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LIFE INSURANCE TRANSFORMATIONS

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ABSTRACT

This paper develops mathematical transformations of variable interest credit and mortality charge scales to simplify calculation of policyholder cash value accumulation and related amounts. These transformations are of theoretical as well as practical interest. Additionally, they suggest new product designs.

INTRODUCTION

Many life insurance products today accumulate policyholder cash values by applying interest credits and mortality charges that are subject to change from month to month. The possibility of fluctuating credits and charges necessitates the use of a complex cash value accumulation formula. A disadvantage of complex formulation is that computation of associated premiums is quite involved. However, mathematical transformations applied to scales of interest credits and mortality charges can remove the complexity from the cash value accumulation formula, thereby greatly easing related premium and other computations.

This paper develops these transformations, illustrates some practical uses, and makes the case that these transformations also are of theoretical interest. Finally, two product possibilities resulting from these transformations are presented.

CASH VALUE ACCUMULATION FORMULAS

The following formula is typically used for accumulating cash values under a life insurance product with variable, nonguaranteed interest credits and mortality charges:

$$[0V + P - Q\,(1/(1+ig) - 0V - P)]\,(1+ic) = 1V. \qquad (1)$$

Formula (1) is more complex than is formula (2), used for fully guaranteed products and taught in life contingencies courses:

$$[(0V + P)\,(1+ig) - Q]\,/\,(1-Q) = 1V. \qquad (2)$$

Definitions of the terms used in these formulas are

$0V$ cash value at the beginning of the period
$1V$ cash value at the end of the period

P premium paid at the beginning of the period, after deducting premium loads
ig guaranteed interest rate
ic current interest rate
Q current mortality charge per \$1 of coverage for the period.

Formula (2) is mathematically more desirable than formula (1) because of computational ease. Resulting from it are many of the relationships between insurance values and annuity values that are taught in life contingencies courses. Commutation functions can be used to calculate values of P to be used with formula (2) to produce any desired plan of insurance.

If formula (1) could be made to look like formula (2), via transformations of ic, ig, and Q, then it would gain the advantages possessed by formula (2).

In the remainder of this paper, the symbol A will represent the single premium for \$1 of life insurance and the symbol a will represent the present value of a life annuity due of \$1 per period. Where the value at a particular age is required, the age will be indicated in parentheses following the symbol.

The Simple Case

Often insurers are called upon to make calculations using formula (1) while assuming that ic will always equal ig. Letting $i = ic = ig$, we obtain a simple case of formula (1) which can be written as

$$(0V + P)\,(1+i) = (1V + Q)\,/\,(1 + Q). \tag{3}$$

But formula (2) can be written as

$$(0V + P)\,(1+i) = Q + (1-Q)\,1V. \tag{4}$$

We require a transformation of Q into Q' such that formula (4), written with Q', is equivalent to formula (3). In other words the following relationship must hold:

$$(1V + Q)\,/\,(1 + Q) = Q' + (1 - Q')\,1V. \tag{5}$$

Formula (5) simplifies to

$$Q' = Q\,/\,(1 + Q). \tag{6}$$

In the simple case where $ic = ig = i$, the transformation (6) of the scale of cost of insurance will simplify the cash value accumulation formula.

To illustrate a practical use of the simple-case transformation, imagine a purely back-end loaded Excess Interest Whole Life product guaranteeing 5.5 percent interest on the cash value accumulation. The insurer wishes to determine gross premiums which will reproduce endowment at age 95 coverage using 5.5 percent and the cost of insurance scale guaranteed for use with the cash value accumulation formula. This might be best accomplished for issue ages 0 through 94 in the following steps:

1. Set $Q(94)$ equal to 1 (to allow for the endowment).
2. Set $A(95)$ equal to 1.
3. Set $Q'(x)$ equal to $Q(x)/(1+Q(x))$ for x equal 0 through 94.
4. Calculate $A(x)$ for x equal 94 through 0, with the recursion formula $A(x) = [Q'(x) + (1-Q'(x))\,A(x+1)]\ /\ 1.055$.
5. Calculate the gross premiums as $dA(x)\ /\ (1 - A(x))$, where d is $.055\ /\ 1.055$.

These steps are illustrated in Table 1 in spreadsheet form. Alternatively, commutation functions could have been calculated from $Q(x)$ and used in computing premiums. Either way, the resulting gross premiums will be exact to as many decimal places as are carried through the process.

TABLE 1

THE SIMPLE CASE

INTEREST RATE 0.055

COI SCALE 1980 CSO MALE ALB

Age	COI Per Thousand	$Q'(x)$	$A(x)$	Endowment at 95 Premium
0	2.63	0.002623	0.043320	2.36
1	1.03	0.001028	0.043193	2.35
2	0.99	0.000989	0.044586	2.43
3	0.97	0.000969	0.046094	2.52
4	0.93	0.000929	0.047707	2.61
5	0.88	0.000879	0.049447	2.71
6	0.83	0.000829	0.051333	2.82
7	0.78	0.000779	0.053371	2.94
8	0.75	0.000749	0.055571	3.07
9	0.74	0.000739	0.057921	3.21
10	0.75	0.000749	0.060412	3.35
11	0.81	0.000809	0.063032	3.51
12	0.92	0.000919	0.065743	3.67
13	1.07	0.001068	0.068503	3.83
14	1.24	0.001238	0.071278	4.00
15	1.42	0.001417	0.074051	4.17
16	1.59	0.001587	0.076815	4.34
17	1.72	0.001717	0.079579	4.51
18	1.82	0.001816	0.082380	4.68
19	1.88	0.001876	0.085249	4.86
20	1.90	0.001896	0.088227	5.04
21	1.90	0.001896	0.091356	5.24
22	1.88	0.001876	0.094664	5.45
23	1.84	0.001836	0.098178	5.68
24	1.80	0.001796	0.101929	5.92
25	1.75	0.001746	0.105928	6.18
26	1.72	0.001717	0.110200	6.46
27	1.71	0.001707	0.114741	6.76
28	1.70	0.001697	0.119549	7.08
29	1.72	0.001717	0.124639	7.42

