

## NET PREMIUMS IN STOCHASTIC LIFE CONTINGENCIES

EDWARD W. FREES

### ABSTRACT

A fundamental problem in actuarial science is the determination of the level of a premium payment plan necessary to meet future contingent obligations. In the modern stochastic version of life contingencies, this net level is determined by the principle that expected value of a certain loss random variable is set equal to zero. This principle is desirable in that it yields the same net premium level as under the traditional deterministic life contingencies. To achieve this reduction to the traditional framework, in stochastic contingencies the actuary must use care in selecting random variables to be used in calculating premiums.

In this paper we use expectations of different random variables to define the notions of retrospective, prospective, individual, and aggregate net premiums. Unlike deterministic contingencies, these premiums are not equivalent. Their respective relationships are discussed in order to provide a deeper understanding of the notion of a net premium. The individual premium turns out to have desirable properties, and therefore this approach is extended from premium to reserving considerations.

---

### INTRODUCTION

In an insurance enterprise, the premium formulation process is complex and must be performed in the presence of the conflicting requirements of several interested parties. A central idea within this process is the notion of a fair or "net" premium. In the framework of stochastic contingencies, the notion of a net premium can be expressed in terms of an expected value of a random variable. In life contingencies, this random variable is typically a simple function of the random time to death. In this paper we investigate the relationships between several net premiums within the stochastic contingencies framework. (See Shapiro [10] for an introduction to the use of net premiums in the general premium formulation process.)

In the theory of modern stochastic contingencies as presented in Bowers, Gerber, Hickman, Jones, and Nesbitt [1], net premiums are determined by equating expected present value of future benefits to expected present value of future payments. Alternatively, one may use as net premium the expected present value of future benefits per unit of present value of future premium

payments. We call the former an aggregate approach and the latter an individual approach. Both approaches are prospective in the sense that expectations are evaluated at contract initiation. If we denote the present value of future benefits as claims and the present value of future premium payments as exposure, then under the individual approach net premiums can be thought of as an expected loss ratio. Here, the loss ratio is claims per unit exposure. Not surprisingly, the individual net premium turns out to be greater than aggregate premiums under general conditions. We also show that reserves, useful in assuring solvency, can be developed under the individual formulation.

Compare the traditional theory of deterministic contingencies as presented by Jordan [7] with the approach presented by Bowers et al. [1]. An interesting, and somewhat perplexing, difference is the active use of prospective and retrospective premiums in the former and the absence of these ideas in the latter. In this paper we define prospective and retrospective aggregate premiums within the stochastic framework. Our notion of prospective premium is the same as the net premium principle in Bowers et al. It turns out that under general conditions, retrospective premiums are smaller than prospective premiums. This difference provides insight into a paradoxical "underwriting gain" noted by Jewell [6].

In the following section the important special case of whole life insurance is discussed. Some general premium relationships which are straightforward to establish for the whole life policy also hold for more general contracts, as is argued in the third section. Several examples and interpretations are provided. We then extend the individual approach to the formulation of reserves. The proofs of the main propositions appear in the Appendix.

#### WHOLE LIFE INSURANCE PREMIUMS

To illustrate the ideas, consider the determination of an annual premium of a whole life policy to a life aged  $x$ . For simplicity, assume that benefits are paid immediately and premiums are paid continuously. Using standard notation as defined in Bowers et al. [1], let  $T = T(x)$  be the random remaining lifetime for a life aged  $x$ ,  $v$  be a constant discount factor, and  $P$  be a fixed, arbitrary premium. The prospective loss of the company,  $L_p$ , is

$$L_p(T, P) = v^T - P \bar{a}_{\overline{T}|},$$

where  $\bar{a}_{\overline{T}|} = (1 - v^T)/\delta$  and  $\delta = -\log(v)$ . Aggregate net premiums are determined under the principle of equivalence, that is, the requirement that the expected present value of losses is zero. With this guiding principle, we have  $E L_p(T, P) = \bar{A}_x - P \bar{a}_x = 0$ , and thus we use as the prospective aggregate net premium

$$P_p = \bar{A}_x/\bar{a}_x = E v^T/E \bar{a}_{\tau_1} \quad (1)$$

To define the individual net premium, we choose  $P$  so that the prospective loss of the company,  $L_p$ , is zero. Thus, the equitable individual premium is a function of  $T$ , i.e.,  $P = P(T) = 1/\bar{s}_{\tau_1}$ . The individual net premium is defined by

