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A GENERALIZED PROFITS RELEASED MODEL FOR THE MEASUREMENT OF RETURN ON INVESTMENT FOR LIFE INSURANCE

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ABSTRACT

The classic insurance product has a stream of book profits that are initially negative, then turn and remain positive. For these products the traditional return on investment (ROI) techniques apply. But for products that have more than one change in sign of book profits or for products that begin with a positive book profit, that is, no initial investment, the traditional approach may not be effective. This paper presents a generalized profits released model, that is, one in which the difficulties of the traditional analysis are resolved.

INTRODUCTION

In his seminal paper "Gross Premium Calculations and Profit Measurement for Nonparticipating Insurance" [1], James C. H. Anderson established the framework for what has come to be known as the profits released model. This may be contrasted with the traditional asset share method in which profits are accumulated and which may be thought of as the profits retained model. The two ideas were partially merged with David S. Lee's paper "A Conceptual Analysis of Nonparticipating Life Insurance Gross Premium and Profit Formulas" [2].

The profits released method, however, presents difficulties for interpretation of ROI and net present value measures for insurance products whose book profit stream has more than one sign change or no initial investment. This can result in either no solutions or multiple solutions, a case that also impairs net present value analysis. Situations like these are mentioned by Donald R. Sondergeld in his paper "Profitability as a Return on Total Capital" [8], and he presents a method to address them. More recently in a discussion to a paper by Bradley M. Smith, "Pricing in a Return-on-Equity Environment" [6], this author identifies these products as yet another reason for disagreement between ROI and the return on equity (ROE) determined from the published GAAP financials of an insurance enterprise. Most recently in "Cash Flow Analysis by the Prudent Banker's Method, or Discounting Turned on Its Head" [4], Claude Y. Paquin presents a product that involves multiple changes in sign of its cash flows and applies to it the same method as appears in Sondergeld [8].

The method as presented in [4] and [8] is incomplete. It does not provide a precise economic interpretation of events, nor does it enlighten us as to the choice and the role of the rate credited on "deposits." That rate has a material impact on the result of the calculation. Even in the citation [9] provided by Mr. Sondergeld for the method, it is presented only as part of a decision rule.

When presented within the context of a more general financial model, the method provides part of the key to resolve these atypical cases into economically meaningful serial transactions. The enhanced viewpoint includes the traditional ROI method as a logical subset.

Part I of this paper presents background information needed for the generalized model. Part II briefly recapitulates the method described by Sondergeld and Paquin, which will be referred to as the accumulating algorithm. Part III constructs an interim model that uses the algorithm to obtain a useful result on ROI. Part IV restates the method totally from a discounting perspective that finally generalizes the traditional profits released model. Part V summarizes the generalized model and presents a comprehensive application.

I. THE PROFITS RELEASED AND PROFITS RETAINED MODELS

Essentially two distinct pricing models are used by actuaries,* informally known as the "profits released" model and the "profits retained" model. Each yields a valid, but different, analysis of the underlying economic reality presented by the insurance transaction. The difference between the two models is where the original investment comes from and where do future earnings go.

In this paper the book profits include investment income only on the reserve and the cash flow and include all taxes on the book profits paid at the insurance company level. The book profits include no component of investment income due to surplus, that is, retained earnings of the book profits or taxes thereon. They are prior to any consideration of taxes at the shareholder level.

In a profits released model the original investment comes from shareholders, by direct investment or by the withholding of shareholder dividends.

*The material in this section is a concise summary of the two pricing models, and the author knows of no specific general publication of these expressions. The key insight provided by the role of surplus in these two models and situations indicating preference of model were earlier shared with the author by Steven D. Sommer. The author gratefully acknowledges his contribution.

Future earnings are transfers back to the shareholders. This is the picture presented by the classical Anderson book profit stream. Note that taxes are paid at the insurance company level. As such, the book profit stream represents a pretax stream to the shareholders. In this model the level of surplus of the insurance company remains constant.

Several profit measures can be calculated based on a profits released model. First, of course, is the ROI earned by the shareholders. (If the book profits contain a charge for "minimum" or "target surplus" and a credit for the aftertax investment earnings on such surplus, then the quantity is usually called ROE.) This quantity can be used by shareholders for comparison with the pretax yield of other investments, enabling them to choose among the alternatives. Clearly this rate should exceed the pretax level of return that the company can earn on financial instruments. It is compensation for the additional risk borne by the shareholders for undertaking the investment in insurance, which typically has greater risk than that found in the investment portfolio.

Second, the "present value of profits" is the net present value of the stream and can be calculated at various investment rates. This quantity represents the present value of additional earnings to the shareholders, that is, the present value of earnings in the stream in excess of the given investment rate used in the discounting. Note that taxes are paid at the insurance company level and not at the shareholder level. This means that the rate used to discount is a pretax rate, specifically a pretax rate to the shareholders.

A special investment rate often used is the insurance company pretax asset earnings rate. The present value of profits based on this rate is an indicator to the company of the value of the service it provides in excess of the return the shareholders would receive if they invested in the same financial instruments as the company. Net present value is useful, because the ROI alone can lead to an incorrect choice between potential products, for example, where the initial investment or duration differs materially.

Third, the "break-even year" can be found. This is the earliest duration in which the net present value of book profits based on an investment rate becomes positive and remains positive when later durations are considered. Again, the most common choice for the investment rate is the insurance company pretax asset earnings rate. The break-even year can be thought of as the year in which the shareholders are as well off as if they had invested in assets similar to those purchased by the insurance company.

Two other measures that can be calculated are present value of profits as a percentage of present value of premiums and the profit per unit in force.

The former is the "profit as a percent of premium" measure, or simply the profit margin. The latter is the ratio of the present value of profits to an annuity due based on interest and survivorship. It represents a "levelized" statutory profit per unit in force each year and is phrased as "x dollars per unit per year." Again, the rate often used for the present value computations is the insurance company pretax asset earnings rate.

The choice of reserve basis can have a major impact on the ROI and present value of profits measures in a profits released model. This impact is a function of both the absolute amount of reserve and the incidence in establishing it within the financial statement.

In a profits retained model the cost of acquiring a block of business comes from the surplus of the insurance company; surplus is reduced. Future earnings are retained by the company, and surplus is repaid and ultimately increased. In this model there are two major profit criteria.

The first is the accumulated value of profits. In this case the cost of acquiring the block is brought forward with interest at the insurance company pretax asset earnings rate, and federal income tax is charged or credited, depending on the sign of the opening value, and then combined with the next book profit and again accumulated. (This treatment of taxes depends on the existence of other taxable income. This step should be modified to reflect a given company's tax situation.) The process terminates with some given duration. Each year the profit stream includes the aftertax investment income on accumulated surplus. At the end of the study the value represents the amount by which surplus has been increased. If the value plus the liability is divided by the survivorship at that point, then this is the traditional asset share.

The second profit criterion is the break-even year, the earliest year that the accumulated surplus becomes positive and stays positive. This is the year that surplus is as well off as if the expense of acquiring the block had remained invested in typical investments. For tax reasons, it is often earlier than the similar quantity in a profits released model.

The profits released model and the profits retained model are frequently thought of as being used by stock companies and mutual companies, respectively. But a stock company might wish to use a profits retained model if it is pricing a participating line of business or if shareholder dividends are reasonably constant. A mutual company might use a profits released model to treat its surplus or part of it as "fixed" and to use funds from it to support a line of business as an investment.

It is not economically meaningful to refer to accumulated profits in a profits released model, nor is it meaningful to refer to an ROI or net present value in the profits retained model. It is possible to make an algebraic computation and obtain a number, but it does not have any economic significance.

Break-even year has sometimes been defined without the added requirement to "remain positive." This reflects pricing products where unusual patterns of book profits do not emerge. The two models should be validated for proper interpretation of profit measures for products with unusual patterns of book profits. The profits retained model does not mislead in these situations. The problem is to generalize the profits released model to include these cases.

II. DEFINITION OF ROI AND THE ACCUMULATING ALGORITHM

Let BP_j be the book profit at duration j, $j = 1, 2, \ldots, n$. Using Promislow notation [5], call the sequence $(BP_1, BP_2, \ldots, BP_n)$ well behaved if there is only one sign change in the sequence of book profits. The occurrence of a zero book profit is a sign change. Note that Mr. Paquin's example in [4] (which will be used later), although stated as cash flows and not book profits, is not well behaved.

The ROI is the value of i such that

$$\sum_{j=1}^{n} BP_j (1 + i)^{-j+1} = 0.$$

This is a discounting process, and if the sequence is well behaved, then it has a unique real valued solution. If i is not equal to -1, then it is algebraically equivalent to

$$\sum_{j=1}^{n} BP_j (1 + i)^{n-j+1} = 0,$$

which has the algebraic form of an accumulation. This can be replaced by the following recursive process. Let OB_1 be the initial outstanding balance, for example, $OB_1 = BP_1$, and let OB_j be the outstanding balance at the end of duration j defined by

$$OB_j = OB_{j-1} (1 + i) + BP_j.$$

The ROI is the value of i such that $OB_n = 0$. In the well-behaved situation with $BP_1 < 0$, that is, the project is a user of shareholder funds, $OB_j < 0$

for $j = 1$ to $n - 1$. This may be called a pure investment project because $OB_j < 0$ for $j = 1$ to $n - 1$ implies there is an amount to be recovered at each duration prior to the last. A less frequently encountered well-behaved situation is found if $BP_1 > 0$, that is, the project is an initial source of funds to the insurance company. In this case $OB_j > 0$ for $j = 1$ to $n - 1$. The product generates an initial positive book profit. The remaining book profits are positive and then may turn negative, but only one sign change occurs. There is no ROI because there is no investment. This case may be called a pure financing project. If a profits retained method is applied and $OB_n > 0$, then surplus is increased by that amount and, in the case of a stock company, could be paid to shareholders. A pure financing project, in which there is no drawdown of surplus, is another example where a stock company might want to use the profits retained model. (The author is utilizing the terms pure investment and pure financing from [9].)

If k represents a rate credited by the company on positive product funds, then the accumulating algorithm used by Sondergeld and Paquin can be stated as follows. Let $OB_1 = BP_1$;

$$OB_j = OB_{j-1}(1 + r) + BP_j$$

where $r = i_{\text{ROI}}$ if $OB_{j-1} < 0$; and $r = k$ if $OB_{j-1} \geqq 0$. This algorithm makes i a function of k. The actuary needs to be aware of the existence of this other assumption in the pricing model. To illustrate, the following tabulation gives the ROI for Mr. Paquin's example under various choices of k:

k	ROI
5%	12.12%
7	13.73
10	15.42
15	17.14

III. THE ACCUMULATING ALGORITHM GENERALIZED

The generalized algorithm is based on a modification of the prior algorithm and the combination of the profits released and the profits retained models.

Let k be the interest rate that the insurance company earns on its invested assets after taxes at the company level. With this choice of k, the algorithm can be used to decompose the book profit stream into consecutive substreams in which the profits released model (in which the project is a user of shareholder funds) and/or profits retained model (in which the project is a source of funds) apply in each substream. The result is that, during each profits retained substream, the accumulated value of surplus is nonnegative and equal to zero at the end of the substream. Each of the profits released substreams has the same ROI, namely, that found in solving the algorithm. Each of the profits retained substreams is a pure financing project, and each of the profits released substreams is an investment project.

In prior formulations [4] and [8], general views were given on the choice of k. In the generalization, k must be the aftertax earned rate on positive funds, so that the accumulated value of surplus in the profits retained substreams equals zero; that is, when a source of funds occurs, there are sufficient future earnings to cover taxes on the investment income on retained earnings and to provide for all future negative book profits within that substream. In the original source of the algorithm [9], no mention was made of taxes, but companies operate in an aftertax environment and this consideration is necessary to fully model the economic transaction.

Future liability flows are provided from the cash flows of the assets purchased by these positive funds. Considerations for the choice of k should reflect the choice of the assets to be used. These assets can be commingled with those of the entire company or segmented for a given product line. The assets can be part of those purchased by the regular investment policy, or dedicated assets can be purchased. If dedicated assets are used, then a range of investment alternatives exists: Treasury instruments, investment-grade corporate bonds, mortgages, high-yield bonds, equities, or other blocks of insurance. Credit risk and maturity structure should be considered. The use of other blocks of insurance should be considered with care, because those "assets" are not carried on the balance sheet of the insurance company, have less certain cash flows, and are less readily marketable than invested assets. For the remainder of this paper, the assumption is that positive funds are used to purchase assets similar in character to the insurance company's invested asset portfolio and k is the aftertax rate earned thereon.

If $BP_1 < 0$, then the sequence begins with a profits released/investment substream. If $BP_1 > 0$, then the sequence begins with a profits retained/pure financing substream. In the latter case the actual shareholder investment occurs at a time later than the issuance of the product. A well-behaved book profit stream has no proper substreams; that is, the entire stream is either profits released or profits retained. Therefore, this method gives the same results on standard book profit streams and so generalizes the classical ROI.

The ROI determined during profits released or investment periods should exceed the pretax value of k to make the insurance project desirable to shareholders.

The algorithm provides two pieces of information. The first item, having chosen k, is the ROI value. The book profits used for the substreams are those of the original book profit stream, except for those durations in which a change from/to profits released/retained occurs. The second piece of information provided by the algorithm, unrecognized before and dependent on tax aspects, is a part of the decomposition of the original book profit into the portion to be retained and the portion to be released.

The rule for the split is given in Table 1. If $BP_1 > 0$, then BP_1 is retained; if $BP_1 < 0$, then BP_1 is released. Without loss of generality, BP_1 is not equal to zero.

TABLE 1

ACCUMULATING ALGORITHM SPLITTING RULE

		Book Profit	
OB_{j-1}	OB_j	Profits Released Portion	Profits Retained Portion
< 0	< 0	BP_j	0
$\geqq 0$	$\geqq 0$	0	BP_j
< 0	$\geqq 0$	$BP_j - OB_j$	OB_j
$\geqq 0$	< 0	OB_j	$BP_j - OB_j$

The original algorithm is stated in terms of a level value for k. This quantity need not be level. The change to the algorithm for non-level k is to define k_j to be the insurance company aftertax asset earnings rate for duration j. The algorithm then becomes: Let $OB_1 = BP_1$;

$$OB_j = OB_{j-1}\ (1 + r) + BP_j$$

where $r = i_{\text{ROI}}$ if $OB_{j-1} < 0$, and $r = k_j$ if $OB_{j-1} \geqq 0$.