TABLE 1—*Continued*

Age	COI Per Thousand	$Q'(x)$	$A(x)$	Endowment at 95 Premium
30	1.75	0.001746	0.130000	7.79
31	1.80	0.001796	0.135640	8.18
32	1.87	0.001866	0.141558	8.60
33	1.95	0.001946	0.147753	9.04
34	2.05	0.002045	0.154233	9.51
35	2.17	0.002165	0.161000	10.00
36	2.32	0.002314	0.168053	10.53
37	2.49	0.002483	0.175387	11.09
38	2.68	0.002672	0.183004	11.68
39	2.90	0.002891	0.190907	12.30
40	3.15	0.003140	0.199091	12.96
41	3.42	0.003408	0.207553	13.65
42	3.71	0.003696	0.216297	14.39
43	4.03	0.004013	0.225330	15.16
44	4.37	0.004350	0.234651	15.98
45	4.73	0.004707	0.244269	16.85
46	5.12	0.005093	0.254193	17.77
47	5.53	0.005499	0.264426	18.74
48	5.97	0.005934	0.274983	19.77
49	6.46	0.006418	0.285869	20.87
50	7.00	0.006951	0.297080	22.03
51	7.63	0.007572	0.308613	23.27
52	8.33	0.008261	0.320441	24.58
53	9.13	0.009047	0.332551	25.97
54	10.01	0.009910	0.344915	27.45
55	10.96	0.010841	0.357518	29.01
56	11.97	0.011828	0.370355	30.66
57	13.04	0.012872	0.383432	32.42
58	14.18	0.013981	0.396756	34.29
59	15.42	0.015185	0.410333	36.28
60	16.80	0.016522	0.424156	38.40
61	18.36	0.018028	0.438203	40.66
62	20.12	0.019723	0.452432	43.08
63	22.09	0.021612	0.466799	45.64
64	24.27	0.023694	0.481262	48.37
65	26.62	0.025929	0.495784	51.26
66	29.13	0.028305	0.510356	54.34
67	31.79	0.030810	0.524979	57.62
68	34.65	0.033489	0.539670	61.12
69	37.81	0.036432	0.554430	64.87
70	41.37	0.039726	0.569230	68.89
71	45.43	0.043455	0.584012	73.19
72	50.08	0.047691	0.598694	77.77
73	55.34	0.052438	0.613173	82.64
74	61.10	0.057581	0.627357	87.77
75	67.25	0.063012	0.641202	93.17
76	73.70	0.068641	0.654710	98.85
77	80.37	0.074391	0.667926	104.86
78	87.32	0.080307	0.680925	111.25
79	94.76	0.086557	0.693785	118.12
80	102.94	0.093332	0.706542	125.52
81	112.09	0.100792	0.719194	133.52
82	122.41	0.109059	0.731707	142.18
83	133.84	0.118041	0.744036	151.54
84	146.12	0.127491	0.756177	161.68

TABLE 1—*Continued*

Age	COI Per Thousand	$Q'(x)$	$A(x)$	Endowment at 95 Premium
85	158.98	0.137172	0.768216	172.79
86	172.21	0.146910	0.780337	185.20
87	185.73	0.156637	0.792818	199.50
88	199.53	0.166340	0.806042	216.65
89	213.69	0.176066	0.820519	238.33
90	228.43	0.185952	0.836939	267.58
91	244.11	0.196212	0.856237	310.50
92	261.43	0.207248	0.879732	381.34
93	282.13	0.220047	0.909326	522.81
94	309.97	0.236623	0.947867	947.87

The General Case

Usually formula (1) is applied with ic not equal to ig. Examples would include illustrations of various prospective interest-crediting rate patterns as part of a sales proposal and day-to-day administration of products using this type of formula.

The following develops the transformations required to simplify formula (1) to formula (2) for the general case where ic is not equal to ig.

Formula (1) can be rewritten as

$$(0V + P)(1+ic) = [1V + Q(1+ic)/(1+ig)]/(1 + Q). \quad (7)$$

As a first simplifying step, let $Q'' = Q(1+ic)/(1+ig)$ and then rewrite (7) as

$$(0V + P)(1+ic) = (1V + Q'')/[1 + Q''(1+ig)/(1+ic)]. \quad (8)$$

We require a transformation such that formula (8) can be rewritten in the form

$$(0V + P)(1+i') = (1V + Q'')/(1 + Q''). \quad (9)$$

Such a transformation would yield the simple case discussed previously; compare formula (3) with formula (9). To obtain the simple case, the following relationship must hold:

$$(1+i')(1+ Q'') = (1+ic)[1 + Q''(1+ig)/(1+ic)]. \quad (10)$$

Formula (10) simplifies to

$$i' = (ic + Q''ig)/(1 + Q''). \quad (11)$$

The transformation (11) gives the simple case, so we also require the transformation (6) on Q'':

$$Q' = Q''/(1 + Q''). \quad (12)$$

To summarize, we have set the following values:

$$Q'' = Q\,(1+ic)\,/\,(1+ig)$$
$$i' = (ic + Q''ig)\,/\,(1 + Q'')$$
$$Q' = Q''\,/\,(1 + Q'').$$

Q'' is an intermediate value which may ease calculations but can be eliminated:

$$i' = [ic\,(1+ig) + Q\,(1+ic)\,ig]\,/\,[1 + ig + Q\,(1+ic)]$$
$$Q' = Q\,(1+ic)\,/\,[1 + ig + Q\,(1+ic)]. \tag{13}$$

In the general case where ic is not equal to ig, the transformations (13) will simplify the cash value accumulation formula. If ic does equal ig, (13) becomes (6), the simple-case transformation.

To illustrate a practical use of the general-case transformation, imagine a purely back-end loaded Excess Interest Whole Life product with a guaranteed interest rate on a fund of 4 percent. A prospective insured aged 35 wishes to see an illustration of whole life coverage with an interest crediting rate of 10 percent. This might be best accomplished for all durations in the following steps:

1. Set $Q(99)$ equal to 1.
2. Set $A(100)$ equal to 1.
3. Set $a(99)$ equal to 1.
4. Calculate $Q'(x)$ as in (13) for x equal 35 through 99.
5. Calculate $i'(x)$ as in (13) for x equal 35 through 99.
6. Calculate $A(x)$ for x equal 99 through 35 with the recursion formula $A(x) = [Q'(x) + (1-Q'(x))\,A(x+1)]\,/\,(1+ i'(x))$.
7. Calculate $a(x)$ for x equal 98 through 35 with the recursion formula $a(x) = a(x+1)\,/\,(1+i'(x))\,/\,(1-Q'(x)) + 1$.
8. Calculate $P(x)$ for x equal 99 through 35 as $A(x)\,/\,a(x)$. $P(35)$ is the illustrative whole life premium.
9. Calculate the fund accumulations at the end of policy year t (t = 0 through 64) as $(P(35+t) - P(35))\,a(35+t)$.

The last formula (for the fund accumulations) will be recognized as a formula for terminal reserves under fully guaranteed life insurance. Under the transformations (6) and (13), fund accumulations are mathematically the same as terminal reserves.

