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INTEGER FUNCTIONS AND LIFE CONTINGENCIES

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ABSTRACT

We study the applications of the integer functions, ceiling and floor, to life contingencies. Various actuarial formulas are derived by applying a theorem of the mean value type for integrals and uniform distribution of deaths assumptions.

I. INTRODUCTION

The simplest version of the Euler-Maclaurin summation formula is

$$\tfrac{1}{2}f(0) + f(1) + f(2) + \ldots + f(k-1) + \tfrac{1}{2}f(k) = \int_0^k f(t)dt + \int_0^k (t - \lfloor t \rfloor - \tfrac{1}{2})f'(t)dt\,,$$

where f is a continuously differentiable function on the interval $[0, k]$ ([6], formula [3.7.22]; [1], Theorem 7.13). The symbol $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. The function $t - \lfloor t \rfloor$ is periodic and sawtooth-shaped.

In this paper we shall show that the integer functions and periodic functions arise naturally in the study of life contingencies. Approximation formulas will be elegantly derived by assuming that deaths are distributed uniformly throughout each year of age or, less restrictively, throughout each month of age.

Notation. Let $\boldsymbol{Z}$, $\boldsymbol{Z}^+$, $\boldsymbol{R}$, and $\boldsymbol{R}^+$ denote the set of integers, positive integers, real numbers, and positive real numbers, respectively.

II. CEILING AND FLOOR

For $t \in \boldsymbol{R}$, let $\lfloor t \rfloor$ denote the greatest integer less than or equal to t and let $\lceil t \rceil$ denote the least integer greater than or equal to t ([18], p. 37). Dr. K. E. Iverson, the originator of APL, calls them the *floor* and *ceiling* of t, respectively. In the mathematical literature, the more commonly used symbol for $\lfloor t \rfloor$ is $[t]$.

If $t \in \boldsymbol{Z}$, then $\lfloor t \rfloor = \lceil t \rceil$; otherwise, $\lfloor t \rfloor + 1 = \lceil t \rceil$. Since $\boldsymbol{Z}$ is a set of measure zero, $\lfloor t \rfloor + 1 = \lceil t \rceil$ almost everywhere. Thus, if $\lfloor t \rfloor$ appears in the integrand of a Riemann integral, replacing it with $\lceil t \rceil - 1$ will not change the value of the integral.

For $m \in \boldsymbol{R}^+$, the graph of $\lfloor mt \rfloor/m$ or $\lceil mt \rceil/m$, as a function of t, resembles an infinite flight of stairs, with step size $1/m$, at an inclination of 45 degrees. As m tends to infinity, $\lfloor mt \rfloor/m$ tends to t from below and $\lceil mt \rceil/m$ tends to t from above.

Definitions. For $s \in \boldsymbol{R}^+$, define

$$\text{(i) } t \bmod s = t - s\lfloor t/s \rfloor , \qquad t \in \boldsymbol{R} ;$$

and

$$\text{(ii) } t \text{ pad } s = s\lceil t/s \rceil - t , \qquad t \in \boldsymbol{R} .$$

The quantity $t \bmod s$ is the (nonnegative) remainder when t is divided by s, and t pad s is the least nonnegative addition to t so that the result is divisible by s. The term *mod*, short for modulo, is standard mathematical terminology. In coining the term *pad*, I am borrowing from computer science, in which the term *padding* means the adding of blanks or nonsignificant characters to the end of a block or record in order to bring it up to a certain fixed size ([3]; [31], p. 30).

Note that $t \bmod s = 0$ if and only if $t/s \in \boldsymbol{Z}$ and if and only if t pad $s = 0$. If this is not the case, then $t \bmod s + t \text{ pad } s = s$.

Although we shall not need these results, it is interesting to know that for $t \notin \boldsymbol{Z}$, the Fourier series

$$\tfrac{1}{2} - \sum_{j=1}^{\infty} \sin(2\pi jt)/\pi j \qquad \text{and} \qquad \tfrac{1}{2} + \sum_{j=1}^{\infty} \sin(2\pi jt)/\pi j$$

converge to $t \bmod 1$ and t pad 1, respectively (see [1], p. 338, No. 11.18a).

III. ANNUITIES-CERTAIN

Let $m \in \boldsymbol{Z}^+$, $k \in \boldsymbol{R}^+$, and $k \bmod 1/m = 0$. Then

$$s_{\overline{k}|}^{(m)} = \int_0^k (1 + i)^{\lfloor mt \rfloor/m} dt$$

and

$$\ddot{s}_{\overline{k}|}^{(m)} = \int_0^k (1 + i)^{\lceil mt \rceil/m} dt .$$

For the integral

$$\int_0^k (1 + i)^t dt = \bar{s}_{\overline{k}|} ,$$

$s_{\overline{k}|}^{(m)}$ and $\ddot{s}_{\overline{k}|}^{(m)}$ are lower and upper Riemann sums.

Since $\lfloor -y \rfloor = -\lceil y \rceil$ and $\lfloor j + y \rfloor = j + \lfloor y \rfloor$ for $j \in \mathbf{Z}$, we also have

$$s_{\overline{k}|}^{(m)} = \int_0^k (1 + i)^{k - \lceil mt \rceil / m} dt$$

and

$$\ddot{s}_{\overline{k}|}^{(m)} = \int_0^k (1 + i)^{k - \lfloor mt \rfloor / m} dt .$$

Multiplying the equations above by v^k, we obtain

$$a_{\overline{k}|}^{(m)} = \int_0^k v^{k - \lfloor mt \rfloor / m} dt = \int_0^k v^{\lceil mt \rceil / m} dt$$

and

$$\ddot{a}_{\overline{k}|}^{(m)} = \int_0^k v^{k - \lceil mt \rceil / m} dt = \int_0^k v^{\lfloor mt \rfloor / m} dt .$$

Remarks. Since $\lceil (-m)t \rceil / (-m) = \lfloor mt \rfloor / m$, we have the relation $a_{\overline{k}|}^{(-m)} = \ddot{a}_{\overline{k}|}^{(m)}$. The formulas $a_{\overline{k}|}^{(m)} = (1 - v^k)/i^{(m)}$ and $\ddot{a}_{\overline{k}|}^{(m)} = (1 - v^k)/d^{(m)}$ immediately imply that $i^{(-m)} = d^{(m)}$. This equation can be derived directly. Recall that $1 + i = (1 + i^{(m)}/m)^m$ and $1 - d = (1 - d^{(m)}/m)^m$. Writing $1 + i = [1 + d^{(m)}/(-m)]^{-m}$, we see that $i^{(-m)} = d^{(m)}$.

For the remainder of this section, assume $k \bmod 1/m \neq 0$. *Question:* How should $a_{\overline{k}|}^{(m)}$, etc., be defined? One way is to define $a_{\overline{k}|}^{(m)}$ by the value of the integral

$$\int_0^k v^{\lceil mt \rceil / m} dt ,$$

that is,

$$a_{\overline{k}|}^{(m)} = a_{\overline{\lfloor mk \rfloor / m}|}^{(m)} + (k \bmod 1/m) v^{\lceil mk \rceil / m} .$$

This definition is the same as the one given by Hart ([14], pp. 104, 285; also see [23], p. 581). Similarly,

$$\ddot{a}^{(m)}_{\overline{k}|} = \ddot{a}^{(m)}_{\overline{\lceil mk\rceil/m}|} - (k \text{ pad } 1/m)v^{\lfloor mk\rfloor/m} .$$

Another way to define $a^{(m)}_{\overline{k}|}$ is by the formula

$$a^{(m)}_{\overline{k}|} = (1 - v^k)/i^{(m)} .$$

This is the approach suggested by Donald ([7], sec. 4.18) and Kellison ([17], sec. 3.6).

It is easy to check that under either definition, $-a^{(m)}_{\overline{k}|}$ is a convex function in t. Thus, in both cases, we can apply Jensen's inequality to obtain $a_x \leqslant a_{\overline{\mathring{e}_x}|}$ ([16], p. 175). See Gerber and Jones (discussion in [22], p. 25).

In order to avoid confusion later, different notation will be used for the second definition. Following Rasor and Greville in [23] and Zwinggi ([32], p. 21), we define

$$\mathring{a}^{(m)}_{\overline{k}|} = (1 - v^k)/i^{(m)}$$

and

$$\mathring{s}^{(m)}_{\overline{k}|} = [(1 + i)^k - 1]/i^{(m)} .$$

Motivated by Nesbitt ([19], p. 137; [23], p. 583), we define

$$\ddot{a}^{\{m\}}_{\overline{k}|} = (1 - v^k)/d^{(m)} .$$

IV. IMMEDIATE APPLICATIONS IN LIFE CONTINGENCIES

For a life aged x, consider the length of time until death as a continuous random variable, and denote it by T. The cumulative distribution function for T is ${}_tq_x$, $t \geqslant 0$, and the probability density function is

$$\frac{d}{dt}\,{}_tq_x = {}_tp_x\,\mu_{x+t} , \qquad t \geqslant 0 .$$

For the rest of this paper, x will denote a fixed age; instead of writing ${}_tp_x\,\mu_{x+t}dt$, we shall always write $d\,{}_tq_x$. The interest rate is assumed to be constant.

