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INDIVIDUAL VERSUS AGGREGATE APPROACH TO FUNDING BENEFITS—AN ILLUSTRATION

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ABSTRACT

Several methods are available for determining the appropriate total funding level for death and retirement benefits provided for a group of individuals. One general approach is to determine the appropriate level for each individual using the traditional equivalence principle (that is, at issue, the actuarial present value of benefits must equal the actuarial present value of net premiums/contributions) and then summing to obtain the total funding level for the group. Another general approach is to determine an average premium/contribution per individual by dividing the total actuarial present value of benefits by the total of the appropriate annuity factors for all individuals in the group. This result may be multiplied by the number of individuals to obtain the total funding level for the group. It is well known to actuaries, especially pension actuaries, that this latter *aggregate* approach does not necessarily yield the same value as the former *individual* approach and that the difference can be significant.

It is the objective of this paper to highlight and analyze the resulting difference in funding level *per participant* in the context of a specific situation. In particular, the paper attempts to explain why the difference can be significant and discusses whether either of the approaches is more appropriate in the context of various criteria, including the expected value and variance of the loss random variable for the benefits provided.

I. INTRODUCTION

The details of the specific situation are summarized in Table 1.

The assumptions used in the initial calculations were as follows:

1. *Interest*—6 percent, since all funds are deposited in an account earning 6 percent.
2. *Mortality*—1980 U.S. Life Tables—Male, as listed in Actuarial Study No. 87 of the Social Security Administration (September 1982). This is

considered appropriate since it represents general population mortality and because the group is 96 percent male.

3. *Retirement Age*—age 60 for those under age 60; age 65 for the seven individuals age 60 and over.
4. *Payment of Death Claims*—immediate.
5. *Distribution of Deaths*—uniform.

Is the funding adequate for the benefits provided? This general question will be addressed by considering the specific question: For the given benefits, what should the monthly contribution be?

Assume that:

1. The contribution is to be the same for all individuals.
2. There is a $10 charge upon entry.
3. This is the initial plan year.

TABLE I

PLAN CHARACTERISTICS

BENEFITS	FUNDING
• $300 at retirement • $1,000 at death prior to retirement	• $10 at entry to plan • $1 per month for all participants
RETIREMENT	INTEREST
• The earliest retirement age is 51 • 391 of the 436 participants: *may* retire after 30 years of service *must* retire at age 60 • Retirement age 65 for the rest	• All funds are deposited in an account earning 6 percent

AGE DISTRIBUTION (AS OF MARCH 1, 1984)*

AGE	NUMBER	AGE	NUMBER	AGE	NUMBER
24	3	38	19	52..........	6
25	10	39	10	53..........	6
26	8	40	15	54..........	6
27	14	41	11	55..........	5
28	18	42	11	56..........	3
29	26	43	12	57..........	5
30	21	44	12	58..........	3
31	12	45	14	59..........	4
32	18	46	13	60..........	3
33	23	47	4	63..........	3
34	16	48	15	64..........	1
35	30	49	8		
36	16	50	12		
37	16	51	4		

*The above includes a total of 17 females age 24–35.

The effects of changing some of the assumptions are discussed in subseqent sections. This paper does not consider the impact of new entrants and legal implications, nor does it comment on the appropriateness of the plan design.

Two reasonable approaches to determining what the monthly contribution for the given benefits should be are outlined in the following.

A. Individual Approach

For each age x, first determine $C(x)$, the monthly contribution for an individual age x, by the traditional equivalence principle. That is, first determine $C(x)$ such that:

$$\$10 + 12 \cdot C(x) \cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|} = \$1{,}000\, \bar{A}^{\,1}_{x:\overline{RA-x}|} + \$300\ {}_{RA-x}E_x \quad (1)$$

where RA = retirement age

Then determine *CInd*, the monthly contribution for each individual using the *individual* approach, as an average of all $C(x)$. That is:

$$CInd = \frac{\underset{x\,=\,\text{all ages in age distribution}}{\Sigma\, N(x) \cdot C(x)}}{\underset{x\,=\,\text{all ages in age distribution}}{\Sigma\, N(x)}} \quad (2)$$

where $N(x)$ = number at age x. Note that this is essentially an "individual level premium" approach for a closed group.

B. Aggregate approach

Applying the traditional equivalence principle with the same contribution, *CAgg*, regardless of age, yields:

$$N(24) \cdot \$10 + N(24) \cdot 12 \cdot CAgg \cdot \ddot{a}^{(12)}_{24:\overline{RA-24}|}$$

$$+ N(25) \cdot \$10 + N(25) \cdot 12 \cdot CAgg \cdot \ddot{a}^{(12)}_{25:\overline{RA-25}|}$$

$$+ \ldots + N(64) \cdot \$10 + N(64) \cdot 12 \cdot CAgg \cdot \ddot{a}^{(12)}_{64:\overline{RA-64}|}$$

$$= N(24)\,[\$1{,}000\, \bar{A}^{\,1}_{24:\overline{RA-24}|} + \$300 \cdot {}_{RA-24}E_{24}]$$

$$+ N(25)\,[\$1{,}000\, \bar{A}^{\,1}_{25:\overline{RA-25}|} + \$300 \cdot {}_{RA-25}E_{25}]$$

$$+ \ldots N(64)\,[\$1{,}000\,\bar{A}^{\,1}_{64:\overline{RA-64}|} + \$300 \cdot {}_{RA-64}E_{64}] \tag{3}$$

Solving for *CAgg*, the monthly contribution for each individual using the *aggregate* approach, yields:

$$CAgg = \frac{\underset{x\,=\,\text{all ages in age distribution}}{\Sigma}\, N(x)\cdot\{(\$1{,}000\,\bar{A}^{\,1}_{x:\overline{RA-x}|} + \$300 \cdot {}_{RA-x}E_x) - \$10\}}{12 \cdot \underset{x\,=\,\text{all ages in age distribution}}{\Sigma}\, N(x) \cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}} \tag{4}$$

Note that if all individuals are the same age a then

$$CInd = \frac{N(a)C(a)}{N(a)} = C(a) = \frac{\$1{,}000\,\bar{A}^{\,1}_{a:\overline{RA-a}|} + \$300 \cdot {}_{RA-a}E_a - \$10}{12 \cdot \ddot{a}^{(12)}_{a:\overline{RA-a}|}}$$

and

$$CAgg = \frac{N(a)\,(\$1{,}000\,\bar{A}^{\,1}_{a:\overline{RA-a}|} + \$300 \cdot {}_{RA-a}E_a) - N(a) \cdot \$10}{12 \cdot N(a) \cdot \ddot{a}^{(12)}_{a:\overline{RA-a}|}}$$

$$= \frac{\$1{,}000\,\bar{A}^{\,1}_{a:\overline{RA-a}|} + \$300 \cdot {}_{RA-a}E_a - \$10}{12 \cdot \ddot{a}^{(12)}_{a:\overline{RA-a}|}}$$

That is, $CInd = CAgg$.

