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# SOME MOMENT INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT

This paper shows how some moment inequalities of probability theory can be applied in three different areas of actuarial science: (1) In the context of the theory of compound interest, a simple and general algorithm for the determination of the yield rate is developed. (2) In the context of life contingencies, some new inequalities for actuarial present values and net single premiums are derived. (3) In the context of risk theory, it is shown that the exponential premium is an increasing function of the parameter.

## SOME FACTS ABOUT MOMENTS

In this section, we discuss some facts concerning the moments of a random variable (or its distribution). Quite surprisingly, these facts can be applied in three areas of actuarial science. The following sections will feature these applications.

Let $X$ be a positive random variable; to fix ideas we assume that its range is bounded, which guarantees the existence of all moments. Then the expression

$$[E(X^t)]^{1/t} \text{ is an increasing function of } t > 0. \tag{1}$$

This statement is hidden in section V.8 of Feller [2]. For completeness we give a proof. Let $0 < s < t$. Then by Jensen's inequality

$$E(X^s) = E[(X^t)^{s/t}] < [E(X^t)]^{s/t},$$

which shows that

$$[E(X^s)]^{1/s} < [E(X^t)]^{1/t}.$$

Since $s$ and $t$ are arbitrary, (1) follows.

Now let $Y$ be a positive random variable. By setting $X = e^{-Y}$ in (1), we see that

$$[E(e^{-tY})]^{1/t} \text{ is an increasing function of } t > 0. \tag{2}$$

It follows that for $s < t < u$

$$[E(e^{-sY})]^{t/s} < E(e^{-tY}) < [E(e^{-uY})]^{t/u}. \tag{3}$$

By taking logarithms, we can rewrite these inequalities as

$$s\,\frac{\ln E(e^{-tY})}{\ln E(e^{-sY})} < t < u\,\frac{\ln E(e^{-tY})}{\ln E(e^{-uY})}\,. \tag{4}$$

Because of (2), the function $[E(e^{-tY})]^{1/t}$ must have a limit for $t \to 0$. In fact, there is an explicit expression for it:

$$\lim_{t \to 0}\, [E(e^{-tY})]^{1/t} = e^{-E(Y)}\,. \tag{5}$$

To verify this formula, one uses the expansion $e^{-tY} \approx 1 - tY$ and the fact that $\lim\,(1 + tx)^{1/t} = e^x$. From (2) and (5) we gather that

$$e^{-tE(Y)} < E(e^{-tY}) \text{ for } t > 0. \tag{6}$$

This inequality can also be obtained directly from Jensen's inequality.

Now let $Z$ be a random variable of bounded range. By setting $X = e^Z$ in (1), we see that

$$[E(e^{tZ})]^{1/t} \text{ is an increasing function of } t > 0.$$

Since the logarithm is an increasing function, it follows that

$$\frac{1}{t}\ln E(e^{tZ}) \text{ is an increasing function of } t > 0. \tag{7}$$

Its limit for $t \to 0$ is particularly simple:

$$\lim_{t \to 0} \frac{1}{t} \ln E(e^{tZ}) = E(Z). \tag{8}$$

For a proof of this formula one interprets the limit as the derivative of the function $\ln E(e^{tZ})$ at $t = 0$. Thus the limit is

$$\frac{d}{dt} \ln E(e^{tZ})\,\Big|_{t=0} = E(Z\,e^{tZ}) \,/\, E(e^{tZ})\,\Big|_{t=0} = E(Z).$$

To facilitate the exposition we have tacitly assumed that $X$, $Y$, and $Z$ are random variables in the proper sense, i.e., that their distributions are not degenerate. The modifications for the case of constant "random" variables are trivial; for example, all inequalities become equalities.

## DETERMINATION OF THE YIELD RATE

We shall propose an algorithm to answer the following classical question: If a series of payments is purchased at a certain price, what is the resulting yield rate? The proposed algorithm seems new, always works, and is simple.

For a certain price $p$, an investor buys a series of $n$ positive payments. Let $q_k$ denote the payment to be received at time $y_k$ ($k = 1,2, \ldots, n$). We denote the price corresponding to the force of interest $\delta$ by $P(\delta)$. Thus

$$P(\delta) = \sum_{k=1}^{n} \exp(-\delta y_k)\, q_k. \tag{9}$$

We are looking for $t$, the solution of the equation

$$P(t) = p,$$

and the corresponding interest rate $i = e^t - 1$.

To establish the connection with the preceding section, we introduce a random variable $Y$ whose distribution is given by the formula

$$Pr(Y = y_k) = q_k/q, \tag{10}$$

where $q = q_1 + \ldots + q_n$ denotes the sum of the payments. Then (9) can be written as

$$P(\delta) = q\, E(e^{-\delta Y}),$$

and we are looking for the solution of the equation

$$E(e^{-tY}) = p/q. \tag{11}$$

From (4) it follows that for $s < t < u$

$$s\,\frac{\ln(p/q)}{\ln(P(s)/q)} < t < u\,\frac{\ln(p/q)}{\ln(P(u)/q)}\,. \tag{12}$$

Thus, if we know that the force of yield $t$ is between $s$ and $u$, we can use (12) to improve the bounds.

If we repeat this process, we obtain the following algorithm. We start with an initial value $\delta_0 > 0$ (which may be less than or greater than $t$) and compute $\delta_1, \delta_2, \ldots$ recursively according to the formula

$$\delta_{k+1} = \delta_k\,\frac{\ln(p/q)}{\ln(P(\delta_k)/q)} \text{ for } k = 0,1, \ldots . \tag{13}$$

If $\delta_0<t$, this gives us an increasing sequence that converges to $t$; if $\delta_0>t$, this produces a decreasing sequence that converges to $t$. Thus, the choice of the initial value is unimportant.

As an illustration, consider a \$1,000 par value bond that has been bought 9 years before its redemption for a price of $p$ = \$1,100. What is the resulting yield rate, if the bond has 12 percent annual coupons?

In this situation $n = 9$, $y_k = k$ $(k = 1, \ldots, 9)$, $q_1 = \ldots = q_8 = 120$, $q_9 = 1{,}120$, $q = 2{,}080$. If it is known that the yield rate of such a bond is somewhere between 10 and 10.5 percent, we may use (12) with $s = \ln 1.1$, $u = \ln 1.105$ to obtain improved bounds for $t$. We get

$$.097406 < t < .097697.$$

Since $i = e^t - 1$, the corresponding interval for the yield rate is

$$10.23\% < i < 10.26\%.$$

In order to get a precise answer, we use the algorithm that is given by formula 13. To demonstrate the power of this algorithm, we use starting values that are unreasonably high ($\delta_0 = \ln 1.2$, corresponding to $i_0 = 20\%$) or low ($\delta_0 = \ln 1.01$, corresponding to $i_0 = 1\%$). The results are displayed in table 1. In each case we find the exact answer, i.e., $t = .097549$ and $i = 10.2465\%$, in just a few steps.