Let us illustrate how the integer functions can be applied in life contingencies:

1. *Insurance*

$$A_x^{(m)} = E(v^{\lceil mT \rceil/m}) = \int_0^\infty v^{\lceil mt \rceil/m}\, d_t q_x\,.$$

$$(I^{(m)}A)_x^{(n)} = E(v^{\lceil nT \rceil/n} \lceil mT \rceil/m)\,.$$

$$(DA)^1_{x:\overline{k}|} = \int_0^k \lceil k - t \rceil v^{\lceil t \rceil}\, d_t q_x \qquad (k \in \mathbf{Z}^+)$$

$$= \int_0^k (k + 1 - \lceil t \rceil) v^{\lceil t \rceil}\, d_t q_x$$

$$= (k + 1) A^1_{x:\overline{k}|} - (IA)^1_{x:\overline{k}|}\,.$$

2. *Life Annuities*

Starting with Jordan's definition of a_x ([16], p. 40), we have

$$a_x = \int_0^\infty {}_{\lceil t \rceil}E_x dt = \int_0^\infty {}_{\lceil t \rceil}p_x\, v^{\lceil t \rceil} dt$$

$$= \int_{t=0}^{t=\infty} \left(\int_{s=\lceil t \rceil}^{s=\infty} d_s q_x \right) v^{\lceil t \rceil} dt$$

$$= \int_{s=0}^{s=\infty} \left(\int_{t=0}^{t=\lfloor s \rfloor} v^{\lceil t \rceil} dt \right) d_s q_x$$

$$= \int_0^\infty a_{\overline{\lfloor s \rfloor}|}\, d_s q_x = E(a_{\overline{\lfloor T \rfloor}|})\,.$$

Using the relation $a_{\overline{\lfloor T \rfloor}|} = (1 - v^{\lfloor T \rfloor})/i$, we immediately obtain

$$a_x = [1 - (1 + i)A_x]/i$$

([16], formula [3.12]; [29], p. 241; [26], sec. 4.5).

Similarly,

$$\ddot{a}_x^{(m)} = \int_0^\infty {}_{\lfloor mt \rfloor/m}E_x dt$$

$$= E(\ddot{a}^{(m)}_{\overline{\lceil mT \rceil/m}|}) = (1 - A_x^{(m)})/d^{(m)} \qquad \text{([16], p. 78, No. 8).}$$

3. *Premium Refund Benefit* ([25], p. 605)

$$\begin{aligned}\bar{A}_x^{PR,m}/\bar{P}(\bar{A}_x) &= E(v^T \bar{a}_{\overline{T \text{ pad } 1/m}|}) \\ &= E[v^T(1 - v^{T \text{ pad } 1/m})]/\delta \\ &= E(v^T - v^{\lceil mT \rceil/m})/\delta = (\bar{A}_x - A_x^{(m)})/\delta\ .\end{aligned}$$

4A. *Complete Life Annuities*

There are several nonequivalent definitions for a complete life annuity $\mathring{a}_x^{(m)}$.

(i) It was suggested by Rasor and Greville in [23] that

$$\begin{aligned}\mathring{a}_x^{(m)} &= E(\mathring{a}_{\overline{T}|}^{(m)}) \\ &= E[(1 - v^T)/i^{(m)}] = (1 - \bar{A}_x)/i^{(m)}\ .\end{aligned}$$

Another way to express this definition is

$$\begin{aligned}\mathring{a}_x^{(m)} - a_x^{(m)} &= E(v^T \mathring{s}_{\overline{T \bmod 1/m}|}^{(m)}) \\ &= E\{v^T[(1 + i)^{T - \lfloor mT \rfloor/m} - 1]/i^{(m)}\} \\ &= A_x^{(m)}/d^{(m)} - \bar{A}_x/i^{(m)} \qquad ([23], \text{p. } 578)\ .\end{aligned}$$

(ii) The definition given by Spurgeon ([28], chap. 9) and Jordan ([16], sec. 7.4) is

$$\begin{aligned}\mathring{a}_x^{(m)} - a_x^{(m)} &= E[v^T(T \bmod 1/m)] \\ &= E[v^T(T - \lceil mT \rceil/m + 1/m)] \\ &= (\bar{I}\bar{A})_x - (I^{(m)}\bar{A})_x + \bar{A}_x/m \qquad ([16], \text{formula } [7.11]).\end{aligned}$$

(iii) A third definition is $\mathring{a}_x^{(m)} = E(a_{\overline{T}|}^{(m)})$. Thus,

$$\begin{aligned}\mathring{a}_x^{(m)} - a_x^{(m)} &= E[v^{\lceil mT \rceil/m}(T \bmod 1/m)] \qquad ([19], \text{p. } 154, \text{III}) \\ &= (\bar{I}A)_x^{(m)} - (I^{(m)}A)_x^{(m)} + A_x^{(m)}/m\ .\end{aligned}$$

4B. *Apportionable Annuities-due*

In analogy to the three definitions above, we shall give three definitions for an apportionable annuity-due, $\ddot{a}_x^{\{m\}}$.

(i) Following Nesbitt (discussion in [23], p. 583; discussion in [19], pp. 137, 153), we define

$$\ddot{a}_x^{\{m\}} = E(\ddot{a}_{\overline{T|}}^{\{m\}}) = (1 - \bar{A}_x)/d^{(m)} .$$

Equivalently,

$$\begin{aligned} \ddot{a}_x^{(m)} - \ddot{a}_x^{\{m\}} &= E(v^T \ddot{a}_{\overline{T \text{ pad } 1/m|}}^{\{m\}}) \\ &= (\bar{A}_x - A_x^{(m)})/d^{(m)} \qquad ([25], \text{p. } 608, \text{eq. } [5]) . \end{aligned}$$

(ii) In [19], Lauer proposed that

$$\begin{aligned} \ddot{a}_x^{(m)} - \ddot{a}_x^{\{m\}} &= E[v^T(T \text{ pad } 1/m)] \\ &= (I^{(m)}\bar{A})_x - (\bar{I}\bar{A})_x . \end{aligned}$$

(iii) Cain (discussion in [19], p. 141) suggested the third definition:

$$\begin{aligned} \ddot{a}_x^{(m)} - \ddot{a}_x^{\{m\}} &= E[v^{\lfloor mT \rfloor/m}(T \text{ pad } 1/m)] \\ &= (1 + i)^{1/m}[(I^{(m)}A)_x^{(m)} - (\bar{I}A)_x^{(m)}] . \end{aligned}$$

This definition is equivalent to $\ddot{a}_x^{\{m\}} = E(\ddot{a}_{\overline{T|}}^{(m)})$.

Remarks. The formulas for definitions (ii) and (iii) involve increasing insurance functions, which will be discussed in Section VIII. Lauer has summarized the six definitions in [19] (p. 154, Table 1). In Section VI, we shall point out that the six expressions in his Table 1 are exact under the mild assumption of a uniform distribution of deaths throughout each $1/m$ year of age.

Rosser (discussion in [19], p. 149) and Isen (discussion in [19], p. 151) observed that under definitions (ii),

$$\ddot{a}_x^{\{m\}} - \mathring{a}_x^{(m)} = (1 - \bar{A}_x)/m = \delta\bar{a}_x/m .$$

Lauer ([19], p. 156) noted that these equations also hold under definitions (i). It was also pointed out ([23], pp. 576, 583) that under definitions (i),

$$i^{(m)}\mathring{a}_x^{(m)} = \delta\bar{a}_x = d^{(n)}\ddot{a}_x^{\{n\}} , \qquad m, n \in \boldsymbol{Z}^+ .$$

Thus, under definitions (i), we have

$$\mathring{a}_x^{(m)} = (1 + i)^{1/m}\ddot{a}_x^{\{m\}} ;$$

this equation is also true under definitions (iii) but false under definitions (ii).

5. *Stationary Population*

We now illustrate the use of the integer functions in solving stationary population problems:

[4], page 35, No. 7: The number of people now living who die before their next birthday.

Solution:

$$\int_{y=0}^{y=\infty}\left(\int_{t=0}^{t=\lceil y\rceil - y} l_{y+t}\mu_{y+t}dt\right)dy = \int_0^\infty (l_y - l_{\lceil y\rceil})dy = T_0 - \sum_{j=1}^{\infty} l_j .$$

[4], page 89, No. 1(d): The number who will die before their 15th birthday out of the people having either an 11th, 12th, 13th or 14th birthday in a calendar year is $4d_{14} + 3d_{13} + 2d_{12} + d_{11}$.