$$P_l = E P(T) = E (1/\bar{s}_{\tau_1}). \quad (2)$$

Under general conditions given in the following section, we have  $P_p \leq P_l$ . For the whole life policy, this is easy to see using a geometric series expansion:

$$\begin{aligned} P_p &= \delta E v^T/(1 - E v^T) = \delta \sum_{j=1}^{\infty} (E v^T)^j \\ &\leq \delta \sum_{j=1}^{\infty} (E v^j) = E (1/\bar{s}_{\tau_1}) \\ &= P_l. \end{aligned} \quad (3)$$

The fact that  $E v^T \leq (E (v^T))^j$ ,  $j \geq 1$ , follows from Jensen's inequality (cf. Gerber [4, equation (1)]). An aggregate premium can be thought of as the ratio of expected benefits to expected premiums, while the individual premium is the expectation of the ratio of benefits to premiums. The difference between the ratio of expectations and expectation of ratios helps to explain otherwise confusing issues in the literature, some of which arise in this paper.

From a retrospective viewpoint, the loss to the company at the instant of expiration,  $L_r$ , is

$$L_r(T, P) = 1 - P \bar{s}_{\tau_1}.$$

Using the principle of equivalence, we have that imposing the requirement  $E L_r(T, P) = 0$  leads to the retrospective aggregate net premium,

$$P_r = 1/E \bar{s}_{\tau_1}. \quad (4)$$

Unlike deterministic contingencies, under the stochastic framework we do not have  $P_p = P_r$ . Under general conditions given in the following section, we have  $P_r \leq P_p$ . For the whole life policy, this is easy to see. Since  $v^T$  and  $\bar{s}_{\tau_1}$  are inversely related, we have that  $v^T$  and  $\bar{s}_{\tau_1}$  are negatively correlated. Thus,

$$E \bar{a}_{\tau_1} = E (v^T \bar{s}_{\tau_1}) \leq E v^T E \bar{s}_{\tau_1} \quad (5)$$

and

$$P_r = 1/E \bar{s}_{\tau} \leq E v^T/E \bar{a}_{\tau} = P_p.$$

If we compare equations (1) and (4), the fact that  $P_p$  and  $P_r$  are not equivalent is not surprising. This highlights the different interpretations of the loss random variables,  $L_p$  and  $L_r$ .  $L_p$  is the loss at contract initiation and  $L_r$  is the loss at contract termination (incurral of loss); these variables thus are related by  $L_p = v^T L_r$ . Note that equality exists in equation (5) when  $v = 1$  or  $i = 1/v - 1 = 0$ .

The notion of a retrospective premium is closely tied to Jewell's [6] underwriting gain, defined to be

$$G(P) = -L_r(T, P) = P \bar{s}_{\tau} - 1. \quad (6)$$

By calculations similar to those in (5), Jewell argued that the expected gain,  $E G(P_p)$ , is nonnegative under mild conditions on the discounting and mortality assumptions. The conditions and benefit structure were subsequently generalized by Chan and Shiu [2] and Ramsay [9]. Jewell interpreted  $E G(P_p)$  to be the expected gain to the company measured at the instant of expiration and noted that the profitability is not affected by this unexpected expectation, which "cannot even be developed through the use of classical expected-value notation" [6, p. 94]. Note that using retrospective premium  $P_r$  in equation (6) produces an expected zero underwriting gain.

#### PREMIUM RELATIONSHIPS FOR GENERAL CONTRACTS

In this section, we establish some relationships between prospective and retrospective aggregate and individual premiums under conditions that hold for several different types of policies. We limit ourselves to policies in which the random time  $T$  serves as a signal to halt the premium payment process and commence the benefit payment process. To this end, for a loss occurring at a fixed time  $t$ , let  $a(t)$  and  $s(t)$  be the present value of a known discounted stream of payments at contract initiation and termination, respectively. Similarly, let  $z(t)$  and  $b(t)$  be the present value of benefits at contract initiation and termination, respectively. In the special case of whole life insurance with constant discount factor  $v$ , we have  $a(t) = \bar{a}_{\tau}$ ,  $z(t) = v^t$ ,  $s(t) = \bar{s}_t$ , and  $b(t) = 1$ . In general, we assume only that premiums and benefits are discounted in the same fashion, so that  $s(t)/a(t) = b(t)/z(t) = F(t)$ , an accumulation factor. With this common factor, prospective and retrospective individual premiums are the same and need not be distinguished.

