EDUCATION COMMITTEE

OF THE

SOCIETY OF ACTUARIES

STRATEGIC DECISION MAKING STUDY NOTE

INTEREST RATE SWAPS–AN EXPOSURE ANALYSIS

by

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SDM-140-14 **Printed in the USA**

Abstract

Vanilla interest rate swaps may be viewed as very simple interest rate derivatives, but the implication of entering into such contracts may not be so readily apparent. In particular, investment managers and asset/liability managers in the insurance industry are often presented with such contracts from investment banks as hedging solutions for insurance liabilities, such as fixed annuities. Within the context of hedging insurance liabilities, if used properly the risk of using interest rate swaps is not as great as if the swap were used for speculative purposes. However, we feel it is important for the potential exposure to interest rate risk inherent in interest rate (IR) swaps, and other interest sensitive financial products, to be analyzed and understood by all practitioners. Though potential counterparties of such deals often measure their exposure to default risk, the magnitude of potential risk due to changing interest rates is not always fully investigated. To quote renowned hedge fund manager George Soros: "the risks involved (in IR swap deals) are not always fully understood, even by sophisticated investors, and I am one of them." In this article, we describe a framework for measuring the potential interest rate exposure of such swaps via modelling of short rates. We will consider generic interest rate swap deals in several different yield curves environments, and under various volatility assumptions, and investigate the potential P&L exposure, and the potential counterparty exposure, under a market-consistent set of yield-curve scenarios. Further, it is our goal to not only present the current analysis but also provide practitioners with the background and tools necessary in order to perform similar analysis. In order to achieve this goal we first provide background on IR swaps, and the various stochastic interest rate models commonly used in industry, as well as several other relevant topics.

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1. Some infamous Swap deals in recent history:

Sallie Mae was one of the first institutions to use interest rate swaps in the early 1980's with the goal of reducing the duration of its liabilities. Since then, there have been many examples of interest rate swap deals which have not produced the desired results for the end-user, largely due to unanticipated interest rate movements. Among them are a vast array of city governments across the US, including New York, Oakland California, and the Alabama school district, as well as many universities including Yale, Georgetown University, and Rockefeller University in New York. Two heavily reported examples are Proctor & Gamble, and Harvard University. In 1994 Proctor & Gamble claims to have lost nearly $157 million on a subset of its IR swap deals. Procter & Gamble had been attempting to protect itself against changes in exchange and sovereign interest rates through the use of plain-vanilla IR swaps. However, when a rising US interest rate regime materialized, contrary to what had been consistent, and adamantly, forecasted P&G suffered a substantial loss. Another notable case is Harvard University's termination of a subset of its LIBOR pegged IR swaps in 2004. While the authors strongly believe that derivatives, such as IR swaps, are powerful tools, which when used properly can effectively transfer risk and enhance the efficiency of markets, it is important for the potential exposure to interest-rate risk to be analyzed. In this article, we provide the necessary background to measure this potential exposure. While this paper focuses on the valuation of interest rate swaps under short rate models, the methods explained can be used for analyzing any portfolio with interest rate risk, be it swaps, insurance liabilities, or corporate debt.

Harvard University

In late 2009 Harvard University, one of the US's oldest[1] and most prestigious institutions of higher learning paid $500 million to terminate a subset of its IR swap portfolio. Harvard entered into the IR swaps to protect against, or hedge against, the potential effect of rising interest rates on its extant variable-rate debt issuances, as well as on its anticipated future debt issuances. As noted in Harvard's 2009 annual report: "*The University has entered into various interest rate exchange agreements in order to manage the interest cost and risk associated with its outstanding debt and to hedge issuance of future debt. The interest rate exchange agreements were not entered into for trading or speculative purposes.*" Many of the interest-rate agreements were entered into during the fiscal year 2004, during a period of low interest rates, in anticipation of the planned Allston Science Center project, for which construction was set to begin in 2007. Additional projects to be supported by the swaps included a new medical research building, a Center for International Studies, and graduate housing. From the end of fiscal year 2004

[1] William & Mary and the University of Pennsylvania, among others, dispute Harvard's claim.

to the end of fiscal year 2005, the notional amount of IR swaps on Harvard's books jumped from \$1,376.6 million to \$3,723.8 million, and the corresponding projected cost to terminate the swaps jumped from \$58.4 million to \$461.2 million. By the end of fiscal year 2008 the notional amount of IR swaps was \$3,524.7 million, and the projected cost to terminate the swaps was \$330.4 million. Between 2005 and 2008 the projected cost to terminate the swaps actually dropped to around \$15 million, but in 2008 the financial crises caused interest rates to drop dramatically, and since Harvard entered the swaps on the pay-fix side, the NPV of the swaps on Harvard's books followed suit. To make matters worse, the terms of the interest rate swaps in which Harvard was involved required the posting of an amount of collateral directly related to the magnitude of the NPV, by the party[2] with negative NPV, which in this case was Harvard. Further, the stipulated form of collateral was cash. Hence, as Harvard's NPV on its IR swap contracts became more and more negative, the amount of cash it had to deliver as collateral rose. This created an additional dimension to Harvard's problems; that of liquidity, or more accurately *il*liquidity. This increased need to generate more and more liquid cash invariably played a roll in Harvard's decision in fiscal year 2009 to pay \$497.6 million to terminate a subset of its interest rate swaps with notional value totaling \$1,138.0 million, three of which were tied to \$431.7 million of bonds the university sold in fiscal years 2005 and 2007.

In general, it is important to note that, if interest rates fall shortly after inception of an IR swap contract, and stay below the rates anticipated by the market at inception, then due to the way IR swaps are priced, which is discussed in the following section, the fixed-rate payor of the IR swap will realize a net loss over the life of the swap. However, just because the NPV of future cash-flows to a counterparty is negative at some point prior to maturity doesn't mean the swap will represent a net loss if held to maturity. In particular, the greater the time remaining till expiration of the swap, the greater the chance that rates will move favorable. Of course, by the same token, the greater the time remaining, the more opportunity for unfavorable rate movements, as well. In the case of Harvard, the exact parameters of the swaps contracts have not been published, and the exact number of swap contracts terminated has not been disclosed. Although, it has been reported that Harvard had at least 19 swaps, as of early 2008, with various counterparties. Hence, due to the uncertainty surrounding the swap contracts terminated by Harvard, including the tenor of each, or length of time payments are required to be exchanged under the swap, it is impossible to determine if Harvard's net loss would have been less if it held the swaps to maturity. But given that the 1-year LIBOR[3] rate has consistently been at historic lows, it seems likely that Harvard's decision to terminate the swaps in fiscal year 2009 was not impulsive.

[2]The parties taking opposing sides in a derivative contract are referred to as *counterparties*.

[3]The LIBOR rate is a composite rate based on the inter-bank deposit rates offered by banks in London, and is similar to the Fed Funds rate in the US.

The proceeding discourse has been included to highlight several points. First, there is a difference between unrealized loss and realized loss. A negative NPV to a counterparty only becomes a realized loss if the contract is terminated or *unwound*. Further, there are many considerations which may compel a counterparty to terminate a swap contract. The immediate cost, as well as the cost in the near future, to honor the contract must be considered, but are usually only secondary considerations. The estimation of these costs, of course, depend on the view of future rate movements. More importantly, collateral requirement must also be considered. As was the case with Harvard, the compounding effect of the need to post collateral can create enough liquidity strain to compel even the most optimistic investor to abandon a deal. Lastly, it is important to note that it is the NPV which often drives the collateral requirements. This is yet another reason the evolution of swaps NPVs is focus of the current research.

The city Of Oakland CA:

IR swaps have been popular not only among companies in the private sector, and universities. One of the largest participants in the IR swap market are states and local governments, or municipalities, which often issue debt to fund infrastructure and development projects. As with universities, there are many examples of municipal governments who claim to have lost money on IR Swaps. The list of cities and municipal governments who have reported large losses on IR swap deals include the San Francisco Bay Area, San Jose, Washington DC, Baton Rouge, Boston, Charlotte, Chicago, Detroit, Los Angeles, New Jersey, New York, and Philadelphia, just to name a few. In fact, some of the most dramatic examples come from the public sector. A particularly vivid example is that of the city of Oakland California, who has recently taken on one of the largest investment banks in the world, Goldman Sachs, in response to the souring of its IR swaps with Goldman. In 1997, during a period of relatively low, but previously rising, interest rates and also strengthening of the US economy, Oakland City locked in a fixed rate of 5.6% in exchange for a variable rate tied to Libor, to protect itself from rising rates. Just like in most of the recent cases, the Oakland city swap deal was going well until the financial crises of 2007 when interest rates dropped to near 0% and stayed there for a record amount of time. In fact, at the time of this writing, some 5 years later, US interest rates are still near historic lows. The city of Oakland claims that falling rates are costing the city $4 million annually swap on the swap deal. Recently, Oakland's city council voted unanimously to authorize the City Administrator to negotiate termination of the swap deal with Goldman, adding that Oakland will cease doing business with Goldman Sachs in the future if Goldman refuses to terminate the swap. However, Goldman Sachs has refused, and CEO Lloyd Blankfein has made several public statements reinforcing Goldman's unwillingness to terminate the swaps, citing their obligation to shareholders. Unlike Harvard, the Oakland City IR swap deals were not collateralized. Hence, the story of Oakland illustrates that even the contracted periodic interest payments on a IR swap, due to unanticipated rate movements, can create enough liquidity strain to compel a counterparty to terminate a swap.

2. Interest Rate swaps:

We first briefly provide the basics of interest rate swaps, for more detail we refer the reader to Hull(2011). In general an interest rate swap is a bilateral contract between counterparties who agree to exchange cash flows based on different indexes at periodic dates in the future. The cash flows exchanged are usually determined by multiplying the rates by a specified amount of a commodity called the Notional principal. A plain-vanilla interest rate swap is a particular type of interest rate swap where fixed payments are exchanged for floating payments usually based on a Libor interest rate. LIBOR rates are the average of the rates a group of international banks in London claim it costs to borrow from one other for durations ranging from overnight to one year. To provide a simple example, consider two firms A and B. Company A borrows from market at LIBOR + 1% while company B borrows from market at 10%. Company A can enter into a swap contract with company B in which company A will pay 8% to company B and will receive LIBOR from company B. After taking into account the swap contract, company A will be making net payments of 9% while company B will be making net payments of LIBOR + 2%. The swap contract between company A and company B will basically transform company A payments from float to fixed and company B payments from fixed to float. It is important to note that company A and company B together will be making the same payments to the market.

Next we discuss the simplistic pricing and valuation of plain-vanilla interest rate swaps (i.e. we ignore Credit Valuation Adjustment (CVA) and Debit Valuation Adjustments (DVA), thereby assuming there is no chance of loss due to counterparty default. That is, we assume both counterparties are default free, or there exist a perfect collateralization, neither of which hold in the real world.) Since an IR swap consists of two streams of coupons, the value of a swap can be cast in terms of the prices of two (default-free) bonds with similar coupon payments between two (default-free) counterparties. To determine the general value of a swap at initiation consider the case of an n-period, plain-vanilla, interest rate swap where cash-flows are exchanged at the end of each period, usually every six months. Next consider two bonds, one with fixed interest payments and one with floating payments. The value of the swap can be viewed as the difference between the value of the two bonds. We consider the value of the swap from the perspective of the floating-rate payer. Let V_{fix} be the value of the fixed bond, and V_{float} be the of value the floating rate bond, then:

$$V_{fix} = \sum_{i=1}^{n} \frac{C}{(1+r_i)^i} + \frac{F_{fix}}{(1+r_n)^n}$$

and

$$V_{float} = \sum_{i=1}^{n} \frac{C_i}{(1+r_i)^i} + \frac{F_{float}}{(1+r_n)^n}$$

where F_{fix} C, and F_{float} C_i, are the Face Amounts and coupons of the fixed rate and floating rate bonds, respectively, and r_i is the interest rate on a zero-coupon bond with maturity i. Then, the value of a 'receive fixed, pay float' swap, at time 0, is $V_{swap} = V_{fix} - V_{float}$. To find the fixed rate for which the swap value will be zero at time zero (i.e. swap rate), we equate the present value of the stream of floating and fixed cash-flows. In other words, if the swap deal were to be considered in isolation, neither the fixed payer nor floating payer should gain from entering the swap. Their desire to enter the swap should be based solely on their particular needs as well as their projections of future interest rates. To derive this swap rate, we cast the formula for the value of the above bonds V_{fix} and V_{float} in common notation of Financial Mathematics. Again, for the sake of simplicity assume that the swap payments are exchanged on a semi-annual basis over n years, and also that the notional amount for the swap is \$1. Next, recall that a semi-annual floating-rate note, or bond, provides interest payments at times $i/2$, for $i = 1, 2, \ldots, 2n$, equal to the forward rate over the period from $(i-1)/2$ to $i/2$. Denote this forward rate by $F[(i-1)/2,\ i/2]$, then:

$$F[(i-1)/2,\ i/2] \;=\; \frac{B[0,(i-1)/2]}{B(0,i/2)} \;-\; 1 \;=\; \frac{B[0,(i-1)/2] - B(0,i/2)}{B(0,i/2)}$$

where $B(0,i)$ is the price of a zero-coupon bond with face amount \$1, maturing in i years in the future. Also, note that to calculate the present-value of the payment at time $i/2$ we can use the above price of a \$1, zero-coupon, bond maturing in $i/2$ years, ie. $B(0,i/2)$, as the discount factor. Hence, we can write the present-value of the stream of payments from the floating rate note as:

$$PV_{float} \;=\; \sum_{i=1}^{2n} F[(i-1)/2,\ i/2]B(0,i/2) \;=\; \sum_{i=1}^{2n} \Big(B[0,(i-1)/2] - B(0,i/2)\Big)$$

which we recognize as a telescoping series, and hence:

$$PV_{float} \;=\; B(0,0) - B(0,n)$$

and similarly, $PV_{fixed} \;=\; \sum_{i=1}^{2n} C \cdot B(0, i/2)$

So, to solve for the fixed coupon rate, which makes the present-value of both streams of payments equal, we solve:

$$B(0,0) - B(0,n) \;=\; PV_{float} \;=\; PV_{fixed} \;=\; \sum_{i=1}^{2n} C \cdot B(0,i/2) \;=\; C \cdot \sum_{i=1}^{2n} B(0,i/2)$$

for C. Hence, since $B(0,0) = 1$, we have:

$$C \;=\; \frac{1 - B(0,n)}{\sum_{i=1}^{2n} B(0,i/2)}$$

However, this only describes how the fixed coupon rate is set at initiation, or time 0, so that the value of the swap is equal to, or close to, zero at time zero. But, now that we see how to describe the value of an IR swap in terms of the value of two bonds, we can investigate the value of the swap at times other than initiation. First, recall that floating-rate notes or floating- rate bonds have a variable coupon rate based on some market reference rate, such as the LIBOR or the Federal Funds Rate. A notable alternative to use of LIBOR rates as the reference rate for the floating leg within an IR swap is the use of an overnight lending rate, such as the Effective Federal Funds rate in the US, which is an average of unsecured overnight lending rates between financial institutions. A swap whose floating rate is indexed to such an overnight rate is called an OIS swap. The fixed rate on an OIS swap, of a given tenor, is called the OIS rate for the given tenor. Unlike the IR swaps discussed so for, cash flows are only exchanged at maturity of an OIS swap. Specifically, the difference between the OIS rate at inception, for the given tenor, and the geometric average of the effective federal funds rate over the same tenor, are exchanged at maturity of the swap. Since the OIS rate is a function of the tenor of the swap, these rates can be used to form an OIS curve. It should be noted that, post the financial crises of 2007, many banks have begun using OIS rates to discount both collateralized and uncollateralized transactions, although this is less often the case for the latter. Due to their derivation from effective federal fund rates, OIS rates are now largely considered a better proxy for the risk-free rate. LIBOR rates, on the other hand, being the short-term, unsecured, borrowing rates of highly-rated banks, reflect an element of credit risk. Also, as discussed below, the *Dodd-Frank act* of 2010 mandates that most swaps be centrally cleared, and that uncleared swaps be collateralized on dealer balance sheets. As a result, it is now commonplace for OTC derivative contracts to include credit support annexes (CSA's), which dictate the parameters of collateral agreements, including the amount and timing of collateral posting, as well the triggers which may require increased collateral requirements. Further, since the required frequency of margining of collateral is often daily, the rate earned on such collateral is an overnight rate, such as the federal funds rate. This is yet another reason why OIS rates are increasingly viewed as the correct rate for valuation. However, the use of OIS rates for discounting significantly complicates the valuation of interest rate derivatives. When OIS discounting is employed, both OIS and Basis curves[4] must be formed simultaneously. Moreover, traditional *bootstrapping* curve-building techniques are no longer applicable to this, so called, *dual curve* problem. Much more can be said about the impact of OIS discounting on derivative valuation, specifically regarding the incorporation of the additional components; CVA, DVA, and even FVA[5], once the primary discounting is complete under the no-default assumption. However, the goal of the current research is to analyze the potential exposure to unanticipated reference-rate changes, inherent in interest rate swaps, and the complexities of OIS discounting are significant enough to warrant a dedicated research project. Hence, for the remainder of this paper, we assume the reference rate is the LIBOR, and that LIBOR rates are used for discounting.

Returning to LIBOR based IR swaps, the value of a floating rate bond at time 0, V_{float}, depends on the current, time 0, LIBOR yield-curve, or the collection of market anticipated yields (or rates) of the LIBOR over various time periods in the future, or "tenors". Hence, since V_{float} depends on the time 0 market anticipated yield-curve, and the price of the swap is set so that $V_{float} \cong V_{fix}$, the price of the swap also depends on the time 0 yield-curve. It may seem obvious, but for the sake of completeness, note that at time 0 there is only one LIBOR yield curve. As time passes, however, the LIBOR yield over the same period of time in the future will mostly likely change. In other words, at time 1, for example, the *realized* LIBOR rate over the next 1-year in the future can either be higher or lower than the LIBOR rate from 1 year in the future to 2 years in the future, that was expected 1-year in the past.

Hence, at times in the future it is very likely that the realized LIBOR rates will differ from what was expected at time 0. When this occurs the *realized* value of the swap at time points in the future will also deviate from the swap values expected at time=0. If the swap was to be terminated at any time in the future, a value equal to NPV (in a world where no replacement cost or friction is considered) computed on the swap at that time will need to be exchanged betweent the counterparties. In addition to above consideration on changes in interest rates, assuming the yield-curve is upward-sloping the expected NPV value of the receive-fix pay-float swap will be negative for the duration of the swap contract. This is because a receive fixed swap, in an upward sloping yield curve environment, initially behaves like receiving a loan from the counterparty which is gradually paid back during the remaining duration of the swap. Note that NPV of the swap at any point in time only factors in the

[4]Basis curves define the basis between LIBOR and OIS rates.

[5]Credit Value Adjustment, Debt Value Adjustment, and Funding Value Adjustment

remaining payments and hence a negative NPV means a net payment in the future. Also, in addition to direct gains or losses in the value of a swap position there are also additional administration fees which are triggered upon termination of a swap position. Further, under ASC 815 (formely FAS 133), unless an interest rate swap qualifies as a hedge, gains or losses must be recorded in earnings, and even if a swap is used as a hedge, ASC 815 only allows hedge accounting if specific prerequisites are satisfied and, moreover, the type of hedge accounting allowed is dependent on the motivation for entering the swap contract. For example if a company issues fixed-rate debt and then enters a pay-float receive-fixed swap in order to replicate the net interest expense of variable-rate debt, then fair-value hedge accounting instead of cash-flow hedge accounting must be followed. In which case the corresponding gains or losses of the interest rate swap are posted to earnings, just as they would be for a derivative which isn't used for hedging. The investigation of the potential gain or loss upon entering such a plain-vanilla interest rate swap, due to changes in the interest, rates is the main focus of this research, which will be presented in section 5, after some more background is covered.

The above description of interest rate swap pricing was predicated on the assumption that swap prices are set so that the value of the swap at time 0 is fairly priced, or that the value of the swap at time 0 is 0. However, this *zero net present value* principle is just a theoretical construct. In practice, the NPV of most swap contracts are positive for the dealer. This is due to the bid-ask spread the dealers quote, which represents the difference between the prices the dealer is willing to pay (purchase), or to receive (sell), respectively, in order to act as counterparty within a contract. Also incorporated into the quoted spread will be some consideration of the collateralization terms, creditworthiness of the company, and potential funding cost for the dealer. One effect of the use of dealers is that swap end-users are not exposed to each other's creditworthiness, instead only to the creditworthiness of the agent. An important consequence of such a dealer-based market, is that the price of swaps is largely dictated by supply & demand. A direct result of which is that, in addition to the assumption of Efficient Markets or Arbitrage-free markets, the accurate pricing of the interest rate risk inherent in these derivatives is implicitly dependent on the assumption that participants in the market have accurately quantified the exposure produced by these derivatives.

In general, derivative trades can be implemented in several different ways, or in several different markets. To this point we have described derivative trades within the bilateral OTC market, where participants directly trade and clear their trades with one another, or with a dealer. However, derivatives trades can also be implemented through what is called a *centralized counterparty*(CCP). Trades through a CCP begin as in the bilateral OTC market, but what would have been a single contract between two parties is essentially broken into two new contracts[6], one between the buyer of the derivative and a ternary counterparty, and one between the seller and the same ternary counterparty. This ternary counterparty is the CCP. This process is similar to how trades take place in an *exchange*-based market, however a CCP may be used in conjunction with decentralized trades, such as occur in the bilateral OTC market. When such a trade takes place the mechanism can be described as decentralized trading with centralized clearing. This procedure has several benefits over the bilateral OTC practice. First, it increases transparency by making information on prices, trading volume, and counterparty exposure available to the public. Second, it simplifies the management of counterparty risk and collateral. Thirdly, since a CCP is theoretically a counterparty to a large number of trades, or potentially even all trades, in a particular derivative, the CCP can more easily perform multilaterally netting, which also ameliorates counterparty and operational risks. Another beneficial aspect of the use of CCP's is that they essentially diversify credit and market risk by spreading it out among its large number of counterparties. Lastly, the use of CCP's may reduce the profiting by market-makers, via price discrimination among its customers, in the bilateral OTC market, which is sometimes purported to exist. The discussion of CCP's is relevant to IR Swaps because the almost 1,000 pages of regulation contained in the Dodd-Frank act[7][8] dictate that most[9] vanilla OTC contracts (including IR swaps) be traded on exchanges and cleared through central counterparties, and that uncleared swaps be collateralized on dealer balance sheets. One of the motivations for requiring derivative trades to be through a CCP is the desire to reduce the required capital held by large banks. In fact, derivative trading through CCP's began before Dodd-Frank, and even before the financial crises. One of the first CCP's for IR swaps was SwapClear[10]. In 2008, two more Clearning houses for swaps were created; CME Cleared Swaps, and the International Derivatives Clearing Group[11] However, we do not wish to give the impression that CCP's are some sort of silver-bullet for the derivatives market. For almost every benefit CCP's provide, there is a corresponding hazard. For one thing, it is important to be cognizant of the fact that CCP's cannot eliminate counterparty risk. Rather they simply convert it into other forms of risk, such as Liquidity and Operational risk. Because of the position the CCP takes within a derivative trade it assumes no market risk, however, it does bare the full counterparty risk. As a result, CCP's will attempt to mitigate this risk by demanding collateral from the end-users of the derivative contract in the form of a *variation* and *initial* margin. Hence, the CCP effectively transforms the counterparty risk to Liquidity risk. Further, since the size of the posted

[6]This process is called *Novation*.

[7]Dodd-Frank was signed into law in July 2010.

[8]Much of Dodd-Frank had its genesis in the 2009 G-20 Pittsburgh communique.

[9]There are exemptions for certain end-users.

[10]SwapClear is a UK-based CCP which was established in September 1999.

[11]CME Cleared Swaps is associated with the Chicago Mercantile Exchange, and IDCG is linked to Nasdaq.

margins only covers the exposure to default up to a certain level, or for a certain percentage of the scenarios in which default occurs, it is possible that the CCP, and its counterparties, could be exposed to the moral hazard of a given counterparty. This is so since, if all margins of the defaulting counterparty are exhausted, the funds of the other counterparties may be used to absorb the remaining losses. Further, some have argued that asymmetric information, or expertise, regarding the valuation of complex derivatives may disadvantage CCP's and lead to adverse selection, similar to what occurs in the insurance market.

Yet another interesting facet of the market's influence on IR swap prices involves the LIBOR rates themselves, or more specifically, the market's view on future LIBOR rates. The mechanism for this influence is the Eurodollar futures market. *Eurodollars* are deposits made to banks outside the U.S., yet which are denominated in U.S. dollars. *Eurodollar futures* are derivative contracts based on Eurodollars, and which are traded on the Chicago Mercantile Exchange(CME). Essentially, Eurodollar futures allow investors to speculate on the future levels of a 3-month European interbank lending rate index. Hence, similar to interest rate swaps, Eurodollar futures allow the 'locking-in' of future interest rates. From their introduction in 1981, through 1996, the index upon which Eurodollar future prices were set was determined by the CME itself, based on information gathered from banks in London. In 1997, however, the CME began using the LIBOR as the index for Euro-dollar futures[12]. Hence, since 1997, Eurodollar future prices have been determined by the market's expectation of the 3-month LIBOR rate at predetermined settlement dates in the future. One of the motivations for the change was the belief that basing Euro-dollar futures on LIBOR rates would facilitate the hedging of IR swaps, and hence bolster the then burgeoning IR swap market as well as increase the trading volume in Eurodollars. The move to a LIBOR-based Eurodollar futures market also, at least potentially, impacts IR swap pricing. Since Eurodollar futures contracts with expiration, or settlement dates, up to 10 years in the future are traded, a 'market-implied' LIBOR yield curve can be educed. Hence, if the IR swap pricing were grossly out of line with this implied LIBOR curve, an arbitrage opportunity would exist. However, for the same reasons given in the previous paragraph, this 'market-consistent' force, or arbitrage relationship, between Eurodollar and IR swap prices will only help move IR swap prices toward a 'more fundamentally correct' level if participants in the Eurodollar futures market have better information regarding future LIBOR rates than those in the IR swap market. Conversely, if participants in the Eurodollar futures market had systematically *less perfect* information regarding future LIBOR rates than those in the IR swap market, this could potentially have a negative effect on participants in the IR swap

[12]Marcy Engel, lawyer for Salomon Brothers Inc. at the time, expressed concern that since the banks in London which set the lending rates are also able to take positions in the eurodollar market, these banks may be tempted to attempt to manipulate the LIBOR rate. It has been reported that since 2008, 19 banks have been investigated for LIBOR fixing, and in 2012, London-based Barclays plc. admitted that employees attempted to manipulate LIBOR rates, and paid a $450 million settlement.

market, or at least to one of the counterparties to a given IR swap deal.

