Another Look at Empirical Estimation of Actuarial Risk Measures

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1. Introduction and Preliminaries

Introduction

- **Tools for measuring "riskiness"**

  Problem of interest: price determination of an insurance risk; *"riskiness" of a risk*: certain properties of claims distribution; consider risk measures defined in terms of expectations with respect to *distorted probabilities*. (See Wang, Young, Panjer (1997), Wang (1998), Artzner (1999).)

- **Unifying representation of risk measures**

$$R(F, \psi) = \int_0^1 F^{-1}(s)\psi(s)\, ds$$

  (See Jones and Zitikis (2003).)

- **Estimators of risk measures**
  - Nonparametric (based on L-statistics)
  - Parametric (based on MLEs)
  - Robust (based on trimmed means)

Risk Measures

- **Proportional Hazard Transform (PHT)**

$$\psi(s) = r(1-s)^{r-1}$$

Here constant r ($0 \leq r \leq 1$) can be interpreted as the degree of distortion.

- **Right-Tail Deviation (RTD)**

$$\psi(s) = r(1-s)^{r-1} - 1$$

For $r = 1/2$ this measure corresponds to Wang's Right-Tail Deviation (Wang (1998)).

- **Wang Transform (WT)**

$$\psi(s) = \exp\{-\lambda\Phi^{-1}(s) - \lambda^2/2\}$$

Here parameter λ reflects the systematic risk and $\Phi(\cdot)$ is the cdf of $N(0, 1)$ distribution.

Interval Estimation

- **General objective**

  For a fixed sample size, favorable statistical procedures are those that yield the *shortest* interval while maintaining the desired (*high*) confidence level.

- **Specific aims**

  – *Convergence rates*

    How fast do the proposed (asymptotic) intervals attain the intended confidence level?

  – *Comparison of procedures at the model*

    Under strict distributional assumptions, how much do we gain if, instead of empirical intervals, parametric or robust confidence intervals are used?

  – *Sensitivity to assumptions*

    How bad are the consequences if the underlying assumptions necessary for the theoretical statements to hold are ignored or cannot be verified?

Simulation Study

- **Contamination model**

$$H_\varepsilon = (1-\varepsilon)\,F + \varepsilon\,G$$

F is *central distribution*, G is *contaminating distribution*, ε is *level of contamination*.

- **Choices for the central distribution F**

  - *Exponential distribution* ($x > x_0$, $\theta > 0$)

$$F_1(x) = 1 - e^{-(x-x_0)/\theta}$$

  - *Pareto distribution* ($x > x_0$, $\gamma > 0$)

$$F_2(x) = 1 - (x_0/x)^\gamma$$

  - *Lognormal distribution* ($x > x_0$, $\mu \in \mathcal{R}$)

$$F_3(x) = \Phi\Big(\log(x - x_0) - \mu\Big)$$

NOTE: Due to x_0, distributions F_1, F_2, and F_3 have the same support.

- **Choices for the contaminating distribution G**
  - *Uniform distribution* $(10\, x_0 < x < 50\, x_0)$
  - *Pareto distribution* $(x > x_0,\ 0 < \gamma_1 < \gamma)$

    NOTE: Since $0 < \gamma_1 < \gamma$, this distribution has heavier upper-tail than Pareto with parameter γ.

- **Choice of parameters**

  Parameters θ, γ, and μ are chosen so that all three distributions have the same level of "riskiness". That is, for selected function ψ,

  $$R(F_1, \psi) = R(F_2, \psi) = R(F_3, \psi)$$

- **Study design**
  - Sample size: $n = 25,\ 50,\ 100,\ 250$.
  - Confidence level: $1 - \alpha = 0.90,\ 0.95,\ 0.99$.
  - Distortion level: $r = 0.50,\ 0.70,\ 0.85,\ 0.95$.
  - Systematic risk: $\lambda = 0.25,\ 0.50$.