Solution: Counting all the deaths that occur during and also after that particular calendar year, we get

$$\int_{y=10}^{y=14}\left(\int_{y+t=\lceil y\rceil}^{y+t=15} l_{y+t}\mu_{y+t}dt\right)dy$$

$$= \int_{10}^{14} (l_{\lceil y\rceil} - l_{15})dy = l_{11} + l_{12} + l_{13} + l_{14} - 4l_{15} .$$

[4], page 90, No. 4(b): How many years do the people who have any birthday from 20th to 29th inclusive during 1966 live from that birthday until December 31, 1975?

Solution:

$$\int_{y=19}^{y=29}\left(\int_{y+t=\lceil y\rceil}^{t=10} l_{y+t}dt\right)dy$$

$$= \int_{19}^{29} (T_{\lceil y\rceil} - T_{y+10})dy = \sum_{j=20}^{29} T_j - Y_{29} + Y_{39} .$$

V. PERIODIC FUNCTIONS

Let h be a locally integrable function defined on $\boldsymbol{R}$. Define the "average value" of h as

$$\text{avg}\,(h) = \lim_{\zeta\to\infty}\left(\int_{-\zeta}^{\zeta} h(t)dt\right)\Big/ 2\zeta .$$

Let h be a periodic function and $\alpha \neq 0$ be a period of h, that is, $h(t + \alpha) = h(t)$, for all t; then, clearly,

$$\text{avg}\,(h) = \left[\int_0^{\alpha} h(t)dt\right] \Big/ \alpha \,.$$

For $m, n \in \mathbf{Z}^+$, the function $\lceil mt\rceil/m - \lceil nt\rceil/n$ is periodic. Since

$$\int_0^1 \lceil mt\rceil/m \, dt = (m + 1)/2m \,,$$

avg $(\lceil mt\rceil/m - \lceil nt\rceil/n) = (n - m)/2mn$. Hence $\lceil mt\rceil/m$ is approximately equal to $\lceil nt\rceil/n - (m - n)/2mn$. Several approximation formulas given by Jordan ([16], sec. 3.5) follow immediately from this observation:

$$\begin{aligned}(I^{(m)}A)_x &= E(v^{\lceil T\rceil}\lceil mT\rceil/m)\\ &\doteqdot E\{v^{\lceil T\rceil}[\lceil T\rceil - (m - 1)/2m]\}\\ &= (IA)_x - [(m - 1)/2m]A_x \qquad ([16], \text{formula } [3.27])\,.\end{aligned}$$

$$\begin{aligned}(I^{(m)}\bar{A})_x &= E(v^T\lceil mT\rceil/m)\\ &\doteqdot (I\bar{A})_x - [(m - 1)/2m]\bar{A}_x \qquad ([16], \text{formula } [3.28])\,.\end{aligned}$$

$$\begin{aligned}(\bar{I}\bar{A})_x &= E(v^T T)\\ &\doteqdot E[v^T(\lceil T\rceil - \tfrac{1}{2})] = (I\bar{A})_x - \tfrac{1}{2}\bar{A}_x \qquad ([16], \text{formula } [3.31])\,.\end{aligned}$$

It has been pointed out in [9] that although Jordan's formula (3.27) in [16] is exact under the uniform distribution of deaths assumption (UDD), formula (3.28) is not. This difference can be explained by the following simple theorem for integrating the product of a periodic function and a step function.

AVERAGE VALUE THEOREM. *Let h be a periodic function. Let I be the union of a set of disjoint intervals, the length of each interval being an integral multiple of the fundamental period of h. Let g be a linear combination of the characteristic functions of the intervals of I. If the functions are integrable, then*

$$\int_{-\infty}^{\infty} h(t)g(t)dt = \text{avg}\,(h) \int_{-\infty}^{\infty} g(t)dt \,.$$

The characteristic function of a set S, χ_S, is defined by $\chi_S(x) = 1$ if $x \in S$, and $\chi_S(x) = 0$ otherwise ([1], definition 10.41).

We call this result the average value theorem because of its similarity to the mean value theorem for integrals. Its proof is obvious, and the condition that the intervals be disjoint is not necessary.

Let us demonstrate that Jordan's formula (3.27) in [16] is exact under UDD:

$$(IA)^1_{x:\overline{k}|} - (I^{(m)}A)^1_{x:\overline{k}|} = \int_0^k (\lceil t\rceil - \lceil mt\rceil/m)v^{\lceil t\rceil}d\,{}_tq_x$$

$$= \int_0^k (\lceil t\rceil - \lceil mt\rceil/m)v^{\lceil t\rceil}\,{}_{\lfloor t\rfloor|}q_x\,dt \qquad \because \text{UDD}$$

$$= \text{avg}\,(\lceil t\rceil - \lceil mt\rceil/m)\int_0^k v^{\lceil t\rceil}\,{}_{\lfloor t\rfloor|}q_x\,dt \qquad \because \text{average value theorem}$$

$$= [(m - 1)/2m]A^1_{x:\overline{k}|}\,.$$

The exact expression for $(I^{(m)}\bar{A})_x$ under UDD will be derived later (Sec. VIII, case 1, $n = \infty$). For an example showing that Jordan's formula (3.28) in [16] is not exact under UDD see [9].

Assume UDD. It is easy to derive the formula $A^{(m)\,1}_{x:\overline{k}|} = s^{(m)}_{\overline{1}|}A^1_{x:\overline{k}|}$ by general reasoning. Now,

$$(A)^{(m)\,1}_{x:\overline{k}|} = \int_0^k v^{\lceil mt\rceil/m}d\,{}_tq_x = \int_0^k v^{\lceil mt\rceil/m}\,{}_{\lfloor t\rfloor|}q_x\,dt$$

$$= \int_0^k (1 + i)^{\lceil t\rceil - \lceil mt\rceil/m}(v^{\lceil t\rceil}\,{}_{\lfloor t\rfloor|}q_x)dt\,.$$

Applying the average value theorem and comparing, we see that

$$\text{avg}\,[(1 + i)^{\lceil t\rceil - \lceil mt\rceil/m}] = s^{(m)}_{\overline{1}|}\,.$$

A generalization of this result will be needed later:

LEMMA 1. *Let $m, n \in \mathbf{Z}^+$ and n mod $m = 0$. Then*

$$\text{(i) avg}\,[(1 + i)^{\lceil mt\rceil/m - \lceil nt\rceil/n}] = i^{(m)}/i^{(n)}\,;$$

$$\text{(ii) avg}\,(v^{\lceil mt\rceil/m - \lceil nt\rceil/n}) = d^{(m)}/d^{(n)}\,.$$

Proof of (i). Since $1/m$ mod $1/n = 0$,

$$\begin{aligned}\text{L.H.S.} &= m\int_0^{1/m} (1 + i)^{1/m - \lceil nt \rceil/n} dt \\ &= m(1 + i)^{1/m} a_{\overline{1/m}|}^{(n)} = m s_{\overline{1/m}|}^{(n)} \\ &= m[(1 + i)^{1/m} - 1]/i^{(n)} = \text{R.H.S.}\end{aligned}$$

Proof of (ii). L.H.S. $= m\ddot{a}_{\overline{1/m}|}^{(n)}$ = R.H.S.

There is an interesting way to relate (i) to (ii). It was shown in Section II that $i^{(-m)} = d^{(m)}$. Set $t = 1 - \tau$; then

$$\lceil mt \rceil/m - \lceil nt \rceil/n = -[\lceil(-m)\tau\rceil/(-m) - \lceil(-n)\tau\rceil/(-n)] .$$

Since $(1 + i)^{-1} = v$, we see that (i) and (ii) would imply each other.

Setting $t = 1 - \tau$ again, we obtain

$$\lceil mt \rceil/m - \lceil nt \rceil/n = -(\lfloor m\tau \rfloor/m - \lfloor n\tau \rfloor/n) ;$$

thus we have

COROLLARY 1. *Let $m, n \in \boldsymbol{Z}^+$ and n* mod $m = 0$. *Then*

$$\text{(i) avg } (v^{\lfloor mt \rfloor/m - \lfloor nt \rfloor/n}) = i^{(m)}/i^{(n)} ;$$

$$\text{(ii) avg } ((1 + i)^{\lfloor mt \rfloor/m - \lfloor nt \rfloor/n}) = d^{(m)}/d^{(n)} .$$

VI. UNIFORM DISTRIBUTION OF DEATHS

With the notation developed in this paper, the assumption that deaths are distributed uniformly throughout each year of age can be expressed as

$$l_{x+t} = (t \text{ pad } 1)l_{x+\lfloor t \rfloor} + (t \text{ mod } 1)l_{x+\lceil t \rceil} , \quad t \in \boldsymbol{R}^+ - \boldsymbol{Z}^+ .$$

A generalization of this assumption is that deaths are distributed uniformly throughout each $1/m$ year of age, for some $m \in \boldsymbol{R}^+$, that is,

$$(1/m)l_{x+t} = (t \text{ pad } 1/m)l_{x+\lfloor mt \rfloor/m} + (t \text{ mod } 1/m)l_{x+\lceil mt \rceil/m} , \quad mt \in \boldsymbol{R}^+ - \boldsymbol{Z}^+ .$$

Let us abbreviate this generalization as UDD(m). Under UDD(m) the differential $d\,{}_tq_x$ becomes $(m\ {}_{\lfloor mt \rfloor/m|1/m}q_x\ dt)$, for $mt \in \boldsymbol{R}^+ - \boldsymbol{Z}^+$. It is clear that if $m > 1$, UDD(m) is a less restrictive assumption than the usual one, which is UDD(1).