To achieve greater generality and thus wider potential usefulness, in the general propositions we allow the force of interest to be a stochastic quantity. Thus,  $a(t)$ ,  $b(t)$ ,  $F(t)$ ,  $s(t)$ , and  $z(t)$ , at each fixed time  $t$ , are potentially

random quantities. (See Panjer and Bellhouse [8] for an argument in favor of assuming a random interest environment.) However, so that the results are easier to interpret, in each corollary we assume the force of interest to be deterministic. Further, in each example the force of interest is assumed to be constant.

For each individual, if the time of loss  $t$  is known, then the equitable premium is

$$P(t) = z(t)/a(t). \quad (7)$$

That is,  $P(t)$  is defined so that the present value of payments,  $P(t) a(t)$ , equals the present value of benefits,  $z(t)$ . Since  $T$  is not known, some alternatives are to use:

- (i)  $P_I = E P(T)$ , its expected value;
- (ii)  $P_p = E z(T)/E a(T)$ , the ratio of expected present values at contract initiation; or
- (iii)  $P_r = E b(T)/E s(T)$ , the ratio of expected present values at contract termination.

Using the whole life example, the use of  $P_p$ ,  $P_I$ , and  $P_r$  notation is suggested by equations (1), (2), and (4). We assume throughout the paper that  $a(t)$ ,  $z(t)$ ,  $s(t)$ ,  $b(t)$ , and the distribution of  $T$  are such that all relevant random variables have finite second moments. To demonstrate a relationship between prospective and individual premiums, we have the following.

*Proposition 1.*

A necessary and sufficient condition that  $P_p \leq P_I$  is

$$\text{cov}(P(T), a(T)) \leq 0. \quad (8)$$

Further,  $P_p = P_I$  if and only if there is equality in equation (8).

Proposition 1 is a characterization in the sense that it provides an exact condition for individual premiums to exceed prospective premiums. An implicit consequence of proposition 1 is that if  $P(T)$  and  $a(T)$  are positively correlated, then  $P_p > P_I$ . If  $a(t)$  is constant, as with single premium plans, then  $P_p = P_I$ . To demonstrate the usefulness of proposition 1, we return to the whole life policy.

*Example 1. — Whole Life Policy*

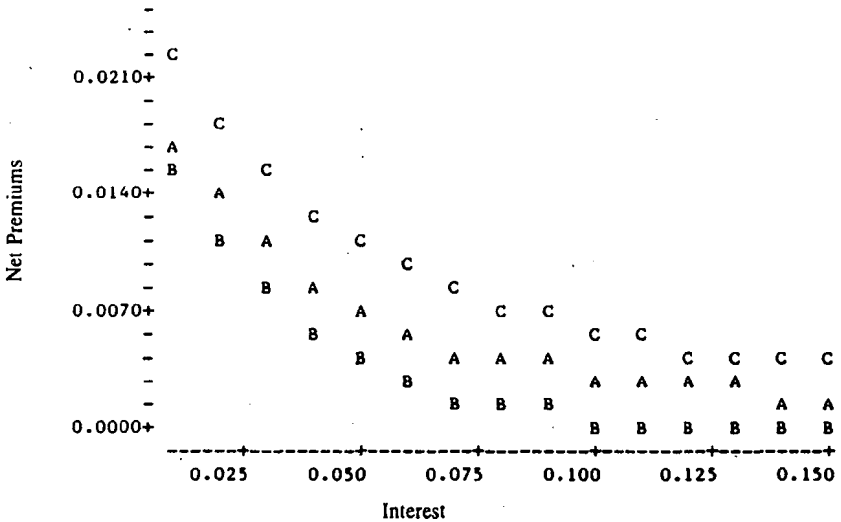
Consider a whole life policy with constant discount factor  $v$ , premiums payable at the beginning of the year, and benefits payable at the end of the

