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REROSHE: THE CONCEPT OF A RISK-FREE EQUIVALENT RETURN ON SHAREHOLDERS' EQUITY

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ABSTRACT

The paper proposes the concept of a risk-free equivalent rate of return on shareholders' equity, or REROSHE, as a means of helping business managers make decisions involving risk. REROSHE is defined as that certain return on an investment of u that will generate the same expected value of utility as will the earnings on a line of business supported by surplus u.

A useful expression for REROSHE is derived using certain simplifying assumptions. This expression is then used to help determine an insurance company's degree of aversion to risk. Examples are given of practical applications of the concept, including the setting of a "break-even" expected ROSHE below which a company should never aim. The same principle can be applied to any return on investment, so that the concept is equally appropriate for mutual life insurance companies and other risk-taking enterprises.

I. DECISION-MAKING IN THE FACE OF RISK

One of the most important, discomforting, and unavoidable responsibilities of a business manager is to make decisions involving risk. Decision-making in the face of risk involves choosing between two or more courses of action, where each course results in a different and uncertain financial outcome—Do we buy an Aaa bond yielding 10 percent, or a Baa yielding 12 percent? One hundred shares of IBM, or a thousand shares of Go-Go, Inc.? When pricing a new individual life product, how conservative should the assumptions be as to future interest rates? To what extent should we invest in short-term instruments to improve a guaranteed investment contract cash-flow mismatch?

Whether it is an investment decision, a more traditional actuarial decision such as the price of an insurance policy, or even a decision about how to deploy available company resources, the underlying problem remains the same: in every case we are asked to choose between two or more futures—the proverbial fork in the road. Although each future is uncertain, the mathematics of probability provide a coherent way for studying such situations. When data are available, a stochastic model may be constructed. If the amount of data is limited, the decision maker must combine all relevant information to form distributions of possible outcomes for each course of action, thus summarizing the information.

Once these distributions are spread out before us—for example, the three distributions shown in Figure 1—how do we choose between them? If they are distributions of earnings, it seems clear that we should choose the one with the highest mean and the lowest variance (distribution C, for example). Unfortunately, such a simple choice is not usually available. Invariably there is a trade-off between expected return and the stability of that return. The choice is usually between distributions such as A and B.

Modern portfolio theory attempts to deal with this problem by looking at all distributions with the same expected gain and choosing the one with the smallest variance. Under this decision criterion, all other distributions, or portfolio choices, are deemed to be "inefficient." But this process only helps us to choose between A and another distribution with the same mean as A but with different variance, not between A and B.

In recent years, increasing use has been made of the return on shareholders' equity, or ROSHE, to help make such decisions. ROSHE, in its most basic form, can be thought of as the ratio of company earnings to the shareholders' equity in that company. Usually this translates to GAAP earnings over GAAP surplus. ROSHE is considered to be a risk-adjusted

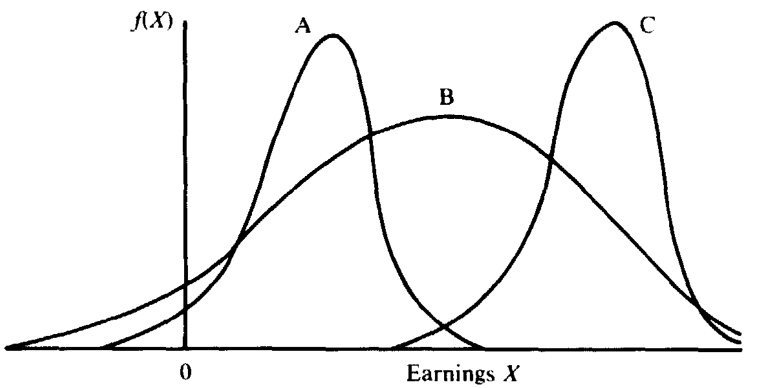

FIG. 1.—Three distributions of earnings

profitability measure. The riskier the line of business, the greater the allocated surplus should be, and the lower the ROSHE for a given level of earnings. For example, one line of business might require $10 million of surplus, but a riskier line might require $20 million; earnings of $1 million in each would produce a ROSHE of 10 percent in the former but only 5 percent in the riskier line. ROSHE can therefore be an effective means of measuring the past performance of a line of business on a risk-adjusted basis.

When we turn our attention to the future and use ROSHE in a prospective sense in planning and pricing, we are talking about *expected* earnings, and we should refer to the *expected* ROSHE, or E (ROSHE):

$$E(\text{ROSHE}) = \mu/u \,, \tag{1}$$

where μ = expected earnings for the line of business and u = surplus allocated to the line.

The difference between the retrospective ROSHE and the prospective E(ROSHE) is often misunderstood. Obviously we want to maximize the former, but should we strive to maximize the latter? Let us analyze the characteristics of E(ROSHE) to see whether it is the ideal prospective profitability measure for planning and pricing purposes.

As in ROSHE, the denominator of E(ROSHE) is risk-adjusted. Required surplus is usually set at a level appropriate to ensure that the risk of ruin is less than some small probability; therefore, the greater the variability in future earnings, the higher the required surplus. Ideally this required surplus is then allocated to the line. The numerator, on the other hand, is not risk-adjusted. It represents the mean or expected value of future earnings, ignoring the possibility of deviations from that mean.

Therefore, E(ROSHE) is risk-adjusted only to the extent that the risk of ruin remains at an acceptably low level. There are several reasons why this is not an ideal criterion.

1. Concentration on ruin potential is concentration on the very event we wish to avoid. By policy it is an extremely unlikely event at the tip of the left-hand tail of the distribution of earnings. For planning and pricing purposes we should be concentrating instead on the more likely nonruinous scenarios, that is, the remainder of the distribution.
2. As J. C. Wooddy [5] has aptly stated, "It is incontrovertible that a primary management objective is to conduct the affairs of an insurance company, including its financial accounting, so as to hold the probability of ruin to a very small number. The philosophy is not so clear when one seeks to make probability statements about an individual line of business or a block of roughly identical policies, let alone a single policy." What is the meaning of ruin for a

line or a block of policies? Should we be so concerned about ruin when measuring expected profitability?

3. Required surplus, although influenced by the riskiness of a line of business, is also affected by considerations outside that line of business. It generally is not sensitive to the riskiness of only a particular policy type; neither is it sensitive to sudden changes in the business or economic environment. It is an allocated portion of overall company surplus, which in turn is based on management decisions that change slowly if at all.

These drawbacks place serious limitations on the usefulness of E(ROSHE) when one is trying to establish or manage a particular block of policies or line of business. It simply does not provide the best index for showing a manager the best course to follow. This is illustrated in the following example.

If we try to use a ROSHE-type analysis to help us choose between distributions A and B from Figure 1, we may or may not feel comfortable with the result. Larger variance usually implies a greater probability of ruin. If we are considering only one year's results (ruin occurs when earnings are less than $-u$), then B would have a higher required surplus. Whether E(ROSHE A) is greater or less than E(ROSHE B) depends on both that excess surplus and the difference between the means of A and B.

Consider the choice illustrated in Figure 2. Both A and B have the same probability of ruin ($X < -u$), therefore the same required surplus. Hence E(ROSHE A) would be greater than E(ROSHE B). But is A preferable to B? What about the possibility of those excess gains under B that are not available with A?

To answer the question "Do we prefer A to B?" we must first ask the question "How risk-averse are we?" Consider the two distributions shown

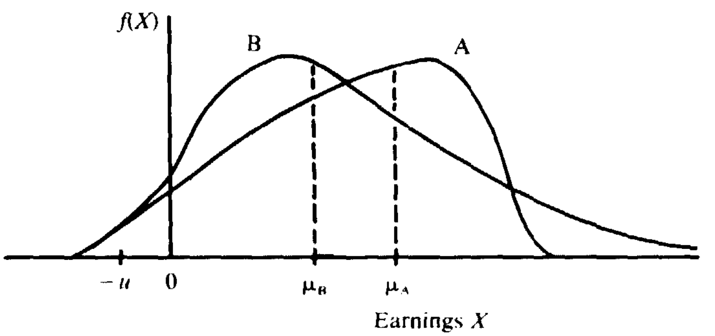

FIG. 2.—Choice in which A = B for $X < -u$

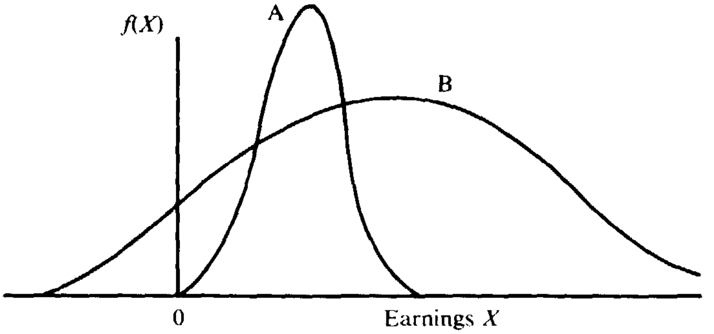

FIG. 3.—Case in which choice depends on attitude toward risk

in Figure 3. (An example of A would be an administrative services only contract with no probability of ruin. This would need very little surplus and have a large ROSHE.) The more averse we are to risk, the better A looks because we avoid the greater probability of small or negative earnings under B. If we are not particularly averse to risk, then B is preferable to A (even though it may have a lower ROSHE) because it has greater expected earnings.