However, for the plan described in Table 1, and the assumptions indicated, $CInd = \$1.76$, $CAgg = \$1.09$, and the ratio of *CInd* to *CAgg* (denoted *CRatio*) is 1.6199 to four decimal places.

II. ANALYSIS

The difference in the values of *CInd* and *CAgg* is significant and raises two main questions:

1. Why are the results different?
2. Which is the "correct" value?

A. *Why the Results are Different*

For both *CInd* and *CAgg*, there is a balance at the initiation of the plan between the actuarial present value of contributions and the actuarial present value of benefits.

If each individual paid *CInd* monthly, then there should be enough to provide $C(x)$ for each individual age x, and equation 1 is satisfied for each individual. If each individual paid *CAgg* monthly, then equation 3 is satisfied, and the actuarial balance is maintained for the group.

Mathematically, it can be shown (see Appendix 1) that

$$CAgg = \frac{\sum_x N(x) \cdot C(x) \cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}}{\sum_x N(x) \cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}},$$

and thus, *CInd* (see equation 2) and *CAgg* are just different weighted averages of the $C(x)$ values. It should not be surprising that the numerical results are different.

Several factors could cause the significant difference in the results. The impacts of the interest assumption, the mortality assumption, the retirement assumption, the size of the group, and the distribution by age are analyzed below.

Table 2 summarizes the impact of a change in the interest assumption.

TABLE 2

IMPACT OF INTEREST ALTERNATIVES

Interest Assumption	*CAgg*	*CInd*	*CRatio*
.03	1.31	2.06	1.5685
.04	1.23	1.95	1.5871
.05	1.15	1.85	1.6043
.06	1.09	1.76	1.6199
.07	1.03	1.68	1.6339
.08	0.97	1.60	1.6466
.09	0.92	1.53	1.6579
.10	0.88	1.47	1.6679

Table 2 indicates that:

- The interest assumption does not have a significant impact on the relationship between *CInd* and *CAgg*.
- As expected, *CAgg* and *CInd* decrease as the interest asumption increases.
- *CRatio* increases slightly, at a slightly decreasing rate, as interest increases (that is, *CAgg* decreases at a slightly faster rate than *CInd*, as interest increases).
- For interest assumptions of .08, .09, and .10, the $1 per month contribution is adequate if *CAgg* is used as the basis for analysis, but is signif-

icantly inadequate if *CInd* is used; this highlights the problem that may develop if one must choose between the aggregate approach or individual approach in making evaluations and recommendations.

Table 3 summarizes the impact of several mortality assumption alternatives.

TABLE 3

IMPACT OF MORTALITY ALTERNATIVES

Mortality Assumption	*CAgg*	*CInd*	*CRatio*
1980 U.S. Life Tables—Male multiplied by:			
1.75	1.41	2.15	1.5180
1.50	1.31	2.02	1.5457
1.25	1.20	1.89	1.5791
1.00	1.09	1.76	1.6199
0.75	0.98	1.63	1.6707
0.50	0.86	1.50	1.7356
0.25	0.75	1.37	1.8212
1980 U.S. Life Tables—Female	0.87	1.51	1.7322

Table 3 indicates that:

- Overall, the mortality assumption does not have a very significant impact on the relationship between *CInd* and *CAgg*.
- A 50 percent improvement or deterioration in mortality at each age has relatively the same impact as a 50 percent improvement or deterioration in interest, (see Table 2) although there is a somewhat greater impact on *CRatio* (1.6199 to 1.7356 and 1.5457 for mortality as compared to 1.6199 to 1.6579 and 1.5685 for interest).
- *CRatio* increases at an increasing rate as mortality improves.
- Use of female mortality has almost exactly the same impact on *CAgg* and *CInd* as a 50 percent improvement in mortality.

The retirement assumption is based on the premise that all individuals will work as long as possible. To analyze the impact of changing the retirement assumption, the following alternatives are considered:

- *Age 51 for those younger than 51; 60 for those aged 51 to 59; and 65 for the rest.* The rationale for this alternative is that it represents the earliest retirement age for those younger than 51 and may be considered the extreme case (although it is not the most extreme since some of those aged 51 to 59 could have 30 years of service before age 60). It also makes the unlikely assumption that all those younger than age 51 were hired at

21. It does provide, approximately, an "early" bound for the range of retirement alternatives.

- *Age 55 for those younger than 55; 60 for those aged 55 to 59; and 65 for the rest.* The rationale for this alternative is that it is more realistic than the previous one (since the average entry age was estimated to be 25), and it provides an intermediate alternative in the range of retirement alternatives.
- *All retire at age 65.* Although this is not possible under the current provisions, it was chosen to provide some information on the impact of increasing the retirement age. It may be considered, approximately, as the "late" bound.

Table 4 summarizes the results for these retirement alternatives:

TABLE 4

IMPACT OF RETIREMENT ALTERNATIVES

Retirement Assumption	*CAgg*	*CInd*	*CRatio*
51/60/65	1.65	3.25	1.9667
55/60/65	1.34	2.56	1.9126
60/65	1.09	1.76	1.6199
65	0.97	1.26	1.2948

Table 4 indicates that:

- The retirement assumption does have a relatively significant impact on the relationship between *CAgg* and *CInd*.
- Changes in the retirement assumption cause less variation in *CAgg* than in *CInd*; thus, the difference between the two, represented by *CRatio*, decreases significantly as the retirement assumption goes from the "early" alternative to the "late" alternative.
- As expected, both *CAgg* and *CInd* decrease as the retirement assumption moves from "early" to "late."