Several examples of the decomposition are presented in Tables 2–7 to illustrate various features. The company's pretax asset earnings rate is assumed to be 10.61 percent and tax rate, 34 percent. This results in an aftertax earnings rate of 7 percent. In each table, the order of information is: duration, book profit, algorithm outstanding balance, and substream book profit stream.

In the example shown in Table 2 Mr. Paquin's cash flows are assumed to be aftertax book profits. The ROI to the shareholders in this example is 13.73 percent (the same as in [4]) pretax. If an ROI computation is performed on the profits released substream, then the ROI will be found to be the same 13.73 percent obtained from the algorithm. If the profits retained substream is accumulated at 7 percent, or equivalently 10.61 percent pretax with a charge for taxes on investment income on retained earnings, then the terminal surplus will be found to be zero. The asterisk denotes the year(s) when the original book profit(s) is (are) split.

TABLE 2

ORIGINAL BOOK PROFIT DECOMPOSITION AT i = 13.73% AND k = 7.00%

			Book Profit	
Duration	Original Book Profit	Outstanding Balance	Profits Released Model	Profits Retained Model
1	−125,138	−125,138.00	−125,138.00	
2	59,135	− 83,183.57	59,135.00	
3	46,986	− 47,618.09	46,986.00	
4	36,013	− 18,142.72	36,013.00	
5*	24,192	+ 3,558.41	20,633.59	3,558.41
6	17,084	+ 20,891.50		17,084.00
7	11,557	+ 33,910.91		11,557.00
8	6,754	+ 43,038.67		6,754.00
9	2,358	+ 48,409.38		2,358.00
10	− 1,087	+ 50,711.03		− 1,087.00
11	− 3,720	+ 50,540.81		− 3,720.00
12	− 7,323	+ 46,755.66		− 7,323.00
13	− 10,132	+ 39,896.56		−10,132.00
14	− 12,735	+ 29,954.32		−12,735.00
15	− 15,210	+ 16,841.12		−15,210.00
16	− 18,020	0.00		−18,020.00

In the example shown in Table 3, Mr. Paquin's original values in durations 13 and 14 are changed to be $30,518.06 and $8,000.00, respectively. The ROI to the shareholders in this example is 20.66 percent, pretax, found by the algorithm, and equals the ROI in each of the two profits released substreams. The two profits retained substreams have accumulated surpluses of zero. Note that the profits released substreams are pure investment projects and the profits retained substreams are pure financing projects.

The example shown in Table 4 illustrates a product whose investment occurs the year following issue; that is, it begins with a positive book profit. The ROI to the shareholders of the insurance company is 10.00 percent pretax. If the company takes the shareholders' investment and merely invests it in assets similar to its existing portfolio, then the pretax return to the shareholders is only 7 percent, because the insurance company must pay taxes on the investment income before paying out gains. But if the shareholders can invest in assets similar to that of the insurance company, then the shareholders can earn 10.61 percent pretax. Hence, this particular product is not desirable for the company's shareholders.

TABLE 3

MODIFIED BOOK PROFIT DECOMPOSITION AT i = 20.66% AND k = 7.00%

			Book Profit	
Duration	Original Book Profit	Outstanding Balance	Profits Released Model	Profits Retained Model
1	−125,138.00	−125,138.00	−125,138.00	
2	59,135.00	− 91,853.70	59,135.00	
3	46,986.00	− 63,842.60	46,986.00	
4	36,013.00	− 41,018.05	36,013.00	
5	24,192.00	− 25,299.45	24,192.00	
6	17,084.00	− 13,441.75	17,084.00	
7	11,557.00	− 4,661.52	11,557.00	
8*	6,754.00	+ 1,129.52	5,624.48	1,129.52
9	2,358.00	+ 3,566.59		2,358.00
10	− 1,087.00	+ 2,729.25		− 1,087.00
11*	− 3,720.00	− 799.70	− 799.70	− 2,920.30
12	− 7,323.00	− 8,287.91	− 7,323.00	
13*	30,518.06	+ 20,518.06	10,000.00	20,518.06
14	8,000.00	+ 29,954.32		8,000.00
15	− 15,210.00	+ 16,841.12		−15,210.00
16	− 18,020.00	0.00		−18,020.00

TABLE 4

FINANCING PROJECT FOLLOWED BY INVESTMENT PROJECT AT i = 10.00% AND k = 7.00%

			Book Profit	
Duration	Original Book Profit	Outstanding Balance	Profits Released Model	Profits Retained Model
1	+ 500.00	+500.00		500.00
2*	−1,000.00	−465.00	−465.00	−535.00
3	100.00	−411.50	100.00	
4	200.00	−252.65	200.00	
5	277.92	0.00	277.92	

The example shown in Table 5 illustrates a pure investment product; it has no proper substreams. The ROI is 12.83 percent, pretax, to the shareholders.

TABLE 5

PURE INVESTMENT PROJECT AT i = 12.83% AND k = 7%

			Book Profit	
Duration	Original Book Profit	Outstanding Balance	Profits Released Model	Profits Retained Model
1	−1,000.00	−1,000.00	−1,000.00	
2	200.00	− 928.26	200.00	
3	300.00	− 747.31	300.00	
4	400.00	− 443.16	400.00	
5	500.00	0.00	500.00	

The example shown in Table 6 illustrates a pure financing product. It begins with a positive book profit followed by negative book profits and has no proper substreams. In this example there is no ROI, because there is no outstanding investment at any duration. However, the product has fiscal merit, because surplus is increased by $44.32 at the end of the project.

TABLE 6

FAVORABLE PURE FINANCING PROJECT AT k = 7.00%

Duration	Original Book Profit	Outstanding Balance	Book Profit	
			Profits Released Model	Profits Retained Model
1	+ 1,000.00	+ 1,000.00		1,000.00
2	− 200.00	+ 870.00		− 200.00
3	− 300.00	+ 630.90		− 300.00
4	− 400.00	+ 275.06		− 400.00
5	− 250.00	+ 44.32		− 250.00

Consider a slight variation on this product. For the example shown in Table 6, suppose the fifth duration book profit was −$300.00 instead of −$250.00. In this event the schema would be as shown in Table 7. This indicates that surplus is reduced by $5.68, that is, that the shareholders now have to invest $5.68. However, because there are no further book profits, the shareholders obtain no return on this investment. This is a product to avoid.

TABLE 7

UNFAVORABLE PURE FINANCING PROJECT AT k = 7.00%

Duration	Original Book Profit	Outstanding Balance	Book Profit	
			Profits Released Model	Profits Retained Model
1	+ 1,000.00	+ 1,000.00		1,000.00
2	− 200.00	+ 870.00		− 200.00
3	− 300.00	+ 630.90		− 300.00
4	− 400.00	+ 275.06		− 400.00
5	− 300.00	− 5.68	− 5.68	− 294.32

The algorithm does not produce an ROI for either of these two examples, because either there is no investment or there is an ultimate investment with no return. But the application of the algorithm results in the correct decision, based on the sign of the terminal outstanding balance, as to the fiscal viability of the project for the shareholders.

Augmenting the accumulating algorithm with the profits released/retained models and splitting the book profits provide a framework for an economic interpretation of the results. The conclusions that can be drawn from the examples only suggest, but do not prove, the utility in applying the algorithm in general for determining ROI. Neither does it provide for other profit measures, nor a systematic approach to the evaluation of projects.

IV. THE GENERALIZED PROFITS RELEASED MODEL

Consider a mirror algorithm derived from the perspective of discounting. The perspective is that of discounting from the last book profit backwards and producing present value balances. The source of this idea is the accumulating algorithm and a technique in finance that eliminates a last-period negative cash flow by charging the prior positive period for an amount that will mature equal to the negative flow. An example of this is in strip-mining, where the land must be restored at the end of the project. This last-period cash flow is met by a performance bond established while positive cash flows still remain. Again, the process uses the aftertax asset earnings rate for positive balances held by the company, because they must both provide for tax on income on retained earnings and cover future obligations.

By means of this discounting approach, the previous results are reproduced and the remaining profits released measures can be calculated. It also has other desirable properties.

Let $\{BP_j\}$ be a sequence of book profits, $j = 1, \ldots, n$. Let $PVB_j(i)$ be the present value balance at duration j at a rate of discount i. Let k_j be the aftertax rate earned by the insurance company on invested assets for duration j. The domain of definition for i and k_j, $j = 1, \ldots, n$, is the open interval $(-1, +\infty)$, that is, i and k_j, $j = 1, \ldots, n$, > -1. The discounting algorithm is defined as follows.

Definition 4.1:

1. $PVB_n(i) = BP_n$;
2. $PVB_j(i) = PVB_{j+1}(i)/(1+r) + BP_j$

where $r = i$ if $PVB_{j+1}(i) > 0$ and $r = k_j$ if $PVB_{j+1}(i) \leqq 0$.

Definition 4.2:

i_{ROI} denotes a value of i such that $PVB_1(i_{\text{ROI}}) = 0$.

The decision rule for splitting the original book profits into the profits released portion and the profits retained portion is given in Table 8.

TABLE 8

DISCOUNTING ALGORITHM SPLITTING RULE

		Book Profit	
PVB_{j+1}	PVB_j	Profits Released Portion	Profits Retained Portion
< 0	< 0	0	BP_j
≧ 0	≧ 0	BP_j	0
< 0	≧ 0	PVB_j	$BP_j - PVB_j$
≧ 0	< 0	$BP_j - PVB_j$	PVB_j

Definition 4.3:

If $BP_n < 0$, then BP_n is retained; if $BP_n > 0$, then BP_n is released.

Without loss of generality, BP_n is not equal to zero. This rule treats any last-period original negative book profit as the end of a profits retained substream and a last-period positive book profit as the end of a profits released substream.

Note that $k_j > -1$ for all j and $i > -1$ imply that the operation of the division by $(1+r)$ cannot, in and of itself, change the sign of a $PVB_j(i)$. In the evaluation of any project, the k_j's are deemed constant with respect to i.

Normalizing Rule:

If $BP_1 > 0$, then consider the quantity $BP_1(1 + k_1) + BP_2$. If that quantity is less than zero, then redefine $\{BP_j\}$ to be $\{BP_1(1 + k_1) + BP_2, BP_3, \ldots, BP_n\}$. If that quantity is greater than or equal to zero, then consider the quantity $BP_1(1 + k_1)(1 + k_2) + BP_2(1 + k_2) + BP_3$. If this quantity is less than zero, then redefine $\{BP_j\}$ to be $\{BP_1(1 + k_1)(1 + k_2) + BP_2(1 + k_2) + BP_3, BP_4, \ldots, BP_n\}$. If this quantity is greater than or equal to zero, repeat the process.

Repeated use of this rule will result either in a sequence of book profits whose first element is negative, that is, an investment, or in the accumulated value of all the book profits at the k_j being nonnegative. If the latter event occurs and the accumulated value is positive, then the project is a pure financing project requiring no shareholder funds and producing positive shareholder value at the termination of the project. If resources are available, then the project is favorable. If resources are constrained, then this positive value should exceed management's expectation of the opportunity cost of not developing alternative projects in order to have merit. If not, then it should be discarded. In the event that the normalizing rule produces a negative quantity prior to duration n, then the new sequence of book profits replaces the original.

Definition 4.4:

$PVB_1(i)$ is said to be independent of i if $PVB_1(i)$ is constant for all values of i in the interval $(-1, +\infty)$.

Consider a project with the following book profits $(-1000, 100, -1000)$, where the aftertax asset earnings rate is 10 percent for all durations. For this case, $PVB_1(i) = (-1000/1.1 + 100)/1.1 - 1000 = -1{,}735.54$ for all i. The class of projects that are normalized and independent of i is non-void.

From this point on, only $\{BP_j\}$ that are normalized will be considered.

Profits released and profits retained substreams will be generated in reverse order. Let i be given and $BP_n > 0$. Then $PVB_n(i) = BP_n > 0$. Let h be chosen to be the first value from the list $n - 1, n - 2, \ldots, 1$, such that $PVB_h(i) \leqq 0$. By the splitting rule, the quantity $BP_h - PVB_h(i)$ is released and $PVB_h(i)$ is retained. This defines a profits released substream, namely, $\{BP_h - PVB_h(i), BP_{h+1}, \ldots, BP_n\}$. If $PVB_h(i) < 0$, then $PVB_h(i)$ begins the next substream, which will be a profits retained substream. If $PVB_h(i) = 0$, then the sign of BP_{h-1} will determine the character of the next substream.

If $BP_n < 0$, then this indicates a profits retained substream. Given i, $PVB_n(i) = BP_n < 0$, let h be the first element from the list $n - 1, n - 2, \ldots, 2$ such that $PVB_h(i) \geqq 0$. By the splitting rule, the sequence $\{BP_h - PVB_h(i), BP_{h+1}, \ldots, BP_n\}$ is a profits retained substream. If $PVB_h(i) > 0$, then $PVB_h(i)$ determines the endpoint of a new substream, which will be a profits released substream. If $PVB_h(i) = 0$, then the sign of BP_{h-1} will determine the character of the next substream.

For the profits retained substreams, the list for h stopped at 2. This is due to the fact that if $PVB_2(i) < 0$, then $PVB_1(i) < 0$ because $BP_1 < 0$ and $k_1 > -1$.

Applying these procedures to $\{BP_j\}$ in reverse order, $\{BP_j\}$ can be decomposed into profits released substreams and profits retained substreams.

Definition 4.5:

A substream is called proper if it does not include BP_1.

Definition 4.6:

The substream containing BP_1 is defined to be a profits released substream, even if the substream's last value is negative.

Proposition 4.1:

Let $\{BP_j\}$ be a normalized stream of book profits that is not independent of i. The present value of each proper profits released substream at i is zero and the present value of each profits retained substream at the applicable k_j is zero.