II. CONSIDERATION OF RISK AVERSION

Utility theory is the mathematical tool that allows us to bring into the analysis the consideration of how risk-averse we are. It assigns to wealth the value, or utility, that a particular decision maker attaches to a particular amount of wealth. If he is "risk-neutral," then his utility curve is linear (Fig. 4). Each additional dollar gained or lost is worth the same no matter how wealthy or poor he is.

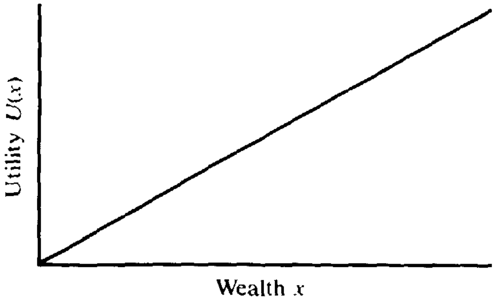

FIG. 4.—Risk-neutral utility curve

A risk-averse individual, however, would have a curve that is concave downward, that is, $U'(x) > 0$ *and* $U''(x) < 0$ (Fig. 5). As this individual gets poorer and poorer, each additional lost dollar is worth more and more to him (as $x \longrightarrow 0$, Δx corresponds to a larger and larger ΔU). Similarly, as he amasses greater wealth, each additional dollar is worth less (he is no longer as worried about the risk of going broke). Risk seekers, on the other hand, have utility curves that are concave upward. They are the gamblers. They will place a bet even though the odds are against them.

The shape of the utility curve determines our degree of aversion to risk. The greater our aversion, the more concave downward the curve. In making decisions about risky events with random outcome X, we can take our degree of risk aversion into account by defining the utility curve $U(x)$ and then considering the expected value of $U(X)$ rather than the expected value of X. (see Bowers et al. [1]).

III. A DECISION MAKER'S UTILITY CURVE

If we assume that our decision maker is risk-averse, then a reasonable choice for his utility function is the exponential

$$U(X) = -e^{-aX}, \tag{2}$$

where X is the amount of one year's earnings and a is a parameter to be determined. The single parameter a gives us all the flexibility we need. It is the measure of our degree of risk aversion. Additional parameters, as in $U(X) = b - ce^{aX}$, merely position the curve and do not affect our later calculations. Functions other than the exponential are possible but generate intractable resulting formulas for REROSHE. When a computer han-

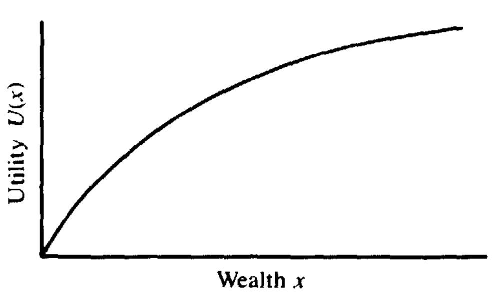

FIG. 5.—Risk-averse utility curve

dles the computational complexities, the same principles can be followed using other utility functions; this presentation considers only the exponential.

When faced with the uncertain future results of a business decision, we now can use the expected value of the utilities of the various outcomes X as the index we want to maximize.

$$\begin{aligned} E[U(X)] &= E[-e^{-aX}] \\ &= -E[e^{-aX}] \\ &= -M_X(-a) , \end{aligned} \tag{3}$$

where M_X is the moment generating function. When the distribution of future earnings can be assumed to be normally distributed with mean μ and variance σ^2, and therefore the moment generating function is $M_x(t) = \exp(\frac{1}{2}a^2\sigma^2 + t\mu)$, then

$$E[U(X)] = -\exp(\tfrac{1}{2}a^2\sigma^2 - a\mu) . \tag{4}$$

IV. THE RISK-FREE EQUIVALENT RATE OF RETURN

We shall now define the risk-free equivalent rate of return on shareholder equity, REROSHE, as that certain return on an investment of u that will generate the same expected value of utility as will the earnings on a line of business supported by surplus (shareholders' equity) u. Using R to designate REROSHE in our calculations, this means that

$$E[U(u + Ru)] = E[U(u + X)] .^{1} \tag{5}$$

Since REROSHE is a *certain* rate, the left-hand E is superfluous. Using equation (2) to give us the value of the left-hand side and equation (4) for the right, we get

$$-\exp[-a(u + Ru)] = -\exp(\tfrac{1}{2}a^2\sigma^2 - a\mu - au) ,$$

[1] This relationship is similar to the indifference relationship, cited by Freifelder [3] as the basic utility theory premium calculation principle,

$$U(W) = \int U(W + P - Z)dF(z) ,$$

which equates the utility of wealth W before issuing a policy with premium P and random claims Z to the utility after. Freifelder in turn cites Pratt and Buhlmann as earlier proponents of this approach.

which is equivalent to

$$-aRu = \frac{1}{2}a^2\sigma^2 - a\mu .$$

REROSHE can therefore be expressed as either

$$\text{REROSHE} = \frac{\mu}{u} - \frac{1}{2}a\frac{\sigma^2}{u} \tag{6}$$

or

$$\text{REROSHE} = \frac{\mu - \frac{1}{2}a\sigma^2}{u} . \tag{7}$$

Equations (6) and (7) lend themselves to interesting interpretations. Equation (6) indicates that REROSHE is equivalent to E(ROSHE) minus a term that is proportional to the degree of risk aversion a and the variance of the return and is inversely proportional to the surplus. This term is our risk-adjustment factor.

Equation (7) indicates that in order to price products to ensure the same risk-free rate of return, the risk charge should be proportional to the variance in that return. This is consistent with the so-called variance principle of actuarial pricing theory. In effect, we should set our risk charges proportional to the variance in claims, not to the standard deviation or some other more arbitrary variable. If our degree of risk aversion a is known, then a risk charge of $\frac{1}{2}a\sigma^2$ will compensate the company exactly on an equivalent utility basis.

It is easy to verify that equation (6) (or the equivalent, eq. [7]) is a *sufficient* as well as a *necessary* condition for REROSHE to be the risk-free rate of return on u that is equivalent to the earnings X, distributed $N(\mu, \sigma^2)$, on a line of business with surplus u. In other words, one can go back from (6) to (5). The same principle can be applied to the return on any equity. We could use REROE for the return on a mutual insurance company's surplus and REROI for the return on a non-insurance-related investment.

V. DETERMINING THE DEGREE OF RISK AVERSION

The main reason why utility theory is not used more frequently in practice is the difficulty in determining the parameters of the utility curve. While we have made certain assumptions about this curve, we still need to determine the value of a that represents our degree of risk aversion.

This is at once the most difficult and the most important element of our analysis. Several methods have been proposed in the literature. We shall describe an additional method that does not require the usual Delphi techniques but is based instead on relationship (6).

The degree of risk aversion to be assigned to a corporation can be determined by analyzing its recent history to determine the combination of risk and return it has assumed. If the corporation has been operating in an efficient manner, it has been choosing between various operating strategies, or mixes of products. During this time period, it could have moved its operating capital into risk-free investments such as Treasury bills instead of utilizing that capital as the surplus required to manage its product line. It chose not to invest in T-bills because the increased risk assumed by issuing insurance policies was balanced by the expected additional profits.

Let us assume that during this time period the corporation exhibited an indifference between (1) investing in T-bills and (2) pursuing the business of insurance with all its associated risks. If we further assume a normal distribution of earnings and exponential utility, equation (6) holds, with

REROSHE = After-tax return on T-bills, i ;

μ/u = Expected rate of return on *total* surplus, μ_T/u_T ;

σ_T^2/u_T = Variance in that return .

Therefore,

$$i = \frac{\mu_T}{u_T} - \frac{1}{2}a\,\frac{\sigma_T^2}{u_T}, \tag{8}$$

or

$$a = \frac{2(\mu_T - iu_T)}{\sigma_T^2}. \tag{9}$$

There are several problems with this approach.