2. Estimation Techniques

Nonparametric Approach

100(1 − α)% confidence interval (based on the empirical estimator of a risk measure) is

$$L_n[X] \pm z_{\alpha/2}\sqrt{\frac{Q_n(\psi,\,\psi)}{n}},$$

where

$$Q_n(\psi,\,\psi) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\Big[c_n(i,j)\,\psi(i/n)\,\psi(j/n)\times$$

$$\times(X_{i+1:n} - X_{i:n})(X_{j+1:n} - X_{j:n})\Big]$$

with $c_n(i,\,j) = \min\{i/n,\,j/n\} - (i/n)(j/n)$ and

$$L_n[X] = \sum_{i=1}^{n} c_{in} X_{i:n}$$

with $c_{in} = \int_{i/n}^{(i\,-\,1)/n} \psi(s)\,ds$, and $z_{\alpha/2}$ is the $\alpha/2$-critical value of $N(0,\,1)$, and $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the ordered values of data $X_1,\ldots,X_n$.

Parametric Methods

- **Exponential distribution** ($x > x_0$, $\theta > 0$)
  - $(1-\alpha)100\%$ confidence interval for θ:
  $$\widehat{\theta}_{\mathsf{ML}}\left(1 \pm z_{\alpha/2}\sqrt{1/n}\right)$$
  - MLE of θ: $\widehat{\theta}_{\mathsf{ML}} = \frac{1}{n}\sum_{i=1}^{n}(X_i - x_0)$
- **Pareto distribution** ($x > x_0$, $\gamma > 0$)
  - $(1-\alpha)100\%$ confidence interval for γ:
  $$\widehat{\gamma}_{\mathsf{ML}}\left(1 \pm z_{\alpha/2}\sqrt{1/n}\right)$$
  - MLE of γ: $\widehat{\gamma}_{\mathsf{ML}} = \left[\frac{1}{n}\sum_{i=1}^{n}\log(X_i/x_0)\right]^{-1}$
- **Lognormal distribution** ($x > x_0$, $\mu \in \mathcal{R}$)
  - $(1-\alpha)100\%$ confidence interval for μ:
  $$\widehat{\mu}_{\mathsf{ML}} \pm z_{\alpha/2}\sqrt{1/n}$$
  - MLE of μ: $\widehat{\mu}_{\mathsf{ML}} = \frac{1}{n}\sum_{i=1}^{n}\log(X_i - x_0)$

NOTE: Corresponding intervals for the risk measures are found by appropriately transforming the above intervals.

Robust Procedures

- **Exponential distribution** ($x > x_0$, $\theta > 0$)
  - $(1-\alpha)100\%$ confidence interval for θ:

$$\widehat{\theta}_{\mathsf{TM}} \left(1 \pm z_{\alpha/2}\sqrt{k/n}\,\right)$$

  - Trimmed Mean (TM) estimator of θ:

$$\widehat{\theta}_{\mathsf{TM}} = \frac{1}{d} \sum_{i\,=\,[n\beta_1]\,+\,1}^{n\,-\,[n\beta_2]} (X_{i:n} - x_0),$$

where

$$d = d(\beta_1, \beta_2, n) = \sum_{j=[n\beta_1]+1}^{n-[n\beta_2]} \sum_{i=0}^{j-1} (n-i)^{-1}$$

and β_1 and β_2 are trimming proportions.

  - Efficiency constants k:

$\beta_1 = \beta_2$	0.00	0.05	0.15	0.45
k	1.00	1.090	1.271	1.946

- **Pareto distribution** $(x > x_0,\ \gamma > 0)$
  - $(1-\alpha)100\%$ confidence interval for γ:

$$\widehat{\gamma}_{\mathrm{TM}} \left(1 \pm z_{\alpha/2}\sqrt{k/n}\right)$$

  - TM estimator of γ:

$$\widehat{\gamma}_{\mathrm{TM}} = \left[\frac{1}{d} \sum_{i\,=\,[n\beta_1]+1}^{n\,-\,[n\beta_2]} \log(X_{i:n}/x_0)\right]^{-1}$$

NOTE: Constants d and k are the same as for the exponential distribution.

