experience and usually will be independent of $(NAR)_{t+1}$ even when defined this way. However, on a valuation net premium basis, or when

${}_{t+1}V_x$ is an experience reserve defined in terms of $\boldsymbol{q}$, it might appear that $A'_x - A_x$ is the correct value of the underwriting requirement, which according to (D.3) supports the use of K rather than $M(\boldsymbol{q}') - M(\boldsymbol{q})$.

Intuitively, once $(NAR)_{t+1}$ is defined, $M(\boldsymbol{q})$ is the experience net single premium for the policy issued to a life with mortality $\boldsymbol{q}$, lapse $\boldsymbol{w}$, and interest i; $M(\boldsymbol{q}')$ is similarly defined but with reference to mortality $\boldsymbol{q}'$. Therefore, $M(\boldsymbol{q}') - M(\boldsymbol{q})$ is by definition the additional experience net single premium associated with forgoing the underwriting requirement under study and consequently incurring mortality at level $\boldsymbol{q}'$ rather than $\boldsymbol{q}$. The key point here is that the elimination of this underwriting requirement had no effect on $(NAR)_{t+1}$. That is, without underwriting, the life with mortality $\boldsymbol{q}'$ would have been issued standard, not substandard, since the extra hazard presumably would not have been recognized.

To formalize this in the context of Dr. Shiu's remark, consider the following. As derived in chapter 5 of Jordan's *Life Contingencies*, the level annual net premium for a life insurance policy can be split into a death benefit component defined on a net-amount-at-risk basis, and a policy reserve increment component. In the case of whole life, one has the following result:

$$P_x = vq_{x+t}(1 - {}_{t+1}V_x) + (v\,{}_{t+1}V_x - {}_tV_x)\,. \tag{D.4}$$

If (D.4) is multiplied by $v^t\,{}_tp_x$ $(w = 0)$ and summed from $t = 0$, one obtains the following identity:

$$A_x = \Sigma_{t\geq 0}\, v^{t+1}\,{}_tp_xq_{x+t}\,(1 - {}_{t+1}V_x) + \Sigma_{t\geq 0}\, v^t\,{}_tp_x(v\,{}_{t+1}V_x - {}_tV_x)\,. \tag{D.5}$$

In other words, A_x can be thought of as the present value of death benefits on a net-amount-at-risk basis, plus the present value of reserve increments for the survivors. Similarly, A'_x could be expressed as in (D.5) with all symbols involving $\boldsymbol{q}'$ primed. In other words, A'_x contemplates reserve values at the ${}_tV'_x$ level.

In the context of the problem at hand, it is assumed that, without underwriting, a standard policy would have been issued and standard reserves maintained, even though the mortality would actually be at the $\boldsymbol{q}'$ level. Consequently, the net single premium for such a policy, utilizing (D.5) as a defining equation, becomes

$$A''_x = \Sigma_{t\geq 0}v^{t+1}\,{}_tp'_xq'_{x+t}(1 - {}_{t+1}V_x) + \Sigma_{t\geq 0}\, v^t\,{}_tp'_x(v\,{}_{t+1}V_x - {}_tV_x)\,. \tag{D.6}$$

Hence, if the underwriting requirement under study is eliminated, the actual increase in the net single premium is given by $A''_x - A_x$, which according to (D.5) and (D.6) becomes

$$A''_x - A_x = M(\boldsymbol{q}') - M(\boldsymbol{q}) - \Sigma_{t\geq 0}\, v^t({}_tp_x - {}_tp'_x)(v\ {}_{t+1}V_x - {}_tV_x)\ , \quad \text{(D.7)}$$

where $w = 0$ in all ${}_tp_x$ and ${}_tp'_x$.

Compared with the intuitive argument above that yielded $M(\boldsymbol{q}') - M(\boldsymbol{q})$ exactly, (D.7) is more explicit in that it displays the fact that, since $\boldsymbol{q}' > \boldsymbol{q}$, there will be fewer lives at future durations for which policy reserves need be maintained, and hence some savings are incurred.

In my paper, the summation in (D.7) was eliminated as a somewhat academic adjustment, since it will usually be quite small and its effect will be totally dominated by the lack of certainty in the predicted future lapse rate vector $\boldsymbol{w}$, which will be used in the actual calculated values of both ${}_tp_x$ and ${}_tp'_x$ in $M(\boldsymbol{q}') - M(\boldsymbol{q})$.

The expression in (D.4) is easily modified for an n-payment whole life policy if ${}_nP_x$ is replaced by 0 for $t \geqslant n$. If this expression is then multiplied by v^t and summed from $t = 0$, one obtains

$$\Sigma_{t\geq 0}\, v^{t+1}q_{x+t}(1 - {}_{t+1}^{\ n}V_x) = c_nA_x\ , \quad \text{(D.8)}$$

where $c_n = \ddot{a}_{\overline{n}|}/\ddot{a}_{x:\overline{n}|}$. Consequently, in the single premium case, $c_n = 1$, and one has Dr. Shiu's interesting development for A_x.

Regarding Dr. Shiu's second point, it was most gratifying to receive alternative derivations of some of the results in the appendix. His suggestions provide quicker verifications of the broader aspects of these results, but may yield somewhat less informative results than the methods employed. For example, utilizing the mean value theorem for the first inequality in Theorem 2 provides less insight into the conditions that result in equality. Similarly, his method of verifying the second inequality in Theorem 2 results in the replacement of strict inequality with "$\leqslant$". However, the power of the techniques of multivariate analysis cannot be overstated and will often easily provide results for which direct verification is unwieldy.

Again, I would like to thank Dr. Shiu for his thought-provoking discussion and the reference to Mr. Shur's paper on a related theme.