Equations 2 and 4 indicate that the size of the group affects the value of *CInd* and *CAgg* only to the extent that the distribution by age is changed. If the group at each age is multiplied by 10, 1,000, or 1,000,000, the values for *CInd* and *CAgg* will remain unchanged. However, the values do vary significantly if the age distribution changes, as shown in Table 5. The values in Table 5 are somewhat unrealistic since only two ages are considered. However, they do provide an indication of the impact of a change in age distribution. Several observations are relevant:

TABLE 5

IMPACT OF AGE DISTRIBUTION ALTERNATIVES

Number of Lives		*CRatio*
Age 27	Age 32	
0	436	1
436	0	1
45	391	1.0015
391	45	1.0018
109	327	1.0032
327	109	1.0036
218	218	1.0045
Age 27	Age 57	
0	436	1
436	0	1
45	391	1.4011
391	45	2.0080
109	327	1.9144
327	109	2.5954
218	218	2.5500
Age 52	Age 57	
0	436	1
436	0	1
45	391	1.0739
391	45	1.0825
109	327	1.1527
327	109	1.1636
218	218	1.2106

- When the ages are young and close together (27/32), the *CRatio* is almost identically 1, regardless of the age distribution.
- When the ages are older and close together (52/57), the *CRatio* does vary by age distribution, and *CInd* can be more than 20 percent greater than *CAgg*.
- When the ages are far apart (27/57), the *CRatio* is affected greatly by the age distribution, and *CInd* can be more than 2.5 times *CAgg*.
- As might be expected, the difference in impact between an age distribution weighted toward one or the other of the pair of ages is not significant for the pairs of ages that are close together, but it is very significant for the

TABLE 6

IMPACT OF AGE DISTRIBUTION

NUMBER AT AGE:		*CAgg*	*CInd*	*CRatio*
27	57			
0	436	8.43	8.43	1.0000
436	0	0.47	0.47	1.0000
45	391	5.43	7.61	1.4011
391	45	0.65	1.30	2.0080
109	327	3.36	6.44	1.9144
327	109	0.95	2.46	2.5954
218	218	1.75	4.45	2.5500

27/57 pair; weighting toward the younger age, compared to the same weighting toward the older age, seems to increase *CRatio*.

To investigate this last point further, Table 6 summarizes the impact of age distribution on *CAgg* and *CInd*.

Perhaps the most significant aspect of the results listed in Table 6 is that a given change in the age distribution seems to have a greater impact on *CAgg* than *CInd*. For example, changing from 25 percent (109) at age 27 and 75 percent (327) at age 57 to 75 percent at age 27 and 25 percent at age 57, reduces *CAgg* by slightly over 70 percent, but it reduces *CInd* by only slightly over 60 percent. This result is the opposite of the result for a change in the retirement assumption for which a given change in the assumption has a greater impact on *CInd* than on *CAgg*.

The preceding analysis indicates that the relationship between *CInd* and *CAgg* is not affected significantly by the interest assumption, the mortality assumption, or the size of the group, but is affected significantly by the retirement assumption and the age distribution. Thus, the significant difference between *CInd* and *CAgg* for the given situation can be attributed to the specific age distribution and retirement assumption. Given a significant difference between *CInd* and *CAgg*, the question of the "correct" value is of particular importance.

B. The "Correct" Value

As indicated previously, both *CInd* and *CAgg* can be considered "correct" in the sense that both are solutions to equations that provide a balance at the initiation of the plan between the actuarial present value of benefits and the actuarial present value of contributions for the group of individuals.

It is troubling to have, for a given situation, two significantly different numerical values identified as providing an actuarial balance. The key distinction is that one represents an actuarial balance for each individual, *provided the appropriate portion of the total contribution is allocated to each individual*, and the other represents an actuarial balance for the group as a whole.

Consideration of loss random variables should help clarify this distinction. For simplicity, the following defines the loss random variable in the context of an annual contribution without entry charge and asssumes an end-of-the-year-of-death payment of claims.

Thus, $(L\text{-}Ind\text{-}Variable)_x$, the loss random variable for an individual age x associated with paying an annual contribution (denoted by $AC(x)$) determined by an equation of the form of equation 1 (without the first term on the left hand side), is defined by:

$$(L\text{-}Ind\text{-}Variable)_x = \begin{cases} \$1{,}000\ v^{K+1} - AC(x) \cdot \ddot{a}_{\overline{K+1}|} & 0 \le K < RA - x \\ \$300\ v^{RA-x} - AC(x) \cdot \ddot{a}_{\overline{RA-x}|} & RA - x \le K \end{cases}$$

where K is the random variable for the number of complete future years lived by (x). Since ${}_{k|}q_x$ is the probability function for the random variable K, the expected value of $(L\text{-}Ind\text{-}Variable)_x$ is given by:

$$E[(L\text{-}Ind\text{-}Variable)_x] = \sum_{k=0}^{RA-x-1} [\$1{,}000\ v^{k+1} - AC(x) \cdot \ddot{a}_{\overline{k+1}|}] \cdot {}_{k|}q_x$$

$$+ \sum_{k=RA-x}^{\omega-1-x} (\$300\ v^{RA-x} - AC(x) \cdot \ddot{a}_{\overline{RA-x}|}) \cdot {}_{k|}q_x$$

$$= \$1{,}000\ A^{\ 1}_{x:\overline{RA-x}|} - AC(x) \cdot \left(\sum_{k=0}^{RA-x-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + \ddot{a}_{\overline{RA-x}|} \cdot {}_{RA-x}p_x \right)$$

$$+ \$300\ v^{RA-x}{}_{RA-x}p_x$$

$$= \$1{,}000\ A^{\ 1}_{x:\overline{RA-x}|} + \$300\ v^{RA-x}{}_{RA-x}p_x - AC(x) \cdot \ddot{a}_{x:\overline{RA-x}|} = 0.$$

Thus, there is an actuarial balance for each individual. Summing over all x, the total expected value for the group is 0. Implicit in the development of this result is the allocation of $AC(x)$ to each individual age x.

However, consider $(L\text{-}Ind\text{-}Level)_x$, the loss random variable for an individual age x associated with paying an annual contribution (denoted *ACInd*) determined by an equation of the form of equation 2:

$$(L\text{-}Ind\text{-}Level)_x = \begin{cases} \$1{,}000\ v^{K+1} - (ACInd) \cdot \ddot{a}_{\overline{K+1}|} & 0 \le K < RA - x \\ \$300\ v^{RA-x} - (ACInd) \cdot \ddot{a}_{\overline{RA-x}|} & RA - x \le K \end{cases}$$

The expected value of $(L\text{-}Ind\text{-}Level)_x$ is *not* 0 for each individual, and the total expected value for the group is approximately $-\$35{,}000$. A gain of \$35,000 is expected over the lifetime of the group, even though the *total paid by the group* is the same (initially) as when each individual pays $AC(x)$. This is due to the different allocation of contribution amounts to each individual: *ACInd*, a *level* amount, as compared to $AC(x)$ a *variable* amount. Since the provisions of the plan indicate each individual is to pay the same amount, $(L\text{-}Ind\text{-}Level)_x$ is the loss random variable for the individual approach consistent with provisions of the plan. For the remainder of this paper $(L\text{-}Ind\text{-}Level)_x$ will be denoted $(LInd)_x$. Thus, the provisions of the plan and the *individual* averaging approach result in a contribution value that is inherently conservative in the sense that it produces an expected gain for the group as a whole.