These steps are illustrated in Table 2 in spreadsheet form. Alternatively, commutation functions could have been calculated using $i'(x)$ and $Q'(x)$ and used in making computations. Either way, the resulting premiums and fund accumulations will be exact to as many decimal places as are carried through the process.

In general, $i'(x)$ varies with the age x. The effect of having ic and ig constant by policy duration is that the $i'(x)$ and $Q'(x)$ columns can be calculated once and used for any issue age. If ic or ig is not constant by policy duration, then $i'(x)$ and $Q'(x)$ must be recalculated each time the issue age changes.

TABLE 2

THE GENERAL CASE
INTEREST RATES

GUARANTEED	0.04
CURRENT	0.10

Age	Current COI per Thousand	$Q'(x)$	$i'(x)$	$A(X)$	$a(X)$	$P(X)$	Fund @ t	t
35	2.00	0.002111	0.099873	0.052458	10.454430	5.02	0.00	0
36	2.06	0.002174	0.099870	0.055704	10.420672	5.35	3.42	1
37	2.14	0.002258	0.099864	0.059222	10.384087	5.70	7.12	2
38	2.24	0.002364	0.099858	0.063020	10.344586	6.09	11.11	3
39	2.36	0.002490	0.099851	0.067108	10.302069	6.51	15.41	4
40	2.50	0.002637	0.099842	0.071497	10.256425	6.97	20.03	5
41	2.65	0.002795	0.099832	0.076199	10.207522	7.47	24.98	6
42	2.82	0.002974	0.099822	0.081238	10.155114	8.00	30.28	7
43	3.01	0.003174	0.099810	0.086632	10.099025	8.58	35.96	8
44	3.24	0.003415	0.099795	0.092398	10.039054	9.20	42.02	9
45	3.50	0.003688	0.099779	0.098540	9.975175	9.88	48.49	10
46	3.82	0.004024	0.099759	0.105072	9.907247	10.61	55.36	11
47	4.19	0.004412	0.099735	0.111980	9.835400	11.39	62.63	12
48	4.60	0.004842	0.099709	0.119262	9.759662	12.22	70.29	13
49	5.04	0.005303	0.099682	0.126926	9.679952	13.11	78.35	14
50	5.50	0.005784	0.099653	0.134992	9.596069	14.07	86.84	15
51	5.96	0.006264	0.099624	0.143491	9.507682	15.09	95.78	16
52	6.45	0.006776	0.099593	0.152477	9.414227	16.20	105.24	17
53	6.97	0.007318	0.099561	0.161984	9.315348	17.39	115.24	18
54	7.56	0.007933	0.099524	0.172052	9.210637	18.68	125.83	19
55	8.25	0.008650	0.099481	0.182692	9.099980	20.08	137.03	20
56	9.03	0.009461	0.099432	0.193893	8.983486	21.58	148.82	21
57	9.90	0.010363	0.099378	0.205657	8.861134	23.21	161.19	22
58	10.88	0.011377	0.099317	0.217992	8.732856	24.96	174.17	23
59	11.99	0.012523	0.099249	0.230892	8.598688	26.85	187.75	24
60	13.25	0.013821	0.099171	0.244345	8.458776	28.89	201.90	25
61	14.69	0.015300	0.099082	0.258326	8.313365	31.07	216.61	26
62	16.31	0.016958	0.098982	0.272795	8.162878	33.42	231.84	27
63	18.11	0.018795	0.098872	0.287718	8.007675	35.93	247.54	28
64	20.09	0.020807	0.098752	0.303067	7.848043	38.62	263.69	29
65	22.25	0.022993	0.098620	0.318822	7.684182	41.49	280.26	30
66	24.56	0.025319	0.098481	0.334974	7.516195	44.57	297.26	31
67	27.04	0.027805	0.098332	0.351544	7.343856	47.87	314.69	32
68	29.79	0.030546	0.098167	0.368555	7.166934	51.42	332.59	33
69	32.89	0.033618	0.097983	0.385979	6.985711	55.25	350.93	34
70	36.45	0.037122	0.097773	0.403753	6.800839	59.37	369.63	35
71	40.58	0.041155	0.097531	0.421764	6.613508	63.77	388.58	36
72	45.26	0.045684	0.097259	0.439846	6.425435	68.45	407.60	37

TABLE 2—*Continued*

Age	Current COI per Thousand	$Q'(x)$	$i'(x)$	$A(X)$	$a(X)$	$P(X)$	Fund @ i	t
73	50.43	0.050638	0.096962	0.457858	6.238089	73.40	426.56	38
74	55.99	0.055909	0.096645	0.475703	6.052470	78.60	445.33	39
75	61.85	0.061401	0.096316	0.493351	5.868894	84.06	463.90	40
76	67.98	0.067079	0.095975	0.510833	5.687038	89.82	482.30	41
77	74.50	0.073042	0.095617	0.528213	5.506229	95.93	500.58	42
78	81.53	0.079388	0.095237	0.545524	5.326138	102.42	518.80	43
79	89.26	0.086265	0.094824	0.562767	5.146733	109.34	536.94	44
80	97.85	0.093789	0.094373	0.579890	4.968557	116.71	554.96	45
81	107.62	0.102196	0.093868	0.596800	4.792568	124.53	572.75	46
82	118.55	0.111419	0.093315	0.613301	4.620797	132.73	590.12	47
83	130.39	0.121198	0.092728	0.629220	4.455047	141.24	606.87	48
84	142.78	0.131203	0.092128	0.644478	4.296105	150.01	622.92	49
85	155.45	0.141202	0.091528	0.659128	4.143395	159.08	638.34	50
86	168.27	0.151088	0.090935	0.673331	3.995240	168.53	653.28	51
87	181.32	0.160920	0.090345	0.687317	3.849173	178.56	668.00	52
88	195.06	0.171028	0.089738	0.701355	3.702364	189.43	682.78	53
89	210.12	0.181832	0.089090	0.715664	3.552436	201.46	697.84	54
90	227.00	0.193611	0.088383	0.730401	3.397629	214.97	713.35	55
91	246.13	0.206557	0.087607	0.745726	3.236080	230.44	729.49	56
92	266.55	0.219925	0.086805	0.761868	3.065091	248.56	746.49	57
93	285.47	0.231915	0.086085	0.779511	2.877095	270.94	765.07	58
94	311.27	0.247684	0.085139	0.800303	2.654244	301.52	786.98	59
95	400.00	0.297297	0.082162	0.825126	2.386077	345.81	813.15	60
96	500.00	0.345912	0.079245	0.847617	2.134559	397.09	836.91	61
97	600.00	0.388235	0.076706	0.869722	1.872022	464.59	860.33	62
98	700.00	0.425414	0.074475	0.896096	1.534759	583.87	888.39	63
99	1000.00	0.514019	0.069159	0.935315	1.000000	935.31	930.30	64
100	—	—	—	1.000000	—	—	1000.00	65

Option B Universal Life

Universal Life insurance products almost always offer the insured the choice of "Option A" or "Option B' coverage (this terminology is common, but not ubiquitous). "Option B" is herein taken to mean that the death benefit equals the face amount plus the fund accumulation. An analogous design is possible with fully guaranteed insurance, but has rarely been marketed, possibly due to administrative difficulties. With the advent of Universal Life, Option B coverage gained a significant portion of the market.