- **Lognormal distribution** $(x > x_0,\ \mu \in \mathcal{R})$
  - $(1-\alpha)100\%$ confidence interval for μ:

$$\widehat{\mu}_{\mathrm{TM}} \pm z_{\alpha/2}\sqrt{K_\beta/n}$$

  - TM estimator of μ $(\beta_1 = \beta_2 = \beta)$:

$$\widehat{\mu}_{\mathrm{TM}} = \frac{1}{n - 2\,[n\beta]} \sum_{i\,=\,[n\beta]+1}^{n\,-\,[n\beta]} \log(X_i - x_0)$$

NOTE: Efficiency constants K_β are: $K_0 = 1$, $K_{0.05} = 1.026$, $K_{0.15} = 1.100$, $K_{0.45} = 1.474$.

3. Comparisons and Conclusions

Comparisons

PHT measure ("clean" data scenario)

TABLE 1. Length (L) and coverage (C) of 95% CI's, for selected F and $\varepsilon = 0$, $r = 0.85$, $n = 100$.

Estimator	F_1 (exponential)		F_2 (Pareto)		F_3 (lognormal)	
	L	C	L	C	L	C
EMPIRICAL	.10	.92	.13	.89	.14	.86
MLE(F_1)	.11	.94	.10	.84	.10	.81
MLE(F_2)	.16	.92	.15	.95	.15	.96
MLE(F_3)	.10	.86	.09	.56	.11	.95
TM(5%,F_1)	.11	.95	.10	.75	.10	.69
TM(5%,F_2)	.18	.84	.15	.95	.15	.97
TM(5%,F_3)	.11	.87	.10	.74	.11	.95
TM(15%,F_1)	.12	.95	.10	.67	.10	.63
TM(15%,F_2)	.21	.77	.17	.95	.16	.97
TM(15%,F_3)	.13	.76	.11	.85	.11	.95
TM(45%,F_1)	.15	.94	.13	.66	.12	.68
TM(45%,F_2)	.29	.75	.22	.95	.22	.98
TM(45%,F_3)	.16	.67	.13	.89	.13	.95

NOTE: Standard errors for all entries are between .0001 and .0009 (for L) and betweeen .001 and .009 (for C).

PHT measure ("contaminated" data scenario)

TABLE 2. Performance of 95% CI's for selected F,
$G = U(10\,x_0, 50\,x_0)$, and $\varepsilon = 0.05$, $r = 0.85$, $n = 100$.

Estimator	F_1 (exponential)		F_2 (Pareto)		F_3 (lognormal)	
	L	C	L	C	L	C
EMPIRICAL	3.66	.19	3.55	.20	3.65	.20
MLE(F_1)	.79	.01	.76	.01	.79	.01
MLE(F_2)	.65	.02	.55	.03	.58	.03
MLE(F_3)	.14	.40	.12	.68	.14	.34
TM(5%,F_1)	.22	.53	.21	.54	.21	.54
TM(5%,F_2)	.30	.32	.26	.49	.27	.51
TM(5%,F_3)	.13	.57	.11	.78	.13	.70
TM(15%,F_1)	.13	.89	.12	.88	.11	.87
TM(15%,F_2)	.24	.44	.20	.80	.19	.87
TM(15%,F_3)	.14	.52	.12	.85	.12	.86
TM(45%,F_1)	.16	.95	.14	.81	.13	.83
TM(45%,F_2)	.33	.54	.25	.89	.24	.94
TM(45%,F_3)	.17	.46	.14	.83	.14	.90

NOTE: Standard errors for all entries are between .0002 and .0151 (for L) and betweeen .001 and .013 (for C).

PHT measure (overall performance)