Proof: Let i be given and consider the case where $BP_n > 0$ and $h > 1$ exists with $PVB_h(i) \leqq 0$ and $PVB_j(i) > 0$ for $j = n - 1, \ldots, h + 1$, that is, a proper profits released substream. By the definition of h:

$$PVB_{h+1}(i) = \sum_{j=h+1}^{n} BP_j/(1 + i)^{j-(h+1)} > 0.$$

$$\begin{aligned} PVB_h(i) &= PVB_{h+1}(i)/(1 + i) + BP_h \\ &= \sum_{j=h+1}^{n} BP_j/(1 + i)^{j-h} + BP_h \leqq 0. \end{aligned}$$

If $PVB_h(i) = 0$, then the above expression can be rewritten:

$$\sum_{j=h}^{n} BP_j/(1 + i)^{j-h} = 0,$$

and so the substream has net present value zero at rate i. If $PVB_h(i) < 0$, then rearranging the terms of the algorithm results in

$$[BP_h - PVB_h(i)] + PVB_{h+1}(i)/(1 + i) = 0.$$

But this is

$$[BP_h - PVB_h(i)] + \sum_{j=h+1}^{n} BP_j/(1 + i)^{j-h} = 0.$$

This states that the sequence $\{[BP_h - PVB_h(i)], BP_{h+1}, \ldots, BP_n\}$ has net present value of zero at rate i. The first term of this sequence is precisely the amount to be included in the profits released substream by the splitting rule.

This argument can be applied to other proper profits released substreams in a similar manner. And it also applies to the profits retained substreams, in which instead of powers of $(1 + i)$, products of $(1 + k_j)$ are used. □

The splitting rule was chosen so that Proposition 4.1 would follow.

Corollary 4.1.1:

If $\{BP_j\}$ is a normalized sequence, not independent of i, and i_{ROI} exists for $PVB_1(i)$, then all profits released substreams have net present value equal to zero at $i = i_{\text{ROI}}$.

Proof: By Proposition 4.1 all proper profits released substreams have net present value equal to zero at a chosen i. The first profits released substream has net present value equal to $PVB_1(i_{\text{ROI}}) = 0$ also. □

Proposition 4.2:

If $\{BP_j\}$ is a normalized sequence and i, an element of $(-1, +\infty)$, exists such that $PVB_1(i) = 0$, then $OB_n = 0$ for that value of i; that is, a solution to the discounting algorithm is a solution to the accumulating algorithm.

Proof: The $\{BP_j\}$ can be decomposed into separate profits released substreams and profits retained substreams. By Proposition 4.1, each of these proper substreams has net present value of zero at i or the associated k_j, respectively. As $PVB_1(i) = 0$, the first profits released substream also has a net present value of zero at i. Applying the accumulating algorithm to BP_1 using i (as $BP_1 < 0$) and noting the results of Proposition 4.1 above, the sign of OB_j will change at the end of the first profits released substream or OB_j will be zero. Due to the definitions of i and $\{k_j\}$, a change in sign of OB_j, and so the use of i or k_j in both algorithms, can only occur due to the sign of BP_j. Therefore, OB_j will follow the same use of i and the $\{k_j\}$ as the discounting algorithm. The net present values of zero for the substreams become accumulated values of zero as the accumulating algorithm is applied and OB_n equals zero. □

In fact, the accumulating algorithm and its splitting rules produce the same profits released and retained substreams.

Proposition 4.3:

If $\{BP_j\}$ is normalized, then $PVB_j(i)$, $j = n, \ldots, 1$, is monotone decreasing on the interval $(-1, +\infty)$. In particular, $PVB_1(i)$ is monotone decreasing on the interval $(-1, +\infty)$.

Proof: Let i_1 and i_2 be elements of $(-1, +\infty)$ such that $i_1 < i_2$.

Case 1: $BP_n > 0$.

$$PVB_{n-1}(i_1) = BP_n/(1 + i_1) + BP_{n-1}$$

and

$$PVB_{n-1}(i_2) = BP_n/(1 + i_2) + BP_{n-1}.$$

$$PVB_{n-1}(i_1) - PVB_{n-1}(i_2) = BP_n\ [1/(1 + i_1) - 1/(1 + i_2)] > 0$$

as $i_1 < i_2$. Thus, $PVB_{n-1}(i)$ is monotone decreasing, in fact, strictly monotone decreasing.

Now $PVB_{n-1}(i_1)$ and $PVB_{n-1}(i_2)$ can be both positive, both nonpositive, or the first is positive and the second is nonpositive; call these cases 1.a, 1.b, and 1.c, respectively.

Case 1.a: $PVB_{n-1}(i_1) > PVB_{n-1}(i_2) > 0$.

$$PVB_{n-2}(i_1) = PVB_{n-1}(i_1)\ /(1 + i_1) + BP_{n-2}$$

and

$$PVB_{n-2}(i_2) = PVB_{n-1}(i_2)/(1 + i_2) + BP_{n-2}.$$

Then

$$PVB_{n-2}(i_1) - PVB_{n-2}(i_2) = PVB_{n-1}(i_1)/(1 + i_1) - PVB_{n-1}(i_2)/(1 + i_2).$$

Because the first numerator is larger than the second and the first denominator is smaller than the second, the difference is greater than zero, or

$$PVB_{n-2}(i_1) > PVB_{n-2}(i_2).$$

Case 1.b: $PVB_{n-1}(i_1) > PVB_{n-1}(i_2)$, $PVB_{n-1}(i_1) \leqq 0$.

$$PVB_{n-2}(i_1) = PVB_{n-1}(i_1)/(1 + k_{n-2}) + BP_{n-2}$$

and

$$PVB_{n-2}(i_2) = PVB_{n-1}(i_2)/(1 + k_{n-2}) + BP_{n-2}.$$

$$PVB_{n-2}(i_1) - PVB_{n-2}(i_2) = [PVB_{n-1}(i_1) - PVB_{n-1}(i_2)]/(1 + k_{n-2}) > 0$$

because the numerator is positive by the assumption for case 1.b and the denominator is positive. Therefore, $PVB_{n-2}(i_1) > PVB_{n-2}(i_2)$.

Case 1.c: $PVB_{n-1}(i_1) > 0$ and $PVB_{n-1}(i_2) \leqq 0$.

$$PVB_{n-2}(i_1) = PVB_{n-1}(i_1)/(1 + i_1) + BP_{n-2}$$

and

$$PVB_{n-2}(i_2) = PVB_{n-1}(i_2)/(1 + k_{n-2}) + BP_{n-2}.$$

$$PVB_{n-2}(i_1) - PVB_{n-2}(i_2)$$
$$= PVB_{n-1}(i_1)/(1 + i_1) - PVB_{n-1}(i_2)/(1 + k_{n-2}) > 0$$

because the first quantity is positive and the second quantity is zero or negative. Therefore, $PVB_{n-2}(i_1) > PVB_{n-2}(i_2)$.

Note that in all cases the inequality was strict, so strictly monotone decreasing was shown for $PVB_{n-2}(i)$.

This method can be repeated, so for case 1, $PVB_1(i)$ is strictly monotone decreasing.

Case 2: $BP_n < 0$.

$$PVB_{n-1}(i_1) = BP_n/(1 + k_{n-1}) + BP_{n-1}$$

and

$$PVB_{n-1}(i_2) = BP_n/(1 + k_{n-1}) + BP_{n-1}.$$

$$PVB_{n-1}(i_1) - PVB_{n-1}(i_2) = 0,$$

so then $PVB_{n-1}(i_1) \geqq PVB_{n-1}(i_2)$ and $PVB_{n-1}(i)$ is monotone decreasing.

Either $PVB_{n-1}(i_1) = PVB_{n-1}(i_2)$ are both nonpositive or both positive. If nonpositive, then the preceding argument results in $PVB_{n-2}(i_1) = PVB_{n-2}(i_2)$ and $PVB_{n-2}(i)$ is monotone decreasing. If both are positive, then

$$PVB_{n-2}(i_1) - PVB_{n-2}(i_2) = PVB_{n-1}(i_1)\,[1/(1 + i_1) - 1/(1 + i_2)] > 0$$

because $i_1 < i_2$. In this case $PVB_{n-2}(i)$ is strictly monotone decreasing.

If both of the $n-2$ values are equal, then repeat the above "equality" argument. If they are not equal, then the argument given in case 1 applies. This results in $PVB_{n-3}(i)$ being monotone decreasing. Therefore, $PVB_1(i)$ is monotone decreasing.□

Corollary 4.3.1:

If $\{BP_j\}$ is normalized and not independent of i, then $PVB_1(i)$ is strictly monotone decreasing.

Proof: In the proof of Proposition 4.1 it was shown that if $BP_n > 0$, then all inequalities were strictly "greater than," and so $PVB_1(i)$ is strictly decreasing. Therefore, consider $BP_n < 0$.

$$PVB_{n-1}(i) = BP_n/(1 + k_{n-1}) + BP_{n-1}$$

If this quantity is positive, then future inequalities are strict, as was shown in Proposition 4.3. If this quantity is nonpositive, then

$$PVB_{n-2}(i) = [BP_n/(1 + k_{n-1}) + BP_{n-1}]/(1 + k_{n-2}) + BP_{n-2}$$

for any value of i in its domain. This argument can be repeated. Either at some point a *PVB* becomes positive, which then results in strict inequality, or the discounting continues to use only the k_j. In the former case, strictly monotone decreasing is established, and in the latter case, $PVB_1(i)$ is independent of i, which contradicts the hypothesis.□

Corollary 4.3.2:

If $\{BP_j\}$ is normalized and independent of i, then $PVB_1(i) = c < 0$ for all i. Thus, such projects are unfavorable.

Proof: $BP_1 < 0$ as $\{BP_j\}$ is normalized. $PVB_1(i) = c$ by hypothesis. If there exists i_0 such that $PVB_2(i_0) > 0$, then by the monotone decreasing property, i can be chosen less than i_0, such that

$$PVB_1(i) = PVB_2(i)/(1 + i) + BP_1 > PVB_2(i_0)/(1 + i) + BP_1 > 0;$$

that is, $PVB_1(i)$ can be made positive. Similarly, i can be chosen greater than i_0, such that if $PVB_2(i) > 0$, then

$$PVB_1(i) = PVB_2(i)/(1 + i) + BP_1 < PVB_2(i_0)/(1 + i) + BP_1 < 0,$$

or if $PVB_2(i) \leqq 0$, then

$$PVB_1(i) = PVB_2(i)/(1 + k_1) + BP_1 < 0.$$

So $PVB_1(i)$ can be shown to assume both a positive and a negative value. But $PVB_1(i)$ is constant; therefore, $PVB_2(i) \leqq 0$ for all i. This implies that

$$c = PVB_1(i) = PVB_2(i)/(1 + k_1) + BP_1 < 0.□$$

Proposition 4.4:

If $\{BP_j\}$ is normalized, then $PVB_1(i)$ is continuous on the open interval $(-1, +\infty)$.

Proof: Let i_0 be an element of $(-1, +\infty)$.

Case 1: Let $BP_n > 0$.

$$PVB_{n-1}(i) = BP_n/(1 + i) + BP_{n-1} = L.$$

$PVB_{n-1}(i)$ is continuous for $i > -1$ and, in particular, for $i = i_0$. If L is not equal to zero, then $PVB_{n-1}(i)$ is bounded away from zero in some neighborhood of i_0. If $L > 0$, then

$$PVB_{n-2}(i) = PVB_{n-1}(i)/(1 + i) + BP_{n-2}$$

and is continuous in that neighborhood and so continuous at i_0. If $L < 0$, then

$$PVB_{n-2}(i) = PVB_{n-1}(i)/(1 + k_{n-1}) + BP_{n-2}$$

and is continuous in that neighborhood (in fact constant) and so continuous at i_0.

Let $L = PVB_{n-1}(i_0) = 0$. Then

$$PVB_{n-2}(i_0) = 0 + BP_{n-2} = BP_{n-2}.$$

Because $PVB_{n-1}(i)$ was shown above to be continuous at i_0 and is monotone decreasing by Proposition 4.3, then there is an open interval about i_0 such that as $i \to i_0^-$, $PVB_{n-1}(i) \to 0^+$, and as $i \to i_0^+$, $PVB_{n-1}(i) \to 0^-$. As $i \to i_0^-$, $PVB_{n-1}(i) > 0$ and

$$PVB_{n-2}(i) = PVB_{n-1}(i)/(1 + i) + BP_{n-2},$$

which approaches BP_{n-2}^+.

As $i \to i_0^+$,

$$PVB_{n-2}(i) = PVB_{n-1}(i)/(1 + k_{n-1}) + BP_{n-2},$$

which approaches BP_{n-2}^-. Thus, the right-hand limit and the left-hand limit of $PVB_{n-2}(i)$ both exist and are equal at i_0 and, in turn, equal the value $BP_{n-2} = PVB_{n-2}(i_0)$. Therefore, $PVB_{n-2}(i)$ is continuous at i_0.

Now $PVB_{n-2}(i)$ is either positive, negative, or zero at i_0. The preceding argument that demonstrates that $PVB_{n-2}(i)$ is continuous can be repeated to show that $PVB_{n-3}(i)$ is continuous at i_0.

Therefore, $PVB_1(i)$ is continuous at i_0.

Case 2: $BP_n < 0$.

$$PVB_n(i) = BP_n < 0$$

and

$$PVB_{n-1}(i) = PVB_{n-1}(i)/(1 + k_{n-1}) + BP_{n-2}.$$

$PVB_{n-1}(i)$ is constant for all i and so continuous at i_0. Depending on the value of $PVB_{n-1}(i)$, $PVB_{n-2}(i)$ will have either the factor $(1 + i)$ or $(1 + k_{n-2})$ in the denominator for any open interval containing i_0. In either event, $PVB_{n-2}(i)$ is continuous in i and so continuous at i_0. $PVB_{n-2}(i)$ is positive, negative, or zero at i_0, so $PVB_{n-3}(i)$ is continuous at i_0 by the same argument used in the prior case. Therefore, by repeated application $PVB_1(i)$ is continuous at i_0.

Because i_0 is an arbitrary element of $(-1, +\infty)$, then $PVB_1(i)$ is continuous on the open interval $(-1, +\infty)$.□

Proposition 4.5:

If $\{BP_j\}$ is normalized and is not independent of i, then $PVB_1(i)$ can be made arbitrarily large positively by taking i sufficiently close to -1, and $PVB_1(i)$ can be made negative by taking i sufficiently large.