year. Define  $K = [T]$  where  $[\cdot]$  is the greatest integer function. Thus,  $a(T) = \ddot{a}_{\overline{K+1}|} = (1 - v^{K+1})/(1 - v)$  and  $P(T) = v^{K+1}/\ddot{a}_{\overline{K+1}|} = 1/\ddot{s}_{\overline{K+1}|}$ . Similarly to equations (1), (4), and (2), we have  $P_p = A_x/\ddot{a}_x$ ,  $P_r = 1/E \ddot{s}_{\overline{K+1}|}$ , and  $P_l = E(1/\ddot{s}_{\overline{K+1}|})$ . (See Figure 1 for a plot of these net premiums for various interest rates using mortality decrements from the 1979-81 Male U.S. Life Tables [1, pp. 55-58].) In the case of  $v < 1$ , similarly to equation (3), with  $\text{cov}(P(T), a(T)) = E v^{K+1} - E \ddot{a}_{\overline{K+1}|} E(1/\ddot{s}_{\overline{K+1}|}) \leq 0$ , we have  $P_p \leq P_l$ . In the case of  $v = 1$ , since

$$\begin{aligned} \text{cov}(P(T), a(T)) &= \text{cov}((K+1)^{-1}, (K+1)) \\ &= 1 - E(K+1)^{-1} E(K+1) \leq 0, \end{aligned} \tag{9}$$

we have  $P_p \leq P_l$ . Further, when  $K$  is not degenerate,  $P_p < P_l$ . The inequality in equation (9) is well known (cf. Shiu [11, p. 97]).

FIGURE 1  
WHOLE LIFE NET PREMIUMS AS A FUNCTION OF INTEREST  
(Mortality Decrement Is 1979-81 Male U.S. Life Table with  $x = 30$ )



A = prospective premiums, B = retrospective premiums, C = individual premiums.

The condition (8) in proposition 1 is precise but may be onerous to check in certain situations. Some stronger sufficient conditions are given in corollary 1.

*Corollary 1.*

Suppose that  $a(t)$  and  $P(t)$  are deterministic and that  $a(t)$  is monotonically increasing in  $t$ . If  $P(t)$  is monotonically decreasing, then  $P_p \leq P_l$ . If  $P(t)$  is monotonically increasing, then  $P_p \geq P_l$ .

The proof of corollary 1 is a direct result of proposition 1 and some standard results in probability (cf. Gurland [5]). To demonstrate the usefulness of corollary 1, we have example 2.

*Example 2. — Pure Endowment Policy*

Consider an  $n$  year pure endowment policy with premium payable continuously. Then,  $z(t) = v^n I(t \geq n)$ ,  $a(t) = \bar{a}_{\overline{t}|} I(t < n) + \bar{a}_{\overline{t-n}|} I(t > n)$  and  $P(t) = z(t)/a(t) = (1/\bar{s}_{\overline{t}|}) I(t > n)$  (cf. [1, p. 90]). It is easy to see that  $a(t)$  and  $P(t)$  are monotonically increasing. Hence,  $E z(T)/E a(T) = P_p \geq P_l = E P(T)$ , by corollary 1.

To demonstrate a case in which corollary 1 is not applicable, we have example 3.

*Example 3. — Increasing Whole Life Policy*

Consider an increasing whole life policy providing  $\$k$  for death in the  $k$ th policy year, with premiums payable continuously. From, for example, Bowers et al. [1, p. 95], we have  $z(t) = [t+1] v^t$ ,  $a(t) = \bar{a}_{\overline{t}|}$  and  $P(t) = [t+1]/\bar{s}_{\overline{t}|}$ . For small interest rates ( $v$  close to 1), it is easy to check that  $P(t)$  is neither monotonically increasing nor decreasing, and thus the conditions of corollary 1 do not hold. For specific interest and mortality assumptions, one must analyze the correlation between  $P(T)$  and  $a(T)$  to determine the relationship between  $P_p$  and  $P_l$  using proposition 1.

To demonstrate a relationship between prospective and retrospective premiums, we have the following.

*Proposition 2.*

Assume that

$$\text{cov}(F(T), a(T)) \geq 0 \text{ and } \text{cov}(F(T), z(T)) \leq 0, \quad (10)$$

then  $P_r \leq P_p$ . If either inequality in equation (10) is strict, then  $P_r < P_p$ . Further, if both inequalities in equation (10) are equalities, then  $P_r = P_p$ .

*Remarks:* In the case of zero interest,  $F(t) = 1$ , and by the third part of proposition 2, we have  $P_r = P_p$ . A set of conditions that is simple to check is provided in corollary 2.