1. The actual return achieved over recent years is not necessarily equivalent to the expected return for the same period.
2. The index of variability over recent years is difficult to measure (the shorter the duration, the fewer the points in the distribution; the longer the duration, the less recent the history) and again not necessarily what was expected.
3. In fact, the corporation did *not* exhibit indifference between T-bills and insurance—it chose insurance. Therefore, this value of a is really a least upper bound, that is, a could have been slightly lower.

So what emerges is not necessarily the corporation's present degree of risk aversion or what it might have thought that characteristic was in the past, but the approximate degree of risk aversion exhibited by the company in recent years. This, then, forms a basis on which to judge the corporation's future actions.

Values of a can be calculated directly from equation (9), or one can graph a variation of equation (8) as a linear relationship between return μ_T/u_T and risk σ_T^2/u_T:

$$\frac{\mu_T}{u_T} = i + \frac{1}{2}a\,\frac{\sigma_T^2}{u_T}\,. \tag{10}$$

The after-tax return on T-bills, i, becomes the y-intercept, and $\frac{1}{2}a$ the slope. Therefore, the greater the slope, the greater the degree of aversion to risk.

This linear equation is similar in concept to what is known in capital asset pricing theory as the capital market line. In this case, however, instead of an individual investor choosing between various investments, we can visualize a corporation choosing between various operating strategies, or mixes of products. Each product it sells requires the investment of a certain amount of capital. There is then an expected rate of *return* μ/u and a degree of variation in that return, or *risk*, σ^2/u.

If we associate a specific degree of risk aversion a to that corporation, then all the points on the risk-aversion line given by equation (10) become of equal value or utility to the corporation. All operating strategies with a combination of risk and return satisfying equation (10) should be equally attractive.

An analysis along these lines was performed for nine stock and eleven mutual insurance companies. Statutory earnings and surplus over the period 1974–80 were used.[2] The value of i for that period was 4.4 percent, the average after-tax return on T-bills.

The resulting combinations of return and risk are illustrated in Figure 6. Risk-aversion lines are drawn for one stock and one mutual company. In the case of the stock companies there are three companies that are more risk-averse than the one on the line, and four that are less risk-averse. The most striking aspect of the graph is the significantly greater

[2] When historical data are analyzed, the effect of growth should first be removed. This was done in our study by performing a linear regression on the earnings pattern and measuring only the variance in actual earnings from this line. The linear regression line can be thought of as expected earnings *with* growth, and therefore variations from that line are variations with the effect of growth removed. Other methods are possible.

degree of risk aversion exhibited by the stock companies in comparison to the mutuals.

Analysis of statutory results in this manner is hampered by the adverse effect of growth on statutory earnings. A further investigation analyzed the *GAAP* earnings of stock insurance companies (life and casualty combined) compared with those of a few sample industrial corporations. The results are illustrated in Figure 7. An analysis of this type can be useful in helping a company decide whether it should aim for a higher or a lower degree of risk aversion in the future.

VI. EXAMPLE

Let us assume that the company on the risk-aversion line in Figure 7 now wants to use the REROSHE concept to analyze the expected future earnings of its three major lines, A, B, and C. With $\mu_T/u_T = 0.16$ and $\sigma_T^2/u_T = 3.28$, its a-value according to equation (9) is 0.071.

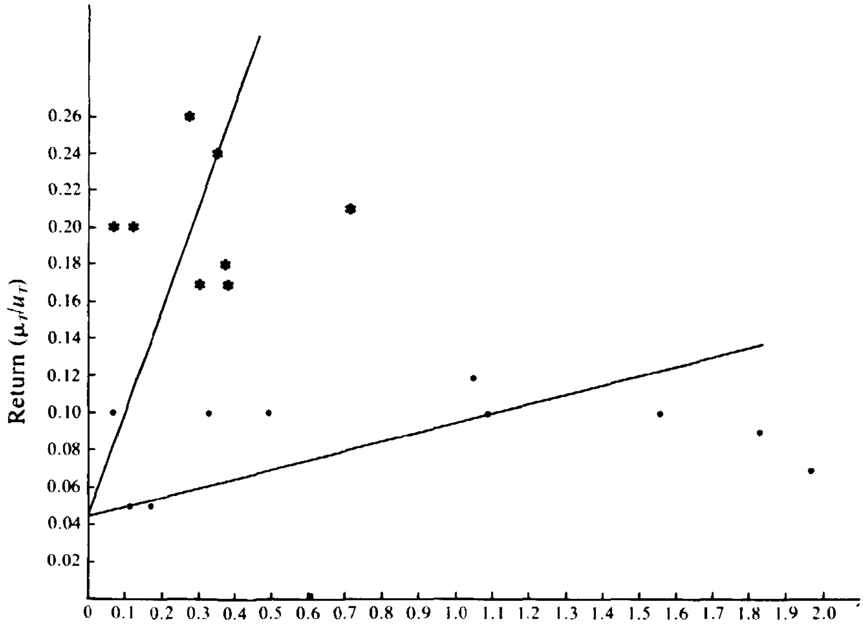

FIG. 6.—Stock(*) versus mutual(•) life insurance companies (statutory basis): degree of risk aversion.

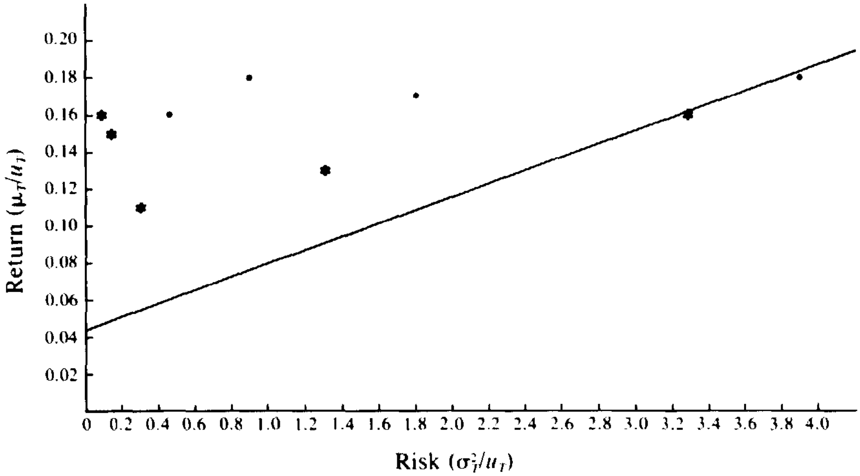

FIG. 7.—Stock(*) insurance companies versus industrials(•) (GAAP basis): degree of risk aversion.

Using equation (6), we can now calculate the REROSHE for lines A, B, and C. We shall assume that the lines have allocated surplus, expected earnings, and variances of earnings as shown in Table 1 and that the *current* value of i is 6.4 percent (the after-tax return if T-bills now yield 14 percent). Lines A and B have REROSHES greater than i and are therefore valid investments of surplus. A is better than B despite its lower expected ROSHE. The REROSHE of line C is below i, and therefore it is a questionable line to be in at the present pricing structure.

In order for these lines to be on an equal footing, they should all be producing the same REROSHE. If we want REROSHE to be at least equal to 6.4 percent, then the lines would have to expect the "break-even" ROSHES shown in Table 2. These values then can be used as minimum expected ROSHES in pricing.

TABLE 1

	LINE		
	A	B	C
a) Allocated surplus, u	\$1,250	\$ 750	\$1,000
b) Expected ROSHE, μ/u	12.0%	14.5%	13.0%
c) Variance of earnings, σ^2	\$ 500	\$1,000	\$2,500
d) REROSHE $= b - \frac{1}{2}(0.071)c/a$	10.6%	9.8%	4.1%

TABLE 2

	Line		
	A	B	C
Break-even ROSHE = $i + \frac{1}{2}a\sigma^2/u$	7.8%	11.2%	15.3%

VII. CONCLUSION

The concept of a risk-free equivalent rate of return can be very helpful in comparing two or more risky alternatives, for it reduces each one to an equivalent risk-free basis and therefore allows for a true risk-adjusted comparison. REROSHE will never be the sole criterion as to which road to take, since it is based on too many simplifying assumptions. But it can be a useful management tool—an index to be taken into consideration. Its main drawback, reliance on mathematical concepts outside the grasp of many managers, should not be allowed to prevent its use entirely. After all, many indexes are presently used by nontechnical managers who are not familiar with the intricacies of their derivation (e.g., CPI, Dow-Jones Average, M-1, Moody's Bond Rating, Statutory Reserves).