The typical Option B accumulation formula is

$$\{0V + P - Q\,[(1 + 0V + P) / (1+ig) - 0V - P]\}\,(1+ic) = 1V. \quad (14)$$

In the simple case, where ic equals ig equals i, (14) becomes

$$(0V + P)\,(1+i+Qi) = 1V + Q. \quad (15)$$

Formula (15) will take the form of formula (3) (the level-death-benefit, simple-case formula) if the following relationship holds:

$$1+i+Qi = (1+i')\,(1+Q). \quad (16)$$

Solving for i',

$$i' = i - Q / (1+Q) \tag{17}$$

With the transformation (17), (15) becomes

$$(0V + P)(1+i') = (1V + Q) / (1 + Q). \tag{18}$$

This is formula (3), except that i is now i', so the level death benefit simple-case transformation (6) will complete the task.

To summarize, if ic equals ig equals i, the transformations

$$\begin{aligned} i' &= i - Q / (1+Q) \\ Q' &= Q / (1 + Q) \end{aligned} \tag{19}$$

allow the Option B accumulation formula (14) to be rewritten as

$$[(0V + P)(1+i') - Q'] / (1-Q') = 1V. \tag{20}$$

This is in the form of (2), and is the formula which greatly eases premium and cash value calculations.

In the general case, ig and ic are not equal. For notational ease, let r equal $(1+ic)/(1+ig)$. Then (14) can be rewritten as

$$(0V + P)[(1+Q)(1+ic) - Qr] = 1V + Qr. \tag{21}$$

Now let Q'' equal Qr. Then we have

$$(0V + P)[(1 + Q''/r)(1+ic) - Q''] = 1V + Q''. \tag{22}$$

Formula (22) simplifies to

$$(0V + P)(1+ic+Q''ig) = 1V + Q''. \tag{23}$$

Letting i'' equal $(ic+Q''ig) / (1+Q'')$, formula (23) becomes

$$(0V + P)(1+i''+Q''i'') = 1V + Q''. \tag{24}$$

Since (24) is in the form of (15), the transformations (19) will complete the task.

To summarize, we have set the following values:

$$\begin{aligned} Q'' &= Q(1+ic) / (1+ig) \\ i'' &= (ic+Q''ig) / (1+Q'') \\ i' &= i'' - Q'' / (1+Q'') \\ Q' &= Q'' / (1 + Q''). \end{aligned}$$

Q'' and i'' are intermediate values which may ease calculations but can be eliminated:

$$\begin{aligned} Q' &= Q\,(1+ic)\,/\,[1 + ig + Q\,(1+ic)] \\ i' &= [(1+ig)\,ic - Q\,(1+ic)\,(1-ig)]\,/\,[1 + ig + Q\,(1+ic)]. \end{aligned} \tag{25}$$

In the Option B general case where *ic* is not equal to *ig*, the transformations (25) will simplify the cash value calculations. If *ic* does equal *ig*, (25) becomes (19).

To illustrate a practical use of the Option B transformation, imagine a purely back-end loaded Universal Life product with a guaranteed interest rate of 4 percent. A prospective insured aged 35 wishes to see an Option B illustration which will provide a cash value at age 65 equal to $2,000 for each $1,000 of face amount at an interest crediting rate of 10 percent. Using annual time periods (monthly periods are treated later), this might be best accomplished in the following steps:

1. Set $A(65)$ equal to 2.
2. Set $a(64)$ equal to 1.
3. Calculate $Q'(x)$ as in (25) for x equal 35 through 64.
4. Calculate $i'(x)$ as in (25) for x equal 35 through 64.
5. Calculate $A(x)$ for x equal 64 through 35 with the recursion formula $A(x) = [Q'(x) + (1-Q'(x))\,A(x+1)]\,/\,(1+i'(x))$.
6. Calculate $a(x)$ for x equal 63 through 35 with the recursion formula $a(x) = a(x+1)\,/\,(1+i'(x))\,(1-Q'(x)) + 1$.
7. Calculate $P(x)$ for x equal 64 through 35 as $A(x)\,/\,a(x)$. $P(35)$ is the premium for the desired coverage.
8. Calculate the fund accumulations at the end of policy year t ($t = 0$ through 29) as $(P(35+t) - P(35))\;a(35+t)$.

These steps are illustrated in Table 3 in spreadsheet form. Alternatively, commutation functions could have been calculated; this approach is illustrated in Table 4. Results are exact with either approach, as evidenced by the agreement of entries in Table 3 and Table 4.

Monthly Application

Universal Life policies generally apply the cash value accumulation formula (1) on a monthly basis. The terms *Q*, *ig*, and *ic* are monthly rates obtained from annual rates, usually by dividing by 12 or by a geometric conversion.

If the transformations developed in this paper had to be applied separately to each month under consideration, the number of computations would increase twelve-fold over the case of annual application. Such an increase would greatly diminish the usefulness of the procedure.