It has been pointed out by Gerber and Jones ([11]; [12]) that under UDD(1) the random variables $\lfloor T \rfloor$ and (T mod 1) are independent; thus $\lfloor T \rfloor$ and $(\lceil mT \rceil/m - \lfloor T \rfloor)$ are also independent. This observation follows easily from the average value theorem.

COROLLARY 2. *For $m \in \boldsymbol{Z}^+$, let h be a periodic function such that $h(t + 1/m) = h(t)$, $t \in \boldsymbol{R}^+$. Assuming UDD(m), we have*

$$\int h(t) f(\lfloor mt \rfloor/m)\, d_t q_x = \operatorname{avg}(h) \int f(\lfloor mt \rfloor/m)\, d_t q_x \,.$$

Proof. Set $g(t) = f(\lfloor mt \rfloor/m) m \;_{\lfloor mt \rfloor/m|1/m}q_x$, and apply the average value theorem.

Recall the six definitions for complete annuities and apportionable annuities-due discussed earlier (Sec. IV, 4). Lauer ([19], p. 154, Table 1) has given an expression for each definition exact under UDD(1). In fact, his six formulas are exact under UDD(m). Note that under UDD (m),

$$i^{(m)} A_x^{(m)} = \delta \bar{A}_x \,.$$

For instance, according to definition (iii) in Section IV, 4A,

$$\begin{aligned} \mathring{a}_x^{(m)} - a_x^{(m)} &= E[v^{\lceil mT \rceil/m}(T \bmod 1/m)] \\ &= A_x^{(m)} \operatorname{avg}(t \bmod 1/m) && \because \text{Corollary 2} \\ &= A_x^{(m)}/2m \,. \end{aligned}$$

If we consider definition (ii) in Section IV, 4B (Lauer's original definition on p. 14 of [19]), then

$$\begin{aligned} \ddot{a}_x^{\{m\}} - \ddot{a}_x^{(m)} &= E[v^T(T \operatorname{pad} 1/m)] \\ &= E[v^{\lceil mT \rceil/m}(1 + i)^{T \operatorname{pad} 1/m}(T \operatorname{pad} 1/m)] \\ &= A_x^{(m)} \operatorname{avg}[(1 + i)^{t \operatorname{pad} 1/m}(t \operatorname{pad} 1/m)] \\ &= A_x^{(m)} \frac{d}{d\delta} \operatorname{avg}[(1 + i)^{t \operatorname{pad} 1/m}] \\ &= A_x^{(m)} \frac{d}{d\delta}(i^{(m)}/\delta) && \because \text{Lemma 1(i)} \\ &= A_x^{(m)}(i^{(m)}/\delta)(1/d^{(m)} - 1/\delta) && \because \text{Lemma 2} \\ &= \bar{A}_x(1/d^{(m)} - 1/\delta) \qquad ([19], \text{p. 15, eq. [7]}) \,. \end{aligned}$$

Lemma 2 is given in Section VIII.

VII. FAMILY INCOME BENEFIT

Following Jordan ([16], sec. 7.6), we let ${}_nF_x$ denote the net single premium for an n-year family income benefit issued at age x and providing an annual income of 1 payable monthly commencing on the date of death and continuing for the balance of the n-year period. Then, with $m = 12$,

$$\begin{aligned}
{}_nF_x &= \int_0^n v^t \ddot{a}^{(m)}_{\overline{\lceil m(n-t)\rceil/m}|}\, d\,{}_tq_x \\
&= \left[\int_0^n v^t(1 - v^{\lceil m(n-t)\rceil/m})\, d\,{}_tq_x\right] \Big/ d^{(m)} \\
&= \left(\bar{A}^{\,1}_{x:\overline{n}|} - \int_0^n v^{t+n-\lfloor mt\rfloor/m}\, d\,{}_tq_x\right) \Big/ d^{(m)} .
\end{aligned}$$

Assuming UDD(m) and applying Corollary 2 and Corollary 1(ii), we have

$$\int_0^n (1 + i)^{\lfloor mt\rfloor/m - t}\, d\,{}_tq_x = (d^{(m)}/d^{(\infty)})\, {}_nq_x .$$

Thus, Jordan's formula (7.25) in [16],

$${}_nF_x = \bar{A}^{\,1}_{x:\overline{n}|}/d^{(m)} - v^n\, {}_nq_x/\delta ,$$

is exact under UDD(m).

In the case of two interest rates, a rate i before death and a rate i' after death, we have

$$\begin{aligned}
{}_nF_x^{i\&i'} &= \int_0^n v^t \ddot{a}'^{(m)}_{\overline{\lceil m(n-t)\rceil/m}|}\, d\,{}_tq_x \\
&= \left[\bar{A}^{\,1}_{x:\overline{n}|} - v'^n \int_0^n (v^t/v'^{\lfloor mt\rfloor/m})\, d\,{}_tq_x\right] \Big/ d'^{(m)} .
\end{aligned}$$

Now put $v'' = v/v'$, that is, $1 + i'' = (1 + i)/(1 + i')$. Then, by assuming UDD($m$) and applying Corollary 2 and Corollary 1(ii) again, we have

$$\begin{aligned}
&\int_0^n (v^t/v'^{\lfloor mt\rfloor/m})\, d\,{}_tq_x \\
&= \int_0^n v^{t-\lfloor mt\rfloor/m}(v''^{\lfloor mt\rfloor/m}\, d\,{}_tq_x) \\
&= (d^{(m)}/\delta)(1 + i'')^{1/m} A''^{(m)1}_{x:\overline{n}|} = (i^{(m)}/\delta)(1 + i')^{-1/m} A''^{(m)1}_{x:\overline{n}|} .
\end{aligned}$$

Thus, under UDD(m),

$$ {}_nF_x^{i\&i'} = \bar{A}^{\,1}_{x:\overline{n}|}/d'^{(m)} - (v'^n i^{(m)}/\delta i'^{(m)}) A''^{(m)\,1}_{x:\overline{n}|} \,. $$

Assuming UDD(1), we have

$$ {}_nF_x^{i\&i'} = (i/\delta d'^{(m)}) A^{\,1}_{x:\overline{n}|} - (v'^n i'' i^{(m)}/\delta i'^{(m)} i''^{(m)}) A''^{\,1}_{x:\overline{n}|} \,. $$

Jordan's formula (7.24) in [16] follows immediately from the relation

$$ i^{(m)}/i'^{(m)} i''^{(m)} = d^{(m)}/d'^{(m)} d''^{(m)} \,. $$

VIII. INCREASING INSURANCE

By assuming UDD(1), we wish to express

$$ (I^{(m)}A)^{(n)\,1}_{x:\overline{k}|} = \int_0^k v^{\lceil nt \rceil/n} \lceil mt \rceil/m \, d\,{}_tq_x \qquad (m,\, n,\, k \in \mathbf{Z}^+) $$

in terms of more standard actuarial symbols. To simplify writing, we set $k = \infty$. Since k appears only as the upper limit of the integral, it will be trivial to transform the formulas for $k = \infty$ to those for $k < \infty$.

First consider $m = n$ (cf. [9], p. 11)

$$ \begin{aligned} (I^{(m)}A)_x^{(m)} &= -\frac{d}{d\delta} A_x^{(m)} \\ &= -\frac{d}{d\delta} [(i/i^{(m)}) A_x] && \because \text{UDD(1)} \\ &= (i/i^{(m)})[(IA)_x - (1/d - 1/d^{(m)}) A_x] \,. \end{aligned} $$

The next result justifies the last step above.

LEMMA 2. $d(i^{(n)}/i^{(m)})/d\delta = (i^{(n)}/i^{(m)})(1/d^{(n)} - 1/d^{(m)})$.

Proof. Consider $d[\log(i^{(n)}/i^{(m)})]/d\delta$. Since $1 + i^{(m)}/m = e^{\delta/m}$, $d(i^{(m)})/d\delta = e^{\delta/m}$. Hence $d(\log i^{(m)})/d\delta = 1/d^{(m)}$.

Recalling the relation $i^{(-m)} = d^{(m)}$ derived earlier, we immediately have

$$ \frac{d}{d\delta} (d^{(n)}/d^{(m)}) = (d^{(n)}/d^{(m)})(1/i^{(n)} - 1/i^{(m)}) \,. $$

When $m \neq n$, we follow Gerber and Jones [12] in considering only the two cases where $n \bmod m = 0$ and $m \bmod n = 0$.