Besides the sort of interline comparisons of profitability illustrated above, the following could be additional uses of these concepts:

1. *Intercompany comparisons of degree of aversion to risk* (as in Figs. 6 and 7). REROSHE should be useful in measuring one's own company's overall aggressiveness or in comparing different companies before purchase of their stock.
2. *Product pricing and management.* The setting of risk charges, and management of various insurance and investment risks, can be facilitated by use of REROSHE. In particular it should be useful in the justification to regulatory authorities of profit margins and noninvestment income pricing in the more volatile property/casualty product areas.
3. *Investment strategy.* As investment strategy is brought more closely into line with the liability characteristics of insurance products, the use of model offices reflecting future cash flows under different interest rate scenarios will be necessary to measure the degree of immunization. Perfect immunization usually will not be possible because of uncertainties in many cash flows and competitive constraints. Instead we again shall be choosing between several investment strategies, each producing its own set of possible future earnings patterns. Such models, if they can produce the variance of each set of earnings patterns, can be easily programmed to generate the REROSHE of each investment strategy, thereby helping us choose one.

It is hoped that this paper will generate discussion and further development of what promises to be a useful management concept.

VIII. ACKNOWLEDGMENTS

The author would like to express his gratitude to James C. Hickman, Robert A. Miller, Cecil J. Nesbitt, and Laurence H. Longley-Cook, for their constructive criticism of early drafts of this paper.

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DISCUSSION OF PRECEDING PAPER

OAKLEY E. VAN SLYKE:

Mr. Longley-Cook is to be congratulated for his excellent contribution to the literature. The principles of decision making using utility concepts have been available to actuaries for many years, and Mr. Longley-Cook's paper makes it clear that the basic concepts are practical.

Utility-adjusted values for potential investment returns are a practical way of making decisions. There are important reasons why these techniques are practical, and why they are better than competing techniques. The method suggested by Mr. Longley-Cook has one flaw, however. A simpler version explained below avoids that flaw. The simpler version can be explained to top management. The actuary can apply the method in practice, regardless of the sophistication of his statistical skills.

It Works

The basic method described by Mr. Longley-Cook works in practice. There are important reasons why it works.

It does what it should. The purpose of all the calculations surrounding a decision is to help us make better, more informed judgments. Proper calculations about risk should give us a realistic assessment of the risk involved in each of the various alternatives available, which Longley-Cook's REROSHE does. So long as the decision at stake is not of such a staggering nature as to make the company's risk-bearing ability different depending on the alternative chosen, the assumptions underlying REROSHE will be valid, and the results reasonable.

It avoids problems of other methods. Longley-Cook cites three major difficulties with the usual formulation for risk-adjusted value, the expected value of the return on shareholders' equity. These criticisms are so valid that the difficulties with using the expected value concept have made it impractical.

It is easy to use. The calculations are straightforward. Also, simple methods of setting the constant for the firm's risk aversion, Longley-Cook's a, are available.

It Has One Flaw

The only difficulty with Longley-Cook's REROSHE is that it relies on the artificial concept of *invested surplus*. The concept of invested surplus is so well established in actuarial literature that we often forget its artificiality. But consider the following:

1. Its name is deceptive.
   a) Surplus is put at risk, not actually invested.

b) All your surplus is at risk, not merely the part arbitrarily allocated to "invested surplus."
c) In the true sense of the word *investment*, only the expenses involved in an option are invested.

2. The concept of invested surplus is of no practical use. Its normal use is to reflect risk. If there is no other adjustment for risk, this is an important use for the concept of invested surplus. Since Longley-Cook is making an explicit adjustment for risk through the calculations surrounding the risk aversion, a, it is no longer necessary to have an artificial concept for risk such as the idea of invested surplus.
3. It is an unnecessary complication. If the calculations were simpler and the results more meaningful with Longley-Cook's formulation, one might as well leave the concept of invested surplus in, since it is so well understood. In actual practice, however, the results of calculations of REROSHE are difficult to communicate, and unless there is widespread agreement about invested surplus, the alternative assumptions about invested surplus lead to additional calculations. It is simpler to leave out the concept of invested surplus.

A Simpler Version Avoids That Flaw

The simpler version is to replace the calculation of REROSHE with a calculation of the risk-adjusted value, or RAV, of the decision. The risk-adjusted value for an option is simply the present value of its cash flow, after each present-value element is adjusted for risk using the method set out by Longley-Cook.

Specifically, the risk-adjusted value is computed as follows:

$$\begin{aligned} \text{RAV} &= -\frac{1}{a} \ln \int_{-\infty}^{\infty} p(x) e^{-ax} dx \\ &= -\frac{1}{a} \ln M_x(-a) , \end{aligned}$$

where x is the amount of profit (loss) discounted to $t = 0$ at a risk-free rate of return. This formula is substantially the same as that cited by Longley-Cook.

The only information needed to compute RAV is the risk aversion, a, the probabilities associated with various outcomes, and a way to estimate the present value of the cost associated with each outcome at a risk-free rate of return.

You can use a return-on-investment criterion instead of a dollar-of-profit criterion if you wish. The risk-adjusted return (RAR) on an option is the interest rate such that the risk-adjusted value of the option is algebraically equal to zero:

$$-\frac{1}{a}\ln\int_{-\infty}^{\infty} p(y, t)\exp\left[-ay/(1 + \text{RAR})^{t}\right]dy\,dt = 0\,,$$

where y is the undiscounted value of the gain (or loss) x at time t.

I recommend you stay with RAV. If you choose to use REROSHE or RAR, you will have the same problems computing a unique interest rate that you have computing return on investment (ROI) in a risk-free environment. These problems have been cited in the literature, and include the following:

1. A more profitable opportunity may look worse.
2. There may be more than one RAR for an option. In particular,
   a) You might not identify all RARs for an option.
   b) What would it mean if one of the risk profile curves (to be explained below) (i) split in two or (ii) split, then rejoined, as your risk capacity increased?
3. It is computationally easier to compute RAV than RAR, because the only way to compute RAR is by computing RAV for various values of RAR in order to find the RAR that makes RAV approximately equal to zero.

The Simpler Version Can Be Explained to Top Management

If we ignore the complications associated with the risk-adjusted return and stay with the simpler calculations of risk-adjusted value, the method can be easily explained to top management by making reference to Figures 1, 2, 3 of this discussion. After appropriate explanations, top management will want to go directly to the diagram in Figure 3. John Cozzolino calls this a risk profile curve.[1]

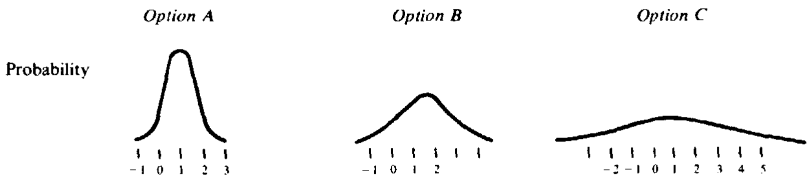

FIG. 1

[1] The mathematical construction presented here is essentially the same as that shown by John M. Cozzolino in "A New Method for Risk Analysis," *Sloan Management Review* (Massachusetts Institute of Technology), Spring 1979. The difference is that we have changed Cozzolino's risk aversion measure, r, to its reciprocal, which we have called risk capacity, c.

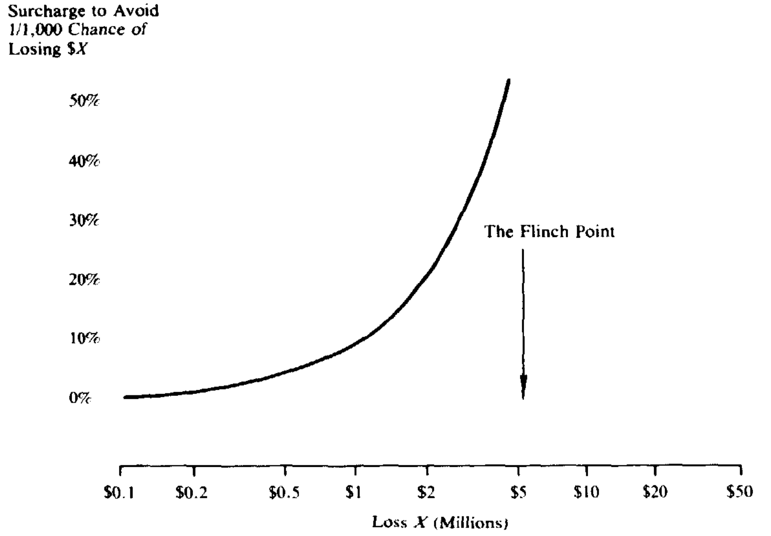

FIG. 2.—Surcharge-for-risk curve. (*Directions:* Move the bottom scale left or right until it is in the right place for your decision. Your risk capacity, c, will be below the line marked as the flinch point.)