TABLE 1

DETERMINATION OF THE YIELD RATE

| $k$ | $\delta_k$ | $i_k$ | $\delta_k$ | $i_k$ |
|---|---|---|---|---|
| 0........ | .009950 | .010000 | .182322 | .200000 |
| 1........ | .092519 | .096934 | .103549 | .109101 |
| 2........ | .097229 | .102112 | .097936 | .102893 |
| 3........ | .097528 | .102443 | .097574 | .102493 |
| 4........ | .097548 | .102464 | .097551 | .102467 |
| 5........ | .097549 | .102465 | .097549 | .102466 |
| 6........ | ....... | ....... | .097549 | .102465 |

If $n = 1$, which is the case for a zero-coupon bond, the random variable $Y$ is a constant, and the inequalities in (12) become equalities. Thus here the solution $t$ is found after one iteration; of course, this result can be confirmed by an elementary calculation.

## INEQUALITIES FOR ACTUARIAL PRESENT VALUES

In this second application, we shall derive some inequalities for actuarial present values (or net single premiums) in the context of life contingencies.

Let $T = T(x)$ denote the future lifetime of $(x)$. We shall consider the net single premium $\bar{A}_x$ to be a function of the underlying force of interest $\delta$. Symbolically, $\bar{A}_x = \bar{A}_x(\delta)$. Since

$$\bar{A}_x(\delta) = E(e^{-\delta T}),$$

a connection with the first section is readily established if we identify $Y$ with $T$. We conclude from (3) that for $s < t < u$

$$[\bar{A}_x(s)]^{t/s} < \bar{A}_x(t) < [\bar{A}_x(u)]^{t/u}. \tag{14}$$

From (6) we gather that

$$\exp(-t\,\mathring{e}_x) < \bar{A}_x(t). \tag{15}$$

Since (6) is a consequence of (3), (14) may be regarded as a generalization of the classical inequality (15).

Since

$$\bar{a}_x(t) = \frac{1 - \bar{A}_x(t)}{t},$$

we can derive the corresponding bounds for $\bar{a}_x(t)$. From (14) it follows that for $s < t < u$

$$\frac{1 - [\bar{A}_x(u)]^{t/u}}{t} < \bar{a}_x(t) < \frac{1 - [\bar{A}_x(s)]^{t/s}}{t}, \tag{16}$$

and from (15) we obtain the classical inequality

$$\bar{a}_x(t) < \frac{1 - \exp(-t\,\mathring{e}_x)}{t} = \bar{a}_{\overline{\mathring{e}_x}|}, \tag{17}$$

which appears in example 5.12 of Bowers et al.[1].

If we know the net single premiums for two given rates of interest, we can use (14) and (16) to get quick estimates for the net single premiums at intermediate interest rates. As an illustration, suppose we know that

$$\bar{A}_{50} = .41272 \text{ for } i = 4\%$$

$$\bar{A}_{50} = .34119 \text{ for } i = 5\% .$$

What can be said about $\bar{A}_{50}$ and $\bar{a}_{50}$, *if* $i = 4.5\%$? By setting

$$s = \ln 1.04,\ t = \ln 1.045,\ \text{and}\ u = \ln 1.05$$

in (14) and (16), we immediately discover that

$$.37039 < \bar{A}_{50} < .37904 \text{ and that}$$

$$14.107 < \bar{a}_{50} < 14.304\ .$$

By varying $Y$, we obtain from (3) a series of analogous inequalities which are displayed in table 2. Note that $K = K(x)$ denotes the curtate lifetime of $x$.

TABLE 2

INEQUALITIES FOR NET SINGLE PREMIUMS

| Choice of $Y$ | Inequality resulting from (3), assuming $s < t < u$ |
|---|---|
| $K+1$ | $[A_x(s)]^{t/s} < A_x(t) < [A_x(u)]^{t/u}$ |
| $\min(T,n)$ | $[\bar{A}_{x:\overline{n}\rvert}(s)]^{t/s} < \bar{A}_{x:\overline{n}\rvert}(t) < [\bar{A}_{x:\overline{n}\rvert}(u)]^{t/u}$ |
| $\min(K+1,n)$ | $[A_{x:\overline{n}\rvert}(s)]^{t/s} < A_{x:\overline{n}\rvert}(t) < [A_{x:\overline{n}\rvert}(u)]^{t/u}$ |
| $\begin{cases} T \text{ if } m < T < m+n \\ \infty \text{ otherwise} \end{cases}$ | $[{}_{m\|n}\bar{A}_x(s)]^{t/s} < {}_{m\|n}\bar{A}_x(t) < [{}_{m\|n}\bar{A}_x(u)]^{t/u}$ |
| $\begin{cases} K+1 \text{ if } m < T < m+n \\ \infty \quad \text{otherwise} \end{cases}$ | $[{}_{m\|n}A_x(s)]^{t/s} < {}_{m\|n}A_x(t) < [{}_{m\|n}A_x(u)]^{t/u}$ |

In the first three cases, one can use the identities

$$\ddot{a}_x(t) = \frac{1 - A_x(t)}{d}\,,$$

$$\bar{a}_{x:\overline{n}\rvert}(t) = \frac{1 - \bar{A}_{x:\overline{n}\rvert}(t)}{t}\,,\ \text{and}$$

$$\ddot{a}_{x:\overline{n}\rvert}(t) = \frac{1 - A_{x:\overline{n}\rvert}(t)}{d}\,,$$

where $d = 1 - e^{-t}$, to obtain the corresponding bounds for the annuity values.

## THE EXPONENTIAL PRINCIPLE OF PREMIUM CALCULATION

In risk theory a principle of premium calculation is a rule to determine the premium, say $\pi$, for any given risk $Z$ (a random variable) to be insured.

The two most simple examples are the variance principle and the standard deviation principle, wherein the loading $\pi - E(Z)$ is proportional to the variance and the standard deviation of $Z$, repectively. However, theoretically the most important principle is the exponential principle (see Gerber [3]) wherein

$$\pi = \frac{1}{a} \ln E(e^{aZ}). \tag{18}$$

Here the parameter $a > 0$ can be interpreted as a measure for the insurer's risk aversion.

Gerber's (1980) proof that $\pi$ is an increasing function of the parameter $a$ is unnecessarily complicated; now we see that this is a direct consequence of (7). Furthermore, we gather from (8) that $\pi \rightarrow E(Z)$ for $a \rightarrow 0$ .