Fortunately, there is a shortcut. The shortcut arises from the fact that, although the accumulation is monthly, the variables *Q*, *ig*, and *ic* are invariably held constant for all 12 months of any particular policy year. (This is not to

TABLE 3

OPTION B UNIVERSAL LIFE

INTEREST RATES

GUARANTEED 0.04

CURRENT 0.10

Age	Current COI per Thousand	$Q'(X)$	$i'(X)$	$A(X)$	$a(X)$	$P(X)$	Fund @ t	t
35	2.00	0.002111	0.097762	0.153585	10.358745	14.83	0.00	0
36	2.06	0.002174	0.097695	0.166841	10.295411	16.21	14.20	1
37	2.14	0.002258	0.097606	0.181360	10.225763	17.74	29.75	2
38	2.24	0.002364	0.097495	0.197249	10.149174	19.44	46.77	3
39	2.36	0.002490	0.097361	0.214624	10.064959	21.32	65.40	4
40	2.50	0.002637	0.097205	0.233611	9.972360	23.43	85.76	5
41	2.65	0.002795	0.097037	0.254353	9.870545	25.77	108.01	6
42	2.82	0.002974	0.096848	0.277014	9.758594	28.39	132.33	7
43	3.01	0.003174	0.096636	0.301766	9.635498	31.32	158.90	8
44	3.24	0.003415	0.096380	0.328797	9.500148	34.61	187.94	9
45	3.50	0.003688	0.096090	0.358295	9.351327	38.31	219.65	10
46	3.82	0.004024	0.095734	0.390476	9.187697	42.50	254.25	11
47	4.19	0.004412	0.095323	0.425546	9.007790	47.24	291.99	12
48	4.60	0.004842	0.094868	0.463744	8.809988	52.64	333.12	13
49	5.04	0.005303	0.094379	0.505343	8.592507	58.81	377.95	14
50	5.50	0.005784	0.093869	0.550655	8.353377	65.92	426.80	15
51	5.96	0.006264	0.093360	0.600031	8.090426	74.17	480.08	16
52	6.45	0.006776	0.092818	0.653882	7.801256	83.82	538.22	17
53	6.97	0.007318	0.092243	0.712626	7.483238	95.23	601.68	18
54	7.56	0.007933	0.091591	0.776727	7.133473	108.88	670.96	19
55	8.25	0.008650	0.090830	0.846652	6.748782	125.45	746.59	20
56	9.03	0.009461	0.089972	0.922886	6.325667	145.90	829.10	21
57	9.90	0.010363	0.089016	1.005977	5.860268	171.66	919.09	22
58	10.88	0.011377	0.087941	1.096524	5.348331	205.02	1017.23	23
59	11.99	0.012523	0.086726	1.195174	4.785165	249.77	1124.23	24
60	13.25	0.013821	0.085350	1.302616	4.165602	312.71	1240.85	25
61	14.69	0.015300	0.083782	1.419593	3.483937	407.47	1367.94	26
62	16.31	0.016958	0.082024	1.546897	2.733874	565.83	1506.36	27
63	18.11	0.018795	0.080078	1.685404	1.908458	883.12	1657.11	28
64	20.09	0.020807	0.077945	1.836080	1.000000	1836.08	1821.25	29
65	22.25	0.022993	0.075628	2.000000				30

say that insurers change crediting rates only on policy anniversaries. It is to say that, in calculating required premiums or making illustrations, midyear changes in these variables are not assumed.) The transformation (6) (13), (19), or (25) needs be applied only once for each policy year to the monthly values of Q, ig, and ic applicable for that year.

When monthly applications of the cash value accumulation formula are in effect, the values of Q' and i' alone will not suffice for making calculations. The present value of \$1 of annual premium payable monthly for one year is not \$1, but instead is

$$\left(\frac{1}{12}\right)\sum_{t=0}^{11}\left(\frac{1-Q'}{1+i'}\right)^{t} \qquad (26)$$

where Q' and i' are based on monthly values of Q, ig, and ic.

TABLE 4

OPTION B UNIVERSAL LIFE
COMMUTATION FUNCTION APPROACH
INTEREST RATES
GUARANTEED 0.04
CURRENT 0.10

Age	Current COI per Thousand	$Q'(X)$	$i'(X)$	$D(X)$	$C(X)$	$P(X)$*	Fund @ t**	t
35	2.00	0.002111	0.097762	1.000000	0.001923	14.83	0.00	0
36	2.06	0.002174	0.097695	0.909021	0.001800	16.21	14.20	1
37	2.14	0.002258	0.097606	0.826317	0.001700	17.74	29.75	2
38	2.24	0.002364	0.097495	0.751136	0.001618	19.44	46.77	3
39	2.36	0.002490	0.097361	0.682792	0.001549	21.32	65.40	4
40	2.50	0.002637	0.097205	0.620663	0.001492	23.43	85.76	5
41	2.65	0.002795	0.097037	0.564185	0.001437	25.77	108.01	6
42	2.82	0.002974	0.096848	0.512843	0.001390	28.39	132.33	7
43	3.01	0.003174	0.096636	0.466171	0.001349	31.32	158.90	8
44	3.24	0.003415	0.096380	0.423743	0.001320	34.61	187.94	9
45	3.50	0.003688	0.096090	0.385172	0.001296	38.31	219.65	10
46	3.82	0.004024	0.095734	0.350110	0.001286	42.50	254.25	11
47	4.19	0.004412	0.095323	0.318235	0.001282	47.24	291.99	12
48	4.60	0.004842	0.094868	0.289258	0.001279	52.64	333.12	13
49	5.04	0.005303	0.094379	0.262915	0.001274	58.81	377.95	14
50	5.50	0.005784	0.093869	0.238967	0.001264	65.92	426.80	15
51	5.96	0.006264	0.093360	0.217197	0.001244	74.17	480.08	16
52	6.45	0.006776	0.092818	0.197407	0.001224	83.82	538.22	17
53	6.97	0.007318	0.092243	0.179416	0.001202	95.23	601.68	18
54	7.56	0.007933	0.091591	0.163062	0.001185	108.88	670.96	19
55	8.25	0.008650	0.090830	0.148195	0.001175	125.45	746.59	20
56	9.03	0.009461	0.089972	0.134680	0.001169	145.90	829.10	21
57	9.90	0.010363	0.089016	0.122394	0.001165	171.66	919.09	22
58	10.88	0.011377	0.087941	0.111225	0.001163	205.02	1017.23	23
59	11.99	0.012523	0.086726	0.101071	0.001165	249.77	1124.23	24
60	13.25	0.013821	0.085350	0.091840	0.001169	312.71	1240.85	25
61	14.69	0.015300	0.083782	0.083449	0.001178	407.47	1367.94	26
62	16.31	0.016958	0.082024	0.075820	0.001188	565.83	1506.36	27
63	18.11	0.018795	0.080078	0.068884	0.001199	883.12	1657.11	28
64	20.09	0.020807	0.077945	0.062578	0.001208	1836.08	1821.25	29
65	22.25	0.022993	0.075628	0.056845	0.001215			30

*$P(x) = 1000\,[C(x) + C(x+1) + \cdots + C(64) + 2D(65)]\,/\,[D(x) + D(x+1) + \cdots + D(64)]$

**$\text{FUND}(t) = (P(x+t) - P(x))\,[D(x+t) + D(x+t+1) + \cdots + D(64)]\,/\,D(x+t)$

Also, the present value of \$1 of death benefit for one year is not $Q'/(1+i')$, but instead is

$$\sum_{t=0}^{11} \left(\frac{Q'}{1+i'}\right)\left(\frac{1-Q'}{1+i'}\right)^{t}. \tag{27}$$

To simplify things, let i'' equal $((1+i')/(1-Q'))^{12} - 1$. Then (26) can be rewritten as

$$\left(\frac{1}{12}\right)(1-v'')\,/\,(1-v''^{\frac{1}{12}}) \tag{28}$$

where v'' equals $1/(1+i'')$. Refer to the expression (28) as $a''^{(12)}$. Then (27) can be rewritten as

$$a''^{(12)}\, v'Q' \tag{29}$$

where v' equals $1/(1+i')$.