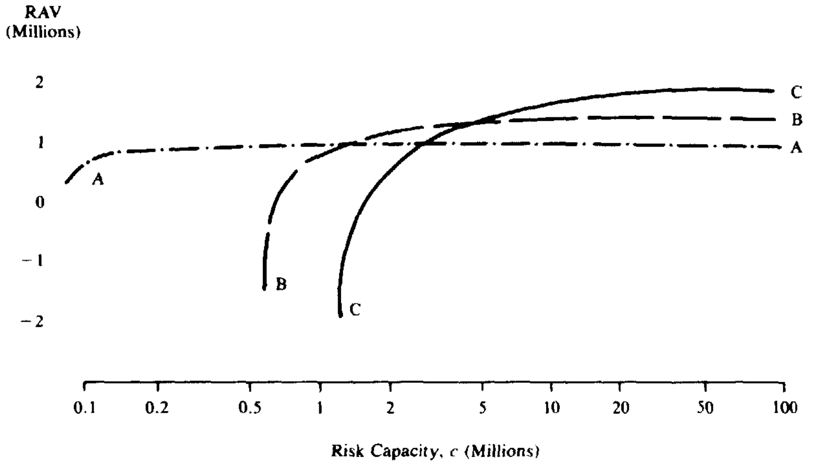

FIG. 3.—Risk profile curves

You Can Do It

You can develop probability distributions about the results of your options. You can do so exhaustively, using Monte Carlo simulation, or the rest of the bag of tricks developed under the heading of collective risk. Or you can just use some standard probability distributions. For example:

Apprehension about Risk	Standard Distribution
Normal concern	Normal
Losses are limited; or possible gains are limited	Gamma
Things could be very bad	Logistic

Naturally, there are a number of other distributions that may be more appropriate than these. You should feel comfortable applying your own knowledge of probability distributions to the problem at hand. But if you do not have much knowledge of probability distributions, you are better off to make a risk adjustment like this explicitly than simply to present top management with uncertainties.

Remember, you are modeling a distribution of the present value of the outcomes at the current risk-free rate of return. You can use the current United States Treasury rates for the term desired, if you want to be accurate. (For example, you may want this level of detail if you are using a Monte Carlo simulation.) Or, if you use one of the standard distributions, you can adjust your estimates of the parameters subjectively to reflect this.

You can get your firm's risk capacity, c, from management or simply by looking at your company's per-life retention. It often is not important to know c precisely, because you can make decisions using the risk profile curves.

The surcharge-for-risk curve (Fig. 2) is simply a plot of

$$\frac{\ln (0.999 + 0.001 e^{x/c}}{0.001^{x/c}} - 1,$$

where c, the risk capacity, is the reciprocal of Longley-Cook's parameter a. Plot the surcharge for risk with a log scale on the abscissa. The shape will then be like $e^{x/c}$ when x/c is about 1.0. This is why the curve bends sharply at $X = c$, your "flinch point."

It is easy to compute the risk profile curve for each option. The curve shows the values of

$$\text{RAV} = -c \ln \int_{-\infty}^{\infty} p(x) e^{-x/c} dx$$

for various values of c. Here are the specific steps:

1. *Monte Carlo simulation.* If your probability gives, say, 1,000 values of the gain, X, just take the following steps:
   - *a*) Adjust each X to be X/c. (If you haven't already done so, this is the time to multiply each X by the present-value factor.)
   - *b*) Exponentiate $-X/c$.
   - *c*) Average them.
   - *d*) Take the log.
   - *e*) Multiply by $-c$.

   Steps b–d: Give extra weight to risk. *Steps a–e: Adjust for your own risk capacity.*

   The result will be the risk-adjusted value for the 1,000 samples in your Monte Carlo simulation.
2. *Standard formula.* The following examples of the moment distributions of standard probability distributions will give you quick formulas for risk-adjusted values if you have assumed the parameters of the distribution of gains.

Distribution	RAV
Normal	$\mu - \frac{1}{2}\sigma^2/c$
Gamma	$\gamma + c\alpha \ln(1 + \beta/c)$
Logistic	$\alpha - c \ln[\Gamma(1 - \beta/c)\Gamma(1 + \beta/c)] \quad (\beta < c)$, where $\Gamma(a)$ is the gamma function.[2]

Summary

Mr. Longley-Cook has set forth a method for calculating risk-adjusted returns on business options that avoids the difficulties associated with the methods commonly in use today. Mr. Longley-Cook's contribution to the literature is important.

A slightly simpler method improves on Mr. Longley-Cook's formulation by setting aside the artificial concept of invested surplus. This version of risk calculations can be explained to top management and is simple enough to apply in practice.

[2] You don't have to look up the gamma function. Rearranging terms, we can write RAV $= \alpha - \beta f(\beta/c)$, where β/c and $f(\beta/c)$ take the following values:

β/c	$f(\beta/c)$	β/c	$f(\beta/c)$
.01	.0165	.75	1.6048
.05	.0822	.85	2.0846
.10	.1651	.90	2.4597
.15	.2486	.95	3.1037
.25	.4200	.99	4.6417
.50	.9032		

From this you can create tables of c and RAV for any values of α and β.

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1. *Introduction*

The purpose of this note is to clarify certain issues and to extend in a practical manner the interesting results of Mr. Longley-Cook on utility assessment. In particular, this discussion will clarify and extend his method of assessing the constant a (the absolute risk aversion measure) to be applicable to return distributions that are other than normally distributed, and to insurance companies that do not invest their total surplus in Treasury bills necessarily, but rather use some of their surplus for the purpose of operating their risky business or another risky investment. Additionally, this discussion will show how to perform the analysis when the exact statistical distribution form is not known with certainty.

2. *Longley-Cook's Determination of the Degree of Risk Aversion*

Formulas (8) and (9) developed by Longley-Cook actually cannot be used to estimate the risk-aversion measure a, for precisely the reasons he gave. In particular, as he noted in Section V, the company in fact did not invest entirely in T-bills; it chose to enter the insurance business. As noted by Longley-Cook, this is a riskier option than buying risk-free bonds, so it can be inferred that the risk aversion of the company is less than that implied by (9). However, the only inference possible from (9) concerns existence of an upper bound for a, and many companies with vastly different risk profiles could obtain the same upper bound.

The easiest way to see that (9) does not exhibit the risk characteristic of the company (even as an approximation for the recent years) is that nowhere does the actual operating policy of the company appear. Indeed, if two companies with distinctly different attitudes toward risk were both faced with an equivalent market having a return distribution with the same mean and variance, equation (9) would assign the same value of a, incorrectly, to both companies. Equation (9) relates the return on T-bills to the return on the surplus for a particular line of insurance, and does not involve the management's attitude toward risk. It follows that the estimate of a, and the corresponding graphical and potential additional uses cited (e.g., intercompany company comparisons of risk, as described in Sec. VII) cannot be supported as developed. A new development can be derived using a similar methodology, however, and this method is presented in the next section.

3. *A New Estimator of the Risk-Aversion Measure*

In many contexts it is more reasonable to model the rate of return on

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investment, rather than the corresponding actual dollar amount, as being random.

Let i denote the rate of return on a safe investment with a known rate of return (e.g., a well-diversified real estate portfolio, or Treasury bills) for the period under investigation, and I the (random) rate of return on the risky insurance business with surplus u_T. If the company acts in such a way as to maximize its end-of-period expected utility, then it will divide its surplus and invest a certain proportion p in the risky business and the remainder, $u_T(1 - p)$, in the safer asset. It will choose p to maximize $E\{U[u_T(1 - p)(i + 1) + u_T p(1 + I)]\}$. For the time being, refer to the safer asset as T-bills, and assume that their rate of return is known. Section 6 of this discussion shows how to generalize the results to possibly stochastic alternative investments that may exhibit dependence with the insurance business.

Using exponential utility $U(x) = -e^{-ax}$ and letting $M_I(t)$ denote the moment generating function of the random variable I, one obtains, for the expected utility,

$$E\{U[u_T + (1 - p)u_T i + pu_T I]\} = -M_I(-apu_T)\exp[-au_T - a(1 - p)u_T i] .$$

Taking the derivative with respect to p yields the first-order condition for p:

$$0 = \exp[-au_T - a(1 - p)u_T i]M_I'(-apu_T)au_T - \exp[-au_T - a(1 - p)u_T i]M_I(-apu_T)au_T i ,$$

or, equivalently, at the extremum p^*,

$$\frac{M_I'(-ap^*u_T)}{M_I(-ap^*u_T)} = i . \tag{1}$$

The second-order condition easily shows p^* to be maximizing.

If the distribution of I, the return on risky funds, is known, one may observe p, the proportion actually invested in the risky business, and equation (1) easily may be solved numerically for a, the risk-aversion coefficient. In the particular case where I has a normal distribution with mean μ and variance σ^2,

$$\sigma^2(-apu_T) + \mu = i ,$$

or

$$\hat{a} = \frac{\mu - i}{\sigma^2 p u_T} . \tag{2}$$

This estimate of a includes the management's attitude toward risk as well as the market-presented potential return distribution, since the actual proportion, p, of dollars invested by the management in the business is given explicitly.