A situation in which market forces may have less of an influence on IR swap prices is when investment banks market IR swap deals directly to insurance companies. However, regardless, it is important that insurance companies perform their own valuation of swap deals. It is this analysis of the interest rate risk exposure for IR swaps which is the main focus of this paper.

3. Interest Rate Models:

As stated above, the goal of the current research is to analyze the potential P&L exposure as well as potential counterparty exposure of interest-rate swaps due to the realization of interest-rates which may differ from the markets expectation upon inception of the swap contract. Of course, the main driver of the interest rate risk of an IR swap is the uncertainty surrounding the level of future interest rates, and many techniques exists to forecast the level of future interest rates. Ideally, the goal when forecasting future interest rates is to correctly, and consistently, anticipate how interest rates will evolve in the future. However, the vast array of continually evolving and changing economic, market, regulatory, etc., factors, which impact the level of future interest rates, imbue their future value with the characteristics of a random process. Hence, if one takes this view, the amount of information available at the current time regarding the level of future interest rates is limited to an understanding of distributional qualities. In this case it is not surprising that the techniques of modern mathematical probability and statistics are often utilized to model future rates. A popular, *forward-looking*, application of this approach is to model the future path of interest-rates as a stochastic process, which evolves according to an Ornstein-Uhlenbeck diffusion process, and is calibrated to produce prices which match the current market prices of liquid securities such as interest rate caps or swaptions. An alternative, *backwards-looking* version of this statistical modeling approach, is accomplished by calibrating the parameters of a given model using a subjectively chosen subset of historical market information. As will be discussed further below, this retrospective approach is implicity predicated on the assumption that the past is likely to repeat itself.

The development of models embracing this mathematical modeling approach has been ongoing since the mid-80's. Originally these models were developed to price option-embedded bonds. Unlike their decedents, these early models were not able to produce prices of non-callable bonds which were consistent with the market, and hence when used for pricing, allowed for arbitrage. In other words, these models where not *arbitrage-free*. These early models assumed the instantaneous forward rates followed a Stochastic Differential Equation (SDE) similar to those of arbitrage-free models, which will be discussed in more more detail below. However, the reason these early models could not calibrate to the current yield-curve, or in other words, were unable to produce the entire current yield curve on an expected basis, is that the coefficients of the model, most importantly the drift term, θ, were assumed to be constant. For example, one of the first such models was the Vasicek model, for which the short-rate satisfies the equation: $dr_t = \kappa(\theta - r_t)dt + \sigma dZ_t$, where Z_t is a Brownian motion. Incidentally, note that although these models are unable to match the entire current yield curve on an expected basis, they are usually able to match the current rates for a few select maturities. To remedy this situation, Arbitrage-free models were subsequently developed. These models are able to match the entire yield curve, on an expected basis, by replacing the constant parameters, specifically the drift parameter θ_t, in the early models with parameters which are a functions of time. Also, Arbitrage-free models utilize the current yield-curve, in the calibration process, in order to facilitate the parameter θ_t being fitted so that the model can match the current yield-curve. For this reason, these models are often described as taking the initial yield-curve as an *input*.

When an Arbitrage-free model is calibrated using the current market prices of available securities, by definition, implicit in the solution is a *risk-neutral* measure, with respect to which the short-rate is a solution to the stochastic differential equation. Technically, *any* set of *martingale* probabilities such that you can replicate all available market prices of securities *in the given market/world* is a risk-neutral measure (or a set of risk-neutral probabilities) *with respect to that particular market/world of prices.* Said another way, a risk-neutral measure is any martingale measure on the given world of securities. The stipulation that a risk-neutral measure is defined relative to a particular world of securities is an important one, and will be discussed further below. In order to explain why the existence of a risk-neutral measure is implicit in the use of arbitrage-free models calibrated using the current market prices, we first provide a brief overview of the concept of risk-neutral probabilities as commonly presented in elementary mathematical finance texts. Most coverage of risk-neutral probabilities, starts off by introducing *Arrow* securities. Arrow securities are hypothetical securities that pay \$1 if a given economic, or market, state arises at a specified time in the future, and pay 0 otherwise. Further, the price of the security at time=0, relative to the \$1 pay-off in the future, provides a measure of the market's view of the likelihood of the corresponding state arising. One motivation for introducing Arrow securities is that if it is assumed that the market is what's called *complete*, then the pay-off of any security in the market can be replicated with a linear combination of the security pay-off's in each state, weighted by the market prices of the Arrow securities corresponding to the same state. An important consequence is that a unique arbitrage-free price for every security in the market can be determined from such a linear combination of Arrow securities. This is so since the set of market(economic) states corresponding to, or triggering, the pay-off's from the set of Arrow securities are assumed to be mutually exclusive and exhaustive, so that by holding all Arrow securities a \$1 pay-off is assured. Hence, if the market price of a given security is other than the price indicated by the linear combinations of security pay-off's weighted by the Arrow securities prices, arbitrage will exist. It is also significant that the heretofore referred to linear combination is weighted by the market *prices* of the Arrow securities. Since probabilities are usually used for weighting it make sense that these Arrow security prices can be viewed as probabilities. In fact, these Arrow security prices are exactly *risk-neutral* probabilities, if the time-value of money is ignored. If the time-value of money, however, is not ignored, or in other words, if there exists a risk-free *bank account*, then the prices of securities can be represented by the exact same linear combination, just with appropriate discounting. If the risk-free interest rate over one period is represented by r, then in this case by holding the whole set of Arrow securities with pay-off's occurring one time period later, the present-value of the the cumulative pay-off is no longer guaranteed to be \$1, but rather $\frac{1}{1+r}$. In other words, the sum of all the time 0 Arrow security prices is $\frac{1}{1+r}$. If we still want to use these prices as probabilities, then we have to adjust them so that they add to 1. This can be accomplished simply by *grossing-up* each individual price by $(1+r)$. So if a_i is the Arrow security price, a.k.a risk-neutral probability, in the 0 risk-free rate situation, then $q_i = (1+r) \cdot a_i$ will be the corresponding risk-neutral probability when the risk-free interest rate is r. If we assume a simple security which only makes payments one time period later, and X_i is the time 1 pay-off of the given security, if state i is realized, then the price of the security at time 0, X_0, is:

$$X_0 \;=\; \frac{1}{1+r} \sum q_i \cdot X_i$$

Returning to the concept of complete markets, if the set of Arrow securities is not rich enough to replicate all security prices, then the set of risk-neutral probabilities which can be used to find the arbitrage-free price of any security are not unique, rather a range of probabilities can serve as risk-neutral. For example, imagine there are only 5 possible states of the market at time 1, but that it's not possible to determine what the price of the Arrow securities corresponding to states 4 and 5 are because, for example, there may not be enough observable securities in the market to make this inference. Then a security which pays \$1 if states 4 or 5 arise, will have an arbitrage-free price of $1 - a_1 - a_2 - a_3$. Further, if another security pays \$1 if states 1, 2, or 5 arise, then its price will be greater than $a_1 + a_2$, but less than $1 - a_3$. More specifically, any combination of prices a_4 and a_5 such that: $a_1 + a_2 < a_4 \;+\; a_5 < 1 - a_3$ can serve as risk-neutral probabilities.

This is a good place to pause and emphasize that the presentation of risk-neutral probabilities so far has actually been developed in an idealistic framework. We started from the assumption that we could observe the market prices today of all Arrow securities, and hence the unique set of risk-neutral probabilities. However, one may correctly ask one's self, if the current market prices of all Arrow securities are known, and the price of any security in the market can be determined from a linear combination of the Arrow security prices, then what is the need for risk-neutral probabilities in the first place? Indeed, this is a prescient observation. In order to gain any advantage from the use of risk-neutral probabilities, securities whose pay-off's cannot be replicated with the known set of Arrow securities must be priced, in which case as we saw above, there will be more than one set of probabilities which qualify as risk-neutral. However, even though the set of risk-neutral probabilities available for pricing a given world of securities may not be unique, it can be shown that the expected price, as measured from time 0, of any security in the given world, under any set of risk-neutral probabilities, will obey an important mathematical property as it moves through time, from time 0 to expiration. Another way to say this is; the expectation, as measured from time 0, of our expectations of the price at future times obeys an important mathematical property. More specifically, using the above reasoning, if X is the pay-off of the security at time n, and P_t is the security price at time t, then using the same reasoning as above, we have:

$$P_t = \frac{1}{(1+r)^{n-t}} E_t^Q(X)$$

where $E_t^Q(\cdot)$ denotes the expectation at time t under a risk-neutral measure. Then using the rules for conditional expectations, specifically the *Tower Property*, which says that: $E[E(X \mid Y) \mid Z] = E(X \mid Z)$, if Z is a function of Y, we have that: (see appendix for details)

$$E_0^Q(P_t) = E_0^Q\Big(\frac{1}{(1+r)^{n-t}} E_t^Q(X)\Big) = \frac{1}{(1+r)^{n-t}} E_0^Q\Big(E_t^Q(X)\Big) = \frac{1}{(1+r)^{n-t}} E_0^Q(X)$$

Now recalling that: $E_0^Q(P_0) = \frac{1}{(1+r)^n} E_0^Q(X)$, we can re-write the above as:

$$E_0^Q(P_t) = \frac{1}{(1+r)^{n-t}} E_0^Q(X) = (1+r)^t \Big[\frac{1}{(1+r)^n} E_0^Q(X)\Big] = (1+r)^t \cdot E_0^Q(P_0).$$

In other words, we see that the *expected* price grows at the risk-free rate through time, where each expectation is calculated as of the same point in time. Recalling the definition of risk-neutral probabilities as *grossed-up*, or *compounded*, state prices ie. $q_i(t) = (1+r)^t \cdot a_i$, for each state i, it makes sense that the expected value (over all states) at time $t > 0$ is also inflated by a factor of $(1+r)^t$. If P_t is considered a stochastic process, which is simply a sequence of random variables indexed by time, then this pattern of growth through time can be described as a *drift* in the process. In this case, we can "correct" for the drift in the process by simply dividing by the growth at each point in time. Assuming the price of the security is known at time 0, we have:

$$E_0^Q\Big(\frac{P_t}{(1+r)^t}\Big) = \frac{1}{(1+r)^t} E_0^Q(P_t) = P_0 = \frac{P_0}{(1+r)^0} \quad \text{for all } t > 0.$$

Hence, letting $S_t = \frac{P_t}{(1+r)^t}$, we have that $E_0^Q(S_t) = S_0$ for all $t > 0$. This implies that the stochastic process S_t has the characteristics of what's called a *Martingale* under the risk-neutral probabilities, and the risk-neutral probabilities are called *Martingale probabilities* for S_t (technically $E(|S_t|) < \infty$ is also required). In fact, as stated above, this criteria can be used to define risk-neutral probabilities: Any set of probabilities under which the stochastic process P_t is a Martingale qualifies as being risk-neutral. Again, it is important to remember that there may be many such sets of risk-neutral probabilities for a given stochastic process, and also to remember the connection between the characteristic of a security price being arbitrage-free and the existence of a risk-neutral measure. *The price of a security, within a given world of securities, is Arbitrage-free if it falls within the range of prices calculated from all extant risk-neutral models for the given world of securities.*

Since the analysis in this paper is focused on IR swaps, and is based on forward yield-curves generated from interest rate models, we first stress that the above risk-neutral framework can be applied to any financial instruments, including bonds. In the case of bonds, it is the interest-rates, which are used for discounting, that are uncertain, not necessarily the pay-off's, or coupons and face value. So, in this case, given the interest rate level at time t, a set of risk-neutral probabilities is assigned to the potential interest rate levels at time $t + 1$. This is done for all times t within the range of analysis, and results in a sequence of possible interest rate paths, through time, with associated probability for each path. The associated probability is found by sequentially multiplying the risk-neutral probabilities for each unit step along the given interest rate path, through time. Then, for each interest rate path, or along each interest rate path, the present value of the bond can be calculated. This gives the present value of the bond assuming the given path of rates is realized. Finally, one can calculate the overall price of the bond as the *expected* present value over the set of future interest rate paths, using the associated probabilities. Next, we reinforce that the concept of Arbitrage-free deals with the relationship between *prices*, not rates. So, even though prices can be inferred from the rates generated by the interest rate models used in this paper, via zero-coupon bond prices, there is a degree of separation between the actual output of said models and the concept of Arbitrage-free, and hence also the concept of risk-neutral. Also, as alluded to above, a risk-neutral measure is defined with respect to a given *world* of securities, or security prices. This world of securities includes the securities used to calibrate the given model and also the securities being priced. Hence, much like the concept of *Independence of events* in elementary probability theory, the qualification of a set of probabilities as being risk-neutral is a joint characteristic of both the measure(probabilities) and the space(set of securities) on which the measure is defined. In particular, while a measure may be considered risk-neutral with respect to a given *world* of securities, it may no longer satisfy the requirements of a risk-neutral measure when applied to a larger universe of securities which contains assets classes whose price information is outside the original world of securities. This last point is especially germane to the above interest-rate models, as there are necessarily only a finite number of market securities with respect to which a model can be calibrated. Hence, such models can only be considered risk-neutral for the purposes of pricing a subset, possibly a small subset, of all securities. Moreover, another consideration which is particularly relevant to arbitrage-free models used for interest-rate modeling is that market price information is only available to calibrate solutions at a small set of durations. Hence, for all durations between these few durations, there is no price information available for calibration, and hence this is an analogous situation to the one described above, where price information for some securities are outside the world of securities used to calibrate the model. However, the SDE essentially constrains the short-rate solution to be such that there is no arbitrage, even at durations between those for which there is market information. Of course, this will only be the case if the short-rate actually does evolve according to the SDE, which is unlikely. After all, it must be remembered that a model is only a model.