Proof: If $PVB_1(i)$ is not independent of i, then at least one $PVB_j(i)$ is positive for some value of $i = i_0$. From the definition of $PVB_1(i)$ and $BP_1 < 0$, then j may be assumed to be 2 or larger. Therefore, $PVB_j(i_0) > 0$. Now

$$PVB_{j-1}(i) = PVB_j(i)/(1 + i) + BP_{j-1}$$

for $i < i_0$ as the $PVB_j(i)$ are monotone decreasing. By taking i sufficiently close to -1, the first term in the expression for $PVB_{j-1}(i)$ can be made arbitrarily large. Because there are only finite steps from $PVB_j(i)$ to $PVB_1(i)$ and the $PVB_s(i)$ are monotone decreasing, then i can be chosen such that all $PVB_s(i)$ for $s = j-1, j-2, \ldots, 1$ are positive.

Consider the second item. If $PVB_2(i)$ is less than or equal to zero for all values of i, then $PVB_1(i) < 0$. Otherwise, there is a value i_0 such that $PVB_2(i_0) > 0$. But the PVB functions are monotone decreasing, and it is possible to choose $i > i_0$ such that if $PVB_2(i) > 0$, then

$$PVB_1(i) = PVB_2(i)/(1 + i) + BP_1 < 0$$

as $BP_1 < 0$, or if $PVB_2(i) \leqq 0$, then

$$PVB_1(i) = PVB_2(i)/(1 + k_1) + BP_1 < 0$$

as $BP_1 < 0$.□

Proposition 4.6:

If $\{BP_j\}$ is normalized and not independent of i, then $PVB_1(i)$ has a unique real root in the interval $(-1, +\infty)$.

Proof: By Proposition 4.5, there exist i_1 and i_2 such that $PVB_1(i_1)$ is positive and $PVB_1(i_2)$ is negative. By Proposition 4.3, $PVB_1(i)$ is monotone decreasing, then $i_1 < i_2$. By Proposition 4.4, $PVB_1(i)$ is continuous, then $PVB_1(i)$ has a root in (i_1, i_2). By Corollary 4.3.1, $PVB_1(i)$ is strictly monotone; therefore, the root is unique.□

Proposition 4.7:

If $\{BP_j\}$ is such that $BP_1 < 0$ and has exactly one sign change among the BP_j, then the return on investment calculated from $\{BP_j\}$ is the same as the i_{ROI} calculated from the discounting algorithm. This indicates that the classical Anderson return on investment is a logical subset of the generalized profits released model.

Proof: By the hypothesis $BP_n > 0$. By considering the present value function applied to the $\{BP_j\}$, as i approaches -1^+, the present value becomes arbitrarily large. Similarly, as i approaches positive infinity, the present value becomes negative as $BP_1 < 0$. Therefore, the classical ROI lies in the interval $(-1, +\infty)$. Because there is only one sign change, this return on investment is unique.

The $\{BP_j\}$ given in the hypotheses is normalized and not independent of i. Therefore, it has unique i_{ROI} in the interval $(-1, +\infty)$ by Proposition 4.6. Apply the discounting algorithm with the value for $i = i_{\text{ROI}}$ to obtain the profits released and profits retained book profits. Because there is only one sign change in the $\{BP_j\}$, if some $PVB_j(i_{\text{ROI}}) \leqq 0$ for $j \geqq 2$, then PVB_1 (i_{ROI}) would be negative.

But $PVB_1(i_{\text{ROI}}) = 0$, so all $PVB_j(i_{\text{ROI}}) > 0$ for $j = 2, \ldots, n$. This means that there is only one substream, and the profits released book profits are precisely the $\{BP_j\}$. Therefore, the net present value of the substream at $i = i_{\text{ROI}}$ is zero by Corollary 4.1.1, and so i_{ROI} must agree with the classical Anderson return on investment. □

The example shown in Table 9 is that shown in Table 3 recalculated by using the discounting algorithm. In this example, the value of i that results in $PVB_1 = 0$ is 20.66 percent; this is the same value as shown in Table 3. The new decision rule applied to the sequence of *PVB*s results in the same split of the original book profits. The ROI value found by the discounting algorithm is the same as the ROI found on the two profits released substreams and in the accumulating algorithm. The present value of surplus on the two profits retained substreams is zero.

TABLE 9

MODIFIED BOOK PROFIT DECOMPOSITION VIA DISCOUNTING ALGORITHM AT $i = 20.66\%$ AND $k = 7.00\%$

			Book Profit	
Duration	Original Book Profit	Present Value Balance	Profits Released Model	Profits Retained Model
1	−125,138.00	0.00	−125,138.00	
2	59,135.00	+150,988.70	59,135.00	
3	46,986.00	+110,828.60	46,986.00	
4	36,013.00	+ 77,031.05	36,013.00	
5	24,192.00	+ 49,491.45	24,192.00	
6	17,084.00	+ 30,525.75	17,084.00	
7	11,557.00	+ 16,218.52	11,557.00	
8*	6,754.00	+ 5,624.48	5,624.48	1,129.52
9	2,358.00	− 1,208.59		2,358.00
10	− 1,087.00	− 3,816.25		− 1,087.00
11*	− 3,720.00	− 2,920.30	− 799.70	− 2,920.30
12	− 7,323.00	+ 964.91	− 7,323.00	
13*	30,518.06	+ 10,000.00	10,000.00	20,518.06
14	8,000.00	− 21,954.32		8,000.00
15	− 15,210.00	− 32,051.12		−15,210.00
16	− 18,020.00	− 18,020.00		−18,020.00

Consider the example shown in Table 10, which uses the example in Table 9 with a discount rate equal to 10.61 percent, or the pretax insurance company earnings rate. The present value of surplus in the two profits retained substreams is zero. The net present value of the sequence (−1,535.35, −7,323.00, +10,000.00) at 10.61 percent is zero. The net present value of the first profits released substream at 10.61 percent is 29,226.12, which equals PVB_1 (0.1061), as shown in Table 10.

TABLE 10

NET PRESENT VALUE OF MODIFIED BOOK PROFITS AT $i = 10.61\%$ AND $k = 7.00\%$

			Book Profit	
Duration	Original Book Profit	Present Value Balance	Profits Released Model	Profits Retained Model
1	−125,138.00	+ 29,226.12	−125,138.00	
2	59,135.00	+170,736.08	59,135.00	
3	46,986.00	+123,437.56	46,986.00	
4	36,013.00	+ 84,560.05	36,013.00	
5	24,192.00	+ 53,695.98	24,192.00	
6	17,084.00	+ 32,633.19	17,084.00	
7	11,557.00	+ 17,198.35	11,557.00	
8*	6,754.00	+ 6,239.68	6,239.68	514.32
9	2,358.00	− 550.33		2,358.00
10	− 1,087.00	− 3,111.91		− 1,087.00
11*	− 3,720.00	− 2,166.65	− 1,553.35	− 2,166.65
12	− 7,323.00	+ 1,718.10	− 7,323.00	
13*	30,518.06	+ 10,000.00	10,000.00	20,518.06
14	8,000.00	− 21,954.32		8,000.00
15	− 15,210.00	− 32,051.12		−15,210.00
16	− 18,020.00	− 18,020.00		−18,020.00

The data in Table 11, with values of profits released book profits through duration j, shows that the break-even year is 5. Per the definition, later durations need to be checked to verify that the present value to duration quantity remains positive. It does, but note that the second period of investment will cause the present value to duration to dip down as the second profits released stream is considered. It returns back to the 29,226 level, however. This will always be the case because the net present value of the second and subsequent profits released substreams is zero at the interest rate used in the algorithm.

TABLE 11

BREAK-EVEN YEAR OF MODIFIED BOOK PROFITS AT i = 10.61% AND k = 7.00%

Duration	Original Book Profit	Present Value Balance	Profits Released Book Profit	Present Value of Profits Released to Duration j
1	−125,138.00	+ 29,226.12	−125,138.00	−125,138
2	59,135.00	+170,736.08	59,135.00	− 71,673
3	46,986.00	+123,437.56	46,986.00	− 33,266
4	36,013.00	+ 84,560.05	36,013.00	− 6,652
5	24,192.00	+ 53,695.98	24,192.00	+ 9,512
6	17,084.00	+ 32,633.19	17,084.00	+ 19,832
7	11,557.00	+ 17,198.35	11,557.00	+ 26,145
8	6,754.00	+ 6,239.68	6,239.68	+ 29,226
9	2,358.00	− 550.33		+ 29,226
10	− 1,087.00	− 3,111.91		+ 29,226
11	− 3,720.00	− 2,166.65	− 1,553.35	+ 28,659
12	− 7,323.00	+ 1,718.10	− 7,323.00	+ 26,243
13	30,518.06	+ 10,000.00	10,000.00	+ 29,226
14	8.000.00	− 21,954.32		+ 29,226
15	− 15,210.00	− 32,051.12		+ 29,226
16	− 18,020.00	− 18,020.00		+ 29,226

Because shareholder surplus is always nonnegative and declining to zero during the profits retained periods, it is excluded from the analyses. This is clear for consideration of net present value and break-even year. And this fact guides us in determining new profit as a percentage of premium and profit per unit in force measures.

Shareholders are affected only by book profits during the profits released period; profits retained periods end with the accumulated value of surplus equal to zero. Premiums are paid in all years. For purposes of determining the profit as a percentage of premium during the profits released periods, consider the premiums paid during these periods. For those years in which the original book profit is split between a profits released substream and a profits retained substream, the total premium should be allocated to the two substreams in proportion to the split in the book profit. Then the present value of profits released profits can be divided by the present value of profits released premiums. In the case of the profit per unit in force, the present value of profits is divided by the present value of an annuity due of 1 based on interest and survivorship. The annuity due in the generalized situation should be based on a profits released annuity due. This can be calculated by taking the normal annuity due and splitting it between the profits released substreams and the profits retained substreams. In those years in which the original profit is split, the proportion is the same as the split profits are to the original book profit. Then the profit per unit in force is the present value of profits released book profits divided by the profits released annuity due.

Even though the period of time covered by the profits released periods may be less than the totality of the project, the duration of the project should still be considered the entire horizon priced because there is risk in those periods.

In deriving a decision rule for projects, if the k_j all equal a constant k, then let K represent the pretax value of k. If $i_{\mathrm{ROI}} > K$, then the project is favorable to the shareholders. If the k_j are not all equal, then let $k_{j_1}, k_{j_2}, \ldots, k_{j_m}$ represent the appropriate values of k during the profits released period. Let $K_{j_1}, K_{j_2}, \ldots, K_{j_m}$ be their pretax counterparts. If $i_{\mathrm{ROI}} > \max \{K_{j_1}, \ldots, K_{j_m}\}$, then the project is favorable; if $i_{\mathrm{ROI}} < \min \{K_{j_1}, \ldots, K_{j_m}\}$, then it is unfavorable. For i_{ROI} falling between the max and the min, a possible decision rule to be used is: If

$$i_{\mathrm{ROI}} > \left[\prod_{h=1}^{m} (1 + K_{j_h})\right]^{1/m} - 1,$$

then the project is favorable.

In this model best estimates are used for future insurance company pretax asset earnings rates and tax rates. Because tax rates and pretax insurance company earnings rates are not known with certainty for many years into the future, it is desirable to put a comfort level on the results to complement the best estimate computation. Recall in Section II how the i_{ROI} increased with increasing values of k. This occurs because fewer funds have to be withheld from shareholders to provide for future obligations if a higher aftertax rate on them can be earned. This releases more funds to shareholders and increases the ROI. Therefore, if one can estimate reasonable lower bounds for the pretax rate and upper bounds for tax rates, then the analysis can be redone with these values to put a reasonable lower bound on the estimate of the ROI for the project.

Perhaps the most well-known example of a project that defies classical analysis is the pump project of Lorie and Savage [3]. Simply stated, if nothing is done, then a well will produce a gain of $10,000 at the end of two years. If a pump costing $1,600 is used, then the $10,000 will be recovered at the end of a single year. In Promislow notation the project can be stated (−1,600, 10,000, −10,000), because there will be no $10,000 at the end of the second year if the pump is used. Classical analysis results in ROIs of 25 percent and 400 percent. In [9] the presence of the two solutions is shown to cause incorrect decisions to be made by using net present values.

As in the original example, taxes are set to zero. The analysis of the problem proposed by Solomon [7] used an asset rate of 23 percent, so that will be used here. Table 12 shows the results of applying the generalized profits released model. The i_{ROI} is 16.87 percent, which is less than 23 percent; thus, the project is unfavorable.

TABLE 12

PUMP PROJECT PROFIT DECOMPOSITION AT i = 16.87% AND k = 23.00%

			Book Profit	
Duration	Original Book Profit	Present Value Balance	Profits Released Model	Profits Retained Model
1	−1,600.00	0.00	−1,600.00	
2	10,000.00	1,869.92	1,869.92	8,130.08
3	−10,000.00	−10,000.00		−10,000.00

In the analysis presented by Solomon, the problem is restated from the perspective of what it is worth to receive the $10,000 one year earlier. Solomon used 23 percent and concluded that it was worth $2,300 more to get the $10,000 one year earlier. From the profits released book profits, if the shareholders either leave the profits released in the company or can earn the same rate, then the $1,869.92 brought forward with interest at 23 percent is the $2,300. Thus, the generalized profits released model results in the same analysis as Solomon's. Solomon proceeds to examine the sequence (−1,600, 0, 2,300), which has an ROI of 19.9 percent. Because this return is less than 23 percent, Solomon would reject the project. This is also the conclusion drawn from the results of the generalized model.

As noted above, multiple roots can invalidate net present value analysis. This happens in the following way. When the net present value of (−1,600, 10,000, −10,000) is computed, it has a negative value (−211.11) at 20 percent and a positive value (175.15) at 30 percent. If one thinks of these two interest rates as rates that can be earned on invested assets, then the conclusion would be that the project is more favorable than an asset earning 30 percent, but less favorable than an asset earning 20 percent. This is unreasonable. The anomaly occurs because the present value function is not monotone decreasing when it has more than one interest rate that makes the net present value zero. This situation cannot happen when the generalized profits released model $PVB_1(i)$ is used.