Case 1: $(1/m) \bmod (1/n) = 0$.

$$\begin{aligned}(I^{(m)}A)_x^{(n)} &= E(v^{\lceil nT\rceil/n}\lceil mT\rceil/m)\\ &= E[(1+i)^{\lceil mT\rceil/m - \lceil nT\rceil/n}(v^{\lceil mT\rceil/m}\lceil mT\rceil/m)]\\ &= (i^{(m)}/i^{(n)})(I^{(m)}A)_x^{(m)}\ ,\end{aligned}$$

by assuming UDD(m) and applying Corollary 2 and Lemma 1(i). Thus, under UDD(1),

$$(I^{(m)}A)_x^{(n)} = (i/i^{(n)})[(IA)_x - (1/d - 1/d^{(m)})A_x]\ .$$

Case 2: $(1/n) \bmod (1/m) = 0$. Assuming UDD(n) and applying Corollary 2, we have

$$\begin{aligned}(I^{(n)}A)_x^{(n)} - (I^{(m)}A)_x^{(n)} &= E[(\lceil nT\rceil/n - \lceil mT\rceil/m)v^{\lceil nT\rceil/m}]\\ &= \operatorname{avg}\,(\lceil nT\rceil/n - \lceil mT\rceil/m)A_x^{(n)}\\ &= [(m-n)/2mn]A_x^{(n)}\ .\end{aligned}$$

Thus, under UDD(1),

$$(I^{(m)}A)_x^{(n)} = (i/i^{(n)})\{(IA)_x - [1/d - 1/d^{(n)} + (m-n)/2mn]A_x\}\ .$$

The formulas above are equivalent to those derived by Gerber and Jones [12].

The increasing insurance formulas exact under UDD(1) can also be derived by "general reasoning." It is easy to come up with the formula

$$(\bar{I}\bar{A})_x = \bar{s}_{\overline{1}|}(IA)_x - (\bar{D}\bar{s})_{\overline{1}|}A_x\ .$$

By considering the area bounded by the graphs $\lceil t\rceil$ and $\lceil mt\rceil/m$, we can write

$$(I^{(m)}A)_x^{(m)} = s_{\overline{1}|}^{(m)}(IA)_x - (D^{(m)}\ddot{s})^{(m)}_{\overline{(m-1)/m}|}A_x\ .$$

If, instead, we consider the area bounded by the graphs $\lfloor t \rfloor$ and $\lceil mt \rceil/m$, then

$$(I^{(m)}A)_x^{(m)} = s_{\overline{1}|}^{(m)}[(IA)_x - A_x] + (I^{(m)}s)_{\overline{1}|}^{(m)}A_x .$$

The case 1 formula can be reasoned similarly:

$$\begin{aligned}(I^{(m)}A)_x^{(n)} &= s_{\overline{1}|}^{(n)}(IA)_x - (D^{(m)}\ddot{s})_{\overline{(m-1)/m}|}^{(n)}A_x . \\ &= s_{\overline{1}|}^{(n)}[(IA)_x - A_x] + (I^{(m)}s)_{\overline{1}|}^{(n)}A_x .\end{aligned}$$

The case 2 formula is a little tricky. Subtracting from the area bounded by the graphs $\lfloor t \rfloor$ and $\lceil nt \rceil/n$ the area bounded by the graphs $\lceil nt \rceil/n$ and $\lceil mt \rceil/m$, we have

$$(I^{(m)}A)_x^{(n)} = \{s_{\overline{1}|}^{(n)}[(IA)_x - A_x] + (I^{(n)}s)_{\overline{1}|}^{(n)}A_x\} - [(m - n)/2mn]s_{\overline{1}|}^{(n)}A_x .$$

IX. LIFE ANNUITIES

Various actuarial writers (Dowling [8], sec. 41; Davis [5], p. 19; Mereu [21], p. 89; Scher [24], p. 374; Charlton [24], p. 378; LeClair [24], p. 385; and Gerber and Jones [11]) have derived expressions exact under UDD(1) for a life annuity payable m times per year. The simplest approach was given by Butcher ([21], p. 108):

$$\ddot{a}_x^{(m)} = (1 - A_x^{(m)})/d^{(m)} = [1 - (i/i^{(m)})A_x]/d^{(m)} ;$$

then the substitution of $1 - d\ddot{a}_x$ for A_x immediately yields

$$\ddot{a}_x^{(m)} = (id\ddot{a}_x - i + i^{(m)})/i^{(m)}d^{(m)} .$$

This method was also used in [12] and [27].

The last equation can also be derived directly. Consider the identity

$$\ddot{a}_{\overline{\lceil mT \rceil/m}|}^{(m)} = [(1 + i)^{\lceil T \rceil - \lceil mT \rceil/m}d\ddot{a}_{\overline{\lceil T \rceil}|} - (1 + i)^{\lceil T \rceil - \lceil mT \rceil/m} + 1]/d^{(m)} .$$

Taking expected values and applying Corollary 2 and Lemma 1(i), we obtain

$$\ddot{a}_x^{(m)} = [(di/i^{(m)})\ddot{a}_x - (i/i^{(m)}) + 1]/d^{(m)} .$$

Formulas for temporary life annuities can be developed with the identity

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x} - {}_nE_x\, \ddot{a}^{(m)}_{x+n} .$$

Thus

$$\ddot{a}^{(m)}_{x:\overline{n}|} = [id\ddot{a}_{x:\overline{n}|} - (i - i^{(m)})(1 - {}_nE_x)]/i^{(m)}d^{(m)} .$$

Another way to derive formulas for temporary life annuities is by means of the minimum operation. For $a, b \in \boldsymbol{R}^+$, let $a \wedge b$ denote the minimum of a and b. Let $m \in \boldsymbol{Z}^+$, $n \in \boldsymbol{R}^+$, and $n \bmod 1/m = 0$; then

$$\ddot{a}^{(m)}_{x:\overline{n}|} = E(\ddot{a}^{(m)}_{\overline{\lceil m(T\wedge n)\rceil/m}|}) = \ldots .$$

Remarks.

(i) $\lceil m(T \wedge n)\rceil/m = (\lceil mT\rceil/m) \wedge n$.

(ii) $A^{(m)}_{x:\overline{n}|} = E(v^{\lceil m(T\wedge n)\rceil/m})$ (cf. [29], sec. 5) .

For $m, n, j, k \in \boldsymbol{Z}^+$, $n \bmod m = 0$ and $j \leq k$, increasing annuity formulas exact under UDD(1) can be developed by means of the identities

$$(I^{(m)}_{\overline{j}|}\ddot{a})^{(n)}_{x:\overline{k}|} = [\ddot{a}^{(m)}_{x:\overline{j}|} - (I^{(m)}_{\overline{j}|}A)^{(n)}_{x:\overline{k}|}]/d^{(n)} \qquad (*)$$

and

$$(I^{(m)}_{\overline{j}|}A)^{(n)}_{x:\overline{k}|} = (I^{(m)}A)^{(n)}_{x:\overline{j}|} + j\,{}_jE_x\, A^{(n)}_{x+j:\overline{k-j}|} .$$

Also see [12], page 45.

When $m = n$ and $j = k$, then (*) can be easily verified. Differentiating the equation

$$A^{(m)}_{x:\overline{k}|} = 1 - d^{(m)}\ddot{a}^{(m)}_{x:\overline{k}|}$$

with respect to δ, we have

$$-(I^{(m)}A)^{(m)}_{x:\overline{k}|} = v^{1/m}\ddot{a}^{(m)}_{x:\overline{k}|} + d^{(m)}(I^{(m)}a)^{(m)}_{x:\overline{k - 1/m}|}$$

$$= -\ddot{a}^{(m)}_{x:\overline{k}|} + d^{(m)}(I^{(m)}\ddot{a})^{(m)}_{x:\overline{k}|} .$$

To prove (*) for the general case, consider the annuity-certain formula

$$(I^{(m)}_{\overline{p}|}\ddot{a})^{(n)}_{\overline{q}|} = (\ddot{a}^{(m)}_{\overline{p}|} - pv^{q})/d^{(n)} ,$$

where $1/m \bmod 1/n = 0$, $p \bmod 1/m = 0$, $q \bmod 1/n = 0$, and $p \leq q$. (Such a formula can be elegantly derived by means of diagrams; see [20].) Setting $p = \lceil m(T \wedge j)\rceil/m$ and $q = \lceil n(T \wedge k)\rceil/n$ and taking expected values, we obtain (*). This proof is a refinement of the technique discussed in [10].

X. MATHEMATICAL SYMBOLISM

A common complaint concerning actuarial mathematics is the vast number and variety of its symbols. Since we are introducing more symbols here, we feel obliged to justify this by concluding our paper with some highlights on the effective use of symbols in mathematics.