$(LAgg)_x$, the loss random variable for an individual age x associated with paying an annual contribution determined by the *aggregate* approach (denoted *ACAgg*) is defined by:

$$(LAgg)_x = \begin{cases} \$1{,}000\ v^{K+1} - (ACAgg) \cdot \ddot{a}_{\overline{K+1}|} & 0 \le K < RA - x \\ \$300\ v^{RA-x} - (ACAgg) \cdot \ddot{a}_{\overline{RA-x}|} & RA - x \le K \end{cases}$$

The expected value of $(LAgg)_x$ is *not* 0 for each individual, but the total expected value for the group *is* 0. This can be seen by the following development:

$$\text{Total Expected Value} = \sum_x N(x) \cdot E[(LAgg)_x]$$

$$= \sum_x N(x) \cdot \left[\sum_{k=0}^{RA-x-1} (\$1{,}000\ v^{k+1} - (ACAgg) \cdot \ddot{a}_{\overline{k+1}|}) \cdot {}_{k|}q_x \right.$$

$$\left. + \sum_{k=RA-x}^{\omega-1-x} (\$300\ v^{RA-x} - (ACAgg) \cdot \ddot{a}_{\overline{RA-x}|}) \cdot {}_{k|}q_x \right]$$

$$= \sum_x N(x) \cdot \left[\$1{,}000\, A^{\,1}_{x:\overline{RA-x}|} - (ACAgg) \sum_{k=0}^{RA-x-1} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_x \right.$$

$$\left. + \$300\, v^{RA-x}\, {}_{RA-x}p_x - (ACAgg) \cdot \ddot{a}_{\overline{RA-x}|} \cdot {}_{RA-x}p_x \right]$$

$$= \sum_x N(x) \cdot [\$1{,}000\, A^{\,1}_{x:\overline{RA-x}|} + \$300\, v^{RA-x}\, {}_{RA-x}p_x$$

$$- (ACAgg) \cdot \ddot{a}_{x:\overline{RA-x}|}]$$

$$= \sum_x [N(x) \cdot (\$1{,}000\, A^{\,1}_{x:\overline{RA-x}|} + \$300\, v^{RA-x}\, {}_{RA-x}p_x)$$

$$- (ACAgg) \cdot N(x) \cdot \ddot{a}_{x:\overline{RA-x}|}]$$

$$= 0$$

Based on the preceding analysis of the expected value of the loss random variables, it might be argued that the *aggregate* approach is more appropriate because it provides an actuarial balance for the group as a whole, while the individual approach provides an expected gain, even though the total contribution is initially calculated on the basis of providing an actuarial balance for each individual. The actuarial imbalance, resulting in an expected gain for the group, is a consequence of the nature of the averaging in the individual approach and the allocation to individuals required by the plan provisions (that is, each pays the same).

To summarize, if either (a) actuarial balance for the group or (b) stability of costs in the context of changing retirement patterns (see Table 4) is an important consideration, *CAgg* is more appropriate than *CInd*. On the other hand, if either (c) conservatism (*CRatio* is always greater than 1) or (d) stability of costs in a changing age distribution is of primary importance, then *CInd* may be the choice.

Although this paper does not consider future years directly, the future impact of the retirement assumptions and the age distribution are related. The age distribution and retirement pattern may change from year to year in the natural course of events as well as through specific employment and retirement policies (for example, early retirement incentives). Unfortunately, it seems that stability of costs is more likely using *CInd* with a changing age distribution and using *CAgg* with a changing retirement pattern.

Another item to consider is the variance of the loss random variables developed previously. The variances of $(LInd)_x$ and $(LAgg)_x$ provide a measure of the variability of the loss associated with each individual. Assuming the losses of the individuals in the group are independent, but identically distributed, then the total variance of loss is just the sum of the individual variances. To the extent relative stability of loss is desirable, the relationship of the $Var[(LInd)_x]$ to $Var[(LAgg)_x]$ can be an important consideration. It can be shown (see Appendix 2) that:

$$\begin{aligned} Var[(LInd)_x] = & \left[\$1{,}000 + \frac{(ACInd)}{d}\right]^2 \left[{}^2A^{\,1}_{x:\overline{RA-x}|} - (A^{\,1}_{x:\overline{RA-x}|})^2\right] \\ & + \left[\$300 + \frac{(ACInd)}{d}\right]^2 \left[{}_{RA-x}{}^2E_x - ({}_{RA-x}E_x)^2\right] \\ & - 2\left[\$1{,}000 + \frac{(ACInd)}{d}\right]\left[\$300 + \frac{(ACInd)}{d}\right] \cdot A^{\,1}_{x:\overline{RA-x}|} \cdot {}_{RA-x}E_x \end{aligned}$$

where ${}^2A^{\,1}_{x:\overline{RA-x}|}$ and ${}_{RA-x}{}^2E_x$ represent net single premiums evaluated at interest rate $j = (1+i)^2 - 1$, i being the interest rate used in evaluating $A^{\,1}_{x:\overline{RA-x}|}$ and ${}_{RA-x}E_x$,

and:

$$\begin{aligned} Var\,[(LAgg)_x] = & \left[\$1{,}000 + \frac{(ACAgg)}{d}\right]^2 \left[{}^2A^{\,1}_{x:\overline{RA-x}|} - (A^{\,1}_{x:\overline{RA-x}|})^2\right] \\ & + \left[\$300 + \frac{(ACAgg)}{d}\right]^2 \left[{}_{RA-x}{}^2E_x - ({}_{RA-x}E_x)^2\right] \\ & - 2\left[\$1{,}000 + \frac{(ACAgg)}{d}\right]\left[\$300 + \frac{(ACAgg)}{d}\right] \cdot A^{\,1}_{x:\overline{RA-x}|} \cdot {}_{RA-x}E_x \end{aligned}$$

As a measure of the relative stability of the potential loss for the total group, consider

$$Total\ VarRatio = \frac{\sum_x N(x) \cdot Var\,[(LInd)_x]}{\sum_x N(x) \cdot Var\,[(LAgg)_x]}.$$

For the given assumptions, the value of this ratio is 1.1356. This indicates that *ACInd* results in a 13.6 percent greater total variance of the loss random variables than *ACAgg*.

Thus, if relative stability of the potential loss for the total group is important, then *CAgg* is the value to use.