If the net present value function is impaired in these situations, then profit criteria derived from it may also be impaired, for example, profit as a percentage of premium and profit per unit. Within the framework of the generalized profits released model, these two profit criteria are always meaningful. If a project $\{BP_j\}$ is well behaved and $BP_1 < 0$, then the value for net present value computed via the classical approach equals the value computed via the generalized profits released model. The break-even years will be the same under both analyses and so will profit as a percentage of premium and profit per unit. These follow by the same reasoning in the proof of Proposition 4.7. The generalized profits released model thus contains the classical case as a subset.

Note that the profit as a percentage of premium measure in the generalized model represents that measure for only those substreams in which the project involves shareholder investment. For a project that has profits retained substreams, the profits released profit as a percentage of the profits released premium is unlikely to equal the classical value computed by using all book profits and all premiums. A similar result holds for profit per unit.

Appendix A contains graphs of $PVB_1(i)$ and the classical net present value function for the examples given in Tables 2, 9, and 12. It also contains the graph of $PVB_1(i)$ and the classical net present value function for the example to be presented in Table 15 of Section V. These graphs demonstrate the monotone decreasing behavior of the present value balance function defined on the open interval $(-1, +\infty)$. Appendix B contains the derivation of an additional book profit splitting rule and profit analysis that can be used under the actuarial constraint of permitting only a single shareholder investment.

V. SUMMARY OF THE GENERALIZED PROFITS RELEASED MODEL

Let $\{BP_j\}$ be a sequence of project book profits and let $PVB_j(i)$ be the present value balance at duration j. Let k_j be the aftertax rate earned by the insurance company on invested assets for duration j. Let i and k_j, $j = 1, 2, \ldots, n, > -1$. Let the discounting algorithm be defined by:

1. $PVB_n(i) = BP_n$;
2. $PVB_j(i) = PVB_{j+1}(i)/(1 + r) + BP_j$

where $r = i_{ROI}$ if $PVB_{j+1}(i) > 0$ and $r = k_j$ if $PVB_{j+1}(i) \leqq 0$.

The project can be analyzed in the following order: Normalize the original book profits, $\{BP_j\}$. That is, if $BP_1 > 0$, then accumulate the original book profits at k_j until the duration in which the sum of the accumulated original book profits and the original book profit produces a net shareholder investment, that is, a resulting net negative book profit, or all BP_j are utilized and the result is greater than or equal to zero.

Case 5.1. If normalizing $\{BP_j\}$ does not produce a net shareholder investment, then this is a pure financing project that is favorable if the accumulated value is positive. If the accumulated value is less than or equal to zero, then the project is unfavorable because it should be riskier than the asset portfolio.

Case 5.2. If $\{BP_j\}$ can be normalized, then $PVB_1(i)$ is either independent of i or not independent of i.

Case 5.2.1. If $PVB_1(i)$ is independent of i, then $PVB_1(i) < 0$ for all i and is unfavorable.

Case 5.2.2. If $PVB_1(i)$ is not independent of i, then there exists a unique ROI for the project. This value can be compared to company hurdle rates and/or the pretax k_j. Meaningful net present value analyses can be performed. $\{BP_j\}$ can be split into profits released and profits retained book profit substreams and analyzed by the rule shown in Table 13. If $BP_n > 0$, then BP_n is released; if $BP_n < 0$, then BP_n is retained.

TABLE 13

DISCOUNTING ALGORITHM SPLITTING RULE

		Book Profit	
PVB_{j+1}	PVB_j	Profits Released Portion	Profits Retained Portion
< 0	< 0	0	BP_j
$\geqq 0$	$\geqq 0$	BP_j	0
< 0	$\geqq 0$	PVB_j	$BP_j - PVB_j$
$\geqq 0$	< 0	$BP_j - PVB_j$	PVB_j

The profits released book profits obtained above can be used to determine the break-even year, the profits released profit as a percentage of profits released premium, and the profits released profit per profits released units in force.

Note that in case 5.1, the distribution to shareholders may have to be deferred until the end of the project because of the incidence of interest income and obligations to be paid.

Consider a comprehensive example. In Table 14, a hypothetical product begins with a profits retained period and switches between profits released and retained. As in prior examples, the insurance company pretax earnings rate is 10.61 percent and the tax rate is 34 percent, resulting in an aftertax rate of 7 percent. The product analysis is based on 10 units with a premium of $40.00 per unit. The in force by duration and the book profits are presented in Table 14.

TABLE 14

SUMMARY EXAMPLE IN FORCE AND BOOK PROFIT DATA

Duration	In Force	Book Profit
1	1.00000	+ 50.00
2	0.99950	−200.00
3	0.87886	+ 20.00
4	0.79018	+ 40.00
5	0.72609	+200.00
6	0.68150	+100.00
7	0.64613	− 70.00
8	0.61229	−100.00
9	0.57993	+ 20.00
10	0.54901	+100.00
11	0.00000	

Because the project begins with a profits retained segment, accumulate the first book profit at the aftertax earnings rate. This results in an accumulated aftertax surplus of $53.50. This can be netted against the second book profit to give a ($146.50). The project duration is 10 years because there is still risk in the first year. When the discounting algorithm is applied to this normalized sequence of book profits, the results are those shown in Table 15. The rate that produces the $PVB_1 = 0$ above is 26.27 percent.

TABLE 15

SUMMARY EXAMPLE PROFIT DECOMPOSITION AT $i = 26.27\%$ AND $k = 7.00\%$

			Book Profit	
Duration	Original Book Profit	Present Value Balance	Profits Released Model	Profits Retained Model
2	−146.50	0.00	−146.50	
3	20.00	+184.99	20.00	
4	40.00	+208.33	40.00	
5	200.00	+212.55	200.00	
6*	100.00	+ 15.85	15.85	84.15
7	− 70.00	− 90.04		−70.00
8*	−100.00	− 21.44	− 78.56	−21.44
9	20.00	+ 99.19	20.00	
10	100.00	+100.00	100.00	

Table 16 presents the algorithm for $i = 10.61$ percent. The present value of profits at 10.61 percent is $75.09. The break-even duration is 5; note that it is 4 durations from the point of shareholder investment.

TABLE 16

SUMMARY EXAMPLE NET PRESENT VALUE AND BREAK-EVEN YEAR ANALYSIS AT $i = 10.61\%$ AND $k = 7.00\%$

Duration	Original Book Profit	Present Value Balance	Book Profit: Profits Released Model	Book Profit: Profits Retained Model	Net Present Value of Profits Released Book Profit to Duration
2	−146.50	75.09	−146.50		−146.50
3	20.00	+245.09	20.00		−128.42
4	40.00	+248.96	40.00		− 95.42
5	200.00	+231.12	200.00		+ 52.09
6*	100.00	+ 34.43	34.43	65.57	+ 75.09
7	− 70.00	− 70.16		− 70.00	+ 75.09
8*	−100.00	− 0.18	− 99.82	− 0.18	+ 20.57
9	20.00	+110.41	20.00		+ 30.44
10	100.00	+100.00	100.00		+ 75.09

Table 17 presents the computations to complete the profit measures. At 10.61 percent, the profits released annuity due is 3.99865, the profits released premium is $1,599.46 for 10 units, and the present value of profits for the 10 units is $75.09. The following summarizes the profit results:

- Project duration is 10 years.
- Shareholder investment occurs at duration 2, terminates at duration 10.
- ROI = 26.27 percent:
- NPV at 10.61 percent = $75.09 (measured at duration 2).
- Break-even occurs at duration 5 (that is, 4 durations from point of net shareholder investment).
- Profit/premium = 4.69 percent (75.09/1,599.46).
- Profit/unit in force = $1.88 (75.09/39.9865).

The net present value of all book profits at 10.61 percent is $70.81. The profit as a percentage of premium quantity based on all book profits and all premiums is 3.38 percent. Note that the profits released present value of profits measured at duration 1 would be $67.89.

TABLE 17

SUMMARY EXAMPLE PROFITS RELEASED DATA

Duration	Percentage of Book Profits Released	Total In Force	Profits Released	
			In Force	Annuity Due
1	0.00	1.00000	0.00000	0.00000
2	100.00	0.99950	0.99950	3.99865
3	100.00	0.87886	0.87886	3.31724
4	100.00	0.79018	0.79018	2.69700
5	100.00	0.72609	0.72609	2.10906
6	34.43	0.68150	0.23461	1.52964
7	0.00	0.64613	0.00000	1.43238
8	99.82	0.61229	0.61121	1.58430
9	100.00	0.57993	0.57993	1.07630
10	100.00	0.54901	0.54901	0.54901

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APPENDIX A

FIGURE A1

GRAPH OF $PVB_1(i)$ FOR ORIGINAL BOOK PROFITS (TABLE 2)

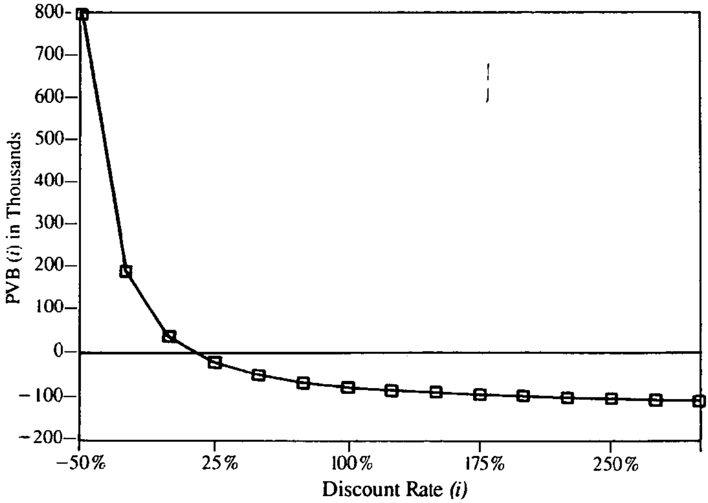


FIGURE A2

CLASSICAL NET PRESENT VALUES FOR ORIGINAL BOOK PROFITS (TABLE 2)

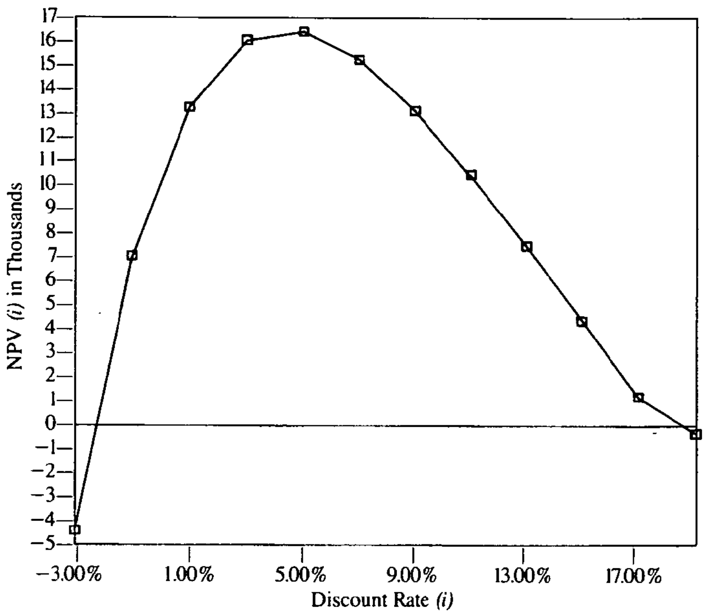


FIGURE A3

GRAPH OF $PVB_1(i)$ FOR MODIFIED BOOK PROFITS (TABLE 9)

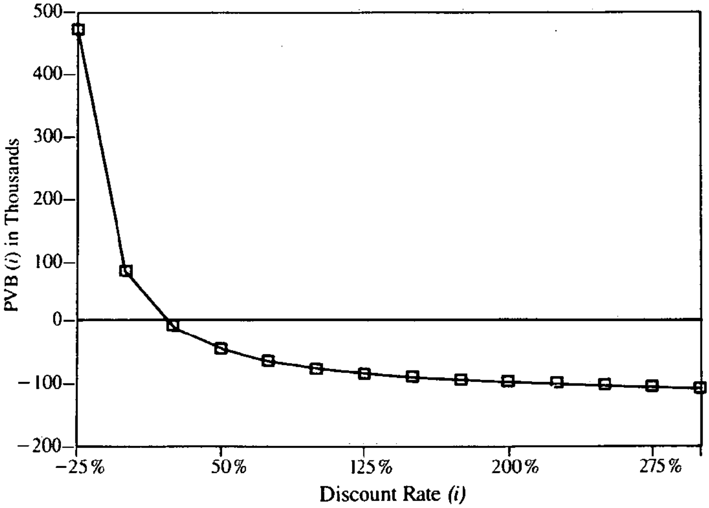


FIGURE A4

CLASSICAL NET PRESENT VALUES FOR MODIFIED BOOK PROFITS (TABLE 9)

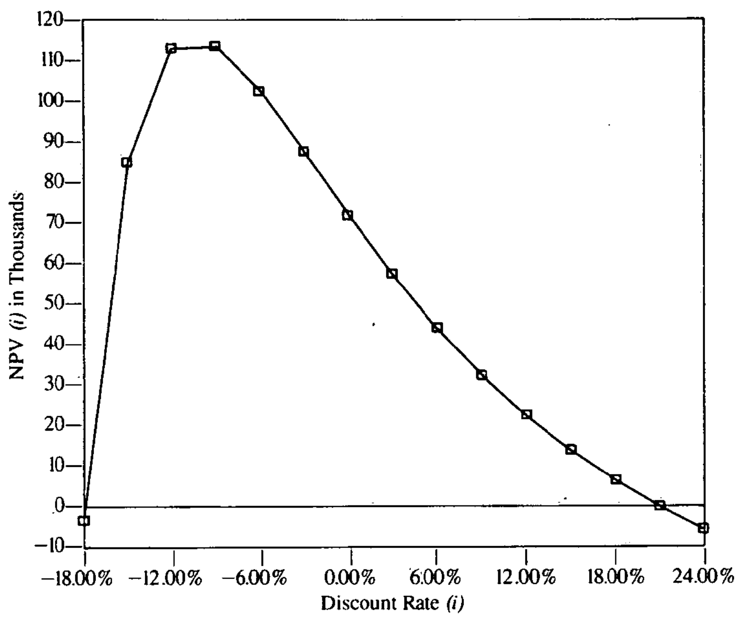


FIGURE A5

GRAPH OF $PVB_1(i)$ FOR PUMP PROJECT PROFITS (TABLE 12)