*Corollary 2.*

Assume that  $a(t)$ ,  $F(t)$ , and  $z(t)$  are deterministic, that  $F(t)$  and  $a(t)$  are monotonically increasing, and that  $z(t)$  is monotonically decreasing. Then  $P_r \leq P_p$ .

For a case in which corollary 2 is not applicable, see the Pure Endowment Policy of example 2. For an application of corollary 2, see the Whole Life Policy of example 1, or see example 4.

*Example 4. — Term Insurance Policy*

Consider an  $n$  year term life insurance policy with premiums payable continuously. Then  $z(t) = v^t I(t < n)$ ,  $a(t) = \bar{a}_{\overline{n}|} I(t < n) + \bar{a}_{\overline{\infty}|} I(t \geq n)$ ,  $P(t) = z(t)/a(t) = 1/\bar{s}_{\overline{n}|} I(t < n)$  and  $F(t) = v^{-t} = (1+i)^t$ . It is easy to see that  $a(t)$  and  $F(t)$  are increasing and that  $z(t)$  and  $P(t)$  are decreasing. Thus,  $P_r < P_p < P_t$ .

## SOME INTERPRETATIONS

In this paper, we use the adjective “net” to refer to cases in which the expected value of some random variable is considered, and not other summary measures of a distribution, such as percentiles, variances, skewness, etc. There is an extensive literature concerning principles of premium calculation, which are defined using various summary measures of a distribution (cf. Gerber [3, Chapter 5] for a nice introduction to this area). By limiting ourselves to expectations of certain random variables, we can take advantage of the law of large numbers to discuss the interpretation of our net premiums in terms of averages of homogeneous random variables.

We first contrast aggregate and individual prospective net premiums. Partition a population of  $n$  individuals into  $m$  strata so that each stratum experiences times until death  $T_{ij}$ ,  $j = 1, \dots, K_i$ ,  $i = 1, \dots, m$ . For convenience we think of each stratum as representing an employer who maintains a contract with an insurance company with identical subcontracts, one for each of the  $K_i$  employees. The random premium level for each employer is

$$P_i = \sum_{j=1}^{K_i} z(T_{ij}) / \sum_{j=1}^{K_i} a(T_{ij})$$

and the level of the total random premium is

$$P = \sum_{i=1}^m w_i P_i.$$



Here,  $\{w_i\}$  is some sequence of weights chosen so that  $w_i \geq 0$  for each  $i$  and  $\sum_i w_i = 1$ . To get the aggregate premium, fix  $m$ , the number of employers, and let the number of employee subcontracts tend to infinity for each stratum, i.e.,  $K_i \rightarrow \infty$  for each  $i$ . Then, under some mild moment and correlation assumptions, by the strong law of large numbers we have that limit  $P_i = E z(T_{i1})/E a(T_{i1}) = P_p$  and thus limit  $P = P_p$ , with probability = 1. Since we are averaging over several subcontracts within a master contract, we have labeled  $P_p$  an aggregate premium. Conversely, now let the number of employers,  $m$ , tend to infinity and keep the number of employees per master contract small. For simplicity, take  $K_i$  identically equal to 1 and  $w_i = 1/m$ . Then again under the strong law of large numbers with some mild assumptions, we have that limit  $P = E (z(T_{11})/a(T_{11})) = P_r$ , with probability = 1. Thus, we have labeled  $P_r$  an individual premium.

The interpretation of aggregate retrospective net premiums is less clear. Here, one could think of a stationary sequence of contracts flowing through a company. At any particular instant in time  $t$  there is a small window,  $[t, t + \Delta t]$ , in which the company experiences losses  $b(T_1), \dots, b(T_r)$  and with associated accumulated premiums  $s(T_1), \dots, s(T_r)$ . Here,  $T_i$  is the time since inception of the  $i$ th contract. If the company could set premiums based on these realized contracts, a candidate might be

$$\lim_{r \rightarrow \infty} \sum_{i=1}^r b(T_i) / \sum_{i=1}^r s(T_i) = P_r,$$

with probability one, by the strong law of large numbers. However, premium determination is typically done at contract initiation and thus on a prospective basis. The quantity  $P_r$  is interesting because it is the smallest premium one can charge and still not suffer Jewell's "expected underwriting loss."