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## DISCUSSION OF PRECEDING PAPER

H.J. BOOM:

Professor Gerber is once again to be complimented on a very interesting contribution. His paper vividly illustrates how a given mathematical result often will lead to surprising consequences if its variables are reinterpreted in an entirely different context.

This discussion will present two other algorithms for obtaining the yield rate involved in a stream of positive cash flows and compare them to the one presented in the paper.

### I. THE "$i$-$j$-$k$" METHOD

If an investor pays a price $p$ for a sequence of $n$ future cash payments $q_r$, to be made at times $r = 1,2, \ldots, n$, his yield $i$ is such that

$$p = \sum_{r=1}^{n} q_r (1+i)^{-r}.$$

It is well known that the investor, with the help of an auxiliary account in which deposits can be accumulated and from which withdrawals can be borrowed, both at rate $i$, can actually realize the exact yield of $ip$ at the end of each period by depositing any excess of $q_r$ over $ip$ to this auxiliary fund, or borrowing any shortage of $q_r$ below $ip$ from it; the fund will then show a balance of $p$, the exact amount originally invested, at the end of the $n$th period. This can easily be shown as follows (Donald [3], p. 89):

$$\begin{aligned}\sum_{1}^{n} (q_r - ip)(1+i)^{n-r} &= \sum_{1}^{n} q_r(1+i)^{n-r} - ips_{\overline{n}|i} \\ &= \sum_{1}^{n} q_r (1+i)^{n-r} - p\,[(1+i)^n - 1] \\ &= (1+i)^n \sum_{1}^{n} q_r(1+i)^{-r} - p(1+i)^n + p = p.\end{aligned}$$

The question arises: at what rate $j$ can the investor achieve a level income

$jp$ if the auxiliary account, operating at a rate $k \neq i$, is still required to have a balance $p$ at the end of the $n$th period? This requires

$$\sum_1^n (q_r - jp)(1+k)^{n-r} = \sum_1^n q_r(1+k)^{n-r} - pjs_{\overline{n}|k} = p, \tag{1}$$

from which we obtain

$$j = \frac{\sum_1^n q_r(1+k)^{n-r} - p}{ps_{\overline{n}|k}} \tag{A}$$

$$= k + \frac{\sum_1^n q_r(1+k)^{-r} - p}{pa_{\overline{n}|k}}. \tag{B}$$

Of course, (A) and (B) will still hold if $i = j = k$; it is easy to see that, if any two of $i$, $j$, and $k$ are equal, all three must be equal. It is therefore to be expected that, as $k \to i$, also $j \to i$.

If we now write $i_m$ for $k$ and $i_{m+1}$ for $j$ in equations (A) and (B), then repeated application of either equation, with $m = 0, 1, 2, \ldots$ successively, will generate a sequence $i_0, i_1, i_2, \ldots$ . Starting with $i_0$, a preliminary estimate for $i$, the sequence, if it converges, will approximate $i$ with any required accuracy.

If we put $q_1 = q_2 = \ldots = q_{n-1}$ and $q_n = q_1 + c$, the investment may, for $c > 0$, be considered as a bond with $n$ coupons $q_1$ and maturity value $c$, purchased for $p$, or, alternatively, for $-q_1 < c$, as a loan of amount $p$, paid off by $n-1$ level payments of $q_1$ followed by a last partial payment (if $-q_1 < c < 0$) or a balloon payment (if $c > 0$); finally, if $c = 0$, the investment is a simple level annuity. Regardless of the value of $c$, as long as $q_1 = q_2 = \ldots = q_{n-1}$, equations A and B can be simplified to:

$$j = \frac{q_1 - (p - c)s_{\overline{n}|k}^{-1}}{p} \tag{A$'$}$$

By substituting $1 - ia_{\overline{n}|i}$ for $v^n$ in the standard formula $p = q_1 a_{\overline{n}|i} + cv^n$ for the price of a straight term bond, Spoerl ([4], p. 192) obtained

$$i = \frac{q_1 - (p - c)a_{\overline{n}|i}^{-1}}{c}. \tag{2}$$

Replacing $i$ in the left-hand side with $j$ $(=i_{m+1})$ and in the right-hand side with $k$ $(=i_m)$ results in

$$j = \frac{q_1 - (p - c)a_{\overline{n}|}^{-1}{}_k}{c} \tag{B$'$}$$

as a companion formula to (A′).

Butcher and Nesbitt ([2], p. 223) obtained formula (A′) from (2) by substitution of $s_{\overline{n}|}^{-1}{}_i + i$ for $a_{\overline{n}|}^{-1}{}_i$. They also showed that, for $p - c < 0$ (discount bonds), formula (B′) generates a monotone sequence ($k < j < i$ or $k > j > i$) converging to $i$, whereas formula (A′) does so for $p - c > 0$ (premium bonds). Furthermore, they gave sufficient (but not necessary) conditions for the convergence of these sequences in the opposite case: for discount bonds the (A′) sequence converges if $c < 2p$ and for premium bonds the (B′) sequence converges if $1/2\,p < c$ (so that for $1/2\,p < c < 2p$ both sequences are always convergent, whether the bond is purchased at a premium or at a discount).

In the special case of the level annuity ($c = 0$), (B′) is not usable and (A′) will always converge.

We now compare the rate of convergence of the algorithms provided by equations (A′) and (B′) to that of Gerber's algorithm by applying them to the same example ($p = 1{,}100$, $q_1 = \ldots = q_8 = 120$, $q_9 = 1{,}120$, i.e., $c = 1{,}000$), with $i_0 = .01$ and $i_0 = .20$ as starting values. The results (we also show in each column at which stage nine-digit accuracy is attained) are exhibited in Table 1.

TABLE 1

| $m$ | Gerber's Equation 13 | | Equation A′ | | Equation B′ | |
|---|---|---|---|---|---|---|
| 0 | $i_0 = .01$ | $i_0 = .20$ | $i_0 = .01$ | $i_0 = .20$ | $i_0 = .01$ | $i_0 = .20$ |
| 1 | .096934 | .109101 | .099387 | .104710 | .108326 | .095192 |
| 2 | .102112 | .102893 | .102379 | .102528 | .102057 | .102966 |
| 3 | .102443 | .102493 | .102463 | .102467 | .102494 | .102431 |
| 4 | .102464 | .102467 | .102465 | .102465 | .102463 | .102468 |
| 5 | .102465 | .102466 | ............ | ............ | .102466 | .102465 |
| 6 | ............ | .102465 | .102465421 | .102465421 | .102465 | ............ |
| 7 | ............ | ............ | ............ | ............ | ............ | ............ |
| 8 | .102465421 | .102465421 | ............ | ............ | .102465421 | .102465421 |