To obtain the general formula in the situation of possibly nonnormal distribution of returns, let

$$\Psi(t) = \frac{M_I'(t)}{M_I(t)} = [\ln M_I(t)]' .$$

Then, from (1),

$$\hat{a} = \frac{-\Psi^{-1}(i)}{p u_T} . \tag{3}$$

If the firm is a utility maximizer, p should be close to p^* and hence a will be well estimated.[1]

4. *The Risk-free Equivalent Rate of Return*

As Longley-Cook does, define the REROSHE as the certainty equivalent rate of return R that corresponds to the expected utility of the chosen strategy; that is, R satisfies

$$\begin{aligned} U[u_T(1 + R)] &= E\{U[u_T(1 - p)(1 + i) + u_T p(1 + I)]\} \\ &= E\{U[u_T + (1 - p)u_T i + p u_T I]\} . \end{aligned}$$

This leads (in the exponential utility case) to the equation

$$-\exp[-au_T - au_T R] = -\exp[-au_T - au_T(1 - p)i]E(e^{-apu_T I}) ,$$

or

$$-au_T R = -au_T(1 - p)i + \ln M_I(-apu_T) ;$$

$$R = (1 - p)i - \frac{\ln M_I(-apu_T)}{au_T} . \tag{4}$$

[1] Bob Witt has pointed out that legal regulations may actually impose constraints upon the proportion p used in equation (1). Accordingly, one should actually perform constrained optimization in practice when implementing our procedure. For ease of presentation we shall not do this here.

For the case of normally distributed I,

$$R = (1 - p)i - \frac{\sigma^2(-apu_I)^2}{2au_I} + \frac{\mu apu_I}{au_I},$$

that is,

$$R = (1 - p)i + p\mu - \frac{\sigma^2 pu_I a}{2}, \tag{5}$$

which is similar to Longley-Cook's equation (6) if $p = 1$.

In practice, of course, μ will be greater than i because of increased necessary returns for increased risk. Thus $(1 - p)i + p\mu > i$. However, the determination of whether or not the REROSHE exceeds i depends upon the risk involved, owing to the final term in (5). With these new determinations of a and R, the potential application given by Longley-Cook now may be realized.

5. *The Case of the Incomplete Statistical Return Information*

If the return on the risky investment, I, is not normally distributed, then formulas (2) and (5) of this paper, and all the formulas of Longley-Cook, are inapplicable. If the exact statistical distribution of returns is known, however, formulas (3) and (4) of this paper may still be calculated. In many situations of practical interest the exact statistical form of the return distribution is not known, although certain pertinent characteristics (i.e, unimodality of the distribution, with mean μ, variance σ^2, and skewness $\rho = E(I - \mu)^3$) may be estimated or forecast. This section shows how to use this information to obtain upper and lower bounds on both the risk-aversion measure a and the risk equivalent return on shareholders' equity R.

The basis of this technique is the theory of Tchebycheff systems expounded in Karlin and Studden [4]. The following result is given in Brockett [1] and in Brockett and Cox [2].

THEOREM. *Suppose that I is unimodal with mode m, mean μ_I, variance σ_I^2, and third central moment (skewness) ρ_I and I is bounded between a and b. Let $h(x) \geq 0$, and let*

$$h^*(x) = \frac{1}{x}\int_0^x h(t + m)dt .$$

1. *If* $h^{(3)}(x) > 0$, *then the best possible bounds* $E[h(I)]$ *using only unimodality, mean, and variance are*

$$h^*(a - m)q + h^*(\xi_1)(1 - q) \leq E[h(I)] \leq h^*(\xi_2)(1 - p) + h^*(b - m)p\,, \tag{6}$$

where

$$\xi_1 = 2\mu_I - 2m - \frac{3\sigma_I^2 - (\mu_I - m)^2}{a + m - 2\mu_I}\,,$$

$$\xi_2 = 2\mu_I - 2m - \frac{3\sigma_I^2 - (\mu_I - m)^2}{b + m - 2\mu_I}\,,$$

$$p = \frac{3\sigma_I^2 - (\mu_I - m)^2}{3\sigma_I^2 - (\mu_I - m)^2 + (b + m + 2\mu_I)^2}\,,$$

$$q = \frac{3\sigma_I^2 - (\mu_I - m)^2}{3\sigma_I^2 - (\mu_I - m)^2 + (a + m - 2\mu_I)^2}\,.$$

2. *If* $h^{(4)}(x) > 0$, *then the best possible bounds on* $E[h(I)]$ *using only unimodality, mean, variance, and skewness are*

$$\begin{aligned} h^*(\eta_1)q + h^*(\eta_2)(1 - q) \\ &\leq E[h(I)] \\ &\leq h^*(a - m)p_1 + h^*(\xi)p_2 + h^*(b - m)(1 - p_1 - p_2)\,, \end{aligned}$$

where

$$\xi = \frac{\rho_V - (a + b - 2\mu_V - 2m)\sigma_V^2}{(a - m - \mu)(b - m - \mu) + \sigma_V^2} + \mu_V\,,$$

$$p_1 = \frac{\sigma_V^2 + (\xi - \mu_V)(b - m - \mu - \mu_V)}{(b - a)(\xi - a - m)}\,,$$

$$p_2 = \frac{\sigma_V^2(b - m - \mu)(a - m - \mu)}{(\xi - b + m)(\xi - a + m)}\,,$$

$$\eta_1 = \frac{\rho_V - \sqrt{(\rho_V^2 + 4\sigma_V^6)}}{2\sigma_V^2} + \mu_V\,,$$

$$\eta_2 = \frac{\rho_V + \sqrt{(\rho_V^2 + 4\sigma_V^6)}}{2\sigma_V^2} + \mu_V\,,$$

$$q = \frac{1}{2} + \frac{\rho_V}{\sqrt{(\rho_V^2 + 4\sigma^6)}}\,,$$

and

$$\mu_x = 2(\mu_l - m) ,$$

$$\sigma_x^2 = 3\sigma_l^2 - (\mu_l - m)^2 ,$$

$$\rho_x = 4\rho_l - 6(\mu_l - m)\sigma_l^2 + 2(\mu_l - m)^2 .$$

Since by equation (1) a is the intersection of $M'_l(-apu_l)$ and $iM_l(-apu_l)$, one may use this theorem and equation (1) to find bounds for a by finding bounds for $M'_l(-apu_l)$ and $iM_l(-apu_l)$, and then finding their intersection.

To find bounds for $M'_l(-apu_l) = E[l \exp(-apu_l l)]$, we let $h_1(x) = h_1(x, a) = x \exp(-apu_l x)$, so that

$$h_1^*(x) = \frac{1}{x}\int_m^{x+m} te^{-apu_l t}dt .$$

The best upper and lower bounds are given by taking the left- and right-hand sides of the bounds in the theorem. The resulting left- and right-hand sides are expectations of $h_1^*(x, a)$ and are functions of a. Call these functions $f_L(a)$ and $f_U(a)$, respectively (see Fig. 1 of this discussion).

Similarly, for $iMl(-apu_l) = E[i \exp(-apu_l l)]$, we have the function $h_2(x) = h_2(x, a) = i \exp(-apu_l x)$, so that

$$h_2^*(x) = i \exp(-apu_l m)\left[\frac{1 - \exp(-apu_l x)}{apu_l}\right] .$$

and the theorem gives bounding functions $g_L(a) \leq iM_l(-apu_l) \leq g_U(a)$ (see Fig. 1).

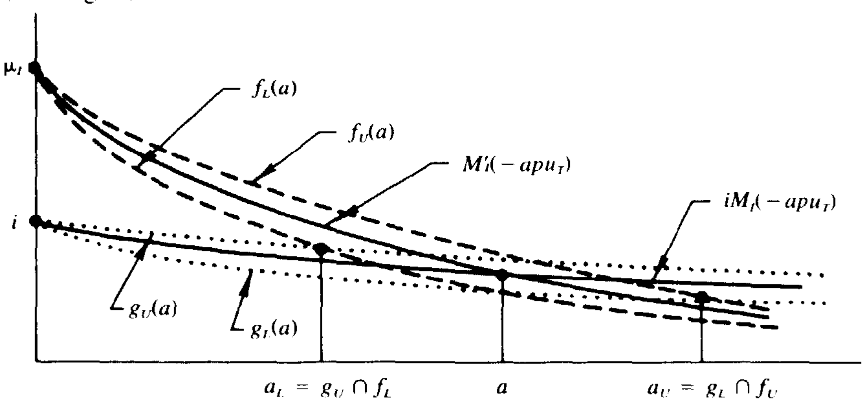

FIG. 1.—Upper and lower bounds for the curves $M'_l(-apu_l)$, $M_l(-apu_l)$ and the risk-aversion coefficient a.