#### EXTENSIONS TO INDIVIDUAL RESERVES

Under the individual approach, the random quantity  $P(T)$  is the equitable level of premiums so that, at contract inception, premium payments exactly meet benefit obligations. Similarly, the notion of a reserve can be defined so that, for times after contract inception, the insurer has an adequate amount of assets available to meet benefit obligations. It is clear that a new concept of a reserve is desirable since, under the broad conditions characterized in proposition 1, we have  $P_i > P_p$ . Thus, at contract inception and using  $P_i$ , the usual reserve, or the expected present value of future benefits minus expected present value of future premiums, is negative:  $E z(T) - P_i E a(T)$

$< 0$ . This fictitious creation of wealth — i.e., negative liabilities — at contract inception would clearly be unacceptable to regulatory authorities.

To motivate a definition of the individual reserve, we begin by considering the expected aggregate reserve at some fixed time  $t > 0$ . We work on the set  $\{T > t\}$  and use the symbol  $E_t$  to denote the expectation conditional on the event  $\{T > t\}$ . At time  $t$ , define  $z(t, T) = z(T) F(t)$  to be the present value of future benefits and  $a(t, T) = (a(T) - a(t)) F(t)$  to be the present value of a future payment stream. The expected aggregate reserve is defined to be

$$\begin{aligned} V_p(t) &= E_t z(t, T) - P_p E_t a(t, T) \\ &= (P_p(t) - P_p) E_t a(t, T). \end{aligned} \quad (11)$$

Here,  $P_p(t) = E_t z(t, T)/E_t a(t, T)$  is the aggregate net premium for a policy issued at time  $t$ . The link, established in equation (11), between premiums and reserves is the so-called premium-difference formula [1, p. 195]. This link suggests the natural definition for the expected individual reserve,

$$V_i(t) = (P_i(t) - P_i) E_t a(t, T) \quad (12)$$

where  $P_i(t) = E_t P(t, T) = E_t \{z(t, T)/a(t, T)\}$ . To complete the definition, in the case where premiums are no longer payable, we have  $a(t, \infty) = 0$  and thus define  $V_i(t) = E_t z(t, T)$ . The motivation behind the definition is as follows. At time  $t$ , to fund the obligation  $z(T)$ , the level of contributions is  $P(t, T) = z(t, T)/a(t, T)$ . The insurer's portion of this is  $P(t, T) - P_i$ , a random quantity. Using the individual approach, we define the level of contributions to the reserve at time  $t$  by  $E_t \{P(t, T) - P_i\} = P_i(t) - P_i$ , which leads to equation (12). With this formulation, it is easy to see that reserves at time 0 are zero.

The problem of comparing individual and aggregate reserves is more complex than in the case of premiums. Even in the basic case of whole life, one can construct simple examples to show that neither reserve dominates the other for all relevant time points. For example, suppose  $i = .06$  and the random remaining lifetime at contract initiation is 10 or 20 years, each with probability one-half, i.e.:  $P(T=10) = P(T=20) = 1/2$ . Then, it is a pleasant exercise to check that  $V_p(5) = .26051 < .40836 = V_i(5)$  and  $V_p(15) = .55259 > .53021 = V_i(15)$ . It would be interesting to establish nontrivial conditions on interest rates and the distribution of  $T$  so that  $V_i(t) \geq V_p(t)$ , for all relevant  $t$ , or  $V_i(t) \leq V_p(t)$ .

## CONCLUDING REMARKS

This paper examined the notion of a net premium in stochastic life contingencies by defining several closely related premium formulations. Aggregate net premiums are calculated using expected benefits per unit of expected premium. Conversely, individual net premiums are calculated as the expected benefit per unit of premium. Even in the case of zero interest, these two concepts are not identical. Under the broad conditions given in proposition 1, we find that aggregate net premiums are less than individual net premiums. Heuristically, there is a reduction of premiums by averaging benefits, averaging payments, and taking the ratio in lieu of averaging the ratios. That is, in most cases the mean of the ratio will be larger because of the highly skewed distribution of the ratio. Prospective and retrospective premiums have been distinguished by examining the expectation of random variables at different points in time. As one would suspect, when the time value of money is constant (zero interest), a consequence of our proposition 2 is that prospective and retrospective premiums are the same.

## ACKNOWLEDGMENTS

Thanks go to my actuarial colleagues at the University of Wisconsin, Professors James Hickman, James Robinson, and Donald Schuette, for their advice and encouragement on this paper. Thanks also go to the anonymous reviewers for their incisive and useful comments.