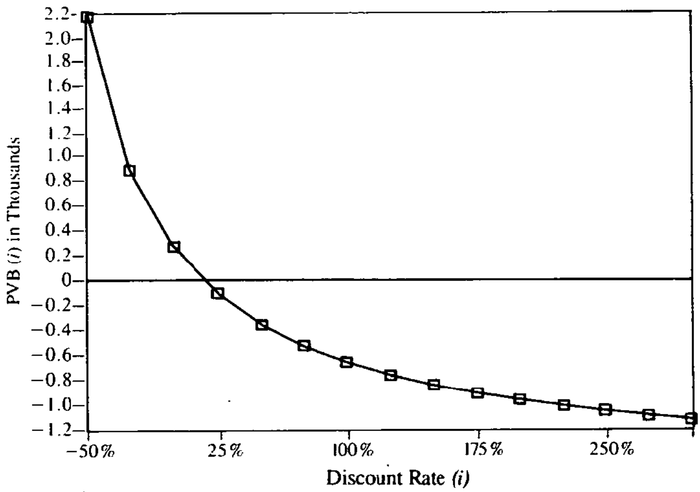


FIGURE A6

CLASSICAL NET PRESENT VALUES FOR PUMP PROJECT PROFITS (TABLE 12)

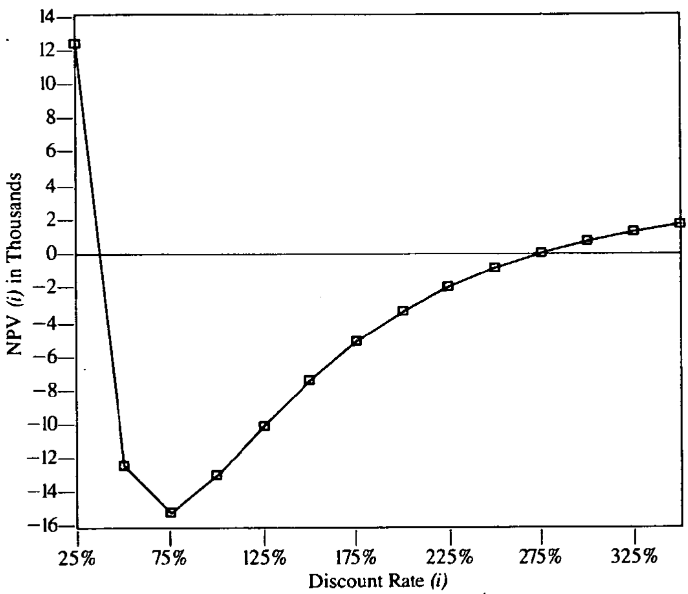


FIGURE A7

GRAPH OF $PVB_1(i)$ FOR SUMMARY EXAMPLE PROFITS (TABLE 15)

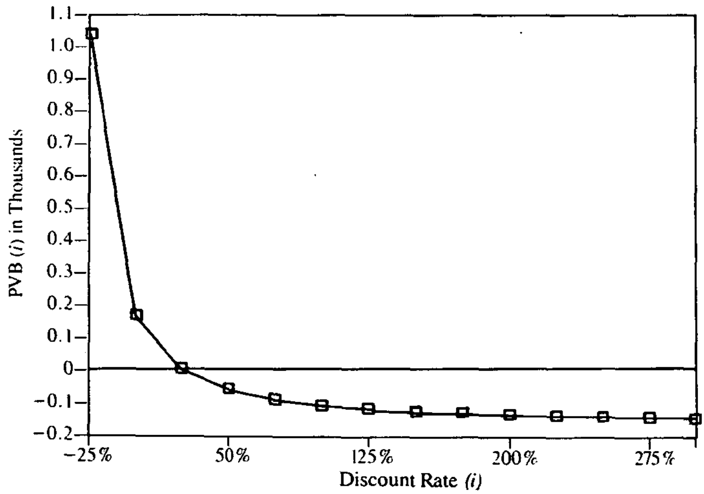
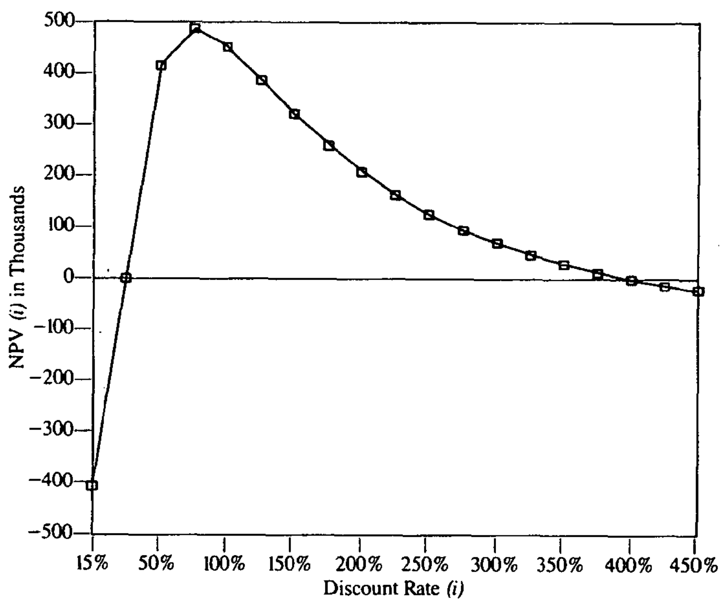


FIGURE A8

CLASSICAL NET PRESENT VALUES FOR SUMMARY EXAMPLE PROFITS (TABLE 15)



APPENDIX B

During discussions of the concepts in this paper, Doug Doll inquired whether the model could be modified in a manner such that it would withhold amounts from positive book profit durations to cover all subsequent future negative book profit durations. An example is given by Table 3. In this case, there is a second shareholder investment that causes a second profits released substream. Can the model be modified so that this example behaves like that in Table 2, that is, so that there is no second period of shareholder investment?

In addition to providing the amount, if any, that needs to be withheld to eliminate any negative book profits after the first, such a modification also would provide the return on investment value and other profit criteria under the restriction of a single shareholder investment. The model can be so modified and the modification is presented below.

Actuarial Constraint: No profits released book profit after the first may be negative.

Without loss of generality, let $\{BP_j\}$, $j = 1, \ldots, n$, be a normalized sequence of book profits, that is, $BP_1 < 0$, and BP_n not equal to zero. Let $\{k_j\}$ be defined as in Section IV. Consider a new algorithm, called the Actuarial Constraint Balance, defined by the following equations:

Definition B.1:

1. $ACB_n = \begin{cases} 0, & \text{if } BP_n > 0, \\ BP_n, & \text{if } BP_n < 0. \end{cases}$

2. $ACB_j = \begin{cases} ACB_{j+1}/(1+k_j) + BP_j, & \text{if } ACB_{j+1} < 0, \\ BP_j, & \text{if } ACB_{j+1} \geqq 0 \text{ and } BP_j < 0, \\ 0, & \text{if } ACB_{j+1} \geqq \text{ and } BP_j \geqq 0. \end{cases}$

The decision rule for splitting the original book profits into the profits released portion and the profits retained portion is given by Table B1 for durations $j = 2, \ldots, n-1$.

Definition B.2: If $BP_n < 0$, then BP_n is retained; if $BP_n > 0$, then BP_n is released.

Definition B.3: ACB_1 is released.

The example given in Table B2 has the same original book profits as those in Table 3 and Table 9. Again, $k_j = 0.07$ for all j, and the tax rate is 34 percent.

Under this formulation of the model, the ACB_j is not a function of any rate other than the k_j. The profits released book profits consists of: (−125,138;

TABLE B1

ACTUARIAL CONSTRAINT SPLITTING RULE

ACB_{j+1}	ACB_j	Book Profit	
		Profits Released Portion	Profits Retained Portion
< 0	< 0	0	BP_j
≧ 0	≧ 0	BP_j	0
< 0	≧ 0	ACB_j	$BP_j - ACB_j$
≧ 0	< 0	0	BP_j

59,135; . . . ; 17,084; 10,982.23; 0; 0; 0; 0; 0; 10,000). The ROI of this sequence of book profits is 20.39 percent. In addition, the other profits released profit criteria can be calculated from the sequence. For example, the break-even duration is 5. Note that $574.77 must be withheld from the duration 7 book profit and set aside to avoid the appearance of a second profits released substream.

TABLE B2

MODIFIED BOOK PROFITS SPLIT BY ACTUARIAL CONSTRAINT SPLITTING RULE AT $k = 7.00\%$

Duration	Original Book Profit	Actuarial Constraint Balance	Book Profit	
			Profits Released	Profits Retained
1	−125,138.00	−125,138.00	−125,138.00	
2	59,135.00	0.00	59,135.00	
3	46,986.00	0.00	46,986.00	
4	36,013.00	0.00	36,013.00	
5	24,192.00	0.00	24,192.00	
6	17,084.00	0.00	17,084.00	
7*	11,557.00	+ 10,982.23	10,982.23	574.77
8	6,754.00	− 615.00	0.00	6,754.00
9	2,358.00	− 7,884.83	0.00	2,358.00
10	− 1,087.00	− 10,959.83	0.00	− 1,087.00
11	− 3,720.00	− 10,563.93	0.00	− 3,720.00
12	− 7,323.00	− 7,323.00	0.00	− 7,323.00
13*	30,518.06	+ 10,000.00	10,000.00	20,518.06
14	8,000.00	− 21,954.32		8,000.00
15	− 15,210.00	− 32,051.12		−15,210.00
16	− 18,020.00	− 18,020.00		−18,020.00

A second example (Table B3) illustrates how a project with multiple initial investments, for example, negative first and second duration book profits, can be accommodated. The same values for k_j and the tax rate are used. Here the ROI of (−1,093.46; 0; 50.00; 200.00; 113.08; 0; 1,636.31) is 11.80 percent. Note that no profits released book profit after the first is negative; $93.46 had to added to the shareholder investment in duration 1 to cover the negative book profit in duration 2. Similarly, $186.92 and $63.39 had to be withheld in durations 5 and 7 to cover future negative book profits.

TABLE B3

MULTIPLE INITIAL INVESTMENT BOOK PROFITS AT k = 7.00%

Duration	Original Book Profit	Actuarial Constraint Balance	Book Profit	
			Profits Released Model	Profits Retained Model
1*	−1,000.00	−1,093.46	−1,093.46	93.46
2	− 100.00	− 100.00	0.00	−100.00
3	50.00	0.00	50.00	
4	200.00	0.00	200.00	
5*	300.00	+ 113.08	113.08	186.92
6	− 200.00	− 200.00	0.00	−200.00
7*	1,700.00	+1,636.31	1,636.31	63.69
8	200.00	− 68.15		200.00
9	− 100.00	− 286.92		−100.00
10	− 200.00	− 200.00		−200.00

DISCUSSION OF PRECEDING PAPER

MARK D.J. EVANS:

Mr. Becker has done a fine job of presenting, refining, and developing techniques for analyzing present values of streams with multiple changes of sign.

He develops a decision rule for situations where 'K' is nonlevel. When i_{ROI} is calculated as a level amount but falls in between the largest value of 'K' and the smallest value of 'K', he suggests a possible decision rule to be based in the following inequality:

$$i_{\text{ROI}} > \left[\prod_{h=1}^{m} (1 + K_{jh})\right]^{1/m} - 1$$

Although, at first glance, this approach may seem reasonable, it does not universally provide reliable results. This is because for some years i_{ROI} will be less than 'K'.

For example, consider the situation described in the table on the following page. The cash flow at the end of each year is shown along with the investment rate, aftertax investment rate, and ROI during the year calculated according to the criteria outlined in Mr. Becker's paper. The table also shows the outstanding balance at the end of each year just after the cash-flow activity for that year has occurred. Now this display suggests a ROI of 6.19 percent. The geometric average of the investment rate is 5.58 percent. Thus, the criteria described would suggest that project is favorable. Yet, if we discount the cash flows at the aftertax investment rates, we find that the present value of the cash flow is actually -69.81. Thus, in fact, the project is unfavorable. The reason for this in this example is fairly obvious. During the first ten years the ROI calculated is less than both the investment rate and the aftertax investment rate, implying that the borrowing rate is less than the lending rate.

Perhaps Mr. Becker's formula could be modified to reflect that the investment rates for different years might be weighted differently depending upon the present value of future cash flows to which that interest rate would serve as a discounting factor. The choice of rates to use in calculating such present values and other difficulties implied in such an approach, in conjunction with the difficulty in demonstrating that any such approach would

NON-LEVEL INTEREST SCENARIO ANALYSIS

Year	Cash Flow	Investment Rate	Aftertax Investment Rate	ROI	Outstanding Balance
0	−1,000				−1,000.00
1	−1,000	10.00%	6.60%	6.19%	−2,061.88
2	−1,000	10.00%	6.60%	6.19%	−3,189.46
3	−1,000	10.00%	6.60%	6.19%	−4,386.81
4	−1,000	10.00%	6.60%	6.19%	−5,658.25
5	−1,000	10.00%	6.60%	6.19%	−7,008.36
6	1,000	10.00%	6.60%	6.19%	−6,442.01
7	2,000	10.00%	6.60%	6.19%	−4,840.62
8	2,000	10.00%	6.60%	6.19%	−3,140.13
9	2,000	10.00%	6.60%	6.19%	−1,334.43
10	2,000	10.00%	6.60%	6.19%	583.00
11	−100	8.00%	5.28%	5.28%	513.78
12	−100	6.00%	3.96%	3.96%	434.12
13	−100	4.00%	2.64%	2.64%	345.58
14	−100	3.00%	1.98%	1.98%	252.43
15	−100	3.00%	1.98%	1.98%	157.43
16	−100	3.00%	1.98%	1.98%	60.54
17	−100	3.00%	1.98%	1.98%	−38.26
18	−100	3.00%	1.98%	6.19%	−140.63
19	−100	3.00%	1.98%	6.19%	−249.33
20	−100	3.00%	1.98%	6.19%	−364.76
21	50	3.00%	1.98%	6.19%	−337.32
22	50	3.00%	1.98%	6.19%	−308.20
23	50	3.00%	1.98%	6.19%	−277.27
24	50	3.00%	1.98%	6.19%	−244.42
25	50	3.00%	1.98%	6.19%	−209.55
26	50	3.00%	1.98%	6.19%	−172.51
27	50	3.00%	1.98%	6.19%	−133.19
28	50	3.00%	1.98%	6.19%	−91.43
29	50	3.00%	1.98%	6.19%	−47.09
30	50	3.00%	1.98%	6.19%	0.00

guarantee universally reliable results, would, in my opinion, render such a modification to the formula as being impractical if not impossible.