III. CONCLUSION

For any funding problem, it is imperative that the funding approach be reviewed regularly (for example, annually) in light of the desired funding pattern objectives, the actual interest and mortality/termination experience, and any relevant legal requirements. These factors may make the mathematical distinction between the individual and aggregate approaches seem rather unimportant; at least the decision of which approach to use may be based on factors other than the inherent mathematical difference. It is important, however, for actuaries to be aware of that difference and the reasons for it. Although the preceding presentation and analysis are limited in that they ignore the impact of new entrants and make only brief reference to future years, the following can be concluded concerning the difference between the individual approach and the aggregate approach:

1. The age distribution and retirement age assumption are important factors in determining the ratio of the amount of contribution *per participant* for the individual approach to the amount of contribution *per participant* for the aggregate approach.
2. The *aggregate* approach represents an actuarial balance for the group as a whole, while the *individual* approach starts with an actuarial balance for each individual but results in an imbalance if it is desired to have each individual contribute the same amount.

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APPENDIX 1

Multiplying (1) by $N(x)$ and summing over all ages in age distribution yield:

$$\sum_x N(x)\cdot[\$10 + 12\cdot C(x)\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}] = \sum_x N(x)\cdot(\$1{,}000\ \bar{A}^{1}_{x:\overline{RA-x}|} + \$300\cdot {}_{RA-x}E_x) \qquad (A1)$$

Equation 3 implies:

$$\begin{aligned}&\sum_x N(x)\cdot\$10 + (CAgg)\cdot\sum_x N(x)\cdot 12\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}\\ &= \sum_x N(x)\cdot(\$1{,}000\ \bar{A}^{1}_{x:\overline{RA-x}|} + \$300\cdot {}_{RA-x}E_x)\end{aligned} \qquad (A2)$$

Thus,

$$\sum_x N(x)\cdot\$10 + (CAgg)\cdot\sum_x N(x)\cdot 12\cdot \ddot{a}^{(12)}_{x:\overline{RA:-x}|} = \sum_x N(x)\cdot[\$10 + 12\cdot C(x)\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}]$$

or

$$CAgg = \frac{\sum_x 12\cdot N(x)\cdot C(x)\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}}{\sum_x 12\cdot N(x)\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}} = \frac{\sum_x N(x)\cdot C(x)\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}}{\sum_x N(x)\cdot \ddot{a}^{(12)}_{x:\overline{RA-x}|}}$$

APPENDIX 2

Simplifying the expression for $(L\text{-}Ind\text{-}Level)_x = (LInd)_x$

$$(LInd)_x = \begin{cases}\left(\$1{,}000 + \dfrac{ACInd}{d}\right)\cdot v^{K+1} - \left(\dfrac{ACInd}{d}\right) & 0 \le K < RA-x\\[2ex] \left(\$300 + \dfrac{ACInd}{d}\right)\cdot v^{RA-x} - \left(\dfrac{ACInd}{d}\right) & RA-x \le K\end{cases}$$

Similarly, $(LAgg)_x$ simplifies to:

$$(LAgg)_x = \begin{cases} \left(\$1{,}000 + \dfrac{ACAgg}{d}\right) \cdot v^{K+1} - \left(\dfrac{ACAgg}{d}\right) & 0 \le K < RA-x \\ \left(\$300 + \dfrac{ACAgg}{d}\right) \cdot v^{RA-x} - \left(\dfrac{ACAgg}{d}\right) & RA-x \le K \end{cases}$$

Let $\left[\$1{,}000 + \left(\dfrac{ACInd}{d}\right)\right] = Y, \qquad \left(\$300 + \dfrac{ACInd}{d}\right) = Z$

and $\left(\dfrac{ACInd}{d}\right) = W.$

Then

$$(LInd)_x = \begin{cases} Y \cdot v^{K+1} - W & 0 \le K < RA-x \\ Z \cdot v^{RA-x} - W & RA-x \le K \end{cases}$$

$$= \underbrace{\begin{cases} Y \cdot v^{K+1} - W & 0 \le K < RA-x \\ 0 & RA-x \le K \end{cases}}_{_1(LInd)_x} + \underbrace{\begin{cases} 0 & 0 \le K < RA-x \\ Z \cdot v^{RA-x} - W & RA-x \le K \end{cases}}_{_2(LInd)_x}$$

$$\begin{aligned} \Rightarrow Var[(LInd)_x] &= Var\,[{}_1(LInd)_x + {}_2(LInd)_x] \\ &= Var\,[{}_1(LInd)_x] + Var\,[{}_2(LInd)_x] + 2Cov\,[{}_1(LInd)_x, {}_2(LInd)_x] \\ &= E[{}_1(LInd)_x^2] - \{E[{}_1(LInd)_x]\}^2 + E[{}_2(LInd)_x^2] - \{E[{}_2(LInd)_x]\}^2 \\ &\quad + 2\{E[{}_1(LInd)_x \cdot {}_2(LInd)_x] - E[{}_1(LInd)_x] \cdot E[{}_2(LInd)_x]\} \end{aligned}$$

But since the probability function for K is ${}_{k|}q_x$,

$$E[{}_1(LInd)_x] = \sum_{k=0}^{RA-x-1} (Y \cdot v^{k+1} - W) \cdot {}_{k|}q_x = Y \cdot A^1_{x:\overline{RA-x|}} - W \cdot {}_{RA-x}q_x$$

$$\begin{aligned} E[{}_1(LInd)_x^2] &= \sum_{k=0}^{RA-x-1} (Y \cdot v^{k+1} - W)^2 \cdot {}_{k|}q_x \\ &= \sum_{k=0}^{RA-x-1} [Y^2(v^2)^{k+1} - 2 \cdot W \cdot Y \cdot v^{k+1} + W^2] \cdot {}_{k|}q_x \\ &= Y^2 \cdot {}^2A^1_{x:\overline{RA-x|}} - 2 \cdot W \cdot Y \cdot A^1_{x:\overline{RA-x|}} + W^2 \cdot {}_{RA-x}q_x \end{aligned}$$

where ${}^2A^1_{x:\overline{RA-x|}}$ is calculated using an interest rate j

such that $v_j = v_i^2$ or $j = (1+i)^2 - 1$

$$E[_2(LInd)_x] = \sum_{k=RA-x}^{\omega-1-x} (Z \cdot v^{RA-x} - W) \cdot {}_{k|}q_x = Z \cdot v^{RA-x}{}_{RA-x}p_x$$

$$- W \cdot {}_{RA-x}p_x = Z \cdot {}_{RA-x}E_x - W \cdot {}_{RA-x}p_x$$

where ω represents the ending age of the mortality table.

$$E[_2(LInd)_x^2] = \sum_{k=RA-x}^{\omega-1-x} (Z \cdot v^{RA-x} - W)^2 \; {}_{k|}q_x$$

$$= \sum_{k=RA-x}^{\omega-1-x} [Z^2(v^2)^{RA-x} - 2 \cdot W \cdot Z \cdot v^{RA-x} + W^2] \cdot {}_{k|}q_x$$

$$= Z^2 \; (v^2)^{RA-x}{}_{RA-x}p_x - 2 \cdot W \cdot Z \cdot v^{RA-x}{}_{RA-x}p_x + W^2{}_{RA-x}p_x$$

$$= Z^2 \; {}_{RA-x}{}^2E_x - 2 \cdot W \cdot Z \cdot {}_{RA-x}E_x + W^2 \cdot {}_{RA-x}p_x$$

where ${}_{RA-x}{}^2E_x$ uses interest rate j as above.