Within industry and academics, especially the latter, it is often admonished that pricing models should rely on risk-neutral measures (a.k.a. risk-neutral probabilities), whereas models used for risk management are often said to not require, and some even claim should not use, risk-neutral probabilities. Rather it is claimed that models used for risk management purposes should rely on, what is often referred to as the *real-world* measure. However, the real-world measure can be considered a theoretical concept, in the sense that knowing the real-world measure would be tantamount to knowing the real chances, ahead of time, that financial security prices will be at given levels in the future. This means that models ostensibly based on the real-world measure can only, at best, be based on an approximation to, or estimate of, the real-world measure, or real probabilities. In practice the *real-world* measure is often estimated by using a subjectively chosen subset of historical market information to calibrate model parameters. Again, when historical information is used in this way, it is important to realize that the assumption is that the past is likely to repeat itself. Furthermore, estimation of the real-world measure usually depends on the subjective choices regarding which historical information is used, as well as over which period this information is gathered, and hence the results of such models are dependent on these subjective choices. Also from recent history we know for certain that the assumption that *the past will repeat itself* is a tenuous one, at best. Events outside of recent past experience, in fact, can and most likely will eventually occur. Instead of relying on subjective views and historical data, risk-neutral models use current market-implied information to calibrate model parameters and, as a result, also adjust automatically in reaction to changing market conditions. However, it should be pointed out that, in practice, much analysis and attention to detail, goes into the selection of historical data and the calibration process, when employing "the" real-world measure. Given this, as well as the practical difficulties in implementing, and interpreting, models based on a risk-neutral measure, some of which are outlined above, the use of historical data to calibrate models is often reasonable, especially if the purpose and scope of the models are amenable to such calibration. In fact, the analysis performed in this research uses historical data to calibrate some of the parameters.

We now investigate the particular arbitrage-free interest rate models relevant to the current work. Specifically the Black-Karasinski model and the more general Hull-White model. Using notation similar to that in the previous section, but allowing the *contracted* time of the swap, t_0, to be other than zero, the forward rate from t_1 to t_2, at contracted at time t_0, is:

$$F(t_0, t_1, t_2) \; = \; \frac{P(t_0, t_1) - P(t_0, t_2)}{(t_2 - t_1)P(t_0, t_2)}$$

where $t_0 \; \leq \; t_1 \; < \; t_2$, and where $P(t_0, t_1)$ is the price of a zero-coupon bond maturing at time t_1, and valued at time t_0. Similarly, the (instantaneous) Short-Rate at time t_0 is:

$$f(t_0, t_0) \; = \; \lim_{t \to t_0} -\frac{\partial Log P(t_0, t)}{\partial t}$$

It is possible to model the evolution of either of the above rates as a stochastic process whose evolution through time is governed by a stochastic differential equation similar to that of the Black-Karasinski model, which is described below. The first model of the evolution of Short rates was the Vasicek model (1977), followed by many others. Short rate models are often preferred to forward rate models because they are more tractable, and easier to understand, but are more limited regarding the volatility structures which may be used. Since forward-rate models are not used in this paper, they will not be discussed further, and the interested reader is referred to Hull(2011). However, as mentioned above, the Heath, Jarrow and Morton forward-rate models, specifically the LIBOR market model are very promising, though they have their own short-comings, and are harder to implement. Some of the most popular short-rate models, to date, are the Hull-White, and the Black- Karasinski models. At the time of this writing, there does not appear to be extensive documentation on the implementation and calibration of the Black-Karasinski model, hence we first discuss this model, and in the next section describe how it is usually implemented in practice, by discretizing and then evaluating the model using a tree structure.

As already mentioned, short rate models specify the behavior of the short-rate through time, which we previously denoted by $f(t_0, t_0)$. To make the derivation, which follows, more clear we simplify notation by simply referring to the short-rate as r_t, and sometimes abuse notation by simply referring to the short-rate as r. The Black-Karasinski model is a specific case of the generalized Hull-White model in which the function of the short-rate $f(r) = ln(r)$ is assumed to follow the following specific form of the Ornstein-Uhlenbeck diffusion process:

$$df(r) \; = \; [\theta(t) - \alpha(t)f(r)]dt \; + \; \sigma(t)dZ_t$$

where Z_t is a Standard Brownian Motion, and $\alpha(t)$, $\sigma(t)$ and $\theta(t)$ are non-random parameters, which are possibly functions of t. For background on Brownian motions the reader is referred to Karatzas & Shreve(1997). Technically the Black-Karasinski model is considered a *single factor* model since the short-term interest rate is assumed to be the only source of uncertainty. The solution to the above SDE, which specifies the Black-Karasinski model, is: (see appendix for details)

$$r_t \;=\; \exp\left[\ln(r_s)e^{-(J(t)-J(s))} \;+\; \int_s^t e^{-(J(t)-J(\tau))}\theta(\tau)d\tau \;+\; \int_s^t \sigma(t)e^{-(J(t)-J(\tau))}dZ_\tau\right]$$

for any $s < t$, and where: $J(t) = \displaystyle\int_0^t \alpha(\tau)d\tau$.

In the above SDE, $\theta(t)$ is the parameter which imbues the model with its initial (at time 0) term structure matching characteristic, $\alpha(t)$ is a *mean-reversion* parameter, which represents the speed at which $ln(r)$ goes to the asymptotic mean, θ, and $\sigma(t)$ represents the instantaneous volatility of the spot rates. $\alpha(t)$ and $\sigma(t)$, taken together, allow the model to be calibrated to current market prices. More specifically, they allow the volatility (term) structure of the model at time 0 to match the market's implied volatility structure. However, by allowing $\alpha(t)$ and $\sigma(t)$ to be functions of time, the model's implied future volatility structure can deviate from the time 0 volatility structure. This is often referred to as the model's implied volatility structure not having a *stationary distribution* through time. Hence, there is a trade-off between matching the volatility term-structure at time 0, and having a non-stationary volatility structure. For this reason, as well as the increased tractability of the model under the assumption of constant parameters, often in practice the parameters $\alpha(t)$, and $\sigma(t)$ are held constant. Again, when this adjustment is made to the model, the volatility structure throughout time remains constant, but the consistency with market prices can be significantly effected.

We now turn our attention from the Black-Karasinski model to the model from which the Black-Karasinski is derived, ie. the Hull-White model. The Hull-White model follows the same SDE, namely:

$$df(r) \;=\; [\theta(t) - \alpha(t)f(r)]dt \;+\; \sigma(t)dZ_t$$

but instead of using $f(r) = ln(r)$, we simply use $f(r) = r$. Hence, in this case, the SDE becomes:

$$dr_t \;=\; [\theta(t) - \alpha(t)r_t]dt \;+\; \sigma(t)dZ_t$$

Then, exactly as for the Black-Karasinski model, we have:

$$r_t \;=\; r_s e^{-(J(t)-J(s))} \;+\; \int_s^t e^{-(J(t)-J(\tau))}\theta(\tau)d\tau \;+\; \int_s^t \sigma(t)e^{-(J(t)-J(\tau))}dZ_\tau$$

for any $s < t$, and again, where: $J(t) = \int_0^t \alpha(\tau)d\tau$. Hence, we can see that the Black-Karasinski model is related to the Hull-White model, only $\ln(r_t)$ is governed by the SDE in the former, and just r_t in the latter.

Now, if we let the coefficients $\alpha(t)$ and $\sigma(t)$ be constant, and put $u = 0$, we have:

$$r_t \;=\; r_0 e^{-\alpha t} \;+\; \int_0^t e^{-\alpha(t-\tau)}\theta(\tau)d\tau \;+\; \sigma\int_0^t e^{-\alpha(t-\tau)}dZ_\tau$$

In general, the price at time 0 of a zero-coupon bond with maturity T is:

$$P(0,T) \;=\; E^Q\Big(\exp\Big[-\int_0^T r_s\; ds\Big]\Big)$$

Hence, we examine $X(t) \;=\; \int_0^T r_s\; ds$

It can be shown that:(see appendix)

$$E\big(X(t)\big) \;=\; \frac{r_0}{\alpha}(1-e^{-\alpha t}) \;+\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t}\theta(\tau)\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)d\tau$$

and:

$$Var\big(X(t)\big) \;=\; \frac{\sigma^2}{\alpha^2}t \;+\; \frac{\sigma^2}{2\alpha^3}\Big(4e^{-\alpha t} - e^{-2\alpha t} - 3\Big).$$

Further, it can be shown that:(see appendix) $\theta(t) \;=\; \alpha f(0,t) \;+\; \frac{d}{dt}f(0,t) \;+\; \frac{\sigma^2}{2\alpha}\Big(1-e^{-2\alpha t}\Big)$.

Hence, by combining the above formula for r_t with that for $\theta(t)$, we see that the Hull-White model has a closed-form solution.

The existence of a closed-form analytical solution is a very attractive feature of the Hull-White model, however one draw-back of this model is that it is possible to have negative interest rates. This draw-back was one of the main motivations for the creation of the Black-Karasinsk model (1991). In general the Black-Karasinsk model is considered a more accurate model of spot rates, however this increased accuracy comes at a cost. Specifically, unlike the Hull-White model, no closed-form analytic solution exists for the Black-Karasinski model, and hence implementation is much more difficult.

In this paper we use the Black-Karasinski model to model the evolution of the short-rate through time. As mentioned above, the Black-Karasinski model is a member of the class of Arbitrage-free models. As described below, the Black-Karasinski model has three parameters; $\alpha(t)$, $\sigma(t)$ and $\theta(t)$, and when such models are described as arbitrage-free it is assumed that $\alpha(t)$, and $\sigma(t)$ are calibrated using market information on interest-rate swaptions or bond options at specific durations. However, it is possible to use historical information to calibrate $\alpha(t)$, and $\sigma(t)$ within an arbitrage-free model. In this case the prices derived from the model are not arbitrage-free, in the strict sense. However, it can be argued that, when this type of calibration is performed, the *real-world* measure is employed. Hence the short-rate process, which is a solution to the SDE, shares some of the attributes of using arbitrage-free models, as well as the attributes associated with using the real-world measure. In order to perform the analysis in the current research, we have employed this *hybrid* approach. This decision has been made for several reasons. First, as already mentioned, arbitrage-free models produce scenarios which, on average, match the current term structure. More importantly, the goal of the current research is to analyze the exposure to counterparties of interest-rate swaps under different yield-curve environments, and also under different volatility assumptions. Hence, it is believed that it will be informative to compare results based on the same yield-curve, yet different volatilities, as well as results based on the same volatility but different yield-curve shapes.

4. Implementation of Interest Rate Models:

In general stochastic interest-rate models, and the Hull-White model and Black-Karasinski models in particular, are often implemented using a trinomial tree. As mentioned above, the purpose of the trinomial-tree structure is to facilitate solving the SDE by discretizing it. Also, the tree-structure allows the model to be fit to the initial term structure, which is the defining characteristic of arbitrage-free models. This is accomplished by strategically *distorting* the tree, as described below. First we recall the SDE that we wish to solve:

$$df(r) \; = [\theta(t) - \alpha(t)f(r)]dt \; + \; \sigma(t)dZ_t$$

Before building the actual tree for $f(r)$, we set the current time to 0, and define a deterministic function g, which satisfies:

$$dg = [\theta(t) - \alpha(t)g(t)]dt$$

Next, we define a new variable: $x(r,t) = f(t) - g(t)$, which now satisfies the simplified diffusion process: $dx = -\alpha(t)xdt + \sigma(t)dz$

The use of the function $g(t)$ is the key to constraining the solution of the SDE so that it matchs the initial term structure, or current yield curve, on an expected basis.

Next, $g(0)$ is chosen so that the initial value $x(r,0)$ is zero. This actually causes the expected value of $x(r,t)$ to be 0 at all future times, since $x(r,t)$ is mean reverting to 0.

Now, instead of building the tree for $f(r)$ directly, which is the object of interest, rather we build a tree for $x(r,t)$, the details of which follow:

In general tree-structures have several parameters; the spacing of the nodes WRT time, the spacing of the nodes WRT interest-rates, and the specific branching process. With regards to the node placement in the time dimension, nodes must be placed at all cash flow payment dates. Once this is accomplished, extra nodes can be added later to increase accuracy. Regarding the spacing of the notes in the interest-rate dimension, at a given time step t_i, nodes are placed at $\pm\Delta x_i, \pm 2\Delta x_i, \ldots, \pm m_i \Delta x_i$, where m_i are the indices of the highest and lowest nodes, in the interest-rate dimension, at time t_i.

A common level of Δx_i that is suggested is: $\Delta x_i = \sigma(t_{i-1})\sqrt{3(t_i - t_{i-1})}$

Based on trial and error, it has been anecdotally determined that the above level of Δx_i will allow enough spacing in the interest-rate dimension to represent the volatility of $x(r, t_i)$.

Next we specify the probabilities of moving from a given node at time step i to each of the 3 possible nodes at time $i+1$. If we are at node $j\Delta x_i$, at time i, then let:

$(k-1)\Delta x_{i+1}$, $k\Delta x_{i+1}$ and $(k+1)\Delta x_{i+1}$, be the three possible nodes to which $x(r, t_i)$ transitions at time $i+1$.

Then, from the form of the simplified diffusion process, denote the expected mean change in $x(r,t)$ over $(t_i,\ t_{i+1})$ by $E(dx) = M$, and denote the second moment of the mean change by $E(dx^2) = V + M^2$.