The most important invention in the history of science is the system of Hindu-Arabic numerals. Although Fibonacci (Leonardo of Pisa), the greatest mathematician of the Middle Ages, published his *Liber abaci* in 1202, even in the sixteenth century only brilliant university graduates were expected to be able to master long division. (In order to cling to Roman numerals, some European countries passed laws forbidding calculations by "algorism.") A. N. Whitehead [30] wrote:

> By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race. . . . Our modern power of easy reckoning with decimal fractions is the almost miraculous result of the gradual discovery of a perfect notation. . . . Symbolism represents an analysis of the ideas of the subject and an almost pictorial representation of their relations to each other. . . . By the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain.

The most important mathematical discovery in the past three hundred years is the calculus. It was independently discovered by Newton, whose fluxion is denoted by $\dot{y}$, and Leibniz, whose notation is $\frac{dy}{dx}$. There was a long and bitter dispute over priority. Let us quote E. T. Bell ([2], p. 114): "The upshot of it all was that the obstinate British practically rotted mathematically for all of a century after Newton's death, while the more progressive Swiss and French, following the lead of Leibniz, and developing his incomparably better way of merely *writing* the calculus, perfected the subject and made it the simple, easily applied implement of research that Newton's immediate successors should have had the honor of making it." The emphasis on the word *writing* is Bell's, not ours.

Bertrand Russell suggested: "A good notation has a subtlety and suggestiveness which at times make it seem almost like a live teacher." This we found to be very true as we developed this paper. Furthermore, when we integrate by parts an integrand involving integer functions, we are immediately faced with a Stieltjes integral. Thus we are led to an interesting investigation of the applications of the Stieltjes integral in life contingencies. Since, in this paper, we wish to keep the mathematics elementary, we shall report the Stieltjes integral results separately.

For readers interested in the symbolism of mathematics we recommend Whitehead ([30], chap. 5), Hammersley in [13], and Iverson (ACM Turing Award Lecture [15]).

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DISCUSSION OF PRECEDING PAPER

C. J. NESBITT:

The author is to be congratulated for his brilliant application of integer functions to the systematic derivation of many of the formulas of actuarial mathematics, ranging from the very simple to the very complex. In doing so, he has partially bridged the gap between fully continuous and fully discrete functions by displaying the latter by integrals with integrands specified by integer functions.

The new textbook that is under preparation contains some elements of these concepts but does not develop them as systematically and as thoroughly as the presentation here. We do adhere consistently to what the author calls the UDD(1) assumption, obtaining such formulas as

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m) = \ddot{a}_{\overline{1|}}^{(m)}\ddot{a}_x - \beta(m)A_x \, , \tag{1}$$

where $\alpha(m) = s_{\overline{1|}}^{(m)}\ddot{a}_{\overline{1|}}^{(m)}$ and $\beta(m) = (s_{\overline{1|}}^{(m)} - 1)/d^{(m)}$. These formulas for $\ddot{a}_x^{(m)}$ agree with those given by the author at the beginning of Section IX. It is important to observe how well such formulas work out in relation to net annual premiums and reserves, in particular, how $P_x^{(m)}$ and ${}_tV_x^{(m)}$ relate to P_x and ${}_tV_x$. We found that the most useful premium relation was the obvious one,

$$P_x^{(m)} = P_x(\ddot{a}_x/\ddot{a}_x^{(m)}) \, , \tag{2}$$

and that by use of (2) and the right-hand member of (1) we can show that

$${}_tV_x^{(m)} = [1 + \beta(m)P_x^{(m)}]\, {}_tV_x \, . \tag{3}$$

This is to be compared with the traditional formula,

$${}_tV_x^{(m)} \doteq \left(1 + \frac{m-1}{2m} P_x^{(m)}\right) {}_tV_x \, .$$

On the basis of the UDD(m) assumption, the author indicates that

$$i^{(m)}A_x^{(m)} = \delta\bar{A}_x \, .$$

By substituting $1 - d^{(m)}\ddot{a}_x^{(m)}$ for $A_x^{(m)}$, and $1 - \delta\bar{a}_x$ for $\bar{A}_x$, and solving for $\ddot{a}_x^{(m)}$, one obtains

$$\ddot{a}_x^{(m)} = (\delta^2\bar{a}_x + i^{(m)} - \delta)/i^{(m)}d^{(m)} . \tag{4}$$

For m a positive integer, UDD(1) implies UDD(m). Hence, under UDD(1), formula (4) holds. This also follows from formula (1).

Additional difficulties appear when one considers $\ddot{a}_{xy}^{(m)}$ and $A_{xy}^{(m)}$. These can be obviated by assuming uniform distribution of termination of the joint life status, but this is different from assuming uniform distribution of death for the individual lives.

These are but a few scattered comments in appreciation of a paper remarkably rich in new concepts. We shall likely hear more of some of them, including $\lceil t \rceil$, t pad s, and UDD(m).

A. D. WILKIE:

I am very pleased to see Mr. Shiu's use of the terms *floor*, *ceiling*, *mod*, and *pad* to assist in the definition of actuarial functions. However, some principles can be used in the construction of an actuarial notation that would assist in choosing between alternative versions of some of the functions.

First, annuities and assurances are contracts that promise to pay specified sums at specified times subject to defined conditions. The symbols a and A should be used to denote the expected present values of such contracts, and the details of the contract should be clear from the notation. The symbols are more than just mathematical functions of a given set of parameters.

Second, it is convenient if an annuity value a at a zero rate of interest has the same value as the corresponding expectation of life e.

Third, it is convenient to be able to use a term-certain $\overline{n}|$ in place of a life (x) with consistency. The term-certain "expires" in exactly n years. This is the justification for using the joint life functions $a_{x:\overline{n}|}$ and $A_{x:\overline{n}|}$ in place of ${}_n a_x$ and ${}_n A_x + {}_n E_x$.

Fourth, it is consistent if $a^{(m)} \longrightarrow \bar{a}$ and $\ddot{a}^{(m)} \longrightarrow \bar{a}$ as $m \longrightarrow \infty$.

Now since $a_x = E(a_{\overline{\lfloor T \rfloor}|})$, it is consistent to set $a_{\overline{K}|} = E(a_{\overline{\lfloor K \rfloor}|}) = a_{\overline{\lfloor K \rfloor}|}$, and, similarly, $a_{\overline{K}|}^{(m)} = a^{(m)}_{\overline{\lfloor mK \rfloor / m}|}$, with no additional fractional payment. If it is now January 1, 1982, and we have promised to pay £100 on each January 1 commencing in 1983 until 5.5 years have expired, the last payment we shall make is £100 on January 1, 1987. We have made no promise to pay £50 on July 1, 1987.

Similarly, we should choose $\ddot{a}_{\overline{k}|} = \ddot{a}_{\overline{\lceil k \rceil}|}$ and $\ddot{a}^{(m)}_{\overline{k}|} = \ddot{a}^{(m)}_{\overline{\lceil mk \rceil/m}|}$, so $\ddot{a}_{\overline{5.5}|} = \ddot{a}_{\overline{6}|}$. The "life" $\overline{5.5}|$, born January 1, 1982, dies July 1, 1987, so it is still alive on January 1, 1987, and six payments are made, from 1982 through 1987.

We can then write $a_x = E(a_{\overline{T}|})$, using this new definition of $a_{\overline{T}|}$; similarly, $\ddot{a}_x = E(\ddot{a}_{\overline{T}|})$, and also $\bar{a}_x = E(\bar{a}_{\overline{T}|})$ as usual.

A complete life annuity contract $\mathring{a}^{(m)}_x$ usually (always?) actually provides a payment of T mod $1/m$ on the death of (x) at time T. The only appropriate definition of $\mathring{a}^{(m)}_x$ is therefore Shiu's definition (ii), used by Spurgeon and Jordan, viz.:

$$\mathring{a}^{(m)}_x = a^{(m)}_x + E[v^T(T \bmod 1/m)] \ .$$

It is then consistent to define $\mathring{a}^{(m)}_{\overline{k}|}$ as the present value of an annuity payable at the rate of $1/m$ for $\lfloor mk \rfloor$ payments, with a final k mod $1/m$ payable at time k. This results in

$$\mathring{a}^{(m)}_{\overline{k}|} = a^{(m)}_{\overline{\lfloor mk \rfloor/m}|} + (k \bmod 1/m)v^k \ .$$

This is equal to none of the definitions of $a^{(m)}_{\overline{k}|}$ or $\mathring{a}^{(m)}_{\overline{k}|}$ given by Shiu, but it does allow one to set $\mathring{a}^{(m)}_x = E(\mathring{a}^{(m)}_{\overline{T}|})$, consistent with Rasor and Greville but with both symbols having different meanings!