Thus $Var\ [(LInd)_x]$

$$= Y^2 \; {}^2A^1_{x:\overline{RA-x}|} - 2 \cdot W \cdot Y \cdot A^1_{x:\overline{RA-x}|} + W^2_{RA-x}q_x$$

$$- (Y \cdot A^1_{x:\overline{RA-x}|} - W \cdot {}_{RA-x}q_x)^2$$

$$+ Z^2 \cdot {}_{RA-x}{}^2E_x - 2 \cdot W \cdot Z \cdot {}_{RA-x}E_x + W^2 \cdot {}_{RA-x}p_x$$

$$-(Z \cdot \; {}_{RA-x}E_x - W \cdot {}_{RA-x}p_x)^2$$

$$+ 2\ [0 - (Y \cdot A^1_{x:\overline{RA-x}|} - W \cdot {}_{RA-x}q_x) \cdot (Z \cdot {}_{RA-x}E_x - W \cdot {}_{RA-x}p_x)]$$

$$= Y^{2} \cdot {}^2A^1_{x:\overline{RA-x}|} - 2 \cdot W \cdot Y \cdot A^1_{x:\overline{RA-x}|} + W^2 \cdot {}_{RA-x}q_x$$

$$-[Y^2 \cdot (A^1_{x:\overline{RA-x}|})^2 - 2 \cdot W \cdot Y \cdot {}_{RA-x}q_x \cdot A^1_{x:\overline{RA-x}|} + W^2 \cdot {}_{RA-x}q_x^2]$$

$$+ Z^2 \cdot {}_{RA-x}{}^2E_x - 2 \cdot W \cdot Z \cdot {}_{RA-x}E_x + W^2 \cdot {}_{RA-x}p_x$$

$$-[Z^2 \cdot ({}_{RA-x}E_x)^2 - 2 \cdot W \cdot Z \cdot {}_{RA-x}p_x \cdot {}_{RA-x}E_x + W^2 \cdot {}_{RA-x}p_x^2]$$

$$-2(Y \cdot Z \cdot {}_{RA-x}E_x \cdot A^{1}_{x:\overline{RA-x}|} - W \cdot Y \cdot {}_{RA-x}p_x \cdot A^{1}_{x:\overline{RA-x}|}$$

$$-W \cdot Z \cdot {}_{RA-x}q_x \cdot {}_{RA-x}E_x + W^2 \cdot {}_{RA-x}q_x \cdot {}_{RA-x}p_x)$$

$$= Y^2 \cdot [{}^2A^{1}_{x:\overline{RA-x}|} - (A^{1}_{x:\overline{RA-x}|})^2]$$

$$+ Z^2 \cdot [{}_{RA-x}{}^2E_x - ({}_{RA-x}E_x)^2]$$

$$+ W^2\ {}_{RA-x}q_x - W^2 \cdot {}_{RA-x}q_x^2 + W^2\ {}_{RA-x}p_x$$
$$- W^2\ {}_{RA-x}p_x^2 - 2W^2 \cdot {}_{RA-x}q_x \cdot {}_{RA-x}p_x$$

$$-2 \cdot Y \cdot Z \cdot A^{1}_{x:\overline{RA-x}|} \cdot {}_{RA-x}E_x$$

$$= Y^2 \cdot [{}^2A^{1}_{x:\overline{RA-x}|} - (A^{1}_{x:\overline{RA-x}|})^2] + Z^2[{}_{RA-x}{}^2E_x - ({}_{RA-x}E_x)^2]$$

$$+ W^2\,[1 - p - (1 - 2p + p^2) + p - p^2 - 2\,(p - p^2)]$$

$-2 \cdot Y \cdot Z \cdot A^{1}_{x:\overline{RA-x}|} \cdot {}_{RA-x}E_x$ where p represents ${}_{RA-x}p_x$

$$= Y^2 \cdot [{}^2A^{1}_{x:\overline{RA-x}|} - (A^{1}_{x:\overline{RA-x}|})^2] + Z^2 \cdot [{}_{RA-x}{}^2E_x - ({}_{RA-x}E_x)^2]$$

$$-2 \cdot Y \cdot Z \cdot A^{1}_{x:\overline{RA-x}|} \cdot {}_{RA-x}E_x$$

Similarly, let $\left[\$1{,}000 + \left(\frac{ACAgg}{d}\right)\right] = Y'$, $\left[\$300 + \left(\frac{ACAgg}{d}\right)\right] = Z'$

and $\frac{ACAgg}{d} = W'$.

Then

$$Var\,[(LAgg)_x] = (Y')^2 \cdot [{}^2A^{1}_{x:\overline{RA-x}|} - (A^{1}_{x:\overline{RA-x}|})^2]$$
$$+ (Z')^2 \cdot [{}_{RA-x}{}^2E_x - ({}_{RA-x}E_x)^2] - 2 \cdot Y' \cdot Z' \cdot A^{1}_{x:\overline{RA-x}|}\ {}_{RA-x}E_x.$$

DISCUSSION OF PRECEDING PAPER

DONALD R. SONDERGELD:

John Maynard Keynes once said: "In the long run, we are all dead." Someone else said: "In the long run, benefit payments plus expenses equal contributions plus investment earnings."

Pension actuaries are well aware that contributions are affected both by the choice of the actuarial funding method and the choice of the actuarial funding assumptions. This paper demonstrates the wide variations that can occur between two specific funding methods referred to as the *individual approach* and the *aggregate approach*. The purpose of my discussion is to help the reader better see why the differences occur.

I would like to change the nomenclature slightly to produce more general formulas that develop average costs per month per individual:

$$C'Ind = \frac{\Sigma W_1(x)\ C(x)}{\Sigma W_1(x)} \text{ and } C'Agg = \frac{\Sigma W_1(x)\ C(x)\ W_2(x)}{\Sigma W_1(x)\ W_2(x)} = \frac{\Sigma W_3(x)\ C(x)}{\Sigma W_3(x)}.$$

In the paper: $C(x)$ = monthly contribution for a person, age x.

$W_1(x) = N(x)$ = number of lives at age x.

$W_2(x) = \ddot{a}^{(12)}_{x:\overline{RA-x}|}$ = monthly annuity due from age x to age RA.