Then letting p_d, p_m and p_u be the probabilities of transitioning to $(k-1)\Delta x_{i+1}$, $k\Delta x_{i+1}$ and $(k+1)\Delta x_{i+1}$, respectively, and equating the mean, and variance of $x(r,t)$ over $(t_i,\ t_{i+1})$ with the above, we get:

$j\Delta x_i + M = k\Delta x_{i+1} + (p_u - p_d)\Delta x_{i+1}$, and:

$$V \ + \ (j\Delta x_i \ + \ M)^2 \ = \ k^2\Delta x_{i+1}^2 \ + \ 2k(p_u - p_d)\Delta x_{i+1}^2 \ + \ (p_u + p_d)\Delta x_{i+1}^2$$

Solving these for p_d, p_m and p_u, we get:

$$p_u \ = \ \frac{V}{2\Delta x_{i+1}^2} + \frac{\alpha^2 + \alpha}{2}$$

$$p_d \ = \ \frac{V}{2\Delta x_{i+1}^2} + \frac{\alpha^2 - \alpha}{2}$$

$$p_m \ = \ 1 \ - \ \frac{V}{\Delta x_{i+1}^2} \ - \ \alpha^2$$

Where $\alpha \ = \ \dfrac{j\Delta x_i \ + \ M - k\Delta x_{i+1}}{\Delta x_{i+1}}$ is the distance from the expected value of $x(r,t)$ to the central node at time $i+1$. Furthermore, it has been shown that, the above probabilities are assured to be positive if: $k \ = \ round\Big(\dfrac{j\Delta x_i \ + \ M}{\Delta x_{i+1}}\Big)$

The above determines the branches and transition probabilities for all nodes. Also the highest and lowest nodes at each future time step, $\pm m_i$ can be iteratively determined by starting from the node $m_0 \ = \ 0$, at time 0, and using the above formulas for k, and p_d, p_m, and p_u to determine $\pm m_1$, then $\pm m_2$, and so on.

At this point, the *base* tree for $x(r,t)$ has been constructed. The last step in the tree-construction process is to *adjust* the tree for $x(r,t)$ to arrive at the tree for $f(r)$.

From the differential equation for the function $g(t)$ (below), it can be seen that $g(t)$ is a function of $\theta(t)$, and recall $\theta(t)$ is used to adjust the model so that the solution matches the initial term-structure on an expected basis. Hence, it can be seen how using $g(t)$ to facilitate initial term-structure matching makes sense.

$$dg \ = \ [\theta(t) - \alpha(t)g(t)]dt$$

To arrive at the tree for $f(r)$, at each node of the trinomial tree for $x(r,t)$, the value of $g(t)$ will be added to all $x(r,t)$ values to arrive at the $f(r_t)$ values.

The main relation between $f(r), x(r,t)$, and $g(t)$ is: $f(r) = x(r,t) + g(t)$.

Now, letting r_{ij} denote the interest rate at node $(i\ ,j)$, using the above, we have:

$r_{ij} = f^{-1}(x_{ij} + g(t_i))$

The actual value of $g(t)$, for each t, is determined so that the *modeled* prices of discount bonds of all maturities are consistent with the corresponding prices based upon the initial term structure observed in the market. To accomplish this, the modeled price P_{i+1}, as measured from node $(0, 0)$, of a discount bond paying \$1 at every node at time $i + 1$, is equated to the price of the corresponding discount bond using the current term structure, $P^M(0, t_{i+1})$.

i.e. we set: $P_{i+1} \ = \ P^M(0, t_{i+1})$, and solve for $g(t_{i+1})$, where:

$$P_{i+1} \ = \ \sum_j Q_{ij} \exp\big[- f^{-1}(x_{ij} + g(t_i))(t_{i+1} - t_i)\big]$$

and:

$$P^M(0, t_{i+1}) \ = \ \exp\left[-R(0, t_{i+1})t_{i+1}\right].$$

where Q_{ij} are called the Arrow-Debreu prices by Hull & White, and represent the value at $(0, 0)$ of a security that pays \$1 at node (i, j) and zero otherwise, and $R(0, t_{i+1})$ is the market observed continuously compound yeild at time 0 on a zero-coupon bond with face value \$1 that matures at t_{i+1}.

For the Black-Karasinski model:

$f^{-1}(x_{ij} + g(t_i)) \ = \ \exp\left(x_{ij} + g(t_i)\right)$

and the equation that needs to be solved for $g(t_{i+1})$ becomes:

$$P_{i+1} \ = \ \sum_j Q_{ij} \exp\big[- \exp\left(x_{ij} + g(t_i)\right)(t_{i+1} - t_i)\big]$$

Unfortunately, unlike the Hull-White model, the above equation must be solved using numerical procedures, such as Newton-Raphson's method. Also, to further complicate things, in order to compute the $g(t_{i+1})$'s, the Q_{ij}'s need to be solved for first. However, an iterative procedure can once again be used. The fact that $Q_{00} = 1$ can be to compute $g(t_0)$, and then this can be used to find the Q_{ij}'s for all j, at $t = 1$, using the formula for P_{i+1}. Then, the Q_{ij}'s for all j, at $t = 1$, can be used to find $g(t_1)$, and so on.

This completes the construction of an interest-rate tree which allows the matching of the initial yield-curve.

5. Exposure analysis:

As stated above, the goal of the current research is to analyze the potential P&L exposure, as well as the potential counterparty exposure, of interest-rate swaps due to the realization of interest-rates which differ from the markets expectation of rates, upon inception of the swap contract. In this section, we perform this analysis under three different yield-curve shapes; upward, inverted, and humped, and also under different volatility assumptions. For each of the six combinations of initial yield-curve shape and volatility, we use the Black-Karasinski model to generate 10,000 forward yield-curves scenarios. Then, as described below, we use these 10,000 simulated forward yield-curves to calculate the trajectories of various statistics, based on the Net Present value of remaining coupons payments.

Before discussing the details of the calculation of these statistics, we describe the implementation of the Black-Karasinski model used to generate the forward yield-curve scenarios. Recalling the SDE which governs the movement of the short-rate under the Black-Karasinski model, there are 3 parameters which must be estimated in order to implement the Black-Karasinski model; α, σ, and $\theta(t)$:

$$d\ln(r) \;=\; [\theta(t) - \alpha\ln(r)]dt \;+\; \sigma dZ_t$$

We estimate the mean reversion parameter, α, from historical data on interest rates. Since we are modelling the movement of 1-year rates, we use yearly movements of this rate, starting from the end of every month in 1990 up to 2011. The first-order auto-correlation of $ln(r_t)$ is used as our estimate of $1-\alpha$. To understand the reasoning behind this, we consider the discrete time version of Black-Karasinski model equation:

$$\ln(r_{t+1}) - \ln(r_t) \;=\; \theta(t) - \alpha\ln(r_t) \;+\; \epsilon_{t+1}$$

Or:

$$\ln(r_{t+1}) \;=\; (1-\alpha)\ln(r_t) \;+\; \theta(t) \;+\; \epsilon_{t+1}$$

Where ϵ_t is drawn from a normal distribution.

An ordinary least-squares estimate of the coefficient $(1-\alpha)$, in the above equation, represents the correlation between $ln(r_t)$ and $ln(r_{t+1})$, or the so called first-order auto-correlation of the time-series $ln(r_t)$. Below is a table listing the first-order auto-correlation of $ln(r_t)$, or rho, for month ends from 1990-2010.

Year/Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
rho = 1-a	0.78	0.79	0.78	0.81	0.80	0.80	0.81	0.82	0.77	0.78	0.78	0.78
a	0.22	0.21	0.22	0.19	0.20	0.20	0.19	0.18	0.23	0.22	0.22	0.22

Depending on the starting month, the historical yearly estimate of α, over the subsequent 20 year period, ranges between 0.18 to 0.23. In the following analysis, we have chosen to use $\alpha = 0.22$, throughout.

For the volatility parameter, σ, we use three values $0.2, 0.35$, and 0.5, under each initial yield-curve. In most modeling situations, one would choose the value of σ to match observed swaption prices in the market, or alternatively, use an estimate based on historical data. However, we wish to make the analysis under each initial yield-curve shape comparable, and using the market implied volatility, by matching observed swaption prices, would likely lead to different values of σ under each of the different yield-curve shapes, and hence cloud the interpretation of the results.

Regarding the $\theta(t)$ parameter, we calibrate the Black Karasinski model such that expected prices of zero swaps match the zero swap prices observed at time 0, and as mentioned above, we consider three different yield curve shapes, which are most commonly observed.

The following simulations and calculations correspond to a 30-year receiver swap, with notional value of \$10,000,000. Further, the NPV calculation at each time, t, considers only the remaining coupon payments. More specifically, the NPV at time t is defined as:

$$\mathrm{NPV}_t = \sum_{i>t} \frac{C}{(1+f_{t,i})^{i-t}} - \sum_{i>t} \frac{F_i}{(1+f_{t,i})^{i-t}}$$

where C is the fixed coupon payment, and F_i is the floating coupon payment, and $f_{t,i}$ is the forward-rate over the period (t, i).

Using this definition of NPV_t, for each level of sigma, and each initial yield-curve shape, we calculate the trajectory of the following statistics, over the term of the each swap. In the table below, we use the notation $\mathrm{NPV}_t^{(i)}$ to denote the value of the NPV_t on the i^{th} out of $10,000$ simulations.

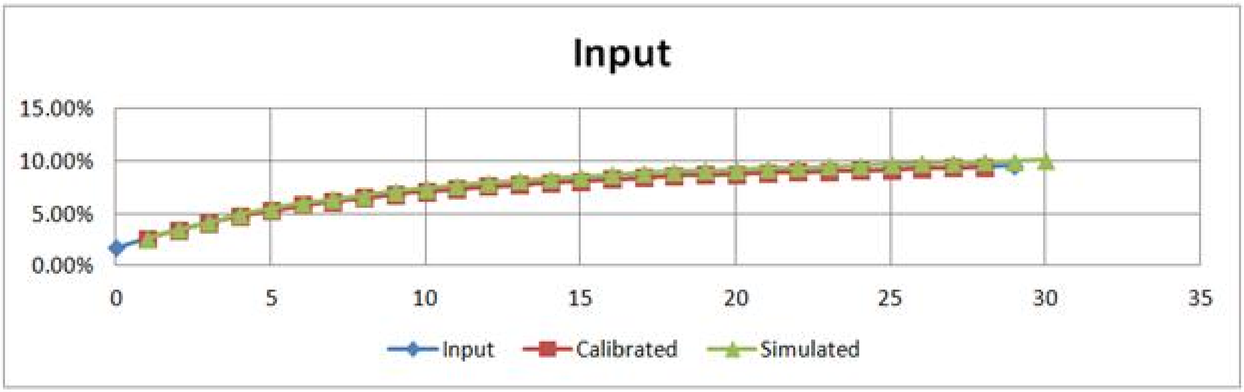
Abbreviation	Name of Statistic	Value of statistic at time $=t$
ENPV	Expected Net Present Value	Based on forward curve at time zero
DENPV	Derived Expected Net Present Value	$\mathrm{Average}(\mathrm{NPV}_t^{(i)})$ over all simulations i.
ELE	Expected Loss Exposure	$\mathrm{Average}(\mathrm{NPV}_t^{(i)})$ over i s.t. $\mathrm{NPV}_t^{(i)} < 0$
PLE	Potential Loss Exposure	$0.5th$ %ile of ordered $\mathrm{NPV}_t^{(i)}$'s, over all i
EGE	Expected Gain Exposure	$\mathrm{Average}(\mathrm{NPV}_t^{(i)})$ over i s.t. $\mathrm{NPV}_t^{(i)} > 0$
PGE	Potential Gain Exposure	$99.5th$ %ile of ordered $\mathrm{NPV}_t^{(i)}$'s, over all i ($\mathrm{VaR}(99.5)$ of $\mathrm{NPV}_t^{(i)}$ values, over i)

Displayed below are three sets of graphs. Each set of graphs corresponds to one of the three common initial yield-curve shapes; upward-sloping, inverted, or humped-shaped. Each set of graphs consist of:

1. One graph of the initial yield-curve.
2. Three graphs of 10,000 simulated forward yield-curves, one for each assumed value of σ.
3. Three graphs of the trajectories of the aforementioned statistics, one for each assumed value of σ.

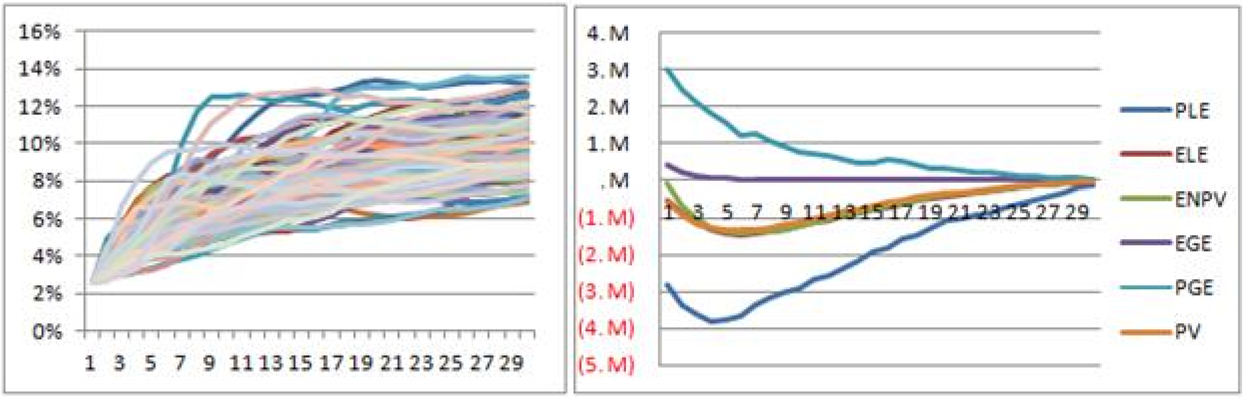
In all graphs, the x-axis denotes time while the y-axis denotes interest rate levels, or the dollar-value NPV of the swap, depending on the graph. The graphs on the left are the forward yield-curves generated by the Black-Karasinski model, based on the corresponding initial yeild curve. Hence, the y-axis is the level of rates, in the graphs on the left. The graphs on the right depict the trajectory of the statistics based on NPV_t, and hence the y-axis is in 1 million dollar units. Note, the plotted NPV's are forward NPV's on the swap, and are not in today's dollar.

