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ALGORITHMS FOR CASH-FLOW MATCHING

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ABSTRACT

Shifts in the term structure of interest rates are the major sources of risk to fixed-income portfolios. Two important portfolio investment strategies in asset/liability management are cash-flow matching and immunization. The cash-flow matching strategy can be enhanced by allowing cash carry-forward and borrowing from future surpluses. Although the mathematical program thus formulated is nonlinear, we show that it can be linearized and solved by standard techniques. The algorithms can easily be implemented on a computer.

I. INTRODUCTION

A major problem facing the insurance industry today is interest rate fluctuations. If the terms of the assets are shorter than those of the corresponding liabilities, reinvestment risk arises because interest rates can fall. On the other hand, if assets are invested longer than liabilities, then disinvestment risk exists because interest rates can rise. C. L. Trowbridge coined the term C-3 risk to denote the risk of losses due to changes in interest rates. Some recent discussions on C-3 risk and valuation can be found in [5], [13] and [25].

Two important methods for insulating a fixed-income portfolio from shifts in the term structure of interest rates are cash-flow matching and immunization. There is extensive literature on immunization theory. Several books that deal solely with immunization theory are [3], [10], [12] and [15], and some recent actuarial papers on immunization and its generalizations are [1], [7], [16], [17], [22], [23] and [24]. Discussions on and applications of cash-flow matching or dedication can be found in [4], [6, chapter 19], [8, chapter 6] and [10, chapter 7].

This paper presents extensions to the classical cash-flow matching technique. The classical formulation is to choose fixed-income investments such that, for each period of the planning horizon, the investment cash flow will

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be sufficient to meet the projected liability payment. Furthermore, the cost of the investment portfolio should be as small as possible. The following notation is given by Elton and Gruber [6, p. 503]:

$L(t)$ = the liability payment at time t,
$C(t,j)$ = the cash flow at time t from a bond of type j,
$P(j)$ = the current price of a bond of type j,
$N(j)$ = the number of bonds of type j to be purchased.

The matching problem is to find the bond portfolio $\{N(j)\}$ such that the total cost

$$\sum_j N(j)\, P(j)$$

is minimized, while the following constraints are satisfied:

$$\sum_j N(j)\, C(t,j) \geq L(t) \quad \text{for all} \quad t$$

and

$$N(j) \geq 0 \quad \text{for all } j.$$

A major variation of the above is to allow cash carry-forward (at low or zero interest); this would lower the cost of the bond portfolio. Obviously, if borrowing also is allowed, that is, negative cash balances are allowed and carried forward, the cost would be lowered further. However, if the interest rate for borrowing is not the same as that for investing, the problem is nonlinear. We give two solutions to this problem.

Benjamin [2] has discussed this formulation, and three related papers in the literature are [14], [20] and [21]. In addition to the references on immunization listed above, some other formulations of the problem of interest rate risk and algorithms for the matching of assets with liabilities can be found in [8], [11], [18], [27], [28], [29], [30], and [31].

In the last section, we also present an algorithm for matching asset and liability cash flows as closely as possible.

II. PROBLEM FORMULATION

For simplicity, the cash flows are assumed to occur at the end of each time period. Hence, the values of t are 1, 2, 3, . . . only. Define

$$\Delta(t) = \sum_j N(j)\, C(t,j) - L(t). \tag{2.1}$$

Let b_t denote the borrowing rate (or financing rate) for period t, that is, from $t-1$ to t, and l_t the lending rate (or investment rate) for period t. Each b_t should be larger than or equal to the corresponding l_t. To be conservative, one would set b_t to be large and l_t to be small.

Let $V(t)$ denote the cash balance at time t. If $V(t)$ is positive, it is carried forward to the end of the next period at rate l_{t+1}; otherwise, at rate b_{t+1}. Then,

$$V(1) = \Delta(1),$$

$$V(2) = \begin{cases} \Delta(2) + (1 + l_2)\, V(1) & \text{if } V(1) \geq 0 \\ \Delta(2) + (1 + b_2)\, V(1) & \text{if } V(1) < 0 \end{cases},$$

$$\vdots$$

$$V(t) = \begin{cases} \Delta(t) + (1 + l_t)\, V(t-1) & \text{if } V(t-1) \geq 0 \\ \Delta(t) + (1 + b_t)\, V(t-1) & \text{if } V(t-1) < 0 \end{cases}, \tag{2.2}$$

$$\vdots$$

$$V(m) = \begin{cases} \Delta(m) + (1 + l_m)\, V(m-1) & \text{if } V(m-1) \geq 0 \\ \Delta(m) + (1 + b_m)\, V(m-1) & \text{if } V(m-1) < 0 \end{cases}.$$

The time $t = m$ is the end of the planning horizon. The problem of interest is:

$$\underset{\{N(j)\}}{\text{Minimize}} \ \sum_j N(j)\, P(j)$$

subject to $V(m) \geq 0$ and $N(j) \geq 0$ for all j. This mathematical program is nonlinear. We present two solutions.

Note that the liability cash flows $\{L(t)\}$ are assumed to be fixed and certain and that the bonds are assumed to be default-free and noncallable. The formulation given in [6, p. 504] is equivalent to the above when $l_t = r$ and $b_t = \infty$.