It is helpful to be clear what payments would be made under an apportionable annuity-due. What may be intended by $\ddot{a}^{\{1\}}_x$, say, is a payment of 1 at the start of each complete year of life of (x), with a final payment of T mod 1 at the start of the final fractional year of (x)'s life. How one is supposed to be able to pay this I don't know, but it is at least clearly defined in retrospect. The only consistent definition of this is Shiu's definition (iii):

$$\ddot{a}^{\{m\}}_x = \ddot{a}^{(m)}_x - E[v^{\lfloor mT \rfloor/m}(T \text{ pad } 1/m)] \ .$$

A more realistic definition is a payment of 1 at the start of each year of life (x) and a refund of T pad 1 on the death of (x) at T. We then have Shiu's definition (ii):

$$\ddot{a}^{\{m\}}_x = \ddot{a}^{(m)}_x - E[v^T(T \text{ pad } 1/m)] \ .$$

For consistency again we should define either

$$\ddot{a}^{\{m\}}_{\overline{k}|} = \ddot{a}^{(m)}_{\overline{k}|} - v^{\lfloor mk \rfloor/m}(k \text{ pad } 1/m)$$

or

$$\ddot{a}^{\{m\}}_{\overline{k}|} = \ddot{a}^{(m)}_{\overline{k}|} - v^k(k \text{ pad } 1/m) ,$$

respectively, where in each case $\ddot{a}^{(m)}_{\overline{k}|} = \ddot{a}^{(m)}_{\overline{\lceil mk \rceil/m}|}$, as defined above.

Why should we not use the symbol $\mathring{a}^{(m)}_x$ instead of $\ddot{a}^{\{m\}}_x$? Or would this imply a final additional payment of T mod $1/m$? Perhaps we should keep $\mathring{a}^{(m)}_x$ for this and use $\mathring{\ddot{a}}^{(m)}_x$ for the apportionable annuity-due. Both converge to $\bar{a}_x$ as $m \longrightarrow \infty$. At zero interest, $\mathring{e}^{(m)}_x = \mathring{\ddot{e}}^{(m)}_x = \bar{e}_x = \mathring{e}_x$, but $\dot{\mathring{e}}^{(m)}_x = \bar{e}_x + 1/m$.

The definition of $a^{(m)}_{\overline{k}|}$ chosen by Shiu and Hart, viz.,

$$a^{(m)}_{\overline{k}|} = a^{(m)}_{\overline{\lfloor mk \rfloor/m}|} + (k \bmod 1/m)v^{\lceil mk \rceil/m} ,$$

at least indicates that a final payment of k mod $1/m$ is made at time $\lceil mk \rceil/m$. The definition chosen by Donald and by Kellison, that is, $a^{(m)}_{\overline{k}|} = (1 - v^k)/i^{(m)}$, denoted $\mathring{a}^{(m)}_{\overline{k}|}$ by Shiu, does not correspond to any uniquely defined set of payments at specific times and in my view should be avoided.

The use of a term-certain $\overline{n}|$ to replace a life (x) allows some interesting notations that have not been widely used. Thus an n-year family income benefit issued at age x and providing an annual income of 1 payable monthly at the end of each policy month for n years but not while (x) is still alive (thus making the same number of payments as Shiu's and Jordan's contract, but on average half a month later) can be denoted by

$$a^{(m)}_{x|\overline{n}|} = a^{(m)}_{\overline{n}|} - a^{(m)}_{x:\overline{n}|} ,$$

comparable to the reversionary annuity to (y) after the death of (x):

$$a^{(m)}_{x|y} = a^{(m)}_{y} - a^{(m)}_{xy} .$$

The family income benefit described by Shiu, and denoted by Jordan ${}_nF_x$, has to have the rather cumbersome symbol $\hat{a}^{(m)}_{x|\overline{n}|}$, which indicates that payments are made at intervals of $1/m$ counting from the death of (x), the first being payable on the death of (x), the last shortly before the expiry of n years. If there is a final fractional payment to be made at the end of the n years, we get the top-heavy symbol $\hat{\mathring{a}}^{(m)}_{x|\overline{n}|}$, where the order in which the "coiffure" is superimposed may or may not be of significance.

A term-certain is also useful, though seldom used, for an annuity guaranteed for n years and life thereafter, viz., $a^{(m)}_{\overline{x:\overline{n|}}}$.

A notation used by King (*Life Contingencies*, 1887), but neglected since his time, is $a_{x|(y)}$, a reversionary annuity payable after the death of (x) to a nominated life who is then aged y. Clearly we do not know at the present time who (y) is. He is perhaps (x)'s successor in office.

Analogously, we can write $a_{x|(\overline{n|})}$ for an annuity that commences on the death of (x) and is payable for n years certain thereafter, being paid on the policy anniversary. Its value is $A_x\ddot{a}_{\overline{n|}}$. If it were paid on the anniversary of the death of (x), it would be denoted $\hat{a}_{x|(\overline{n|})}$, with the value $\bar{A}_x a_{\overline{n|}}$.

If we use $\overline{n|}$ as a life, we must be careful about Shiu's implication (top of p. 572) that we can ignore the case where (x) dies in exactly an integral number of years, that is, $T \in Z$. If n is integral, or $n \bmod 1/m = 0$, then $\overline{n|}$ will "expire" exactly when a payment may be due. I assume that if a life annuitant dies on a payment date, he receives the payment if the annuity is "in arrears" ($a^{(m)}_x$) but not if it is "in advance" ($\ddot{a}^{(m)}_x$). Actual life offices may adhere to this practice, or may be more generous. But it allows the definitions of $a_{\overline{n|}}$ and $\ddot{a}_{\overline{n|}}$ to be as expected.

British government securities provide a practical example of yet another type of annuity-certain for an irregular period. Each of these pays interest on defined half-yearly payment dates and is redeemable on one such date; but each is usually issued on a date that is not a payment date, and the first interest payment is for a fractional amount corresponding to the first fractional period. Such an $m$thly annuity for k years has the value

$$v^f(f + a^{(m)}_{\overline{n|}}) \ ,$$

where $f = k \bmod 1/m$ and $n = \lfloor mk \rfloor/m$, so $k = n + f$. I don't know what symbol to give this. Would $\underset{\circ}{a}^{(m)}_{\overline{k|}}$ do, indicating that the apportionment is initial, not final?

Such an annuity-certain has no practical life annuity equivalent. One could write $\underset{\circ}{a}^{(m)}_x = E(\underset{\circ}{a}^{(m)}_{\overline{T|}})$ but one cannot actually agree to make payments on the preanniversaries of the death of (x), commencing with a suitable first fractional payment, without more prescience than an insurer deserves!

(AUTHOR'S REVIEW OF DISCUSSION)

ELIAS S. W. SHIU:

I would like to thank Dr. Nesbitt and Mr. Wilkie for their discussions. Their remarks add considerably to the paper's perspective.

Dr. Nesbitt's equation (3) is an elegant formula, and the following is a useful reformulation: Let t and n be two positive integers and $t < n$; then under UDD(1)

$$ {}_t^nV^{(m)}(\ ) - {}_t^nV(\ ) = \beta(m)\,{}_nP^{(m)}(\ )\,{}_tV^{1}_{x:\overline{n}|}\,, \tag{1} $$

"where the sets of parentheses have been left blank to indicate that the benefit involved is unspecified, emphasizing that the relationship exhibited is independent of the particular benefit" (Scher [4], p. 614).

Equation (1) is derived in the same way as Dr. Nesbitt's equation (3); it follows from

$$ {}_nP(\ ) = (\ddot{a}^{(m)}_{x:\overline{n}|}/\ddot{a}_{x:\overline{n}|})\,{}_nP^{(m)}(\ ) $$

and

$$ \ddot{a}^{(m)}_{x:\overline{k}|} = \ddot{a}^{(m)}_{\overline{1}|}\ddot{a}_{x:\overline{k}|} - \beta(m)A^{\ \ 1}_{x:\overline{k}|}\,, \quad k \in \boldsymbol{Z}^{+}\,. \tag{2} $$

To prove (2), consider the identity

$$ \ddot{a}^{(m)}_{\overline{\lceil m(T\wedge k)\rceil/m}|} = (d\ddot{a}_{\overline{\lceil T\wedge k\rceil}|} + v^{\lceil T\wedge k\rceil} - v^{\lceil m(T\wedge k)\rceil/m})/d^{(m)}\,. $$

Taking expectations, we have

$$ \begin{aligned} \ddot{a}^{(m)}_{x:\overline{k}|} &= (d\ddot{a}_{x:\overline{k}|} + A_{x:\overline{k}|} - A^{(m)}_{x:\overline{k}|})/d^{(m)} \\ &= \ddot{a}^{(m)}_{\overline{1}|}\ddot{a}_{x:\overline{k}|} + (A^{\ \ 1}_{x:\overline{k}|} - A^{(m)\,1}_{x:\overline{k}|})/d^{(m)}\,. \end{aligned} $$