## II. EXTENSION OF WANG'S METHOD FOR ANNUITIES

Wang ([5], p. 235) presents a Newton-Raphson approach to obtain the yield rate for a level annuity, i.e., for the special case mentioned previously, with $q_1 = q_2 = \ldots = q_n$. We adapt his procedure to the more general case $q_1 = q_2 = \ldots = q_{n-1} \neq q_n$. Then $p = (q_n - q_1)(1+i)^{-n} + q_1 a_{\overline{n}|i}$. If we let $p/q_1 = a$, $(q_n - q_1)/q_1 = c$ and $1 + i = x$, this becomes

$$a - cx^{-n} - \frac{1-x^{-n}}{x-1} = 0.$$

We multiply by $x - 1$ and put

$$g(x) \equiv -cx^{-n+1} + (c+1)x^{-n} + ax - a - 1,$$

so we have to solve $g(x) = 0$, for which we use the Newton-Raphson algorithm (Burden and Faires [1], pp. 42–43) given by

$$x_{m+1} = x_m - \frac{g(x_m)}{g'(x_m)}. \tag{3}$$

Differentiating:

$$g'(x) = (n-1)cx^{-n} - n(c+1)x^{n-1} + a,$$

so that (3) becomes

$$x_{m+1} = x_m + \frac{cx_m^{-n+1} - (c+1)x_m^{-n} - ax_m + a + 1}{(n-1)cx^{-n} - n(c+1)x_m^{-n-1} + a}. \tag{C$'$}$$

(Note that putting $c = 0$ will result in Wang's original algorithm for the level annuity.)

We now compare the results of (C′) with those of Gerber's equation 13, again for the same example, in Table 2.

TABLE 2

| $m$ | Equation 13 | | Equation C′ | |
|---|---|---|---|---|
| 0 | $i_0 = .01$ | $i_0 = .2$ | $i_0 = .01$ | $i_0 = .2$ |
| 1 | .096934 | .109101 | .021853 | .117362 |
| 2 | .102112 | .102893 | .056738 | .103450 |
| 3 | .102443 | .102493 | .173448 | .103356 |
| 4 | .102464 | .102467 | .112911 | .102470 |
| 5 | .102465 | .102466 | .102989 | .102469 |
| 6 | ................ | .102465 | .102467 | .102465421 |
| 7 | ................ | ................ | .102465421 | ................ |
| 8 | .102465421 | .102465421 | ................ | ................ |

It is remarkable that the Newton-Raphson method, which converges quadratically, does not become accurate to six significant digits any earlier than the other methods considered here; however, when it does reach this stage, it is actually accurate to (at least ) nine digits!

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GRAHAM LORD:

Mr. Gerber has provided an elegant adaptation of inequalities from probability theory to various actuarial problems. His principal result, that $[E(e^{-tT})]^{1/t}$ is an increasing function of $t > 0$, has applicability to the calculation of internal rates of return, to the comparison of net single premiums, and to the development of properties satisfied by premiums determined by the exponential principle as it is defined in Risk Theory. The latter application shows the elegance of the inequality by furnishing an immediate proof that $\pi(a)$ is increasing in $a$.

As a practical tool, the method does have value as evidenced by the net single premium dominance relationships summarized in Table 2. The bounds on a premium for an unusual though constant interest rate are direct and simple to calculate. However, when the inequalities and their derivative methodologies are applied iteratively, it appears that they are globally less efficient than standard algorithms.

Take the example of the \$1,000 par value bond bought at a price of \$1,100 and paying nine annual coupons of 12 percent. The determination of the yield correct to six decimal places by the recursive formula 13 requires five or six iterations depending on the starting value. (See Table 1.) But if instead of (13), Newton-Raphson is used, without any enhancements such as localization to improve efficiency, convergence to the same yield from the same starting values is obtained in one fewer iteration. Here the difference in the number of steps taken by the two methods is negligible. If, however, it is necessary to know the yield to double the precision, the difference between formula 13 and Newton-Raphson becomes significant: with the same starting values as before, the latter converges in five iterations—less than half the number of steps required by formula 13. The reason for this marked difference in speed is the fact that the convergence of the inequality-derived method, formula 13, is of first-order or linear (as can be shown by elementary algebra), whereas that of Newton-Raphson is of second-order. (See, for example, Stephen G. Kellison's *Fundamentals of Numerical Analysis*, R.D. Irwin, 1975, pages 254–55.) Though the difference in the orders of convergence implies the overall superiority of Newton-Raphson, there are instances when (13) will perform more efficiently. An easy example is the case of a zero-coupon bond.

ERIC S. SEAH:

I would like to focus my discussion on the algorithm given in the paper (which will be referred to as Gerber's algorithm) for determining the yield rate of a stream of positive payments at a given price. We present recursive *APL* programs for both Gerber's and Newton-Raphson's algorithms, and compare the number of iterations required for some typical situations using various initial estimates.

We will follow the notation of the paper. Newton-Raphson's algorithm calls for the generating of a sequence $\{\delta_i | i = 0,1,\ldots\}$, with the iteration formula $\delta_{i+1} = \delta_i - f(\delta_i)/f'(\delta_i)$, where $f(x) = p - P(x)$. For a detailed description of this algorithm, refer to [2]. It is well known that the sequence $\{\delta_i\}$ of Newton-Raphson's algorithm is quadratically convergent ([1], pp. 52–56). Gerber's algorithm is linearly convergent with asymptotic constant given by the following formula:

$$\lim_{i \to \infty} | \delta_{i+1} - \delta | / | \delta_i - \delta | = 1 - \delta \cdot [P'(\delta)/P(\delta)]/\mathrm{Log}_e(p/q).$$

Thus, in general, a sequence generated by Newton-Raphson's algorithm would be expected to converge more rapidly than the one generated by Gerber's algorithm. However, there are situations where Gerber's algorithm outperforms Newton-Raphson's algorithm.

Recursive *APL* programs for the two algorithms are listed in the appendix of this discussion. For simplicity, we assume the time unit for the streams of payments runs from 0 in increments of integral value 1. For both algorithms, we stop the iteration process when the difference of the two consecutive iterated values is less than $10^{-6}$. Since $q_k \geq 0$, it follows that $f'(x) > 0$. Thus $\delta_i$ is defined for all $i$ under Newton-Raphson's algorithm. Gerber's algorithm can always be applied, except when $\delta_0 = 0$.

The following tables give the number of iterations for three streams of cash flows: coupon bond, level annuity, and zero-coupon bond. The first example is identical to Gerber's. Various $\delta_0$'s are used.