I would propose, in such a situation, that one must use an ROI at each year that is not less than the investment rate but that could vary from year to year. Development of such an approach, however, probably far exceeds the intended scope of Mr. Becker's paper.

JOHN T. GILCHRIST:

Mr. Becker has given us an admirable tool for the consideration and treatment of policies that produce positive and negative statutory gains. In addition, he has brought to the forefront the situation of current profits offsetting later losses, a situation that valuation actuaries, accountants and regulators may need to confront and possibly address.

Many cash-flow and asset-share-type projections assume that all net funds received will be retained. Operations are thus improved by the interest assumed earned on the previously accumulated surplus. If such interest is deducted, the profits may well become losses.

The question now facing all of us—valuation actuaries, accountants and regulators—is what do we do about it, if anything? Can a valuation actuary conscientiously sign a certification as to reserves when he/she knows that statutory losses are inevitable? Or can he/she take comfort in assuming that in aggregate all will be well? Likewise, GAAP and the regulator. And how will management and the Internal Revenue Service react to setting aside additional monies? Do we violate the fundamental actuarial principle that future losses from whatever cause are to be prefunded?

Mr. Becker's methodologies will undoubtedly help us in quantifying the problem under different prefunding assumptions. His paper represents a great step forward; we are all indebted to him.

CLAUDE Y. PAQUIN:

I admire the author's initiative in developing and articulating a theory that seems to integrate all familiar current approaches to profit measurement in the individual life and health insurance business. His reference to the pump project analysis is an effective way to illustrate that if you ask the wrong question, you're sure to get the wrong answer.

The first question a practical person would ask in solving this problem is whether it would be possible to obtain the $1,600 for the pump somewhere, and if so, at what cost. If the $1,600 is unavailable, there is no problem left to resolve. Given the availability of competing lenders and investors in our capitalistic society, it should be possible to raise the $1,600 capital for the pump, but at a price to be determined from shopping around. If a banker be found who is willing to lend the $1,600 at 15 percent annual interest, then the well owner has progressed to the point of being able to compare prospective net proceeds of $8,160 at the end of one year (after repayment of the $1,600 loan with $240 interest) versus $10,000 in two years. So he asks himself what he'd have to earn on the $8,160 from the end of year one to the end of year two so it would become equivalent to $10,000 at the end of year two, assuming he has no pressing liquidity need during all that time. The answer is 22.549 percent. If he's confident he can exceed that rate of return on his $8,160 during year two, he gets both the loan and the pump, trusting he'll thus have more than $10,000 at the end of year two; otherwise, he does without both the loan and the pump.

If a classical analysis of the foregoing problem results, as asserts the author (absolutely correctly, mathematically speaking), in annual returns on investment of 25 percent and 400 percent, then we have to question the value of classical analysis in solving practical problems. It is interesting to note the tradeoff for the well owner: in year one, he could pay 400 percent interest (on his $1,600 loan) in exchange for receiving 400 percent interest (on his then $2,000 net deposit) in year two. Such may be the result of classical analysis, but what does it mean? What is needed is a practical, business-oriented analysis.

The main message of my paper [1] was to caution against potentially delusive theoretical approaches to practical problems. I focused on the sort of counsel an actuary might endeavor to provide to a client, corporate manager or investor not himself an actuary. Mr. Becker indicates that I applied the same method as appears in a paper by Mr. Sondergeld [2]. It is important to point out the totally different outlooks of the Sondergeld paper and mine.

The Sondergeld paper was devoted to describing a method of measuring the contribution to profitability of life insurance products by determining a rate of return on total capital. This rate was shown to vary annually, as did the capital, but the Sondergeld paper explained, without emphasis, that a single overall internal rate of return could be computed that would make the discounted or accumulated value of all transfers equal to zero. At that point, the paper referred to a two-rate accumulative approach when the investment becomes negative, with a comment on using first the "rate one is willing to pay to borrow money." There can be a vast difference between a borrowing rate (as alluded to by Mr. Sondergeld) and a rate payable on forcedly received deposits (my approach). By borrowing at 10 percent and reinvesting at 12 percent, all manners of interesting (yet generally misleading) results are possible, so borrowing was out of the question for the hypothetical prudent banker referred to in my paper.

Whether profits can be computed by discounting or by accumulating is as irrelevant as debating the use of the retrospective or prospective methods in computing *classical* statutory reserves: The result from using old mortality tables is always retrospective, but it is always prospective for its anticipating future mortality, no matter what the computational method. If future mortality and interest do not conform to the assumptions rooted in the past and stilled by statute, the result is always wrong in spite of being theoretically right. (Yet it may be, practically speaking, right enough.)

The message that I have aimed to convey has in essence been a plea for practicality. I have no war against *profits released* or *profits retained*, but I urge healthy skepticism toward all the results that purport to emerge from the actuarial exercises that spawn esoteric profit measurements. Too often, in actuarial computations, so-called *ill-behaved* cash flows are transformed into *well-behaved* book profits by the insertion of reserves in the computational process. These reserves, generally based on outdated assumptions prescribed by statutes, are often unrealistic, with the result that a purported investment in reserves presents much less risk for the insurer than parting with good commission money paid to the selling agent. In my simple example, I had thus left out the complexity of reserves on purpose, to concentrate on pure cash flows [3]. Income tax was left out for essentially the same reason and also because recent U.S. experience indicates it is apt to change with some frequency and in midstream. Important as it is, the tax tune is generally independently called by its own piper.

In his generalized profits-released model, Mr. Becker proposes the use of successive one-year accumulations starting from the bottom end of the sequence of figures, ratcheted up one step at a time to a present or discounted value at the time of origin. This is discounting turned on its head one step at a time. As a mathematical technique, it is clever enough and fine so far as it goes. As a conceptual technique, it is difficult to explain or follow. All roads may lead to Rome, but some are more direct and easier both to explain and to take.

Shareholders, states the paper, are affected only by book profits during the profits-released period, while profits-retained periods end with the accumulation value of surplus equal to zero. Such a Delphic statement runs counter to the prudent (and reasonable) banker's expectation of making money on both his loans and his deposits; it points to the artificiality by which some profit objectives are expressed. Breaking even within five years, or earning seven percent of premiums as profit, shouldn't be absolute goals, as there are times when they make no sense. It's no wonder ten-year endowment plans don't sell any more if insurers seek to keep seven percent of the premiums as profits! Actuaries who view their work from the perspective of a consumer understand this.

The passage of time will tell us how much this paper increases our understanding of the measurement of profitability. A la Yogi Berra, I am inclined to believe, on this subject, that healthy skepticism is healthy.

END NOTES

1. PAQUIN, C.Y. "Cash Flow Analysis by the Prudent Banker's Method, or Discounting Turned on Its Head," *TSA* XXXIX (1987):177–82.
2. SONDERGELD, D.R. "Profitability As a Return on Total Capital," *TSA* XXXIV (1982):415–29, Discussion, 431–33.
3. When an insurance product is ceded to an alien reinsurer not subject to the same rules as the cedent on the computation of reserves, it is hardly wise to burden the analysis with reserve amounts that might vary from country to country and possibly from one reinsurer to another, depending upon the flexibility in determining reserves offered by each one's own regulatory environment. An effort is now under way to giving extraterritoriality to U.S. reserving standards for reinsurance abroad through the imposition of *mirror reserving*. If computational and regulatory overkill should become the norm, the nascent movement toward transnational reinsurance will undoubtedly be stifled. Yet, as the September 1985 Mexico City earthquake reminded us, transnational reinsurance makes eminent sense, even for life reinsurance.

BRADLEY M. SMITH:

David Becker is to be congratulated for this significant contribution to the actuarial literature on profit quantification and measurement. It is clear that this topic is of interest to many members of the Society, as witnessed by the increased level of attention given it at Society meetings and within Society literature over the last few years. This increased exposure can only lend insight into what is a difficult topic.

In this discussion I would like to make a few points that, although addressed in other actuarial writings, cannot be damaged by additional emphasis herein. Mr. Becker discusses the differences between the profits-released and profits-retained models and the different profit objectives that are implied by each. In doing so, he emphasizes the differences between the two models. Although rather basic, it is important to remember that the two models are tied together by the following relationship:

Pretax Present Value of Profits Released equals
Pretax Present Value of Profits Retained
(that is, Accumulated Statutory Surplus) when discounted using the investment earnings rate

The discussion of an ill-behaved stream of profits (that is, one with more than one change in sign) is enlightening. However, many times an analysis of the profitability of a line of business or product with an ill-behaved stream of profit emergence merits additional review.

Theoretically, the stream of profit emerges continuously. The calculation of ROI is simplified by assuming that profit emerges periodically. Profits emerging within the accounting period are accumulated at the investment earnings rate to the end of the period or discounted at the investment earnings rate to the beginning of the period. As some expense is incurred (that is, issue and underwriting expense, agency development expense, direct response solicitation costs) prior to receiving any incremental revenue, it is difficult to imagine a product or product line that does not require some initial investment. Typically, this occurs in an actuarial analysis because too broad of an accounting period was used in the simplification of the continuous profit stream.

A stream of statutory profits that turns negative after the initial investment has been made is equally troublesome. By definition the prospective present value of statutory profits must be positive, if the reserve is to make good and sufficient provision to cover future unmatured contractual obligations and expenses associated with meeting those obligations. It can be argued that this requirement is applicable to the company as a whole and need not be applied on a policy-by-policy basis. While true, I know few actuaries who would be comfortable developing a new product that would result in such an undesirable valuation position. In general, products that produce an ill-behaved profit emergence do so because the reserving methodology is inappropriate or the accounting period used to simplify the continuous profit emergence is too short. Given the assumption that the prospective present value of profits cannot be negative, extending the accounting period or increasing the reserve will necessarily result in the elimination of negative profit emergence in accounting periods after the initial investment is made.

Mention has been made in this paper and in prior literature on this subject of the use of cash flows in the calculation of return on investment (ROI). This is acceptable if used as a simplifying assumption in order to illustrate a point. However, it is quite dangerous when calculating the return on statutory investment associated with a life insurance product. Because insurance is a regulated industry, assets that support a certain level of reserve must be held by an insurance company. This represents an additional cost of investing in the issuance of the product, as the return on the assets held to support this reserve is generally (hopefully) less than the return anticipated when investing in new business. This opportunity cost will result in a lower ROI.

The author states that, "Considerations for the choice of k should reflect the choice of assets used." I agree with this statement but would extend it to say that considerations for the minimum acceptable ROI (that is, hurdle rate) should reflect the choice of assets used. An investment strategy that includes investment in lower-quality instruments or results in exposing the company to significant disintermediation risk should result in a higher hurdle rate than one that does not expose the company to these additional risks, because the anticipated return is less certain due to the acceptance of these risks.

Finally, I would like to comment on the concept of the hurdle rate appropriate for a particular block or line of business reflecting the risks associated with it. I submit that this does not exist. Rather, a hurdle rate exists that is appropriate for a particular block or line of business *for a particular company* given its financial position and the risks that it currently accepts. Different types of business offered by insurance companies have risks that offset each other. Thus, a company that can minimize certain risks through the acceptance of other risks that are offsetting may be able to accept a lesser return than a company that does not do so. Likewise, companies have differing costs of capital and thus may require a different return from the same or similar blocks of business. Thus, the level of the hurdle rate for a particular block or line of business must reflect much more than the risks associated with that particular block or line of business.

Again, congratulations and thanks to the author for this timely and insightful contribution to the actuarial literature.

(AUTHOR'S REVIEW OF DISCUSSION)

DAVID N. BECKER:

I want to express my appreciation to Messrs. Evans, Gilchrist, Paquin, and Smith for contributing discussions to this paper. The discussions and the review of the discussions provide opportunities to question and emphasize the material presented and to discover new insights into it.

Mr. Evans' example is useful in probing the limits of reliability of the geometric mean test. It is also valuable because it demonstrates the utility of examining the actual profits-released book profits and of testing more than one profitability "benchmark."

There are several possible benchmarks for accepting or rejecting projects. First, one could require that the present value of the project, $PVB_1(i)$, be greater than or equal to zero at a specified hurdle rate, for example, 15 percent. This is the same as requiring i_{ROI} to equal or exceed 15 percent. Second, one could look for a positive $PVB_1(i)$ at the company's cost of capital. Third, one could test to determine whether the i_{ROI} exceeds the pretax earnings rate of the company's invested assets. It is often of value to consider more than one of these tests.

The third test is based on the pretax earnings rate of the company's invested assets. If the company chooses to invest a project's retained funds in assets of a different character, the yields may be different. To determine generalized net present values or the generalized return on investment, the aftertax earned rate on retained funds should be used in the algorithm. These rates do not necessarily equal the aftertax earned rate on the company's invested assets. When the third test is used, the comparison is between the generalized return on investment and one of: the pretax invested asset earnings rate, if level; or, if such rates are nonlevel, the maximum of the year-to-year rates, or the weighted geometric mean of such rates during the profits-released periods.

Mr. Evans' example employs the third test, using the geometric mean criterion. For the test to be valid, it is assumed that the interest rates are those of the company's invested assets and that these are also the rates earned by retained funds.

This example presents a series of cash flows during a period in which the income on any funds retained varies downward over time. Based on the rates used, the pretax rates are 10 percent for 10 years, followed by 8 percent, 6 percent, 4 percent, and 3 percent thereafter and a tax rate of 34 percent.

The i_{ROI} provided by the algorithm is 6.19 percent.