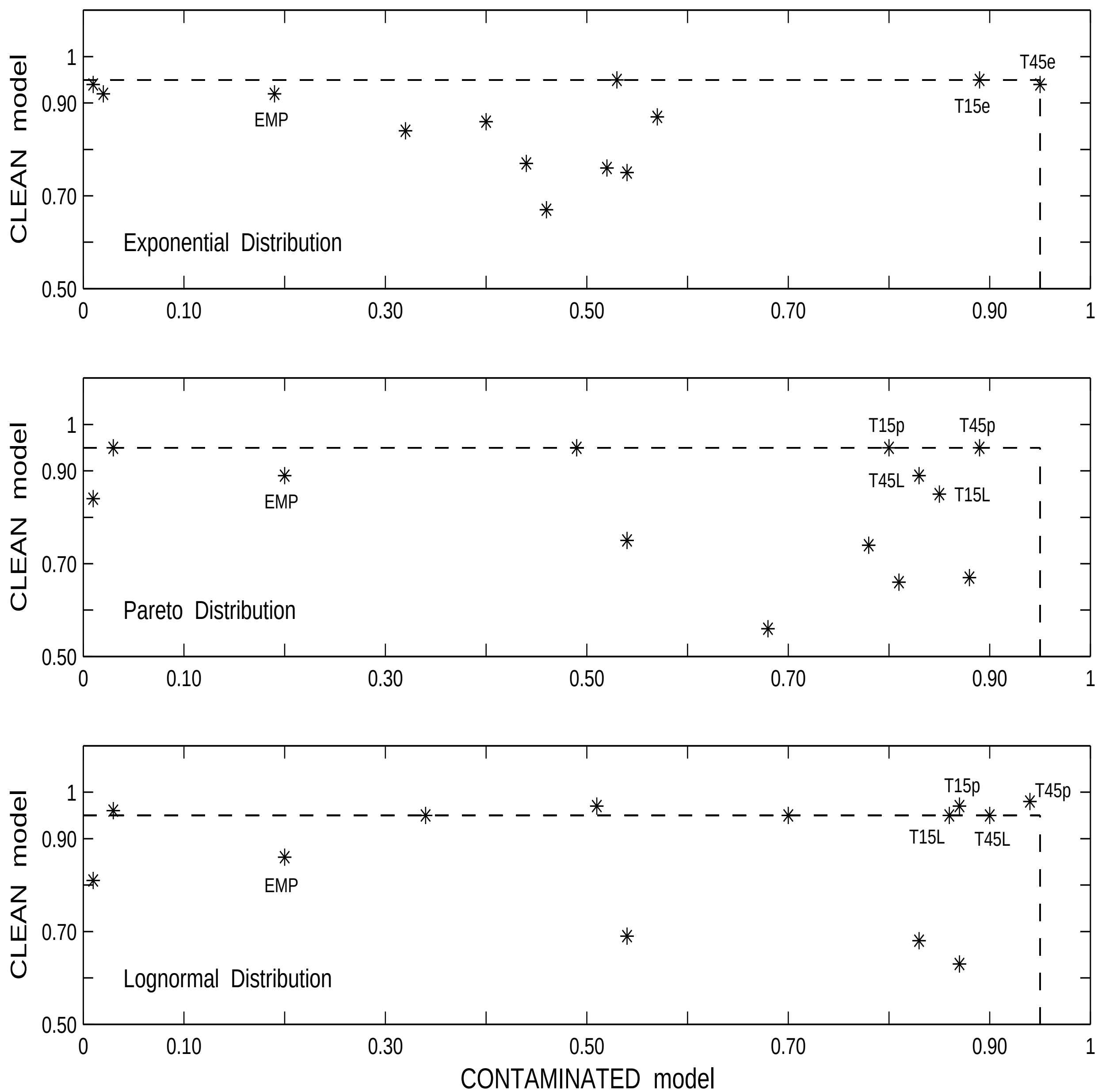

FIGURE 1. Proportions of coverage of 95% CI's for selected F, $G = U(10\,x_0, 50\,x_0)$, $r = 0.85$, $n = 100$, $\varepsilon = 0$ ("clean" model) and $\varepsilon = 0.05$ ("contaminated" model).

WT measure (overall performance)