These considerations suggest the following commutation function definitions:

$$\begin{aligned} D(0) &= 1 \\ D(x) &= D(x-1)(1-Q'(x-1))^{12} \,/\, (1+i'(x-1))^{12} \\ D(x)^{(12)} &= D(x)\, a''^{(12)} \\ C(x)^{(12)} &= 12\, D(x)\, a''^{(12)}\, v'\, Q' \end{aligned} \tag{30}$$

where $i'(x-1)$ is i' for age $x-1$ and $Q'(x-1)$ is Q' for age $x-1$.

Table 5 illustrates these commutation functions for a hypothetical Universal Life policy. For computational ease, the additional functions $N(x)^{(12)}$ and $M(x)^{(12)}$ can be calculated in the usual way.

Once these commutation functions have been calculated, they can be used with traditional actuarial formulas to calculate premiums and fund accumulations on an exact basis for any desired plan of insurance. Fund accumulations are obtained using formulas analogous to traditional formulas for reserves.

Front-End Loads

The front-end loads most commonly used with life insurance products incorporating fund accumulations are similar to those sometimes used in pricing fully guaranteed products: dollar amounts per policy, dollar amounts per unit of face amount, and percentages of premiums paid. These types of loads (also referred to as expense charges) can be handled with little trouble when employing the transformations developed earlier.

For example, assume a loading of \$60 per policy per year, charged monthly, for the first 5 years and a zero loading after 5 years. The exact annual premium required for whole life coverage would be calculated in two steps. First, the net premium is determined using methods discussed earlier. Second, the level cost of the 5-year load is determined and added to the net premium. The level cost is equal to

$$\frac{60 \sum_{t=0}^{4} D(x+t)^{(12)}}{\sum_{t=0}^{\infty} D(x+t)}$$

where the terms are defined as in (30).

TABLE 5

OPTION B
UNIVERSAL LIFE
MONTHLY FORMULA APPLICATION

INTEREST RATES	ANNUAL	MONTHLY
GUARANTEED	0.04	0.00327374
CURRENT	0.10	0.00797414

Age	Monthly Current COI per Thousand	$Q'(x)$	$i'(x)$	$i''(x)$	$\overset{(12)}{a''}(x)$	$D(x)$	$\overset{(12)}{D}(x)$	$\overset{(12)}{C}(x)$
0	0.18	0.000181	0.007792	0.100008	0.957613	1.000000	0.957613	0.002062
1	0.07	0.000070	0.007903	0.100003	0.957615	0.909085	0.870553	0.000729
2	0.07	0.000070	0.007903	0.100003	0.957615	0.826438	0.791410	0.000663
3	0.06	0.000060	0.007914	0.100003	0.957615	0.751305	0.719462	0.000516
4	0.06	0.000060	0.007914	0.100003	0.957615	0.683003	0.654054	0.000469
5	0.06	0.000060	0.007914	0.100003	0.957615	0.620911	0.594594	0.000427
6	0.06	0.000060	0.007914	0.100003	0.957615	0.564463	0.540538	0.000388
7	0.05	0.000050	0.007924	0.100002	0.957616	0.513147	0.491397	0.000294
8	0.05	0.000050	0.007924	0.100002	0.957616	0.466496	0.446724	0.000267
9	0.05	0.000050	0.007924	0.100002	0.957616	0.424087	0.406112	0.000243
10	0.05	0.000050	0.007924	0.100002	0.957616	0.385533	0.369192	0.000221
11	0.05	0.000050	0.007924	0.100002	0.957616	0.350483	0.335628	0.000201
12	0.06	0.000060	0.007914	0.100003	0.957615	0.318621	0.305116	0.000219
13	0.07	0.000070	0.007903	0.100003	0.957615	0.289655	0.277378	0.000232
14	0.08	0.000080	0.007893	0.100003	0.957615	0.263322	0.252161	0.000241
15	0.09	0.000090	0.007883	0.100004	0.957615	0.239382	0.229236	0.000247
16	0.10	0.000100	0.007873	0.100004	0.957615	0.217620	0.208396	0.000249
17	0.10	0.000100	0.007873	0.100004	0.957615	0.197835	0.189450	0.000227
18	0.11	0.000111	0.007863	0.100005	0.957615	0.179850	0.172227	0.000227
19	0.11	0.000111	0.007863	0.100005	0.957615	0.163499	0.156569	0.000206
20	0.11	0.000111	0.007863	0.100005	0.957615	0.148635	0.142335	0.000187
21	0.11	0.000111	0.007863	0.100005	0.957615	0.135122	0.129395	0.000170
22	0.11	0.000111	0.007863	0.100005	0.957615	0.122838	0.117631	0.000155
23	0.11	0.000111	0.007863	0.100005	0.957615	0.111670	0.106937	0.000141
24	0.10	0.000100	0.007873	0.100004	0.957615	0.101518	0.097215	0.000116
25	0.10	0.000100	0.007873	0.100004	0.957615	0.092289	0.088377	0.000106
26	0.10	0.000100	0.007873	0.100004	0.957615	0.083898	0.080342	0.000096
27	0.10	0.000100	0.007873	0.100004	0.957615	0.076271	0.073038	0.000087
28	0.10	0.000100	0.007873	0.100004	0.957615	0.069337	0.066398	0.000079
29	0.10	0.000100	0.007873	0.100004	0.957615	0.063033	0.060362	0.000072
30	0.10	0.000100	0.007873	0.100004	0.957615	0.057303	0.054874	0.000066

Radix = 1.000000

The level cost can be determined using the transformed values i', Q', and a'' (the a'' value being useful whenever the load is applied monthly or premiums are paid monthly).

A type of load sometimes used with fund accumulation products, but not having a counterpart in fully guaranteed products, is the crediting of a lower rate of interest on the first \$x of the fund. This type of load raises a number of practical problems that may not be subject to simplification by mathematical transformations.