III. FIRST SOLUTION

Our first solution is to reformulate the problem as a mixed integer 0–1 linear program. Let M be a large positive number. Let $Z_2, Z_3, Z_4, \ldots Z_m$ be $(m-1)$ variables, each of which is either 0 or 1. Then, the nonlinear constraint (2.2) can be "linearized" as

$$\begin{aligned}
0 &\le V(t-1) && + MZ_t \\
V(t) &\le (1 + l_t)\, V(t-1) + \Delta(t) && + MZ_t \\
-V(t) &\le -(1 + l_t)\, V(t-1) - \Delta(t) && + MZ_t \\
0 &\le -V(t-1) && + M(1 - Z_t) \\
V(t) &\le (1 + b_t)\, V(t-1) + \Delta(t) && + M(1 - Z_t) \\
-V(t) &\le -(1 + b_t)\, V(t-1) - \Delta(t) && + M(1 - Z_t)
\end{aligned}$$

Thus, we have a mixed integer 0–1 linear program, for which many computer packages are available.

IV. SECOND SOLUTION

By introducing twice as many extra variables as in Section III, we can in fact transform the nonlinear problem formulated in Section II into a linear program: Find nonnegative numbers $\beta_1, \beta_2, \ldots, \beta_{m-1}, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}$, $N(1), N(2), \ldots$ such that

$$\sum_j N(j)\, P(j)$$

is minimized, and

$$\begin{aligned}
\Delta(1) &= \lambda_1 - \beta_1 \\
(1 + l_2)\, \lambda_1 - (1 + b_2)\, \beta_1 + \Delta(2) &= \lambda_2 - \beta_2 \\
&\ \ \vdots
\end{aligned}$$

$$(1 + l_t)\,\lambda_{t-1} - (1 + b_t)\,\beta_{t-1} + \Delta(t) = \lambda_t - \beta_t$$

$$\vdots$$

$$(1 + l_{m-1})\,\lambda_{m-2} - (1 + b_{m-1})\,\beta_{m-2} + \Delta(m-1) = \lambda_{m-1} - \beta_{m-1}$$

$$(1 + l_m)\,\lambda_{m-1} - (1 + b_m)\,\beta_{m-1} + \Delta(m) \geq 0$$

To understand the equivalence, consider the nonnegative number β_t as money borrowed at time t from income at time $t+1$ and λ_t as money carried forward from time t to time $t+1$. Because each borrowing rate is greater than the corresponding lending rate, it holds that, when the linear program is solved, at most one of β_1 and λ_1 can be nonzero; at most one of β_2 and λ_2 can be nonzero; and so on. One might impose an upper bound on β_t to limit the amount of borrowing that is allowed.

V. CONSTRAINT ON CAPITAL

The mathematical program formulated in Section II minimizes the total cost subject to the constraint that the cash balance at time m is nonnegative. However, this minimum cost may turn out to be larger than the amount of funds available to back up the liabilities. (The difference between the minimum cost and the available funds can be regarded as a measure of C-3 risk.) In this case, a more appropriate formulation would be:

$$\underset{\{N(j)\}}{\text{Maximize}}\ V(m)$$

subject to

$$\sum_j N(j)\,P(j) \leq K$$

and

$$N(j) \geq 0 \quad \text{for all } j.$$

Here, K is the amount of funds available to back up the liabilities. One seeks the bond portfolio that maximizes the final accumulated balance. The algorithms in Sections III and IV can easily be modified to solve this problem.

VI. CHEBYSHEV APPROXIMATION

Although the formulations in Sections II and V can be solved to give optimal solutions with respect to cost or final accumulated balance, one may prefer that the liability cash flows be as "closely" matched with the asset cash flows as possible. A way to accomplish this objective is to make the largest difference between the asset and liability cash flows as small as possible. Thus, the bond portfolio is determined by minimizing

$$\underset{t}{\text{maximum}} \{|\Delta(t)|\}, \tag{6.1}$$

where $|\Delta(t)|$ is the absolute value of the difference between the asset and liability cash flow in period t. In mathematical approximation theory, this is known as the Chebyshev problem. Before 1960, "there were plenty of existence-and-uniqueness theorems, but no one knew how to compute the answers" [9, p. 8]. Stiefel [26] was able to point out that, by introducing a new variable x, this nonlinear problem can be solved by the method of linear programming:

$$\text{Minimize } x \tag{6.2}$$

subject to

$$-x \leq \Delta(t) \leq x, \quad \text{for all } t,$$

and

$$N(j) \geq 0, \quad \text{for all } j.$$

Instead of (6.1), one may choose to minimize

$$\underset{t}{\text{maximum}} \{|w(t)\Delta(t)|\}.$$

An example of a weight function $w(t)$ is the discount function $(1 + i)^{-t}$. Also, to ensure that cost is taken into account, one may change (6.2) to

$$\text{Minimize } x + \zeta \sum_j N(j)\, P(j), \tag{6.3}$$

where ζ is a positive parameter to be chosen to control the relative emphasis between cost minimization and the matching of the asset and liability cash flows. The objective function (6.3) is inspired by a time-honored actuarial technique—the Whittaker-Henderson graduation method.

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