$W_3(x) = W_1(x)W_2(x)$

C' *Ind* and C' *Agg* are obviously different weighted averages of the $C(x)$ values. If $C(x) < C(x+1)$, and

$W_2(x) > W_2(x+1)$, then

$C'\ Agg < C'\ Ind$, as the aggregate method will give greater weight to the smaller $C(x)$ at the lower ages. This was generally the case in the examples in the paper. See my Supplementary Table I for which Mr. Luckner provided the information.

In the paper, $W_1(x) = N(x)$ and $W_2(x) = \ddot{a}^{(12)}_{x:\overline{RA-x}|}$ · $W_2(x) > W_2(x+1)$ for $x<59$. Although this relationship does not hold at age 59, W_2 generally gives heavier weight to the younger ages. In the paper, $C'\ Ind$ = \$1.76 and $C'\ Agg$ = \$1.09. $C(x)$ generally increased for this benefit plan as x

SUPPLEMENTARY TABLE I

x	$N(x)$	$\ddot{a}^{(12)}_{x:\overline{RA-x}\rceil}$	$C(x)$	x	$N(x)$	$\ddot{a}^{(12)}_{x:\overline{RA-x}\rceil}$	$C(x)$	x	$N(x)$	$\ddot{a}^{(12)}_{x:\overline{RA-x}\rceil}$	$C(x)$
20....	0	15.03	.34	35....	30	12.68	.77	50....	12	7.26	2.56
21....	0	14.93	.36	36....	16	12.44	.82	51....	4	6.72	2.86
22....	0	14.83	.37	37....	16	12.18	.88	52....	6	6.15	3.22
23....	0	14.72	.39	38....	19	11.91	.94	53....	6	5.54	3.69
24....	3	14.60	.41	39....	10	11.62	1.01	54....	6	4.90	4.30
25....	10	14.47	.43	40....	15	11.33	1.09	55....	5	4.21	5.14
26....	8	14.34	.45	41....	11	11.01	1.17	56....	3	3.48	6.38
27....	14	14.20	.47	42....	11	10.67	1.26	57....	5	2.70	8.43
28....	18	14.05	.50	43....	12	10.32	1.36	58....	3	1.86	12.48
29....	26	13.89	.53	44....	12	9.95	1.48	59....	4	.97	24.56
30....	21	13.71	.56	45....	14	9.56	1.60	60....	3	4.14	5.66
31....	12	13.53	.60	46....	13	9.15	1.74	61....	0	3.43	6.93
32....	18	13.34	.63	47....	4	8.71	1.91	62....	0	2.67	9.01
33....	23	13.13	.68	48....	15	8.25	2.09	63....	3	1.85	13.10
34....	16	12.91	.72	49....	8	7.77	2.30	64....	1	.96	25.22

x = age; $N(x)$ = number of persons age x; $C(x)$ = monthly contribution for each person age x.

$$C'\ Ind = \frac{\Sigma N(x)C(x)}{\Sigma N(x)} = \frac{767.37}{436} = 1.76$$

$$C'\ Agg = \frac{\Sigma N(x)\ddot{a}^{(12)}_{x:\overline{RA-x}\rceil}\,C(x)}{\Sigma N(x)\ddot{a}^{(12)}_{x:\overline{RA-x}\rceil}} = \frac{5{,}262.56}{4{,}843.94} = 1.09$$

increased through age 59, decreased at age 60, but then increased from age 61 to age 64. Therefore, the weighting factor W_2 would cause $C'\ Agg$ to be less than $C'\ Ind$. Although that result could be predicted, the ratio of $C'\ Ind$ to $C'\ Agg$ of 1.6 needed to be calculated.

Irrespective of whether the benefit formula is the fixed benefit formula of this example, a final salary formula, or a career average salary formula—it is still possible to define $W_1(x)$ as the present value of future projected salaries. Then, the C' calculation produces a cost rate per individual, as a percentage of salary. Costs for each individual are then produced by multiplying this rate by his or her current salary.

Alternatively, $W_1(x)$ can be defined as something else, such as projected benefits. Twenty-five years ago, I was responsible for valuing many defined benefit retirement plans. I routinely calculated costs on three bases to see the effect on an employer's costs of changing the W_3 weights:

(a) $W_3(x) = N(x)\ \ddot{a}^{(12)}_{x:\overline{RA-x}\rceil}$,
(b) a weighting using projected benefits, and
(c) $W_3(x)$ = present value of projected salaries

Mr. Luckner pointed out that if all people were the same age, then C' *Agg* = C' *Ind*. This is also true for all ages if we define $W_3(x) = W_1(x) = N(x)$.

The author states: "This paper does not consider the impact of new entrants and legal implications, or the appropriateness of the plan design." This is almost like comparing a Porsche with a Ford, and ignoring cost. The choice of the funding method, the weights, new entrants, and so on do affect important things such as the employer's cash flow and federal income taxes. Thus, if there is a "correct" C' *Ind* or C' *Agg*, it is dependent upon many factors that were not considered in the paper. Also, with Financial Accounting Standard No. 87, the accounting profession has indicated that pension expense shown in the generally accepted accounting principles income statement must be calculated using the projected unit credit method.

In practice, there are few, if any, pension plans where the contribution is the same for everyone fixed at inception of the plan—with no provision for change. Also, the funding methods in use make provisions for the handling of actuarial gains and losses in renewal years.

However, the paper illustrates the necessity for the pension actuary to understand the impact of the choice of weights used in the funding method.

PAULETTE TINO:

Mr. Luckner's paper is the by-product of an academic project probably directed toward the use of expected value and variance of loss variables. In that context the consideration of whether a particular method satisfies basic funding principles is not a critical issue. The absence of this point of view might be troubling to pension actuarial students reading the paper. Therefore, I should like to address the question of the relative appropriateness of the two methods examined in the paper from the practitioner's point of view. In this regard, I should note that the following represents my own personal views and not an official position of my employer, the Internal Revenue Service.

We shall assume that the plan calls for level contributions and ignore the \$10 advance contribution payable at entry.

In the Abstract of the paper two funding methods are described. The first method is the individual level premium, which leads to the determination of a monthly rate of contribution (equal to the normal cost rate), $C(x)$, per equation 1 of the paper. Equation 1 can be rewritten $PVB_x = 12 \cdot C(x) \cdot TA_x$. PVB_x is the present value of future benefits at age x, and TA_x, the temporary annuity at the same age (with monthly payments of 1/12 payable at the beginning of the month). The second method is the aggregate method, under which the level monthly rate of contribution per employee (equal to the normal cost rate), *CAgg*, is determined by equation 4 of the paper. Equation 4 can be rewritten $\Sigma PVB_x = 12 \cdot CAgg \cdot \Sigma TA_x$. The two methods are reasonable.