FORMULA VARIATIONS

Not all life insurance products incorporating fund accumulations use formula (1) or formula (14) for Option B Universal Life. In fact, formulas (1) and (14) as presented here may be the most difficult ones used in practice. For example, the following formula is sometimes used:

$$[0V + P - Q\,(1 - 0V - P)]\,(1+ic) = 1V.$$

This formula is equivalent to formula (1) with *ig* equal to zero. The transformations presented earlier can be used with this formula if *ig* is set equal to zero, which is a simplification.

Quarterly, Semiannual, and Irregular Premiums

Monthly premium payments were discussed earlier, and (28) defined a variable a'' for use in calculating commutation functions:

$$a''^{(12)} = \left(\frac{1}{12}\right)(1-v'')\,/\,(1-v''^{\frac{1}{12}}).$$

This definition can be generalized as follows:

$$a''^{(n)} = \left(\frac{1}{n}\right)(1-v'')\,/\,(1-v''^{\frac{1}{n}}). \qquad (31)$$

Now the definition of $D(x)^{(12)}$ in (30) can be generalized:

$$D(x)^{(n)} = D(x)a''^{(n)}. \qquad (32)$$

This generalization fully allows for modal premiums in the fund accumulations. It can even allow for continuous premiums, by letting n approach infinity. Note that the generalization of C is not necessary, because assessment of the cost of insurance charge is, in practice, always on either a monthly or an annual basis.

Irregularly timed premium payments could be included in commutation function formulas by estimating the function D at nonintegral values of x. This is unappealing in that it introduces an element of approximation into a process which up to now has been exact. Perhaps transformation techniques should not be used where irregular premium payments are involved.

Irregular Current Interest

When the interest-crediting rate to the fund varies by duration, a complication is introduced which is analogous to that of split-interest cash value and reserve calculations for fully guaranteed life insurance. The complication is that the transformations must be recalculated for each issue age. If many issue ages are involved, this may make using transformations less practical than iterative methods.

ACTUARIAL THEORY

Life insurance products that accumulate policyholder cash values by applying interest credits, mortality charges, and expense charges that can vary from month to month are sometimes referred to as "unbundled." This means that the changes in cash value can be allocated precisely to interest, mortality, and expense elements. Yet it is widely theorized that the same mechanism is at work in fully guaranteed products, except that it is kept under wraps.

With the transformations presented earlier, the equivalence of unbundled and fully guaranteed products is established mathematically. The theory is proven.

Letting n approach infinity in (31) suggests the idea of a continuous Universal Life policy. The following continuous commutation functions could be defined:

$$\bar{D}(x) = D(x)\,\bar{a}'' \tag{33}$$

$$\bar{C}(x) = D(x)\,\mu\,\bar{a}'' \tag{34}$$

where a'' equals $(1 - v'') \,/\, \ln(1 + i'')$ and μ is the continuous counterpart to the annual cost of insurance (COI) rate.

A simpler route to a continuous policy is to start with (1) and let the time interval between $0V$ and $1V$ approach zero. This causes the left-hand side of (1) to become

$$[0V + P\,dt - \mu\,dt\,(1/e^{igc\,dt} - 0V - P\,dt)]\,[e^{icc\,dt}] \tag{35}$$

where icc is the continuous counterpart to the annual current interest-crediting rate, igc is the continuous counterpart to the annual guaranteed interest rate, and time dt is measured in units of one year. Formula (35) represents $0V$ plus the differential of $0V$, so the derivative of $0V$ with respect to t is found by subtracting $0V$, dividing by dt, and then letting dt approach zero. The result is

$$\frac{d}{dt}\,0V = 0V\,icc + P - \mu\,(1-0V). \tag{36}$$

This result holds regardless of the relationship between ig and ic; in other words, at the limit, the general case always reduces to the simple case.

Further, this result takes the same form as the derivative of the reserve for fully guaranteed insurance, meaning that, at the limit, the simple case transformation is unnecessary. This could have been anticipated by noting that

$$\mu_x = \lim_{t\to 0} \frac{{}_tq_x}{t} = \lim_{t\to 0} \frac{{}_tq_x}{t(1+{}_tq_x)}. \tag{37}$$

These considerations establish the equivalence of a continuous unbundled policy and a continuous fully guaranteed policy.

In practice, a continuous Universal Life policy would be administered by crediting interest daily and charging a cost of insurance daily.

PRODUCT DESIGN IMPLICATIONS

Pac Man Universal Life

Since it is possible to make the Universal Life fund accumulation formula look exactly like the traditional fully guaranteed cash value formula, it is also possible to make the latter look exactly like the former. This suggests that in-force traditional policies can be converted to Universal Life policies. In addition to gaining flexibility, such conversions would defend against replacement by turning the tables on the replacing company. The strategy of turning the tables is sometimes referred to as the Pac Man strategy, since it plays a crucial role in the video game by that name.

The simplest implementation of such conversions would be to modify (1) to be

$$[0V + P - Q\,(1/(1+ic) - 0V - P)]\,(1+ic) = 1V. \tag{38}$$

With (38) the simple-case transformation is sufficient, but must be applied in reverse:

$$Q = Q' \,/\, (1 - Q') \tag{39}$$

where Q' is the mortality rate underlying the cash value formula of the old policy and Q is the annual COI rate to be used with the new policy. The company must guarantee that *ic* will never be less than the cash value interest rate of the old policy. Finally, a flexible loading formula is needed which will duplicate the loading of the old policy (gross premium less nonforfeiture factor). One approach would be to deduct the policy fee from the fund at the beginning of every year and to convert the rest of the load to a percentage of premium paid.

Such conversions should be acceptable to state insurance departments since flexibility for the policyholder is added and the equivalence to the old policy cash value accumulation formula can be demonstrated.

The expense of implementing a large-scale conversion might be offset by the administrative cost savings from becoming a one-product company.

Cash Management Account Universal Life

The possibility of continuous Universal Life policies was previously pointed out. The differential equation (36) indicates that, for a continuous policy, the fund accumulation will equal the accumulation of premiums at continuous interest less the accumulation of cost of insurance charges at continuous interest.

This suggests the creation of an insurance fund side-by-side with an investment fund of the type used by brokerage firms for clients with idle cash. Such investment funds have interest credited daily at market rates. The insurance fund assets would be indistinguishable from the investment fund assets, and the same daily rate of interest would be credited to deposits to either fund. The only difference is that the insurance fund is charged each day for a cost of insurance.