Using the symbol $\cap$ for the intersection of two curves, we have the inequalities

$$a_2 = g_U \cap f_L \leq g_U \cap m_I'(-apu_T) \leq iM_I(-apu_T) \cap M_I'(-apu_T)$$
$$= a \leq g_L \cap M_I'(-apu_T) \leq g_L \cap f_U = a_U \ .$$

Thus, using only unimodality and the first few moments of the return distribution makes it possible to estimate a. Typically these will be rather tight bounds, since a will be close to zero (cf. Keeler, Newhouse, and Phelps [5], who argue that $a = 0.0005$ for buyers of health insurance, or Friedman [3], who estimates $a = 0.0025$). The bounding curves using k moments of the return distribution have the first k derivatives at zero agreeing with the curve to be bounded, so if the true intersection point a is close to zero, the upper and lower bounds a_L and a_U will be close to zero also.

Obtaining the appropriate bounds on R, the risk equivalent return of shareholders' equity, is simple matter using the bounds already obtained for $M_I(-apu_T)$ together with equation (4).

6. *The Case of Two Stochastic Alternatives*

Longley-Cook has pointed out in a private communication that the insurance company may not actually select a proportion of their surplus to invest in T-bills for risk management purposes as assumed in Section 3. Accordingly, this section shows how to generalize the previous results to a choice between two stochastic alternatives with the return on the first being I_1, and the return on the insurance business being denoted I_2. Again, as in Section 3, the end-of-period expected utility obtained by investing a proportion p in the insurance business and $(1 - p)$ in the alternative portfolio is

$$E\{U[u_T + (1 - p)u_T I_1 + pu_T I_2]\}$$
$$= E\{-\exp[-au_T - a(1 - p)u_T I_1 - apu_T I_2]\}$$
$$= -M(-au_T(1 - p), -au_T p)\exp(-au_T) \ ,$$

where $M(t_1, t_2)$ is now the joint moment generating function for the pair (I_1, I_2).

From this equation the first-order condition for optimal division p^* of wealth is easily obtained:

$$0 = E\{(I_1 - I_2)\exp[-au_T(1 - p)I_1 - au_T PI_2]\} \ . \qquad (7)$$

This reduces to (1) if I_1 is identically equal to i.

Inserting the observed value of p yields the estimate for a as before. In the case of independent returns I_1 and I_2, equation (7) reduces to

$$\frac{M'_{I_1}(-au_I(1-p))}{M_{I_1}(-au_I(1-p))} = \frac{M'_{I_2}(-apu_I)}{M_{I_2}(-apu_I)} . \tag{8}$$

In the case where (I_1, I_2) has a bivariate normal distribution with means (μ_1, μ_2), variances (σ_1^2, σ_2^2), and correlation coefficient ρ, the joint moment generating function is given by

$$M(t_1, t_2) = \exp\left[t_1\mu_1 + t_2\mu_2 + \tfrac{1}{2}(t_1^2\sigma_1^2 + 2\rho t_1 t_2\sigma_1\sigma_2 + t_2^2\sigma_2^2)\right] .$$

Accordingly, the first-order condition for p becomes equivalent to the equation

$$0 = \mu_1 au_I - a\mu_2 u_I - a^2u_I^2(1-p)\sigma_1^2 + a^2u_I^2p + \rho a^2u_I^2\sigma_1\sigma_2(1-2p) ,$$

and hence the estimate of the risk-aversion coefficient is

$$\hat{a} = \frac{\mu_2 - \mu_1}{u_T\sigma_2^2p + \rho u_T\sigma_1\sigma_2(1-2p) - u_T\sigma_1^2(1-p)} . \tag{9}$$

Again, this reduces to (2) if $\sigma_1^2 = 0$, $\mu_1 = i$.

Following the logic of Section 4 will also yield the certainty equivalent rate of return, R, in the stochastic choice situation.

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ELIAS S. W. SHIU:

In Section 2 of his paper [7], Pratt considers the following equations:

$$U(u + \mu - \alpha) = E[U(u + X)] , \quad (1)$$

$$U(u + \beta) = E[U(u + X)] , \quad (2)$$

and

$$U(u) = E[U(u + X - \gamma)] . \quad (3)$$

Equation (2) is equation (5) of the paper, and equation (3) is the equation in the footnote on the same page. Pratt calls α the *risk premium*, β the *cash equivalent*, and γ the *insurance premium*.

Applying the Taylor expansion formula to both sides of (2), we have

$$U(u + \mu) + (\beta - \mu)U'(u + \mu) + \ldots$$
$$= E[U(u + \mu) + (X - \mu)U'(u + \mu) + \tfrac{1}{2}(X - \mu)^2U''(u + \mu) + \ldots] .$$

Thus

$$(\beta - \mu)U'(u + \mu) \doteq \tfrac{1}{2}\sigma^2U''(u + \mu) ,$$

or

$$\beta \doteq \mu - \tfrac{1}{2}r(u + \mu)\sigma^2 , \quad (4)$$

where

$$r(x) = -U''(x)/U'(x) .$$

The function $r(x)$ is sometimes called the ***Arrow-Pratt absolute risk-aversion index***; various properties of $r(x)$ are given in [7] and in Section 4.5 of [6]. If REROSHE is defined as β/u, then formula (4) can be used to generalize formula (6) of the paper. Note that a variant of (4) also appears as exercise 10.a in chapter 1 of [4].

In the example given in Section VI the C line of business has a REROSHE less than the after-tax return on T-bills. Thus it seems that the company should close down this line of business completely and use the surplus for the expansion of lines A and B. However, there is another way to analyze the problem, following the approach of Markowitz's portfolio theory (cf. [10], p. 179, exercise 22). Consider a company with n lines of business and a total surplus of value u_t. Let R_i denote the rate of return on shareholders' equity for the ith line of business; R_i is a random variable. One wishes to determine the amount of surplus, u_i, to support the ith line of business so that the expected utility

$$E[U(\Sigma_i u_i + R_i u_i)] \tag{5}$$

is maximized.

Let

$$E(R_i) = r_i \; , \qquad \mathrm{Var}\,(R_i) = s_i^2 \; ,$$

and

$$\mathrm{Cov}\,(R_i, R_j) = \rho_{ij} s_i s_j \; .$$

If the R_i's are normally distributed, then expression (5) becomes

$$-\exp\,[\tfrac{1}{2}a^2\,\Sigma_i\Sigma_j\,\rho_{ij}s_is_ju_iu_j - a\,\Sigma_i\,(1 + r_i)u_i] \; ,$$

by equation (4) of the paper. Hence one should minimize

$$\tfrac{1}{2}a^2\,\Sigma\Sigma\rho_{ij}s_is_ju_iu_j - a\,\Sigma_i\,(1 + r_i)u_i \tag{6}$$

subject to the constraints that

$$u_1 \geqslant 0, \; . \; . \; . \; , u_n \geqslant 0 \tag{7}$$

and

$$u_1 + u_2 + \; . \; . \; . \; + u_n = u_I \; .$$

This is a *quadratic programming* problem (cf. [2], chap. v).

However, if we ignore the positivity constraints (7), the problem can be solved by the method of *Lagrange multipliers*. For simplicity, consider the case where the R_i's are independent random variables. Then expression (6) reduces to

$$\tfrac{1}{2}a^2\,\Sigma s_i^2u_i^2 - au_I - a\,\Sigma r_iu_i \; .$$

By the method of Lagrange multipliers one obtains the equalities

$$a^2s_1^2u_1 - ar_1 = a^2s_2^2u_2 - ar_2 = \; . \; . \; . \; = a^2s_n^2u_n - ar_n \; ,$$

or

$$\frac{u_1 - r_1/as_1^2}{1/s_1^2} = \frac{u_2 - r_2/as_2^2}{1/s_2^2} = \ldots = \frac{u_n - r_n/as_n^2}{1/s_n^2} .$$

Using the fact that

$$\frac{a}{b} = \frac{c}{d}$$

implies

$$\frac{a}{b} = \frac{c}{d} = \frac{a + c}{b + d} ,$$

then

$$\frac{u_j - r_j/as_j^2}{1/s_j^2} = \frac{\Sigma u_i - (1/a)\Sigma(r_i/s_i^2)}{\Sigma(1/s_i^2)} \quad \text{for } j = 1, 2, \ldots, n .$$

Thus the *optimal* amount of surplus to be allotted to support the jth line of business is

$$u_j = \frac{r_j}{as_j^2} + \frac{1}{s_j^2}\left[\frac{u_T - (1/a)\Sigma(r_i/s_i^2)}{\Sigma(1/s_i^2)}\right] .$$

One might question the validity of the assumption that the probability distribution of future earnings is a normal distribution. In the business of insurance, profits are bounded above by the total premiums received, while losses can be catastrophic. Thus a more appropriate probability density function is one with a truncated right-hand tail and a long, thin left-hand tail. A good candidate for such a function is the *negatively skew lognormal distribution* ([1], Sec. 2.9). It is interesting that Cozzolino and Zahner [5], using the *principle of maximum entropy*, have proved that the distribution of future stock prices is lognormal if the investor's utility function is logarithmic.