EXAMPLE 1

COUPON BOND
$y_k \leftarrow$ 1 2 3 4 5 6 7 8 9
$q_k \leftarrow$120 120 120 120 120 120 120 120 1120
$q \leftarrow$ 2080
$p \leftarrow$ 1100
$\delta$ is 0.0975489

| $\delta_0$ | NUMBER OF ITERATIONS | |
|---|---|---|
| | Newton-Raphson's | Gerber's |
| − 10.000000 | 95 | 6 |
| − 1.000000 | 14 | 6 |
| 0.009950 | 5 | 6 |
| 0.100000 | 3 | 4 |
| 0.182322 | 5 | 6 |
| 0.693147 | 34 | 6 |
| 1.000000 | 80 | 6 |

EXAMPLE 2

LEVEL ANNUITY
$y_k \leftarrow$ 1 2 3 4 5 6 7 8 9 10 11 12
$q_k \leftarrow$ 250 250 250 250 250 250 250 250 250 250 250 250
$q \leftarrow$ 3000
$p \leftarrow$ 1400
$\delta$ is 0.133175

| $\delta_0$ | NUMBER OF ITERATIONS | |
|---|---|---|
| | Newton-Raphson's | Gerber's |
| − 10.000000 | 124 | 8 |
| − 1.000000 | 16 | 8 |
| 0.100000 | 4 | 7 |
| 0.126151 | 3 | 6 |
| 0.200000 | 5 | 7 |
| 0.693147 | 24 | 8 |
| 1.000000 | 58 | 8 |

EXAMPLE 3

| |
|---|
| ZERO-COUPON BOND |
| $y_k \leftarrow 1\ 2\ 3\ 4\ 5\ 6$ |
| $q_k \leftarrow 0\ 0\ 0\ 0\ 0\ 1700$ |
| $q \leftarrow 1700$ |
| $p \leftarrow 1000$ |
| $\delta$ is 0.088438 |

| $\delta_0$ | Number of Iterations | |
|---|---|---|
| | Newton-Raphson's | Gerber's |
| −10.000000 | 65 | 2 |
| −1.000000 | 11 | 2 |
| 0.100000 | 3 | 2 |
| 0.200000 | 5 | 2 |
| 0.693147 | 39 | 2 |
| 0.900000 | 130 | 2 |

Judging from these results, we would conclude that Gerber's algorithm works quite well. Although Newton-Raphson's algorithm in general requires fewer iterations when $\delta_0$ is close to $\delta$, it nevertheless requires a lot more iterations when $\delta_0$ is far away from $\delta$ (refer to all three examples). This is because, when $\delta_i$ is negative and far away from $\delta$, $|f'(\delta_i)| >> |f(\delta_i)|$, thus causing $\delta_{i+1}$ to inch slowly towards $\delta$ (if we start with $\delta_0 >> \delta$, $\delta_1$ will be $< 0$). The following sequence of iterated values using Newton-Raphson's algorithm (Example 1 with $\delta_0 = -1.0$) illustrates this phenomenon:

| $i$ | $\delta_i$ | $i$ | $\delta_i$ |
|---|---|---|---|
| 0 .................. | −1.0000000 | 8 .................. | −0.0956956 |
| 1 .................. | −0.8877458 | 9 .................. | 0.0018087 |
| 2 .................. | −0.7751942 | 10 .................. | 0.0690339 |
| 3 .................. | −0.6622467 | 11 .................. | 0.0946412 |
| 4 .................. | −0.5487865 | 12 .................. | 0.0975171 |
| 5 .................. | −0.4347253 | 13 .................. | 0.0975490 |
| 6 .................. | −0.3201847 | 14 .................. | 0.0975490 |
| 7 .................. | −0.2060739 | .................. | .................. |

These observations suggest we could use Gerber's algorithm (or alternatively the Bisection algorithm) to obtain a reasonably good approximation, and then employ Newton-Raphson's algorithm to get to the final answer. As an illustration, we perform three iterations of Gerber's algorithm for Example 2 with $\delta_0 = -1.0$, and get $\delta_3 = 0.132246$. Now applying Newton-Raphson's algorithm with $\delta_0 = 0.132246$, we obtain $\delta_3 = 0.133175$. Thus, a total of six iterations are required, as compared to eight and sixteen iterations under the Gerber's and Newton-Raphson's algorithms, respectively (refer to the table for Example 2).

In [3], Silver studied the computation of the interest rate $i$ of a $n$-year level annuity and derived the following inequality:

$$i \geq [1 - (a_{\overline{n}|}/n)^2] / [a_{\overline{n}|} + (a_{\overline{n}|}/n)^2],$$

where $a_{\overline{n}|}$ is the price for a stream of $n$ annuity payments of 1, with the first payment commencing one year after the price is paid. It was shown that, under normal situations ($0.01 \leq i \leq 0.21$ and $10 \leq n \leq 60$), this lower bound for $i$ proves to be an extremely good initial approximation for Newton-Raphson's algorithm. Using Example 2 again, we have $i = 0.1344538$ (hence $\delta_0 = 0.1261513$), and Newton-Raphson's algorithm converges after three iterations (refer to the table for Example 2). Thus, Newton-Raphson's algorithm, using the lower bound above as the initial approximation, appears to be a more efficient way of determining the yield rate of level annuities.

One could also compare the number of operations required for each iteration under Gerber's and Newton-Raphson's algorithms. In this respect, they are quite comparable. Newton-Raphson's algorithm involves evaluating $f(x)$ and $f'(x)$, and Gerber's algorithm requires calculating $P(x)$ and the logarithm of $P(x)/q$ (the logarithm, in general, is approximated by some minimax polynomial on the computing machines).

In conclusion, while Gerber's algorithm works rather well with the problem of finding the yield rate of a stream of positive payments at a given price, Newton-Raphson's algorithm can be applied to more general situations involving negative cash flows.

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## APPENDIX

1. Gerber's algorithm:
   - GERBER△MAIN is the main program which calls recursive subprogram GERBER to produce the sequence of iterated values based on the initial estimate XO.
   - Lines [1] and [3] of GERBER△MAIN define the stream of positive payments (*QK*) and the given price (*PVAL*). Example 1 is being shown here.
   - Function *P* gives the present value of the stream of positive payments using the yield rate *X*.

```
      ∇ GERBER∆MAIN X0:C
[1]     QK←(8⍴120),1120
[2]     YK←⍳⍴QK
[3]     PVAL←1100
[4]     Q←+/QK
[5]     C←⍟PVAL÷Q
[6]     TOL←1E¯6
[7]     GERBER X0
      ∇
```

```
      ∇ Z←GERBER X
[1]     →(TOL>|X-Z←X×C÷⍟(P X)÷Q)/0
[2]     Z
[3]     Z←GERBER Z
      ∇
```

```
      ∇ Z←P X
[1]     Z←+/QK××-YK×X
      ∇
```