For the *upward-sloping* yield-curve case, we used the following initial yield-curve:

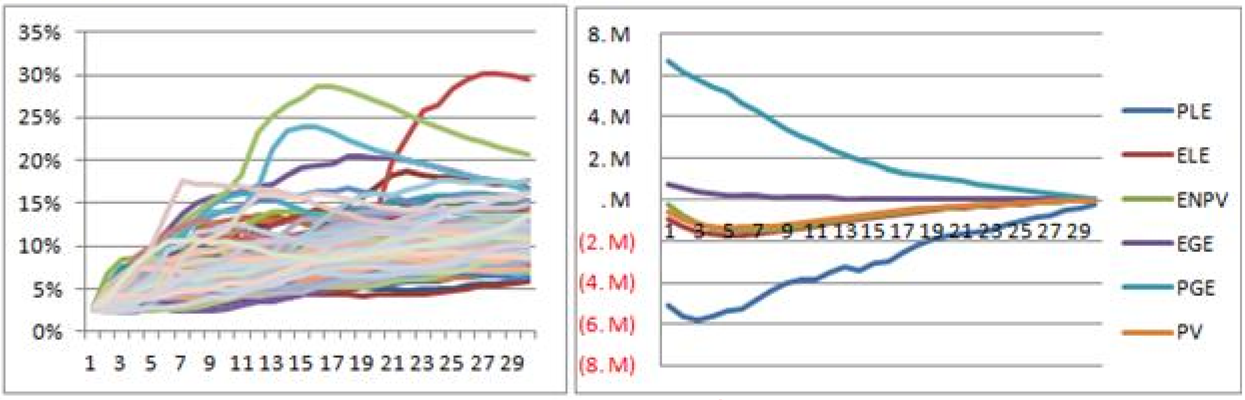


which resulted in the following simulations:

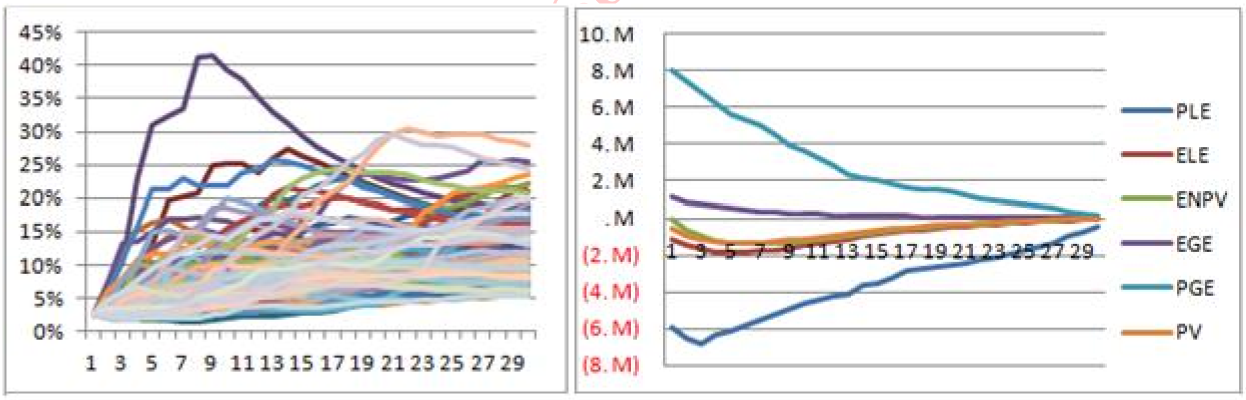
Sigma = 0.2



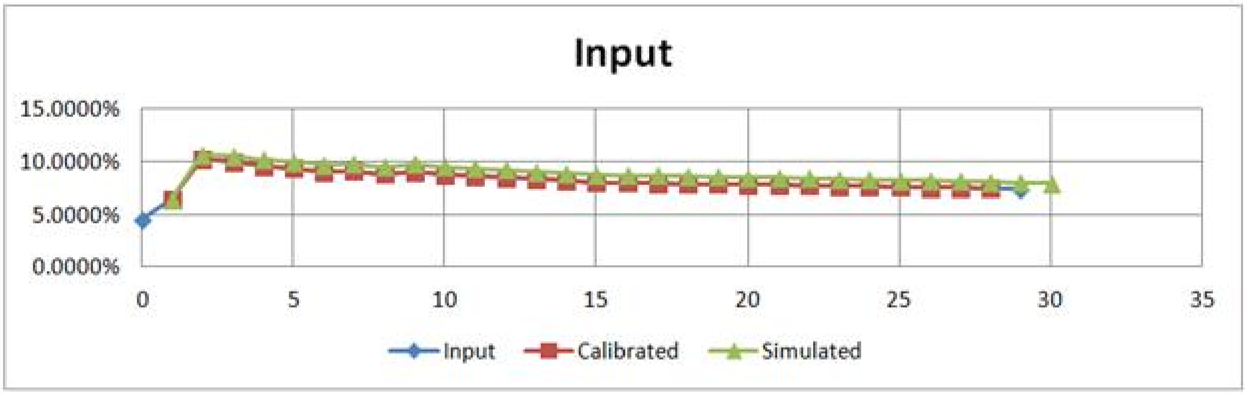
Sigma = 0.35



Sigma =0.50

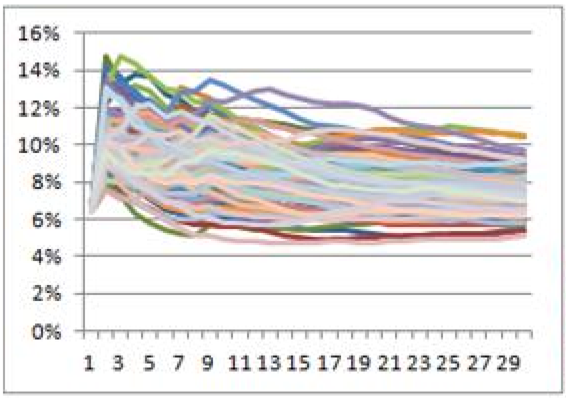


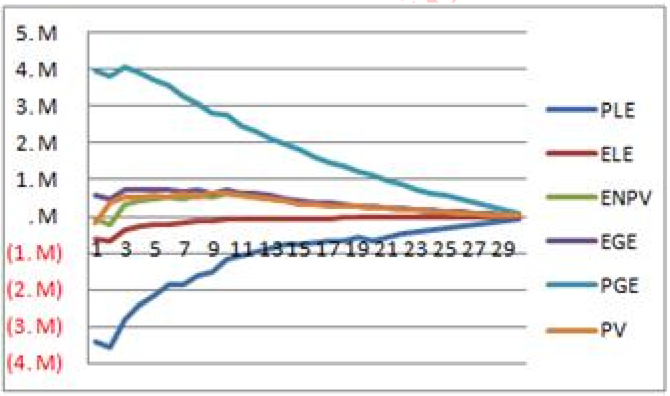
For the *hump-shaped* curve case, we used the following for of the initial yield-curve:



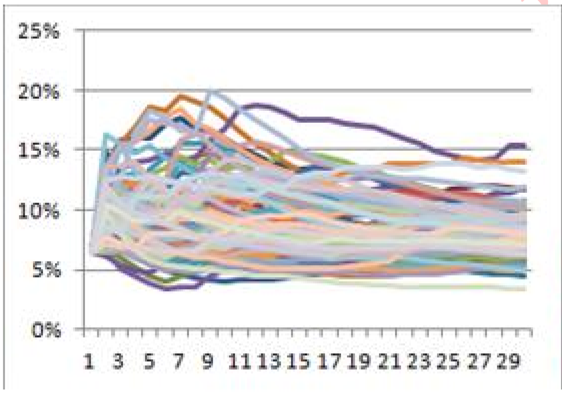
which resulted in the following simulations:

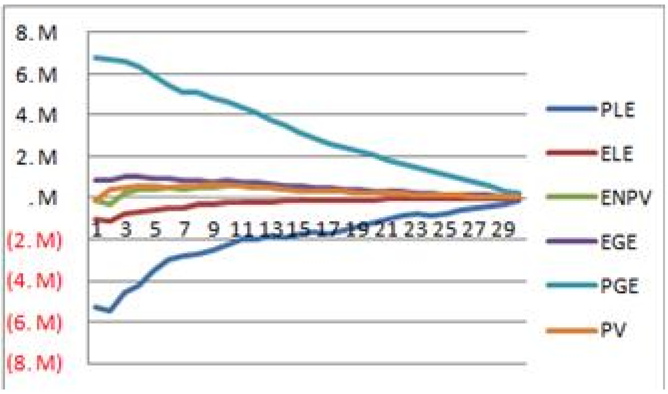
Sigma = 0.20



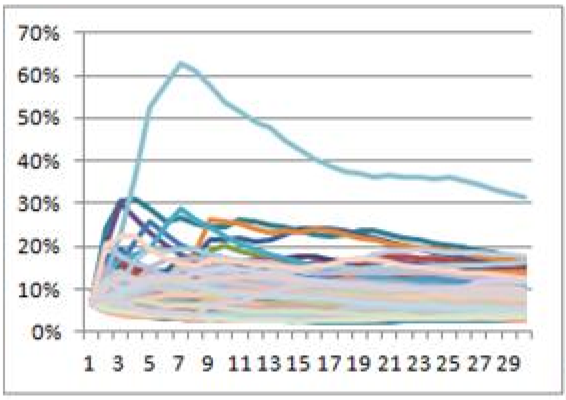


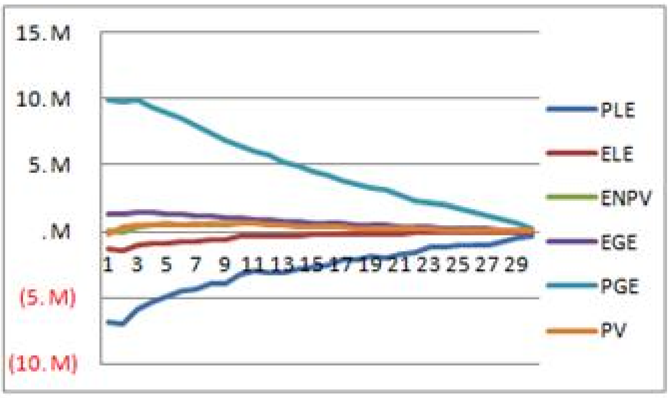
Sigma = 0.35



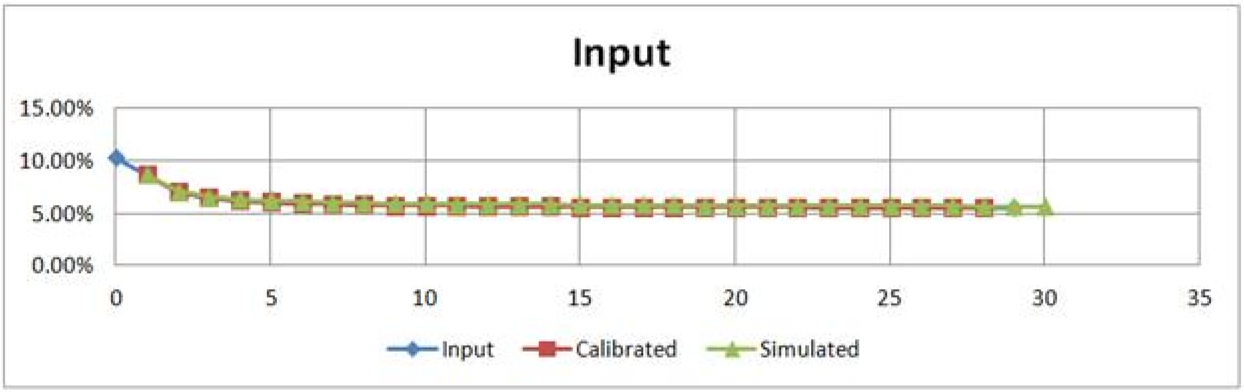


Sigma = 0.50



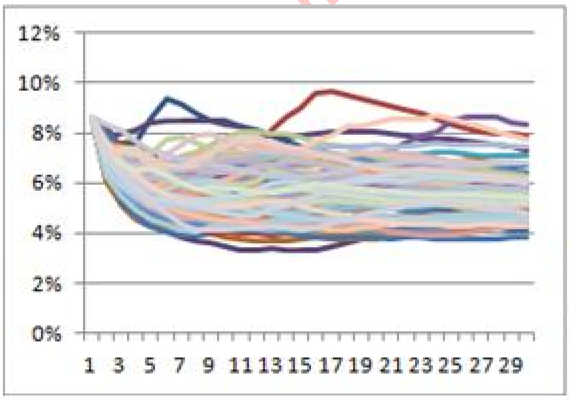


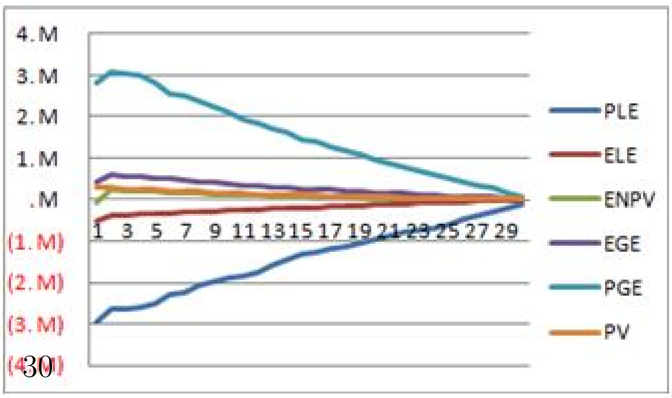
And for the *inverted* curve case, we used the following for of the initial yield-curve:



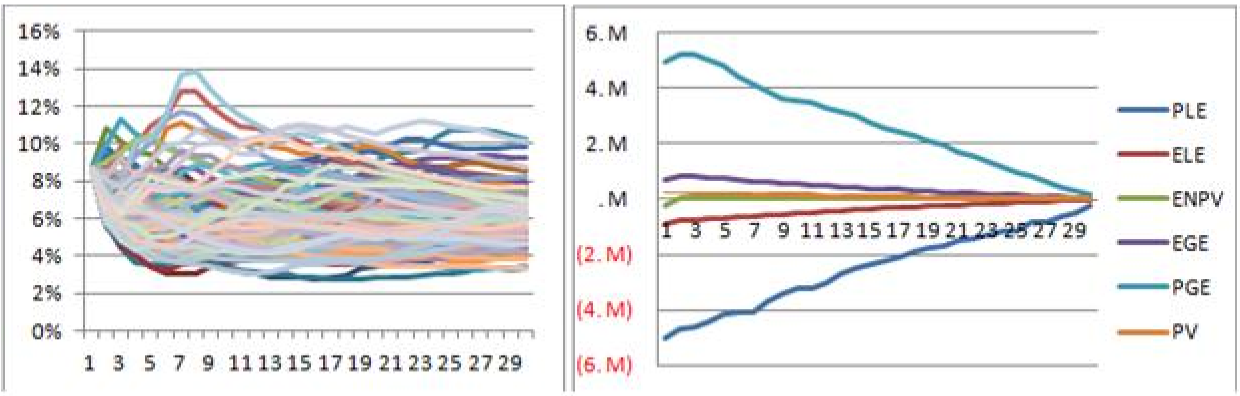
which resulted in the following simulations:

Sigma = 0.20

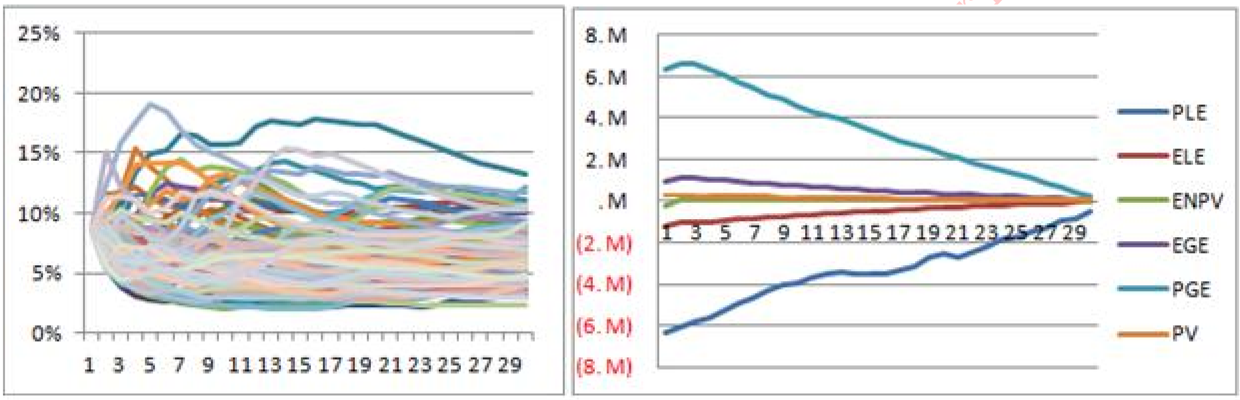




Sigma = 0.35



Sigma = 0.50



Positive NPV on the swap will give rise to counterparty exposure, while negative NPV will be observed as a mark-to-market (MTM) loss on the swap. ENPV is the NPV of the swap based on the forward curve at time zero. DENPV, on the other hand, is the expected NPV of the swap based on the simulated yield curves. The ENPV and DENPV curves should follow very closely, and any observed difference in the graphs below is due to limitation in the number of simulations. Both expected NPV trajectories, start off at zero and then after an initial pull to positive or negative territory, fall toward zero. Depending on the shape of the curve, there is a point where both curves reach their maximum absolute value and then start pulling back to zero. However, one must be careful in interpreting the ENPV curve. The observed pattern of the ENPV, discussed above, is due to the mismatch between the fixed payments and floating payments of the swap, and does not represent a realized loss or gain for either counterparty, rather, it represents the difference in the mark-to-market values of the cash coupons exchanged *up to that point in time*. The difference between Potential Loss Exposure and the ENPV, on the other hand, represents a clear potential for a loss in MTM value of the swap, due to adverse movements in the yield-curve. From the graphs, one can see that a higher value of σ gives rise to higher volatility in rates, as expected, and hence also a higher potential loss in mark-to-market value of the swap.

The difference between Potential Gain Exposure and the ENPV represents the potential gain the receiving counterparty can realize as a net profit. It is essential to understand that any NPV value above the expected NPV can be interpreted as a gain for the receiving counterparty, due to changes in the yield-curve, and its subsequent impact on the mark-to-market of the swap. However, we will only consider positive NPV where there will be a credit exposure to the counterparty. As is the case with PLE, higher values of Sigma result in higher PGE. Expected Loss Exposure and Expected Gain Exposure are defined as the expected NPV, but conditioned on negativity, or positivity, of NPV, respectively. The difference between Expected Loss Exposure and Expected NPV can be viewed as the MTM loss one would expect under the simulated rate scenarios, given there will be a loss. Expected Gain Exposure, on the other hand, can be viewed as the expected counterparty exposure the receiving party will see on the swap, given there is a counterparty exposure.

By comparing the results from the hump-shaped, inverted, and upward-sloping yield-curves, one can study the impact of the yield-curve shape on these measures. ENPV falls below zero, and then slowly increases toward 0, in an upward-sloping curve, while under the special case of a hump-shaped yield curve the ENPV initially moves above zero, and then slowly falls toward 0. Under an inverted yield-curve the ENPV stays close to zero. As can be observed from the results, the PLE indicates lower potential NPV values under the upward-sloping curve, compared to under the humped-shaped or inverted yield-curves. Also, the PLE generally rebounds more slowly toward 0 under the upward-sloping yield-curve. At the same time, PGE indicates higher potential NPV values under the humped-shaped yield-curve, compared to the NPV under the upward-sloping or inverted yield-curves. In all three interest rate environments, PLE and PGE follow similar trends, exposing the receiving party to a potentially large MTM loss, or a potentially large counterparty exposure early into the contract.

6. Conclusion:

In this paper we have presented several historically notable interest rate swap deals which have been reported to have produced significant losses as a result of large, unanticipated, interest rate movements, and the resulting liquidity and capital constraints. In order to provide the reader with the background necessary to make an informed investigation of such cases, we have reviewed the basic structure of interest rate swaps, and the mathematical foundation of the valuation of such vanilla interest rate swaps. We have provided a comprehensive treatment of the Hull-White and Black-Karasinski interest rate models, including their theoretical and mathematical details, as well as their implementation through the use of tree-structures. It is hoped that the depth of this coverage is sufficient to provide practitioners with the background necessary to implement these models, for their own purposes. We have explained that the above interest rate models belong to the class of arbitrage-free interest rate models, and have included an in-depth discussion of modeling under risk-neutral verses real-world measures, or probabilities. An effort has been made to clarify the subtleties between, and limitations of, these two approaches, especially as they pertain to models which are the solution to stochastic differential equations, which is often absent in coverage of this material in the literature. We discuss that while risk-neutral models are generally used for pricing and less often for risk management purposes, they do have several benefits, including the ability to provide an objective framework from which exposures can be analyzed. We do acknowledge that, when performing such exposure analysis, many in industry rely on real-world measures, or as we point out, approximations to the real-world measure based on historical data. We note that implicit in the use of such techniques is an element of subjective view on the market, and the assumption that history will repeat itself, which may often be desirable characteristics. In this paper we use a hybrid approach to model the potential exposure generated from interest rate swaps. We employ the Black-Karasinski interest rate model, which is the solution to a stochastic differential equation with respect to a risk-neutral measure, yet for which some of the parameters of the model are calibrated using historical data. We argue that such a model can also be used for exposure analysis, with the caveat that one should never rely on a single model, and in particular, when performing capital modeling or risk management, models calibrated using various subsets of historical data should be among those employed. Hence, the techniques presented in this paper should be viewed as one of many potential modeling techniques in the analyst's toolkit.

Finally, we analyze the potential exposure of vanilla interest rate swaps under the most common yield curve shapes, and also under several different volatility assumptions, in order to provide a comparison of the potential interest rate exposure that may be generated over the term of a contract. To perform this analysis many forward yield-curves are simulated using an arbitrage-free interest rate model, which is commonplace in industry, and which produces simulations which are consistent with the initial yield-curve, on an expected basis. The analysis highlights the significant marked-to-market(MTM) or counterparty exposure that could arise due to changing interest rates, and illustrates the importance of measuring this potential exposure, which as illustrated, may result in significant MTM losses and necessitate substantial collateral posting. In fact, the amount of collateral posting may be so significant as to force an end-user to unwind the contract, in which case the MTM loss will be realized. Also, we highlight the potential for significant MTM gains, which on the other hand, will create a counterparty exposure if the swap is not collateralized.

It is our hope that this paper provides the reader with an understanding of the details, some of which are potentially subtle, surrounding the analysis of these seemingly benign interest rate derivatives, and will aid the practitioner in managing the risks of these very beneficial and ubiquitous financial products.

7. appendix:

6.A Iterated Expected Values:

Note that $E_0^Q\Big(E_t^Q(X)\Big) \;\cong\; E^Q\Big(E^Q(X \mid \mathfrak{F}_t) \;\Big|\; \mathfrak{F}_0\Big) \;=\; E^Q(X \mid \mathfrak{F}_0) \;=\; E_0^Q(X)$,

and where $\mathfrak{F}_t$ is a concept from mathematical probability called a *Filtration*. A filtration can be thought of as the amount of *information* available at time t regarding the evolution of the process up-to, and including, time t. Hence, the expected value conditional on the filtration at time t represents expected values looking forward from time t, and based on the information available at time t.

6.B Solution to Black-Karasinski SDE:

In order to solve the SDE: $\quad d\ln(r_t) \;=\; [\theta(t) - \alpha(t)\ln(r_t)]dt \;+\; \sigma(t)dZ_t$

let:

$J(t) = \displaystyle\int_0^t \alpha(\tau)d\tau$, and $Y_t = \ln(r_t)$,

then:

$$d(e^{J(t)}Y_t) \;=\; e^{J(t)}J'(t)Y_t dt \;+\; e^{J(t)}dY_t \;=\; e^{J(t)}\alpha(t)Y_t dt \;+\; e^{J(t)}\Big([\theta(t) - a(t)Y_t]dt \;+\; \sigma(t)dZ_t\Big) \;=$$

$$=\; e^{J(t)}\Big(\theta(t)dt \;+\; \sigma(t)dZ_t\Big).$$

Then integrating both sides from s to t, where $s < t$:

$$\int_s^t d(e^{J(\tau)})Y_\tau d\tau \;=\; \int_s^t e^{J(\tau)}\theta(\tau)d\tau \;+\; \int_s^t \sigma(\tau)e^{J(\tau)}dZ_\tau \;.$$

$$\Rightarrow\quad e^{J(t)}Y_t - e^{J(s)}Y_s \;=\; \int_s^t e^{J(\tau)}\theta(\tau)d\tau \;+\; \int_s^t \sigma(\tau)e^{J(\tau)}dZ_\tau \;.$$

$$\Rightarrow\quad Y_t \;=\; e^{(J(s)-J(t))}Y_s + \int_s^t e^{J(\tau)-J(t)}\theta(\tau)d\tau \;+\; \int_s^t \sigma(\tau)e^{J(\tau)-J(t)}dZ_\tau \;.$$

$$\rightarrow\quad Y_t \;=\; e^{-(J(t)-J(s))}Y_s \;+\; \int_s^t e^{-(J(t)-J(\tau))}\theta(\tau)d\tau \;+\; \int_s^t \sigma(\tau)e^{-(J(t)-J(\tau))}dZ_\tau.$$

Hence:

$$\ln(r_t) \;=\; e^{-(J(t)-J(s))}\ln(r_s) \;+\; \int_s^t e^{-(J(t)-J(\tau))}\theta(\tau)d\tau \;+\; \int_s^t \sigma(\tau)e^{-(J(t)-J(\tau))}dZ_\tau.$$

$$\rightarrow \;\; r_t \;=\; \exp\Big[\ln(r_s)e^{-(J(t)-J(s))} \;+\; \int_s^t e^{-(J(t)-J(\tau))}\theta(\tau)d\tau \;+\; \int_s^t \sigma(\tau)e^{-(J(t)-J(\tau))}dZ_\tau\Big]$$

for $t > s$.