Each month, the client would receive a summary showing the activity in the two funds. The insurance fund activity could be shown as the accumulation at interest of the beginning fund plus deposits, less the accumulation at interest of the insurance charges.

DISCUSSION OF PRECEDING PAPER

ERIC SEAH AND ELIAS S. W. SHIU:

The recursive formulas considered in this paper are of the form

$$a(k)V(k) + b(k)P = c(k)V(k+1) + d(k), \tag{D.1}$$

where $a(k)$, $b(k)$, $c(k)$ and $d(k)$ are known functions in k. Given a pair of boundary values $V(m)$ and $V(n)$, $m < n$, we wish to find the level premium P for which (D.1) holds for all k between m and $n-1$.

Formula (D.1) is a first-order linear difference equation. Analogous to the case of first-order linear differential equations which are solved by the method of integrating factors, first-order linear difference equations are solved by the method of summation factors; see [3]. Transforming (D.1) as

$$e(k)V(k) + f(k)P = e(k+1)V(k+1) + g(k) \tag{D.2}$$

and summing (D.2) from $k = m$ to $k = n-1$, we obtain

$$e(m)V(m) + P\sum_{k=m}^{n-1} f(k) = e(n)V(n) + \sum_{k=m}^{n-1} g(k). \tag{D.3}$$

Hence, the level premium P is given by

$$P = \frac{e(n)V(n) - e(m)V(m) + \sum_{k=m}^{n-1} g(k)}{\sum_{k=m}^{n-1} f(k)}. \tag{D.4}$$

In the remainder of this discussion we shall assume that $V(m) = 0$. We write $P(m)$ for P to emphasize that there is a different level premium for each m, $m=n-1, n-2, n-3, \ldots$. The definitions:

$$e(k) = \prod_{j=k}^{n} \frac{a(j)}{c(j)}, \qquad k \le n, \tag{D.5}$$

$$f(k) = e(k+1)b(k)/c(k), \qquad k<n, \tag{D.6}$$

and

$$g(k) = e(k+1)d(k)/c(k), \qquad k<n, \tag{D.7}$$

transform (D.1) into (D.2). Hence, (D.4) becomes

$$P(m) = \frac{\dfrac{a(n)}{c(n)} V(n) + \displaystyle\sum_{k=m}^{n-1} \frac{d(k)}{c(k)} \prod_{j=k+1}^{n} \frac{a(j)}{c(j)}}{\displaystyle\sum_{k=m}^{n-1} \frac{b(k)}{c(k)} \prod_{j=k+1}^{n} \frac{a(j)}{c(j)}}. \tag{D.8}$$

Formula (D.8) can be elegantly implemented in *APL*, as follows:

```
      ∇ PREN;U;S;T;NUM;DENON
[1]     U←(⍳S)∘.≤⍳S←¯1+⍴A
[2]     DENON←+/U×(S,S)⍴(¯1↓B÷C)×T←1↓⌽×\⌽A÷C
[3]     NUM←(VN×(¯1↑A)÷¯1↑C)++/U×(S,S)⍴(¯1↓D÷C)×T
[4]     1000×NUN÷DENON
      ∇
```

In this program A, B, C and D are vectors that correspond to $a(k)$, $b(k)$, $c(k)$ and $d(k)$ in formula (D.1). The *APL* global variable *VN* is $V(n)$. For example, to derive the last column of Table 1, we set:

$A \leftarrow$ 1.055 1.055 . . . 1.055
$B \leftarrow$ 1.055 1.055 . . . 1.055
$Q \leftarrow$ 0.00263 0.00103 0.00099 . . . 0.28213 0.30997 1.0
$C \leftarrow 1/(1 + Q)$
$D \leftarrow Q/(1 + Q)$
$VN \leftarrow 1.0$

In this example, $0 \le m < n = 95$. Upon invoking *PREM*, the computer will return a vector of length 95, giving the premium rates at issuance age 0, 1, 2, . . ., 94.

Remarks. (i) Net premium reserves satisfy the formula [1, equation (7.8.2)]

$$({}_kV + \pi_k)(1 + i_k) = p_{x+k}\,{}_{k+1}V + q_{x+k}\, b_{k+1}. \tag{D.9}$$

If $\pi_k = Pg(k)$ for a known function $g(k)$, then (D.9) can be expressed in the form (D.1) and the method above can be applied to solve for the net premiums π_k.

(ii) The symbol P is used to denote the "premium . . . after deducting premium loads." Calling it "gross" premium may create confusion.

(iii) Formula (36) in the paper is sometimes called *Thiele's differential equation* [2, p. 70]. Variations of the differential equation can be found in exercises 7.32 and 7.41 of [1].

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LAWRENCE SILKES:

Mr. Eckley's paper introduces some interesting transformations of the mortality and interest elements in life insurance policies. These transformations enable the actuary to perform Universal Life calculations in the traditional manner.

The introduction of the Universal Life product by the insurance industry coincided with a switch from thinking prospectively to thinking retrospectively on the part of the actuary, regulator, and auditor.

Prospective analysis is cumbersome because it requires the projection of premiums and costs over all future periods. Once such projections are made, commutation functions can greatly simplify further calculations. But prospective thinking kept life insurance products static because of the necessity of reconfiguring all calculations in order to make changes.

Another important change in actuarial thinking is that traditional reserve factors are not asset accumulations. A net premium calculation is an idealized equation of positive and negative cash flows. A reserve factor is the difference between accumulated net premiums and accumulated tabular claims; another idealization.

A subtle aspect of reserve factors that has great implications is that they are calculated on a per-survivor basis. The various recursion formulas for reserves, one of which is discussed at the beginning of Mr. Eckley's paper, illustrate this aspect. Survivorship, in addition to premiums and interest, causes traditional reserve factors to increase over time. But where is the survivorship element in the typical Universal Life accumulation formula? One way to view Mr. Eckley's paper is as an answer to this question.

(AUTHOR'S REVIEW OF DISCUSSION)

DOUGLAS A. ECKLEY:

Professors Seah and Shiu have offered evidence that the transformations developed in the paper can, after all, be programmed and used efficiently. Their discussion strikes me as an example of actuaries using their mathematical expertise to solve practical problems. I do feel that the transformations can be of practical use in certain situations, such as the calculation of illustrative or guideline premiums, so I am thankful to Professors Seah and Shiu for their contribution.

Mr. Silkes takes a step back and views the paper in the context of retrospective versus prospective thinking. He sees the transformations as the link between these types of thinking. Perhaps that is both their appeal and their bane: It is reassuring to know that the link exists, but the simplicity of the retrospective approach may eventually make a relic of the prospective approach. I thank Mr. Silkes as well.