## REFERENCES

1. BOWERS, N. L., GERBER, H. U., HICKMAN, J. C., JONES, D. A., AND NESBITT, C. J. *Actuarial Mathematics*. Itasca, Ill.: Society of Actuaries, 1986.
2. CHAN, F. Y., AND SHIU, E. S. W. "Underwriting Gain As a Random Variable," *Actuarial Research Clearing House* (1982.2):5-9.
3. GERBER, H.U. *An Introduction to Mathematical Risk Theory*. Homewood, Ill.: Richard D. Irwin, 1979.
4. GERBER, H.U. "Some Moment Inequalities and Their Applications," *TSA XXXVIII* (1987): 75-81.
5. GURLAND, J. "Inequalities of Expectations of Random Variables Derived by Monotonicity or Convexity," *American Statistician* 22 (1968): 26-27.
6. JEWELL, W. S. "Models in Insurance: Paradigms, Puzzles, Communications and Revolutions," *Transactions of the Twenty-First International Congress of Actuaries* 5 (1980): 87-141.
7. JORDAN, C. W. *Life Contingencies*. Itasca, Ill.: Society of Actuaries, 1967.

8. PANJER, H. H., AND BELLHOUSE, D. R. "Stochastic Modelling of Interest Rates with Applications to Life Contingencies," *Journal of Risk and Insurance* 47 (1980): 91-110.
9. RAMSAY, C. M. "The Optimality of the Net Single Premium in Life Insurance," *Journal of the Institute of Actuaries* 113 (1986): 499-508.
10. SHAPIRO, R. D. "The Process of Premium Formulation," *TSA XXXIV* (1982): 435-445.
11. SHIU, E. S. W., in discussion of GERBER, H. U., "Some Moment Inequalities and Their Applications," *TSA XXXVIII* (1987): 96-102.

## APPENDIX

*Proof of Proposition 1:*

From equation (8) and the definitions of  $P_p$  and  $P_I$ ,

$$\begin{aligned} P_p/P_I - 1 &= E \{P(T) a(T)\} / \{E P(T) E a(T)\} - 1 \\ &= \text{cov} (P(T), a(T)) / \{E P(T) E a(T)\}. \end{aligned}$$

The result of the proposition is immediate. ‡

*Proof of Proposition 2:*

By equation (10), we have  $E F(T) E a(T) \leq E\{F(T) a(T)\}$  and  $E\{F(T) z(T)\} \leq E F(T) E z(T)$ . Thus,

$$\begin{aligned} P_r/P_p &= E\{F(T) z(T)\} E a(T) / \{E z(T) E(F(T) a(T))\} \\ &= \{E\{F(T) z(T)\} / \{E z(T) E F(T)\}\} \\ &\quad \{E F(T) E a(T) / E(F(T) a(T))\} \\ &\leq 1, \end{aligned}$$

which is sufficient for  $P_r \leq P_p$ . The other cases are immediate. ‡

## DISCUSSION OF PRECEDING PAPER

COLIN M. RAMSAY:

I would like to congratulate Dr. Frees on his paper exploring some of the alternatives to the traditional net premium  $P_p$  given in his Equation (1). However, care must be taken when interpreting his retrospective premium,  $P_r$ , as given in Equation (1). Because  $P_r$  satisfies that equation, it is determined by summing monies (that is, losses) without considering the time value of money.

For example, suppose  $T$  is an integer-value random variable with

$$Pr[T = t] = 1/3 \text{ for } t = 1, 2, \text{ and } 3,$$

and interest is at 10 percent per annum. The retrospective loss at time  $T$  will be

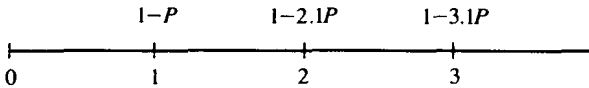
$$L_r(T, P) = 1 - P s_{\overline{T}|i}$$

The expected loss is given by

$$E[L_r(T, P)] = \frac{1}{3} [(1 - P) + (1 - 2.1P) + (1 - 3.31P)].$$

See the diagram below. Summing the financial losses at the end of years 1, 2, and 3 violates the concept of the time value of money, even though it is quite sound mathematically.

This dilemma arises because  $L_r(T, P)$  is a function of the random time of death  $T$ . A similar criticism can be applied to the concept of the expected underwriting gain at death in Equation (6).