6.C Closed form solution for Hull-White model:

In the constant coefficient version of the Hull-White SDE the short-rate which solves the SDE has form:

$$r_t \;=\; r_s e^{-\alpha\cdot(t-s)} \;+\; \int_s^t e^{-\alpha\cdot(t-\tau)}\theta(\tau)d\tau \;+\; \sigma\int_s^t e^{-\alpha\cdot(t-\tau)}dZ_\tau$$

for $t > s$. In particular, for $s = 0$, we have:

$$r_t \;=\; r_0 e^{-\alpha t} \;+\; \int_0^t e^{-\alpha\cdot(t-\tau)}\theta(\tau)d\tau \;+\; \sigma\int_0^t e^{-\alpha\cdot(t-\tau)}dZ_\tau$$

Since, the price at time 0 of a zero-coupon bond with maturity T is:

$$P(0,T) \;=\; E^Q\Big(\exp\Big[-\int_0^T r_y\, dy\Big]\Big) \;=\; E^Q\Big(e^{-X(T)}\Big),$$

we investigate: $X(T) = \int_0^T r_y\, dy$.

$$X(t) \;=\; \int_0^t r_y\, dy \;=\; \int_0^t \Big[r_0 e^{-\alpha y} \;+\; \int_0^y e^{-\alpha\cdot(y-\tau)}\theta(\tau)d\tau \;+\; \sigma\int_0^y e^{-\alpha\cdot(y-\tau)}dZ_\tau\Big]dy \;=$$

$$=\; \int_0^t r_0 e^{-\alpha y}dy \;+\; \int_0^t\int_0^y e^{-\alpha\cdot(y-\tau)}\theta(\tau)d\tau dy \;+\; \sigma\int_0^t\int_0^y e^{-\alpha\cdot(y-\tau)}dZ_\tau dy.$$

Now, integrating the first term, and reversing the order of integration in the next two terms, we have:

$$X(t) \;=\; \frac{r_0}{\alpha}(1 - e^{-\alpha t}) \;+\; \int_{\tau=0}^{\tau=t}\int_{y=\tau}^{y=t} e^{-\alpha\cdot(y-\tau)}\theta(\tau)dyd\tau \;+\; \sigma\int_{\tau=0}^{\tau=t}\int_{y=\tau}^{y=t} e^{-\alpha\cdot(y-\tau)}dydZ_\tau.$$

Then, noting that:

$$\int_{\tau=0}^{\tau=t}\int_{y=\tau}^{y=t} e^{-\alpha\cdot(y-\tau)}\theta(\tau)dyd\tau \;=\; \int_{\tau=0}^{\tau=t} e^{\alpha\tau}\theta(\tau)\Big(\int_{y=\tau}^{y=t} e^{-\alpha y}dy\Big)d\tau \;=\; \int_{\tau=0}^{\tau=t} e^{\alpha\tau}\theta(\tau)\Big(\frac{e^{-\alpha\tau}-e^{-\alpha t}}{\alpha}\Big)d\tau \;=$$

$$=\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t} e^{\alpha\tau}\theta(\tau)\Big(e^{-\alpha\tau}-e^{-\alpha t}\Big)d\tau \;=\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t}\theta(\tau)\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)d\tau.$$

and:

$$\int_{\tau=0}^{\tau=t}\int_{y=\tau}^{y=t} e^{-\alpha\cdot(y-\tau)}dydZ_\tau \;=\; \int_{\tau=0}^{\tau=t} e^{\alpha\tau}\Big(\int_{y=\tau}^{y=t} e^{-\alpha y}dy\Big)dZ_\tau \;=\; \int_{\tau=0}^{\tau=t} e^{\alpha\tau}\Big(\frac{e^{-\alpha\tau}-e^{-\alpha t}}{\alpha}\Big)dZ_\tau \;=$$

$$=\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t} e^{\alpha\tau}\Big(e^{-\alpha\tau}-e^{-\alpha t}\Big)dZ_\tau \;=\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t}\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)dZ_\tau.$$

Therefore:

$$X(t) \;=\; \frac{r_0}{\alpha}(1 - e^{-\alpha t}) \;+\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t}\theta(\tau)\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)d\tau \;+\; \frac{\sigma}{\alpha}\int_{\tau=0}^{\tau=t}\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)dZ_\tau.$$

Note that the only randomness in the formula for $X(t)$ is in the last term, since Z_τ is a Brownian-motion. Therefore, $X(t)$ is a Gaussian process, and we can figure out its mean and variance.

The integral in the last term in the equation for $X(t)$ can be written:

$$\int_{\tau=0}^{\tau=t} f(\tau)dZ_\tau, \text{ where } f(\tau) \;=\; 1-e^{-\alpha\cdot(t-\tau)} \in L^2([0,t])$$

Hence, $\int_{\tau=0}^{\tau=t}\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)dZ_\tau$ is a *Weiner integral*, and so we have:

$$\int_{\tau=0}^{\tau=t}\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)dZ_\tau \;\sim\; N\Big(0,\; \int_0^t f(\tau)^2d\tau\Big).$$

Therefore:

$$E\big(X(t)\big) \;=\; \frac{r_0}{\alpha}(1-e^{-\alpha t}) \;+\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t}\theta(\tau)\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)d\tau \;+\; \frac{\sigma}{\alpha}E\Big[\int_{\tau=0}^{\tau=t}\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)dZ_\tau\Big] \;=$$

$$=\; \frac{r_0}{\alpha}(1-e^{-\alpha t}) \;+\; \frac{1}{\alpha}\int_{\tau=0}^{\tau=t}\theta(\tau)\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)d\tau.$$

and:

$$Var\big(X(t)\big) \;=\; \frac{\sigma^2}{\alpha^2}Var\Big[\int_{\tau=0}^{\tau=t}\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)dZ_\tau\Big] \;=\; \frac{\sigma^2}{\alpha^2}\int_0^t\Big(1-e^{-\alpha\cdot(t-\tau)}\Big)^2 d\tau \;=$$

$$=\; \frac{\sigma^2}{\alpha^2}t \;+\; \frac{\sigma^2}{2\alpha^3}\Big(4e^{-\alpha t}-e^{-2\alpha t}-3\Big).$$

Now we recalling that $P(0,t) = E^Q\Big(\exp\Big[-X(t)\Big]\Big)$, we can use the fact that $X(t)$ is a Gaussian process together with the mean and variance above, to find an expression for $P(0,t)$. Recall, for a Gaussian random variable X: $M_X(t) = E\big(e^{Xt}\big) = e^{\mu t+\frac{1}{2}\sigma^2t^2}$, so:

$$P(0,t) = E^Q\Big(e^{-X(t)}\Big) = M_{X(t)}(-1) = \exp\Big[-E\big(X(t)\big)+\frac{1}{2}Var\big(X(t)\big)\Big].$$

Therefore:

$$\ln\big(P(0,t)\big) = -E\big(X(t)\big)+\frac{1}{2}Var\big(X(t)\big).$$

Also:

$$f(0,t) = -\frac{d}{dt}\ln\big(P(0,t)\big) = -\frac{d}{dt}\Big[-E\big(X(t)\big)+\frac{1}{2}Var\big(X(t)\big)\Big] = \frac{d}{dt}E\big(X(t)\big)-\frac{1}{2}\frac{d}{dt}Var\big(X(t)\big).$$

where $f(0,t)$ is the instantaneous forward rate, and so:

$$\frac{d}{dt}f(0,t) = -\frac{d^2}{dt^2}\ln\big(P(0,t)\big) = -\frac{d^2}{dt^2}\Big[-E\big(X(t)\big)+\frac{1}{2}Var\big(X(t)\big)\Big] = \frac{d^2}{dt^2}E\big(X(t)\big)-\frac{1}{2}\frac{d^2}{dt^2}Var\big(X(t)\big).$$

and from the above equations for $E\big(X(t)\big)$ and $Var\big(X(t)\big)$:

$$\alpha\frac{d}{dt}E\big(X(t)\big) = r_0\alpha e^{-\alpha t} \;+\; \alpha\int_0^t\theta(\tau)e^{-\alpha\cdot(t-\tau)}d\tau,$$

and:

$$\frac{d^2}{dt^2}E\big(X(t)\big) = -r_0\alpha e^{-\alpha t} \;-\; \alpha\int_0^t \theta(\tau)e^{-\alpha\cdot(t-\tau)}d\tau \;+\; \theta(t).$$

Therefore: $\alpha\frac{d}{dt}E\big(X(t)\big) \;+\; \frac{d^2}{dt^2}E\big(X(t)\big) \;=\; \theta(t).$

Also:

$$-\frac{1}{2}\alpha\frac{d}{dt}Var\big(X(t)\big) = -\frac{\sigma^2}{2\alpha} \;+\; \frac{\sigma^2}{\alpha}e^{-\alpha t} \;-\; \frac{\sigma^2}{2\alpha}e^{-2\alpha t},$$

and:

$$-\frac{1}{2}\frac{d^2}{dt^2}Var\big(X(t)\big) = \frac{\sigma^2}{\alpha}e^{-2\alpha t} \;-\; \frac{\sigma^2}{\alpha}e^{-\alpha t}.$$

Therefore: $-\frac{1}{2}\alpha\frac{d}{dt}Var\big(X(t)\big) \;-\; \frac{1}{2}\frac{d^2}{dt^2}Var\big(X(t)\big) \;=\; -\frac{\sigma^2}{2\alpha}\Big(1-e^{-2\alpha t}\Big).$

So, all together:

$$\begin{aligned}\alpha f(0,t)+\frac{d}{dt}f(0,t) &= \left(\alpha\frac{d}{dt}E\big(X(t)\big)-\alpha\frac{1}{2}\frac{d}{dt}Var\big(X(t)\big)\right)+\left(\frac{d^2}{dt^2}E\big(X(t)\big)-\frac{1}{2}\frac{d^2}{dt^2}Var\big(X(t)\big)\right) = \\ &= \left(\alpha\frac{d}{dt}E\big(X(t)\big)+\frac{d^2}{dt^2}E\big(X(t)\big)\right)-\left(\alpha\frac{1}{2}\frac{d}{dt}Var\big(X(t)\big)+\frac{1}{2}\frac{d^2}{dt^2}Var\big(X(t)\big)\right) = \\ &= \theta(t)-\frac{\sigma^2}{2\alpha}\Big(1-e^{-2\alpha t}\Big).\end{aligned}$$

Therefore: $\theta(t) \;=\; \alpha f(0,t) \;+\; \frac{d}{dt}f(0,t) \;+\; \frac{\sigma^2}{2\alpha}\Big(1-e^{-2\alpha t}\Big).$

REFERENCES:

P. Billingsley, (1995). *Probability and measure*, Third edition. Wiley Series in Probability and Mathematical Statistics.

Black, F., and P. Karasinski (1991). *Bond option pricing when the short rates are lognormal*, Finan. Analysts J., 47(4), 52-59.

Cox, J. C., J.E. Ingersoll, and S. Ross, (1981). *The relation between forward prices and futures prices.* J. Financial Economics 9, 321-346.

Damiano Brigo and Fabio Mercurio, (2006). *Interest Rate models: Theory and Practice.* Springer.

Duffie, D. and H. Zhu, (2009): *Does a central clearing counterparty reduce counterparty risk?*, Working Paper, no 46, Rock Center for Corporate Governance.

Harvard University Financial Report, Fiscal year 2009.
http://vpf-web.harvard.edu/annualfinancial

John C. Hull, (2011). *Options, Futures, and Other Derivatives.* Prentice Hall; 8th edition.

John Hull and Alan White, (2000). *The General Hull-White Model and Super Calibration.* University of Toronto, working paper.

K. Ito and H.P. McKean, Jr., (1974). *Diffusion processes and their sample paths.* Springer-Verlag; 2nd edition.

Karatzas and S. E. Shreve, (1988). *Brownian motion and stochastic calculus.* Springer Verlag.

Aisha Khan, Eric Guan and Ser-Huang Poon. *Short Rate Models: Hull-White or Black-Karasinski.* Manchester Business School Working Paper No. 562 September 2008.

John Lauerman and Michael McDonald. *Harvard's Bet on Interest Rate Rise Cost $500 Million* *to Exit.* Bloomberg.com, October 17, 2009.

Lieppold, M., and Z. Wiener, (1999). *Algorithms behind Term Structure Models of Interest Rates I. Valuation and Hedging of Interest Rates Derivatives with the Ho-Lee Model*, Working Paper, The Hebrew University of Jerusalem.

Massimiliano De Santis, (2011). *Demystifying Financial Derivatives: Interest Rate Swaps and Municipal Derivatives.* NERA Economic consulting .

Oakland City Memo: Scott Johnson to City Council, Questions on Goldman Sachs Rate Swap, March 02, 2012.

Pirrong, C. (2009): *The economics of clearing in derivatives markets: netting, asymmetric information, and the sharing of default risks through a central counterparty.* http://ssrn.com/abstract=1340660

Halah Touryalai. *City Of Oakland Taps Occupy Wall Street To Take On Goldman Sachs.* Forbes online magazine, July 11, 2012.

Mayra Rodriguez Valladares. *Futures and Futurization: Full Steam Ahead on Dodd-Frank.* MRV Associates. November 19, 2012.