By UDD(1),

$$ A^{(m)\,1}_{x:\overline{k}|} = (i/i^{(m)})A^{\ \ 1}_{x:\overline{k}|}\,. $$

Thus

$$ \begin{aligned} \ddot{a}^{(m)}_{x:\overline{k}|} &= \ddot{a}^{(m)}_{\overline{1}|}\ddot{a}_{x:\overline{k}|} + A^{\ \ 1}_{x:\overline{k}|}[1 - (i/i^{(m)})]/d^{(m)} \\ &= \ddot{a}^{(m)}_{\overline{1}|}\ddot{a}_{x:\overline{k}|} - \beta(m)A^{\ \ 1}_{x:\overline{k}|}\,. \end{aligned} $$

Mr. Wilkie has provided a comprehensive discussion on the definitions of annuity symbols. However, I cannot quite agree that the definition

$$ a^{(m)}_{\overline{k}|} = (1 - v^{k})/i^{(m)}\,, \quad k \in \boldsymbol{R}^{+}\,, \tag{3} $$

should be avoided. This definition is mathematically elegant, and it is consistent with formulas such as

$$a_{\overline{k}|}^{(m)} = v^k s_{\overline{k}|}^{(m)} , \qquad \ddot{a}_{\overline{k}|}^{(m)} = (i^{(m)}/d^{(m)}) a_{\overline{k}|}^{(m)} , \qquad \frac{1}{a_{\overline{k}|}^{(m)}} = i^{(m)} + \frac{1}{s_{\overline{k}|}^{(m)}} ,$$

and so on.

It is interesting to note that Todhunter ([5], chap. 3, art. 16) gave two definitions for $a_{\overline{k}|}^{(m)}$; he suggested that formula (3) be used "for purposes of theory," while "in practice"

$$a_{\overline{k}|}^{(m)} = a_{\overline{\lfloor mk \rfloor / m}|}^{(m)} + (k \bmod 1/m) v^k .$$

With the electronic computer, yesterday's theory can become today's practice. Formulas such as (3) can be evaluated readily using a pocket calculator with an exponentiation key.

In the rest of this review we shall expand on Mr. Wilkie's remarks that a term-certain $\overline{n}|$ be used in place of a life (x). For a positive number n, let us define

$$_tp_{\overline{n}|} = \begin{cases} 1, & t < n \\ 0, & t \geq n , \end{cases}$$

and

$$_tq_{\overline{n}|} = 1 - {}_tp_{\overline{n}|} .$$

Following Jordan ([3], formulas [9.1] and [9.2]), we have

$$_tp_{x:\overline{n}|} = {}_tp_x \, {}_tp_{\overline{n}|} ,$$

and

$$_tq_{x:\overline{n}|} = 1 - {}_tp_{x:\overline{n}|} .$$

Applying the method of Stieltjes integration, we obtain

$$\bar{A}_{x:\overline{n}|} = \int_0^\infty v^t \, d \, {}_tq_{x:\overline{n}|}$$

(cf. [1], p. 223; [2], p. 2).

By Jordan's formula (11.1),

$$
\begin{aligned}
{}_tq^{1}_{x:\overline{n}|} &= \int_0^t {}_sp_{\overline{n}|}\, d\,{}_sq_x \\
&= \int_0^{t\wedge n} d\,{}_sq_x \\
&= {}_{t\wedge n}q_x \\
&= \begin{cases} {}_tq_x\,, & t < n \\ {}_nq_x\,, & t \geqslant n\,. \end{cases}
\end{aligned}
$$

Thus

$$
\bar{A}^{1}_{x:\overline{n}|} = \int_0^\infty v^t\, d\,{}_tq^{1}_{x:\overline{n}|}\,.
$$

Similarly,

$$
\begin{aligned}
{}_tq_{x:\overline{n}|}^{\;\;1} &= \int_0^t {}_sp_x\, d\,{}_tq_{\overline{n}|} \\
&= \begin{cases} 0\,, & t < n \\ {}_np_x\,, & t \geqslant n\,; \end{cases}
\end{aligned}
$$

and

$$
\bar{A}_{x:\overline{n}|}^{\;\;1} = \int_0^\infty v^t\, d\,{}_tq_{x:\overline{n}|}^{\;\;1}\,.
$$

Unlike ${}_tq_{x:\overline{n}|}$, the contingent probability ${}_tq^{1}_{x:\overline{n}|}$ as a function in t is not a cumulative distribution function. However, it has interesting applications. Consider the following problem ([3], p. 266, No. 9):

Find the value of an annuity of \$1.00 per annum payable to (y), the first payment to be made at the end of the tth year succeeding the year in which (x) dies, provided (x) dies within n years, the annuity to be void if (x) lives beyond n years.

Solution:

$$
\sum_{j=1}^{\infty} {}_jq^{1}_{x:\overline{n}|}\; {}_{j+t}E_y = {}_tE_y \sum_{j=1}^{\infty} {}_{j\wedge n}q_x\; {}_jE_{y+t}
$$

$$= {}_tE_x \sum_{j=1}^{\infty} (1 - {}_{j\wedge n}p_x)\, {}_jE_{x+t}$$

$$= {}_tE_x(a_{x+t} - a_{x:x+t:\overline{n}|} - {}_np_x\, {}_{n|}a_{x+t}) \,.$$

The symbol

$$\bar{A}^{\,2}_{z\alpha} \tag{4}$$

denotes the net single premium for an insurance of 1 payable immediately at the death of (z), provided that the status (α) fails before (z) (Jordan [3], p. 233). If the subscript α denotes the compound status $(\overset{1}{x}:\overline{n}|)$, then (4) is the net single premium for an insurance of 1 payable immediately at the death of (z), provided that (x) dies within n years and before (z). By Jordan's formula (11.31), expression (4) becomes

$$\begin{aligned} \int_0^{\infty} v^t\, {}_tq^{\,1}_{x:\overline{n}|}\, d\,{}_tq_z &= \bar{A}_z - \int_0^{\infty} v^t\, {}_{t\wedge n}p_x\, d\,{}_tq_z \\ &= \bar{A}_z - \bar{A}^{\,1}_{z:x:\overline{n}|} - {}_np_x\, {}_{n|}\bar{A}_z \,. \end{aligned} \tag{5}$$

On the other hand, if α denotes the status $(x:\overline{n}|)$, then (4) is the net single premium for an insurance of 1 payable immediately at the death of (z) provided that (x) dies within n years and before (z), or that both (x) and (z) survive for n years. Applying Jordan's formula (11.31) again, we have

$$\begin{aligned} \bar{A}^{\,2}_{z:x:\overline{n}|} &= \int_0^{\infty} v^t\, {}_tq_{x:\overline{n}|}\, d\,{}_tq_z \\ &= \bar{A}_z - \int_0^{\infty} v^t\, {}_tp_{x:\overline{n}|}\, d\,{}_tq_z \\ &= \bar{A}_z - \bar{A}^{\,1}_{z:x:\overline{n}|} \end{aligned} \tag{6}$$

(cf. [3], p. 235, first paragraph). Let us illustrate the application of formulas (5) and (6) with a question from the 1970 Part 4 Examination ([6], p. 56, No. 1):

> A family policy is issued on the lives of a man age 40, his wife age 35 and their son age 15. At issue, the policy provides \$10,000 of whole life insurance on the man, \$2,500 of endowment insurance on his wife which matures at her age 60, and \$1,000 of term insurance on the son which expires at his age 25.
>
> At the moment the first benefit (death or endowment) is paid under the policy, the amount payable under any other insurance and/or endowment benefit then in

force is doubled. At the moment the second benefit (death or endowment) is paid under the policy, the amount payable under any other benefit then in force becomes three times the original amount.

If all death benefits are payable at the moment of death, find the part of the net single premium for the policy which applies to the life insurance benefit on the man only. Express your answer in terms of pure endowments, probability symbols and continuous insurances payable at the first death.

Solution:

$$\text{N.S.P.} = 10{,}000(\bar{A}_{40} + \bar{A}^{2}_{40:s} + \bar{A}^{2}_{40:w}) ,$$

where the subscript $s = (^{1}_{15:\overline{10}|})$ and the subscript $w = (_{35:\overline{25}|})$.

Since

$$\bar{A}^{2}_{40:s} = \bar{A}_{40} - \bar{A}^{1}_{40:15:\overline{10}|} - {}_{10}p_{15}\ {}_{10|}\bar{A}_{40}$$

and

$$\bar{A}^{2}_{40:\overline{w}|} = \bar{A}_{40} - \bar{A}^{1}_{40:35:\overline{25}|} ,$$

we have

$$\text{N.S.P.} = 10{,}000(3\bar{A}_{40} - {}_{10}E_{40:15}\,\bar{A}_{50} - \bar{A}^{1}_{40:15:\overline{10}|} - \bar{A}^{1}_{40:35:\overline{25}|}) .$$

In conclusion, I wish to thank Dr. Nesbitt and Mr. Wilkie again for their interest in the paper and their stimulating discussions.

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