2. Newton-Raphson' algorithm:
   - NEWTON∆MAIN is the main program which calls recursive subprogram NEWTON to produce the sequence of iterated values based on the initial estimate XO.
   - Lines [1] and [3] of NEWTON∆MAIN define the stream of positive payments ($QK$) and the given price ($PVAL$). Example 1 is being shown here.
   - Function $F$ calculates the value $f(X) = PVAL - P(X)$ for a given yield rate $X$.
   - Function $FP$ returns the first derivative of $f(X)$ for a given yield rate $X$.

```
      ∇ NEWTON∆MAIN X0
[1]   QK←(8⍴120),1120
[2]   YK←⍳⍴QK
[3]   PVAL←1100
[4]   Q←+/QK
[5]   TOL←1E¯6
[6]   NEWTON X0
    ∇
```

```
      ∇ Z←NEWTON X
[1]   →(TOL>|X-Z←X-(F X)÷FP X)/0
[2]   Z
[3]   Z←NEWTON Z
    ∇
```

```
      ∇ Z←F X
[1]   Z←PVAL-P X
    ∇
```

```
      ∇ Z←FP X
[1]   Z←+/QK×YK×*-YK×X
    ∇
```

MARK D. EVANS:

Professor Gerber has developed some interesting new actuarial tools. Perhaps a few comments concerning the determination of yield rates will be helpful.

Comparing the iteration method utilizing formula 13 to results obtained by a modification of the secant method permits some interesting observations. This modification of the secant method uses the previous two values of $\delta_k$, which minimizes the absolute value of $p - P(\delta_k)$. This suggests combining the two methods: using the moment formula for two iterations and then switching to the modified secant method. The results are shown in the following table, which is based upon Table 1 in the paper.

| $k$ | Moment $i_k$ | Modified Secant $i_k$ | Combined $i_k$ |
|---|---|---|---|
| 0 | .200000 | .01 and .20 | |
| 1 | .109101 | .143598 | |
| 2 | .102893 | .078828 | .1029 and .1091 |
| 3 | .102493 | .078220 | .102451 |
| 4 | .102467 | .102991 | .102465 |
| 5 | .102466 | .102453 | |
| 6 | .102465 | .102465 | |

The moment method works well at getting within a few basis points of the yield but has difficulty when it is very close to the solution. Switching to the modified secant method after two iterations solves this difficulty. This is because the moment method gives successive iterations with errors decreasing by roughly constant magnitude while the modified secant method, being exact for linear equations, has an error roughly related to $(\delta - \delta_k)^2$ of Taylor's formula, permitting rapid convergence near the root.

Frequently one must determine yield rates from cash flows where some future cash flows are positive and others are negative. Consider a series of payments purchased for $p = 1{,}000$ where $q_1 = \ldots = q_5 = 420$ and $q_6 = \ldots = q_{10} = -200$. This series of payments has two yield rates. Given starting points of $-.05$ and 0, the modified secant method obtains a yield of $-.037223$ after five iterations, while the moment method requires fourteen iterations from a starting interest rate of $-.05$. Given starting points of .20 and .25, the modified secant method obtains a yield of .212646 after four iterations, while the moment method diverges, given a starting interest rate of .20.

The poor performance of the moment method when negative payments are involved is consistent with Professor Gerber's condition that $Y$ must be positive. The importance of this condition is emphasized by considering the following limit:

$$\lim_{q \to 0} \frac{\ln (p/q)}{\ln [P(\delta_k)/q]}.$$

HO KUEN NG:

Professor Gerber has given an interesting and simple algorithm for the yield rate of a series of payments. We will look at it from another perspective.

Although the algorithm is derived using moment inequalities, it belongs to the wide class of fixed-point algorithms. In this discussion we will disregard the case when the yield $t$ is 0. (Note that in the case $t = 0$, the algorithm gives $\delta_1 = 0$.) With the notations used in the paper, and letting $g(x) = x \dfrac{\ln(p/q)}{\ln[P(x)/q]}$, we have $P(t) = p$ if and only if $g(t) = t$. Thus, the theory of fixed-point algorithms suggests that we try equation 13. This turns out to be a good choice among all possible fixed-point functions because of its convergence properties and simplicity.

Next we consider the convergence properties of the proposed algorithm.

With $g(x) = x \dfrac{\ln(p/q)}{\ln[P(x)/q]}$, we obtain

$$g'(x) = \ln(p/q) \left( \frac{1}{\ln[P(x)/q]} - \frac{x[P'(x)/P(x)]}{\{\ln[P(x)/q]\}^2} \right).$$

Note that $P'(t) = -\sum \exp(-ty_k)\, q_k y_k < 0$, and $\ln(q/p) > 0$. Thus $g'(t) < 1$.

On the other hand, (4) shows that if $s < t < u$, then $g(s) < t < g(u)$, that is, $g(s) < g(t) < g(u)$. With this and the differentiability of $g$ except at 0, checking the left- or right-hand derivative of $g$ at $t$ gives $g'(t) \geq 0$.

Since $0 \leq g(t) < 1$, we see that $t$ is a so-called point of attraction.

In fact, the algorithm does even better than predicted by the theory of fixed-point algorithms. Professor Gerber's algorithm is of a nonlocal nature. Since $P(x)$ is continuous, the equation $P(x) = p$ has a unique positive solution. Starting with any point $\delta_0 \neq 0$, the algorithm must converge to the yield $t$. However, this derivation gives some interesting results.

We observe that $-P'(t)/P(t)$ is the duration $D$ of the series of payments. It follows that $D \leq \ln(q/p)/t$. Also, the rate of convergence of the algorithm is linear, except in the case $D = \ln(q/p)/t$, when the rate will be faster. Finally, in the linear convergence case, since $0 < g'(t) < 1$, procedures such as the Aitken's $\Delta^2$-process can be used to speed up the convergence to the yield rate.

ELIAS S.W. SHIU:

Dr. Gerber is to be congratulated for another elegant mathematical paper. I would like to present an alternative derivation of (1) by means of Hölder's Inequality, which states that

$$\int_\Omega |fg| \, d\mu \leq \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^{p'} \, d\mu\right)^{1/p'},$$

where

$$p > 1$$

and

$$p' = p/(p-1).$$

For $0 < s < t$, consider

$$f = |h|^s,$$

$$g \equiv 1$$

and

$$p = t/s.$$

By Hölder's Inequality,

$$\begin{aligned}\int_\Omega |h|^s d\mu &\leq \left(\int_\Omega |h|^t \, d\mu\right)^{1/p} \left(\int_\Omega 1 \, d\mu\right)^{1/p'} \\ &= \left(\int_\Omega |h|^t \, d\mu\right)^{s/t} [\mu(\Omega)]^{(t-s)/t}.\end{aligned}$$