In the Introduction to the paper, the first method is modified by equation 2, which defines a monthly rate of contribution, *CInd*, applicable to all employees, as the sum of the first monthly contribution divided by the total number of employees:

$$CInd \cdot \Sigma N(x) = \Sigma C(x) \cdot N(x).$$

CInd is treated in Part B of the Analysis section of the paper as a level contribution. It is also stated in several parts of the paper that "for both *CInd* and *CAgg*, there is a balance at the initiation of the plan between the actuarial present value of contributions and the actuarial present value of benefits." Those are the two points I would like to discuss.

Before showing why a contribution at the rate *CInd* is defective, let us recall in what manner a contribution at the rate $C(x)$ is adequate. For this we return to equation 1, $PVB_x = 12 \cdot C(x) \cdot TA_x$. This equation tells us that, at each time t, a contribution $C(x)$ is paid for each of the $N(x) \cdot {}_tP_x$ active survivors and is deposited in the fund to grow with interest. From the fund, at each time t, a death benefit of \$1,000 is paid on account of each of the $N(x) \cdot {}_tp_x \cdot \mu_{x+t}$ employees dying in active service. When the retirement age is attained, there is just enough money left in the fund to pay \$300 to each of the $N(x) \cdot {}_{RA-x}p_t$ employees reaching that age. Equation 1 merely sums up the present value of the projected benefits and the present value of the projected contributions and states that those present values are equal. From this analysis, we can see that if *CInd* (a constant) were, at each time t a contribution is paid, the average $\overline{C_t(x)}$ of the $C(x)$ contributions paid at that time, then substituting *CInd* for $C(x)$ in equation 1 probably would not be valid for any age x. But the equation obtained by summing up the $N(x)$ equations 1 would be valid. However, $\overline{C_t(x)}$ is not level, and *CInd* is equal to $\overline{C_t(x)}$ only at time zero. This is shown by writing the algebraic expression of $\overline{C_t(x)}$:

$$\overline{C_t(x)} = \frac{\Sigma N(x) \cdot {}_tp_x \cdot C(x)}{\Sigma N(x) \cdot {}_tp_x}$$

where the summations are for all values of x except, when $x \leq 59$, for x such that $x + t \geq 60$, and when $60 \leq x \leq 64$, for x such that $x + t \geq 65$.

In the calculation of $\overline{C_t(x)}$, the weights applied to the $C(x)$'s are no longer $N(x)$ but $N(x)\cdot{}_tp_x$, and at each integral value of t, survivors from certain age x groups who are retiring—those associated, at that time, with the highest $C(x)$—are dropped from the calculation. Therefore

1. $\overline{C_t(x)}$ has to be very sensitive to any change in assumption (e.g., turnover, retirement age) affecting the projected number of active employees at $x+t$, and more important
2. $\overline{C_t(x)}$ is a decreasing function of t.

Payments at the rate *CInd* will be too big: if contributions at the rate *CInd* are collected at time t, after allocating a portion $N(x)\cdot{}_tp_x\cdot C(x)$ of the total collection to the $N(x)\cdot{}_tp_x$ active survivors of each age x group, a surplus is left. Therefore, using *CInd*, equation 1 will not be satisfied at any age, nor will it be satisfied when each side of the equation is summed up for all employees.

In order to complete the analysis let us see in what manner *CAgg*, a level rate per employee, is an adequate rate. Again, all the benefits payable at any time t, and the contributions collectable at any time t, have been projected. Focusing on contributions, we know by equation 4 that ($\Sigma PVB_x = 12\cdot CAgg \Sigma TA_x$) that *CAgg* is an adequate contribution. Therefore *CAgg* must be less than *CInd*, and we can write: $\Sigma C(x)\cdot N(x) > CAgg\cdot \Sigma N(x)$. It follows that, at the outset and for a certain time t_0 thereafter, less contributions will be collected using the *CAgg* approach than using the $C(x)$ approach, and that for $t>t_0$ the situation will be reversed. In contrast with the situation when the $C(x)$ approach is used, we are no longer assured, with the *CAgg* approach, that the benefits expected to be paid to any age x group will be exactly funded by the time the survivors retire. In fact, it is not until after $t=t_0$ that the accumulated deficiencies will start to be repaid. However, by the time the survivors of the youngest age x group retire, all benefits will be exactly funded. The *CAgg* approach works because *CAgg* is a weighted average of the $C(x)$'s (as shown in section II.A of the paper) that takes into account the length of time over which a particular contribution at the rate $C(x)$ is made together with the timing and the amount of each contribution.

To summarize, the Individual Approach develops an excessive rate of contribution and therefore unacceptable normal costs. As a consequence, it develops unacceptable accrued liabilities (the accrued liability calculated as an accumulation of past *CInd* normal costs will exceed the accrued liability calculated prospectively as the present value of future benefits minus the present value of future *CInd* normal costs). This imbalance between present value of benefits and present value of contributions is convincingly demonstrated in the paper by the computation of the expected value of $(L-Ind-Level)_x$.

These considerations show that the *CInd* approach is flawed and should not be used by a practicing pension actuary.

(AUTHOR'S REVIEW OF DISCUSSION)

WARREN R. LUCKNER:

I would like to thank Mr. Sondergeld and Ms. Tino for their discussions. As an actuary who has not been closely involved with pension actuarial work, I found their clarifications and insights educational, and I appreciate their help in analyzing the illustration.

Mr. Sondergeld furthers the analysis by identifying more general formulas and additional possible weights to use. Mr. Sondergeld also correctly points out the fact that items not considered in the paper (e.g., new entrants, legal implications, plan design, and accounting requirements) do have an important practical impact on the funding approach and level. Because the class during which this illustration was discussed was a course in pension mathematics, the analysis in the paper focuses on the theoretical question of *why* the results using two seemingly reasonable approaches are different.

Ms. Tino's first sentence is partly correct: I did want to try to take advantage of the "new" approach to life contingencies by incorporating the use of expected values and variances of loss random variables in my analysis. However, the main impetus was to try to determine why two seemingly reasonable approaches provide apparently significantly different results. Ms. Tino's discussion helps answer that question by pointing out some problems with using the individual approach as defined in the paper. The problems develop from the desire that the contribution be the same for all individuals each month, and from the lack of recalculation of *CInd* at any time after the initiation of the plan.

Despite the problems with the individual approach, the paper's main point is valid: There are inherent differences between an individual and aggregate approach, and it is important for actuaries to be aware of that difference and the reasons for it. As Mr. Sondergeld observes, the paper "illustrates the necessity for the pension actuary to understand the impact of the choice of weights used in the funding method."