ELIAS S.W. SHIU:

The remarks below are motivated by Dr. Frees's elegant inequalities

$$E\left(\frac{1}{\bar{s}_{\overline{T}|i}\right) \geq \frac{E(v^T)}{E(\bar{a}_{\overline{T}|i})} \geq \frac{1}{E(\bar{s}_{\overline{T}|i})} \tag{D.1}$$

and his reference to stochastic interest rates.

Let  $P(s, t)$  denote the price, at time  $s$ , of a pure discount bond paying 1 at time  $t$ ,  $s \leq t$ . One learns in Compound Interest [4, section 1.10] that, if  $\delta(\tau)$  is the instantaneous rate of return or force of interest at time  $\tau$ ,  $s \leq \tau \leq t$ , then

$$P(s, t) = \exp\left[-\int_s^t \delta(\tau) d\tau\right]. \quad (\text{D.2})$$

However, at time  $s$ , the function  $\delta(\tau)$ ,  $\tau \geq s$ , is not known. One might postulate that  $\delta(\tau)$  is a stochastic process and the price  $P(s, t)$  is an average value or an expectation. How does one define this expectation? There are several possibilities:

$$P(s, t) = E\left\{\exp\left[-\int_s^t \delta(\tau) d\tau\right] \mid \delta(s)\right\}, \quad (\text{D.3})$$

$$P(s, t) = \exp\left\{-\int_s^t E[\delta(\tau) \mid \delta(s)] d\tau\right\} \quad (\text{D.4})$$

and

$$P(s, t) = \frac{1}{E\left\{\exp\left[\int_s^t \delta(\tau) d\tau\right] \mid \delta(s)\right\}}. \quad (\text{D.5})$$

Each of these three conditional expectations seems to provide a reasonable model for explaining the relationship among the returns on bonds of different maturities. Each has been called (by different authors) the *Expectations Hypothesis of the Term Structure of Interest Rates*.

If  $\delta(\tau)$  is a deterministic function, each of (D.3), (D.4) and (D.5) reduces to (D.2). When  $\delta(\tau)$  is stochastic, consider the positive random variable

$$X = \exp\left[-\int_s^t \delta(\tau) d\tau\right]$$

for a given initial value  $\delta(s)$ . Then, the arithmetic, geometric and harmonic means inequalities

$$E(X) \geq \exp\{E[\log_e(X)]\} \geq [E(X^{-1})]^{-1}$$

show that (D.3), (D.4) and (D.5) are three distinct propositions. Indeed, Cox, Ingersoll and Ross [2] prove that only Formulation (D.3) is sustainable in a continuous-time rational expectations equilibrium (also see Chapter 18 of [3]).

Evaluation of (D.3) as a function space integral is discussed in [1].

#### REFERENCES

1. BEEKMAN, J.A., AND SHIU, E.S.W. "Stochastic Models for Bond Prices, Function Space Integrals and Immunization Theory," *Insurance: Mathematics and Economics* VII (1988):163-75.
2. COX, J.C., INGERSOLL, J.E., JR., AND ROSS, S.A. "A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates," *Journal of Finance* XXXVI (1981):769-99.
3. INGERSOLL, J.E., JR. *Theory of Financial Decision Making*. Totowa, New Jersey: Rowan and Littlefield, 1987.
4. KELLISON, S.G. *The Theory of Interest*. Homewood, Illinois: Irwin, 1970.

#### (AUTHOR'S REVIEW OF DISCUSSION)

EDWARD W. FREES:

I thank Professors Ramsay and Shiu for their discussions, which amplify and extend certain aspects of the paper.

Professor Ramsay emphasizes the point that the retrospective premium does not make sense except in the most contrived examples. These premiums are useful to include in the paper because they complete some mathematical excursions, but should not be used in practice. His example is simple and to the point.

Professor Shiu suggests extensions to a stochastic interest structure in which, for simplicity, the time until loss is considered fixed. Probably the most important frontier of research in life contingency models is the introduction of a stochastic interest environment. Actuaries need such models in order to talk, and gain credibility, with other financial analysts. As Hickman\* noted, "interest rate variation and resulting risk is a fact of business life."

\*HICKMAN, J.C. "Why not random interest?" *The Actuary* 19, No. 2 (1985):1-4.