Thus, for $0 < s < t$,

$$\left(\int_\Omega |h|^s \, d\mu\right)^{1/s} \leq \left(\int_\Omega |h|^t \, d\mu\right)^{1/t} [\mu(\Omega)]^{(1/s - 1/t)},$$

which implies that, for a nonnegative random variable $X$,

$$[E(X^s)]^{1/s} \leq [E(X^t)]^{1/t}.$$

In fact, the last inequality holds as long as $s \le t$. It is not necessary to assume that they are both positive numbers ([10], p. 455; [1], section 16). An immediate consequence of this result is the ordering among the harmonic mean, geometric mean, and arithmetic mean:

$$[E(X^{-1})]^{-1} \le e^{E[\ln(X)]} \le E(X).$$

For a nonnegative random variable $X$ and $0 < s < t$, the inequality

$$[E(X^s)]^{1/s} \le [E(X^t)]^{1/t}$$

implies that $[E(X^s)]^{t/s} \le E(X^t)$ and $E(X^s) \le [E(X^t)]^{s/t}$. It is possible to obtain sharper inequalities. For instance, given $0 < a < b$ and $r \ge 1$, we have

$$\begin{aligned} b^r &= [a + (b - a)]^r \\ &\ge a^r + (b - a)^r, \end{aligned}$$

which is a sharper bound than $b^r \ge a^r$. Tong ([12]; [13], Lemma 2.3.1) has applied this observation to obtain the following moment inequality. For a nonnegative random variable $X$ and a real number $k \ge 2$,

$$E(X^k) \ge [E(X)]^k + [\mathrm{Var}(X)]^{k/2}.$$

Instead of comparing $[E(X^s)]^{1/s}$ with $[E(X^t)]^{1/t}$, we may consider comparing $\phi^{-1}(E[\phi(X)])$ with $\Psi^{-1}(E[\Psi(X)])$, where $\phi$ and $\Psi$ are strictly-monotonic continuous functions. Define

$$\lambda(x) = \phi[\Psi^{-1}(x)]$$

and

$$V = \Psi(X).$$

Without loss of generality, assume that $\phi$ is an increasing function. Then

$$\phi^{-1}(E[\phi(X)]) \ge \Psi^{-1}(E[\Psi(X)]) \qquad \text{(D.1)}$$

if and only if

$$E[\lambda(V)] \ge \lambda[E(V)],$$

which holds if $\lambda$ is a convex function. If (D.1) holds for each random variable $X$, then the function $\lambda$ is necessarily a convex function ([6], p. 70).

The approximation

$$e^{-tY} \approx 1 - tY$$

is used in the proof of Dr. Gerber's equation (5). For this approximation to be valid, we need the random variable $Y$ to be of bounded range. It is possible to prove equation (5) without this assumption of bounded range, but the proof is quite complex ([6], p. 139).

Motivated by the development in the paper, let me now present an application of Jensen's Inequality to the theory of immunization. Using the notation in the paper, let $\{q_k\}$ be a stream of cash flows to occur at times $\{y_k\}$. Given a force of interest $\delta$, the value of the cash flows evaluated at time $\tau$ is

$$V(\delta;\tau) = \sum \exp[\delta(\tau - y_k)]q_k.$$

If the force of interest changes from $\delta$ to $\delta + \epsilon$, then the value of cash flows changes to

$$\begin{aligned} V(\delta + \epsilon; \tau) &= \sum \exp[(\delta + \epsilon)(\tau - y_k)]q_k \\ &= \sum \exp[\epsilon(\tau - y_k)]\exp[\delta(\tau - y_k)]q_k. \end{aligned}$$

Note that the function

$$f(y) = \exp[\epsilon(\tau - y)]$$

is a convex function. Introduce a random variable $W$ whose distribution is given by

$$\begin{aligned} Pr(W = y_k) &= \exp[\delta(\tau - y_k)]q_k/V(\delta; \tau) \\ &= \exp[\delta(-y_k)]q_k/V(\delta; 0). \end{aligned} \tag{D.2}$$

By Jensen's Inequality

$$E[f(W)] \geq f[E(W)].$$

It is easy to check that

$$E[f(W)] = V(\delta + \epsilon; \tau)/V(\delta; \tau)$$

Also,

$$f[E(W)] = 1$$

if

$$\tau = E(W).$$

Thus, if $\tau$ is the Macaulay-Redington duration of the stream of positive cash flows $\{q_k\}$,

$$V(\delta + \epsilon; \tau) \geq V(\delta; \tau).$$

This result can easily be generalized to the case where yield curves are not flat. Consider

$$V(\delta; \tau) = \sum \exp[\int_{y_k}^{\tau} \delta(t)\, dt] q_k,$$

$$V(\delta + \epsilon; \tau) = \sum \exp\{\int_{y_k}^{\tau} [\delta(t) + \epsilon(t)]\, dt\} q_k$$

and

$$f(y) = \exp[\int_{y_k}^{\tau} \epsilon(t)\, dt].$$

Let the $\tau$ be the duration of the cash flows $\{q_k\}$ computed with the force-of-interest function $\delta(t)$. Since

$$f''(y) = f(y)\{[\epsilon(y)]^2 - \epsilon'(y)\},$$

we have

$$V(\delta + \epsilon; \tau) \geq V(\delta; \tau)$$

if

$$[\epsilon(y)]^2 \geq \epsilon'(y)$$

for all positive $y$, and

$$V(\delta + \epsilon; \tau) \leq V(\delta; \tau)$$

if

$$[\epsilon(y)]^2 \leq \epsilon'(y)$$

for all positive $y$.

This generalizes a theorem of Fisher and Weil [3]. This generalization can also be derived by the Mean Value Theorem; see [11].

My final comment is motivated by the result

$$\frac{d}{dt} \ln(e^{tZ})|_{t=0} = E(Z)$$

and an article by C.L. Trowbridge [14]. Put

$$E(Z) = \mu.$$

Let

$$E[e^{t(Z-\mu)}] = 1 + \mu_2 t^2/2 + \mu_3 t^3/3! + \mu_4 t^4/4! + \dots$$

and

$$\ln[E(e^{tZ})] = \kappa_1 t + \kappa_2 t^2/2 + \kappa_3 t^3/3! + \kappa_4 t^4/4! + \dots$$

The cumulants and the moments about the mean are related by the formulas ([9], p. 73):

$$\begin{aligned}
\kappa_1 &= \mu, \\
\kappa_2 &= \mu_2, \\
\kappa_3 &= \mu_3, \\
\kappa_4 &= \mu_4 - 3(\mu_2)^2, \text{ and so on.}
\end{aligned}$$