This paper has pointed out an interesting application of utility theory to insurance. Utility theory is a powerful tool and can be used to solve many insurance problems. Some brilliant applications of utility theory to modern insurance methods, particularly those of reinsurance, can be found in Borch [3] (also see [9], chap. 6). The recent book by Schoemaker [8] on utility theory has an interesting chapter about an experimental study on insurance decisions.

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DONALD R. SONDERGELD:

I would like to congratulate the author on an excellent addition to actuarial literature.

The major yardstick my company uses to measure the expected profitability of new products is the internal rate of return (IRR). This is the return we expect to earn on the after-tax investment of surplus in new business. The investment includes something I call statutory benchmark surplus. This was described in my paper *Profitability as a Return on Total Capital*, published in Volume XXXIV of the *Transactions*.

I believe we should price our insurance products and services with the expectation of producing a higher return than we could get by otherwise investing capital and surplus in securities. For illustration, let us assume that our minimum target rate is 12 percent. Does this mean that we should price each of our products with an expected IRR of only 12 percent? No. I have argued that for those lines of business that are more volatile, we need a higher expected IRR. Others have said that the lines of business, or products, whose earnings fluctuate a great deal have a heavier benchmark surplus component that adjusts automatically for the greater volatility. The concepts in the Longley-Cook paper appear to buttress my thinking.

The paper deals with the determination of an expected risk-free equivalent return on GAAP capital and surplus for one calendar year at a time. I wonder whether Mr. Longley-Cook could comment on how this might be extended to the determination of a risk-free equivalent statutory IRR on products that cover many policy years. I would like, for example, to be able to tell management that a particular product should produce an expected IRR of 22 percent but that the risk-free equivalent IRR is 12 percent.

(AUTHOR'S REVIEW OF DISCUSSION)

ALASTAIR G. LONGLEY-COOK:

I am delighted that my paper has generated four discussions. It is this kind of generalization of the basic concept, extension into parallel applications, and comparison with other research that I hoped my paper would elicit.

Looking at the four discussions together, it is apparent that return on surplus, or allocated surplus, means different things to different people. Let me expand upon what I believe ROSHE and REROSHE mean.

The surplus of a life insurance company is an accounting item. It is the difference between the assets and the liabilities on the balance sheet. Once the accounting methods have been established for calculating the assets and liabilities, the amount of surplus is the residual item.

The investment department of a life insurance company does not invest surplus. It invests positive cash flow. It does not choose how much of that positive cash flow to invest in Treasury bills on the basis of the company's degree of risk aversion. Instead, the amount of T-bills in an investment portfolio is determined largely by the degree of liquidity the company believes it needs.

ROSHE, since it is calculated entirely from accounting items, is also an accounting item, and is related only indirectly to any investments (in the usual meaning of the word) that the company might be making at the time. ROSHE is earnings divided by surplus, and, strictly speaking, that is all it is.

It does, however, become a useful index in comparing one company with another or one time period with another within one company. Once ROSHE is applied in this manner, it is easy to view it as the return on an "investment" of an amount of money equal to surplus; but it is an investment in the business, not an investment in the usual sense of the word, by the financial department of the company.

With these thoughts in mind I do not see the problem Mr. Van Slyke finds in using return on surplus in this manner.

My paper uses a comparison between the exhibited ROSHE of the company and the return on T-bills to determine the company's degree of risk aversion. The implication is that the company had a choice of pursuing various lines of business with various degrees of return and risk. Comparing a particular company's return on a line of business with risk with the return on an investment with no risk (T-bills) is merely a way of comparing all of the companies with a common index. The use of T-bills bears no relation to the actual amount of T-bills a company actually may have invested its cash flow in during that period of time.

Nor do I think it possible that a company would drop everything and invest all its surplus in T-bills. The comparison with investment of T-bills is merely a comparison with a fixed point—a fixed star, as it were, from which ships can chart their progress.

For this reason I do not agree with Professor Brockett's contention that my method cannot be used to estimate the degree of risk aversion of a company. One has only to look at the record of various companies' returns as compared with the variability in those returns (as exhibited in my graphs) to see that there is a wide dispersion among such companies. More important, the greater return that one company is earning over another is not always commensurate with the additional variability in that return, that is, different companies seem to have varying degrees of aversion to such variability (which I am using as a measure of risk).

Comparing the degrees of risk aversion that companies have exhibited may therefore be achieved by comparing their return/risk data points with a single index point (the return on riskless T-bills). None of those companies (despite Professor Brockett's contention) maximized their end-of-period expected utility by dividing their surplus and investing a certain proportion, p, in a risky business and the remainder in a safer asset.

Another layer of confusion emerges as we try to apply the REROSHE approach to a line within a company. Here we are dealing with allocated surplus. Surplus is usually allocated to a line on the basis of some formula (e.g., x percent of premiums + y percent of reserves). This formula should and usually does bear a strong relationship to the "risk of ruin" within that line. I think this is appropriate, and I believe that the earnings of that line divided by the allocated surplus of that line can be a meaningful measure of performance. I do not believe, as Mr. Shiu indicates, that such surplus should be, or ever is, allocated on the basis of a methodology that ensures the maximization of expected utility.

Let us take one final step and descend to the level of an individual product within a line.

Theoretically, the same REROSHE techniques could be applied if one felt confident that the allocated surplus at that level was a meaningful number and that one could measure adequately the variance in earnings and the degree of risk aversion to be assumed at that level. Except perhaps for major product lines, I believe we begin to get on thin ice at this point.

The other problem that can emerge at this level results from applying these techniques to the usual methods of measuring profitability. One of the most commonly used measures of profitability at this level is present value of book profits (PVBP) or internal rate of return (IRR)—the latter being that discount rate which sets the PVBP equal to zero. It is the application of REROSHE techniques to these measures which Mr. Sondergeld raises the possibility of, and for which Mr. Van Slyke provides detailed methodology.

Mr. Van Slyke has ably illustrated the application of REROSHE techniques to profitability measures of this type. The reader should be cautioned, however, that the words *profit* and *surplus* are not used as synonyms.

First, when dealing with PVBP and IRR, we are commonly talking about statutory profits as opposed to the GAAP profits which are commonly used in ROSHE.

Second, IRR is commonly defined as being equivalent to return on investment (ROI). It is usually the case for such products as whole life insurance that the first-year book profit is negative and the remaining are positive. The ROI or IRR is therefore equivalent to the rate of return on an investment of the first year's statutory loss, where the returns are equal to the statutory gains in the following years. This first-year "invested surplus" is not the same as that contained in ROSHE, where the "invested surplus" is the surplus allocated to support that product from a risk standpoint. Mr. Sondergeld explores this further in his paper, cited in his discussion.

Once these differences are straight in one's mind, one can see that Mr. Van Slyke's methodology is mathematically parallel to mine.

Instead of my criterion (5),

$$E[U(u + Ru)] = E[U(u + X)] ,$$

Mr. Van Slyke has used the following criterion:

$$E[U(RAV)] = E[U(x)] \ ,$$

where RAV is defined as "the present value of . . . cash flow, after each present-value element is adjusted for risk." In other words, RAV is a risk-free equivalent present value of cash flow. If instead of cash flow we use book profit, then we have a criterion that allows us to solve for the risk equivalent PVBP. For a product line such as whole life insurance with extremely uneven book profits, this can be an important profitability measure. In a similar vein, a risk equivalent IRR can be calculated by solving for that discount rate that sets the REPVBP equal to zero. Mr. Van Slyke refers to this value as RAR. This, I believe, is the measure Mr. Sondergeld was seeking. Again, RAR $\neq$ REROSHE just as IRR $\neq$ ROSHE.

How useful these concepts will actually be in practice when, say, one is pricing out a new life insurance product I am not sure. I believe the problems mentioned earlier about assumptions as to variability and risk aversion at this level begin to get significant. For my own part, I would prefer to apply the REROSHE technique as originally described in my paper to an entire division or line of business, set a minimum E(ROSHE) standard, as described in the paper, and then use a model-office projection to ensure that the E(ROSHE) of the new product is indeed greater than that minimum.