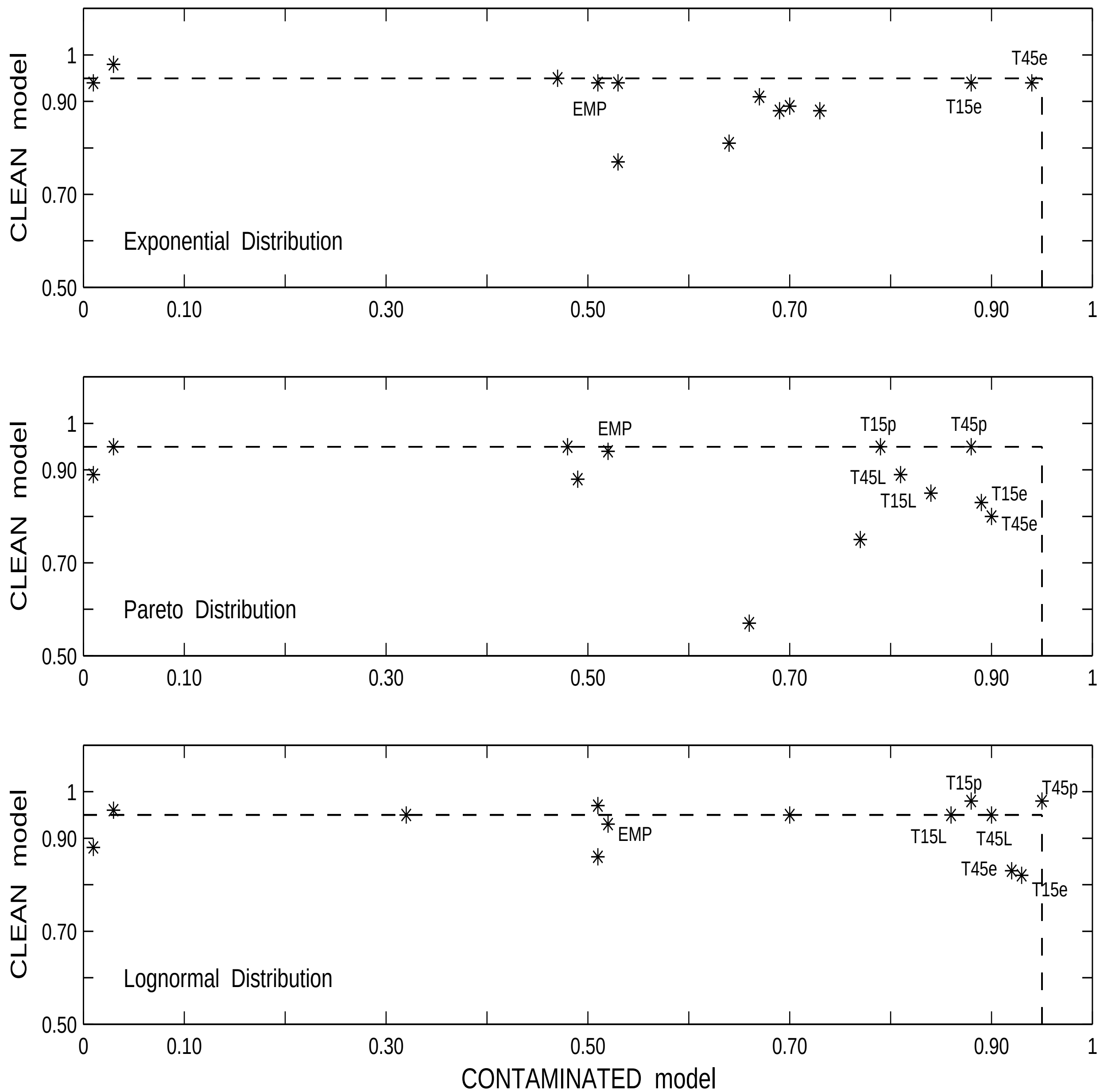

FIGURE 2. Proportions of coverage of 95% CI's for selected F, $G = U(10\,x_0, 50\,x_0)$, $\lambda = 0.25$, $n = 100$, $\varepsilon = 0$ ("clean" model) and $\varepsilon = 0.05$ ("contaminated" model).

Conclusions

- Convergence of the proportion of coverage of the empirical intervals is slow and depends on the function ψ. For "light" ψ, the coverage levels of these intervals get reasonably close to the nominal level for $n \geq 100$ and for all distributions F that we considered. For "severe" ψ, however, their performances are unacceptable even for $n = 1500$. Parametric and robust intervals attain the intended confidence levels for all ψ and F, and for sample sizes as small as $n = 50$.

- At the assumed model F, robust and parametric intervals perform better than empirical intervals with respect to the coverage criterion. Also, for $n \geq 250$, parametric intervals dominate robust and empirical counterparts with respect to the length criterion.

- When the assumed model F is contaminated or misspecified, both parametric and empirical procedures perform poor. In such situations, only sufficiently robust estimators, designed for model F, yield intervals with *consistently* satisfactory performance.