Let $W$ be the random variable defined by (D.2). Thus,

$$V(\delta + \epsilon; 0)/V(\delta; 0) = E(e^{-\epsilon W}).$$

Let $\tau = E(W)$ be the duration of the cash flows computed with $\delta$. Then the relation

$$\mu = \kappa_1$$

means that

$$\begin{aligned}
\tau &= -\frac{d}{d\epsilon}\ln[V(\delta + \epsilon; 0)/V(\delta; 0)]\,\Big|_{\epsilon=0} \\
&= -\frac{d}{d\epsilon}\ln[V(\delta + \epsilon; 0)]\,\Big|_{\epsilon=0} \\
&= -\frac{d}{d\delta}\ln[V(\delta; 0)] \qquad\qquad (\text{D.3}) \\
&\left( = s - \frac{d}{d\delta}\ln[V(\delta; s)] \quad \text{for each real number } s \right).
\end{aligned}$$

Expression D.3 explains why Sir John Hicks ([7], p. 186) called $\tau$ the *elasticity* of the capital value with respect to the discount ratio $e^{-\delta}$. Also, L. Fisher [2], then unaware of Hicks's elasticity, had shown that the Macaulay-Redington duration had the properties of an elasticity.

Following [4], let us denote the variance of $W$ by $M^2$, i.e.,

$$M^2 = \sum (y_k - \tau)^2 \, Pr(W = y_k).$$

Then, the equations $\kappa_2 = \mu_2$ and $\kappa_3 = \mu_3$ immediately yield the two formulas pointed out by Trowbridge [14]:

$$\frac{d}{d\delta}\tau = -M^2 \tag{D.4}$$

and

$$\frac{d}{d\delta}M^2 = -\sum (y_k - \tau)^3 \, Pr(W = y_k).$$

Formula D.4 is known in the immunization literature ([5], p. 40; [8], pp. 148–49).

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## (AUTHOR'S REVIEW OF DISCUSSION)

HANS U. GERBER:

I would like to thank the discussants for their comments, which I enjoyed. Most of these comments concern the first application (determination of the yield rate). Alternatives to the algorithm that is given by formula 13 are the Newton-Raphson method (as pointed out by Mr. Boom, Mr. Lord, and Mr. Seah), the secant method (suggested by Mr. Evans) and the most intriguing "*i-j-k*" method (described by Mr. Boom).

As pointed out by Mr. Evans and Mr. Seah, the assumption of *positive* payments is crucial. If negative payments are admitted, the equation $P(t) = p$ might have several solutions, and not all of them can be obtained from applying formula 13. However, the notion of a yield rate may not be that meaningful in such a situation.

The algorithm that is based on formula 13 can be interpreted as a fixed-point algorithm (see the discussions of Mr. Ng and Mr. Seah), which leads to first order convergence (also observed by Mr. Evans and Mr. Lord). Alternatively, the algorithm has the following two attractive interpretations:

For the first interpretation, we introduce the function

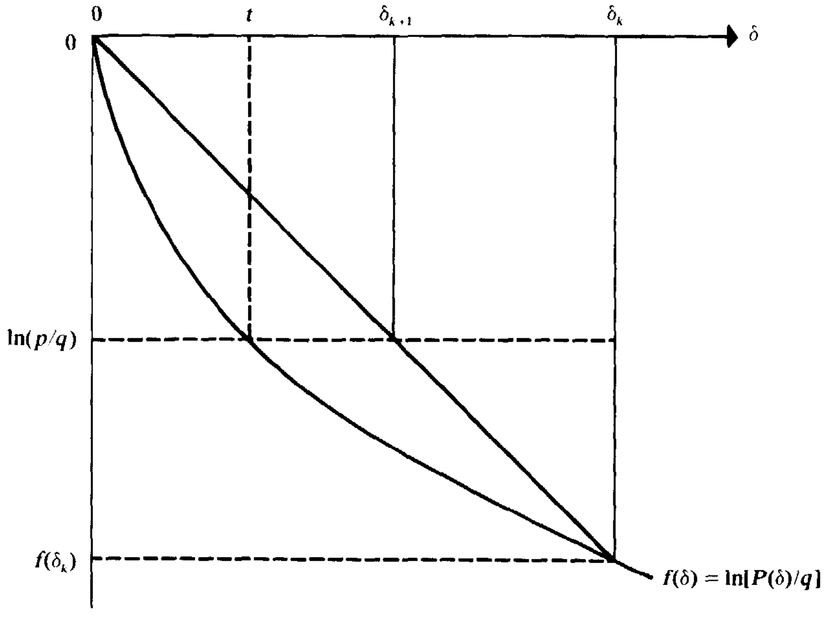
$$f(\delta) = \ln[P(\delta)/q].$$

Note that $f(0) = 0$, $f'(\delta) = P'(\delta)/P(\delta)$ is negative, and that

$$f''(\delta) = P''(\delta)/P(\delta) - [P'(\delta)/P(\delta)]^2$$

can be interpreted as a variance and is therefore positive. We seek $t$, the solution of the equation $f(t) = \ln(p/q)$. Given an approximation $\delta_k$, we use the secant between the origin and the point with coordinates $\delta_k$ and $f(\delta_k)$ to determine a new approximation $\delta_{k+1}$. This procedure is illustrated in the following figure and leads to formula 13.

INTERPRETATION OF FORMULA 13




For the second interpretation we assume an approximation $\delta_k$. Then the idea is to replace the payments $q_1, \ldots, q_n$ by a single payment of $q$ at time $\tau_k$ that is equivalent under the force of interest $\delta_k$. Thus $\tau_k$ is determined from the condition that

$$\exp(-\delta_k \tau_k)\, q = P(\delta_k),$$

which leads to

$$\tau_k = -f(\delta_k)/\delta_k.$$

Now $\delta_{k+1}$ is determined such that $p$ is the present value of the payment of $q$ at time $\tau_k$:

$$p = \exp(-\delta_{k+1} \tau_k)\, q.$$

This gives

$$\delta_{k+1} = -\ln(p/q)/\tau_k = \delta_k \ln(p/q)/f(\delta_k),$$

which is formula 13.

Recently I learned that essentially this derivation has been given by Jaumain [2] and that the resulting algorithm has been explored by De Vylder [1]. Thus I propose that the algorithm that is based on formula 13 be called Jaumain's algorithm.

I would like to add some comments about the random variable $W$ whose distribution is given in formula D.2 of Mr. Shiu's discussion. This distribution can be interpreted as the Esscher transform of the distribution that is defined in formula 10. There is a connection with the function $f$ that has been introduced here: the expectation of $W$ is $-f'(\delta)$, and its variance is $f''(\delta)$.

As is indicatd by the comments of Mr. Shiu, there are models and problems of finance that go beyond the classical theory of compound interest. It is important that actuaries examine and understand these new models, so they do not have to rely on the judgment of others. I look forward to hearing more from Mr. Shiu about this topic.

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