The risk in the geometric mean criterion is that the geometric mean weights each year of each period equally. If there are major differences between the profits-released periods, the rule could lead to an invalid decision to accept or reject the project. This is exactly what Mr. Evans' example demonstrates.

The geometric mean of the pretax rates during the profits-released periods is approximately 5.99 percent. Because $i_{\text{ROI}} = 6.19$ percent, the rule would accept the project. But, in fact, the present value of the profits-released cash flows at the after tax rate is -69.81. That this quantity is negative indicates the decision to accept the project was incorrect.

At this point I believe a minor modification is needed. In comparing the profitability of alternative choices, the appropriate rates for discounting the profits-released cash flows are the pretax rates, not the aftertax rates. If the present value of the profits-released cash flows is computed at the pretax rates, that present value is -852.56. This is because taxes in the profit study reflect taxes at the insurance company level, not the shareholder level. If the shareholders could have invested in other instruments earning at the pretax rate, their decision to go with the insurance product or an alternative should be based on an analysis that uses those pretax rates. This changes the magnitude, not the sign of the result.

Now consider Table 18, which completes the remaining computations. The tax rate equals 34 percent.

TABLE 18

MR. EVANS' CASH FLOW DECOMPOSITION AT $i = 6.19\%$ AND VARYING k

Pretax K	Duration	Aftertax k	Original Cash Flow	Discount Algorithm	Cash Flow: Profits Released	Cash Flow: Profits Retained
10 %	0	6.60 %	−1,000.00	0.00	−1,000.00	
10	1	6.60	−1,000.00	1,061.88	−1,000.00	
10	2	6.60	−1,000.00	2,189.46	−1,000.00	
10	3	6.60	−1,000.00	3,386.81	−1,000.00	
10	4	6.60	−1,000.00	4,658.25	−1,000.00	
10	5	6.60	−1,000.00	6,008.36	−1,000.00	
10	6	6.60	+1,000.00	7,442.01	1,000.00	
10	7	6.60	+2,000.00	6,840.62	2,000.00	
10	8	6.60	+2,000.00	5,140.13	2,000.00	
10	9	6.60	+2,000.00	3,334.43	2,000.00	
8	10	5.28	+2,000.00	1,417.00	1,417.00	583.00
6	11	3.96	− 100.00	− 613.78		−100.00
4	12	2.64	− 100.00	− 534.12		−100.00
3	13	1.98	− 100.00	− 445.58		−100.00
3	14	1.98	− 100.00	− 352.43		−100.00
3	15	1.98	− 100.00	− 257.43		−100.00
3	16	1.98	− 100.00	− 160.54		−100.00
3	17	1.98	− 100.00	− 61.74	− 38.26	− 61.74
3	18	1.98	− 100.00	40.63	− 100.00	
3	19	1.98	− 100.00	149.33	− 100.00	
3	20	1.98	− 100.00	264.76	− 100.00	
3	21	1.98	+ 50.00	387.32	50.00	
3	22	1.98	+ 50.00	358.20	50.00	
3	23	1.98	+ 50.00	327.27	50.00	
3	24	1.98	+ 50.00	294.42	50.00	
3	25	1.98	+ 50.00	259.55	50.00	
3	26	1.98	+ 50.00	222.51	50.00	
3	27	1.98	+ 50.00	183.19	50.00	
3	28	1.98	+ 50.00	141.43	50.00	
3	29	1.98	+ 50.00	97.09	50.00	
3	30	1.98	+ 50.00	50.00	50.00	

In Mr. Evans' example, the cash flows begin with duration 0. Without loss of generality the algorithm can be restated to begin with duration 0. In the following, the symbol $PVB_0(i)$ is used to denote the generalized net present values and i_{ROI} is the value of i such that $PVB_0(i) = 0$.

Note that the present value of the profits-retained cash flows at the aftertax rates (k) is zero. And the present value of each profits-released cash flow substream at 6.19 percent is zero.

When both of the profits-released cash-flow substreams are directly examined, the relative imbalance of the magnitude of each substream can be observed. This suggests that a geometric mean (with equal weighting) is not appropriate. The net present value of the first substream at the pretax rate for that substream (level 10 percent) measured at the beginning of that substream is -793.15, and the net present value of the second substream at the pretax rate for that substream (level 3 percent) is 67.18. The magnitude of difference would suggest that the project be rejected, despite the results of the geometric mean. This demonstrates the advantage of examining all the information produced by the generalized profits-released model.

Computing the present value of the project, that is, $PVB_0(I)$, at i equal to a given hurdle rate, for example, the company's cost of capital, gives another profitability benchmark. In this case, if the company's cost of capital were 12 percent, then $PVB_0(0.12) = -1{,}241.38$. In this case the project would be rejected because it did not meet the acceptance criterion of providing value in excess of the company's cost of capital. This is especially useful here because it helps resolve a difficult situation. If the company's hurdle rate is 15 percent, the project would again be rejected, because $PVB_0(0.15) = -1{,}666.35$. The magnitude of $PVB_0(0.15)$ is consistent with the fact that $PVB_0(i)$ is monotonic decreasing.

This example further suggests a way to modify the geometric mean criterion to improve its predictive capability.

Suppose there are m profits-released substreams. Let

$$\{K_{j_{1_h}}, \ldots K_{j_{m_h}}\}$$

be the company's invested asset pretax K_j during each of the m profits-released substreams, that is, for $h = 1, \ldots, m$.

Let NPV_h be the classical net present value of the profits-released book profits during the h-th substream calculated at the K_j for that substream. Note that within a given substream, the classical net present value results in an economically meaningful number. Let NPV_h^0 be the present value of NPV_h at duration 0 computed by discounting NPV_h at the K_j.

Let

$$w_h = |NPV_h^0| / W,$$

where

$$W = |NPV_1^0| + \ldots + |NPV_m^0|.$$

Let K^h be the geometric mean of the pretax K_j during the h-th substream.

The revised decision criterion is that a project is acceptable if

$$i_{\text{ROI}} > i_W = W_1 K^1 + \ldots + W_m K^m;$$

otherwise, reject the project.

In Mr. Evans' example, $NPV_1^0 = -793.15$, $NPV_2^0 = 19.33$, and the weighted average rate, i_w, is 9.83 percent. Because i_{ROI} is 6.19 percent, the project would be rejected.

It is possible to calculate a geometric weighting for i_W. This formula would be

$$i_W = [(1 + K^1) \,\hat{}\, W_1 \, x \ldots x \, (1 + K^m) \,\hat{}\, W_m] - 1.$$

If this formula were used in the above example, then $i_W = 9.83$ percent.

This question arises from a comparison of the i_{ROI} to the pretax asset earnings rates for the project. If those rates had been level, K, then one would only have needed to compute $PVB_0(K)$ to check for a positive present value balance (accept) or a negative balance (reject). The difficulty occurs because the rates are nonlevel. The *PVB* algorithm doesn't allow for a nonlevel discount rate. If it did, that would be preferable to using the geometric mean criterion.

Consider the following revision. Let $\{i_1, i_2, \ldots, i_n\}$ be a set of discount rates to be used in computing the present value balance of a project. Let K_j and k_j be the pretax and aftertax rates, respectively, earned by a project's retained funds. Define the present value balance of the project, *PVB*, at $\{i_1, i_2, \ldots, i_n\}$ by:

$$PVB_n = BP_n;$$

$$PVB_j = PVB_{j+1}/(1 + r) + BP_j$$

where

$$r = k_j, \text{ if } PVB_{j+1} \leqq 0,$$

and

$$r = i_j, \text{ if } PVB_{j+1} > 0.$$

This calculates the present value balance for a project in which the discount rate varies over time. In Mr. Evans' example, let $i_j = K_j$ for $j = 0$ to 30; that is, the discount rates equal the pretax asset earnings rate.

In this case $PVB_0 = -850.54$; thus the project would be rejected.

It is important to emphasize the fact that when multiple classical "return on investment" values exist, classical net present value analysis is not economically meaningful. This was shown by the pump project example near the end of Section IV of the paper. The classical net present value of a stream discounted at a rate i should be replaced by the quantity $PVB_1(i)$. This is the generalized net present value, and it gives the same result as the classical computation in those situations in which the flows are well-behaved.

The difference between the generalized net present value, $PVB_1(i)$, and the classical net present value is demonstrated by comparing the graphs for the four examples in Appendix A. In each set, the graph of the classical net present value function is not monotonic decreasing. The failure of the classical net present value function to be monotonic decreasing leads to the erroneous conclusion that a project can be both superior to an alternative investment earning rate i and inferior to a second alternative investment earning rate j, where i is greater than j. This contradiction indicates the failure of the classic net present value to be economically meaningful in the general situation.

Mr. Gilchrist has identified an important question of reserve adequacy for products that have patterns of future statutory book profits that include losses. The generalized method and the special modification in Appendix B assist in determining the amounts that need to be allocated to prevent future negative book profits. I thank Doug Doll who, after reading a preliminary version of the paper, asked if the method could be modified to examine this situation; this resulted in the material presented in Appendix B.

The issue raised by Mr. Gilchrist also has been considered indirectly by recent modifications to New York Regulation 126. Earlier versions of this regulation used a positive surplus at the end of the test period as an acceptance criterion for adequacy of the current reserves. This test is a profits-retained model; it assumes that all earnings are retained and reinvested. But if shareholder dividends are paid, the amount of future surplus available will differ from the projection. The regulation now requires that an estimate for dividends be reflected in the projection.

Mr. Paquin's discussion raises several questions. The first question refers to the difference between the method used by Mr. Sondergeld [2] and the method used by Mr. Paquin [1]. Although the methods are algebraically the same, the "outlook" of the methods rests on the interpretation of the rate to be used when the project's outstanding balance is positive, that is, when it becomes a source of funds to the company. Mr. Sondergeld refers to it as a rate on borrowed money, and Mr. Paquin refers to it as a rate payable on forcedly received deposits. Neither of these descriptions is definitive, so it difficult to conclude that the outlooks are totally different. The resolution of this issue lies in the fact that neither approach is a complete analysis of the economic transaction. My paper provides the needed economic structure. Within that structure the role of the disputed quantity is unambiguously the aftertax earned rate on the assets backing the project's retained funds. These funds and the associated aftertax investment income are used to offset future negative project book profits that would otherwise have required additional shareholder investment.

Second, Mr. Paquin indicates that whether "profits" can be computed by discounting or by accumulating is as irrelevant as debating the use of the retrospective or prospective methods in computing classical statutory reserves. The issue here isn't "profits," but profitability, specifically, how to determine profitability. The analogy with retrospective and prospective reserve methods is flawed. A reserve is a quantity at a given point in time. Measures of profitability are numeric indicators that provide economic interpretations of the series of "profits" of a project over time.

Further, these profits may be retained by the company or released. One has to choose the manner in which to view the transaction. This "view" represents how the company or product line will be managed. Once the view is chosen, measures of profitability can be developed to assist management in deciding between alternative projects. The "profits-released" view and the "profits-retained" view are independent. The resulting profit measures are different. This is described more fully in Section I of the paper.

Third, Mr. Paquin asserts that ill-behaved cash flows are turned into well-behaved book profits by the insertion of reserves and that income tax was left out in his analysis because it is apt to change with some frequency. It is not necessarily true that the insertion of reserves will result in well-behaved book profits. Both traditional and flexible premium products exist in which the incidence of profits was marked by multiple sign changes. In these cases nonforfeiture and reserve values were computed in strict accordance with the prevailing laws. Dynamic studies of in-force single-premium deferred annuities and excess interest whole life have revealed patterns of most likely book profits that had unusual (and unfavorable) patterns. Mr. Gilchrist and Mr. Smith also commented on this in their discussion.

If items that are apt to vary with some frequency are left out of profit studies, one might also exclude lapses, premium persistency, and expenses. One of the actuary's functions is to provide management with information about the expected change in the economic condition of the organization as a result of proposed projects. If the information fails to account for major items of income and expense, management cannot make an informed decision. In addition, regulatory solvency considerations and limitations on shareholder dividends must be considered. Thus, reserve liabilities and the accounting structure cannot be ignored.

Mr. Smith makes several valuable observations in his discussion. One of these is that if a project displays unusual patterns of book profits, consideration should immediately be focused on the accuracy of the model and the correctness of the reserves. If these two issues cannot be satisfactorily resolved, the investigation should go deeper.

Although unusual, it is possible for unusual patterns of book profits to emerge. Consider a product with a heavily front-ended premium scale. A product of this type can result in an initial statutory gain followed by losses for a period of time. This can also happen in flexible premium products if the initial premium received is sufficiently larger than the commissionable premium. A modest surrender charge amortized quickly can then result in losses followed by gains in a profitable product.

A level commission on a term product could result in future losses if the differential between future premiums and mortality is too small. Any expense increasing by duration that is funded via a level premium might result in such behavior. Persistency bonuses to producers or to policyholders could cause an isolated future negative book profit. Some of these may be more likely in the current environment of thinner profit margins.

Most of these situations do not pose a question of solvency, but may cause misleading profit study results if examined using the classical tools.

Another area of concern expressed by Mr. Smith is the situation in which the statutory profits turn negative after some point. Putting aside questions of reserves being adequate in the aggregate, is it wise to continue to value the block according to traditional methods? This is the same question raised by Mr. Gilchrist in his discussion and by Mr. Doll privately. Should an additional amount be set aside to provide for the future negative statutory book profits? The analytical tools provided in valuation actuary software and the methods in this paper can help quantify the additional reserve required.

Another application of the generalized methodology is in appraisals. An appraisal has indicated, for example, that the mean profits for an in-force block of single-premium deferred annuities would be positive for the first eight years and then turn negative for the remainder of the projection period. The appraisal showed larger classical net present values for larger discount rates.

If an additional assumption of a tax rate of 34 percent is assumed, the aftertax classical net present values in thousands of this block at various discount rates would be as follows:

Discount Rate	Classical NPV
11%	453
13	502
15	529
17	541

If the generalized net present value methodology is used with the assumption that the company can earn 10 percent pretax on its assets, the values become:

Discount Rate	Generalized NPV
11%	136
13	133
15	130
17	127

In this case part of the third-year positive book profit and the remaining five years of positive book profits were required to mature all the future negative book profits. This example is of concern to both the regulator, company management, and potential purchasers.

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