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PROBABILISTIC CONCEPTS IN MEASUREMENT OF ASSET ADEQUACY

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ABSTRACT

The probabilistic concepts underlying measurement of asset adequacy are discussed in a simplified way. The basic theory involves complex probability distributions of C-1, C-2, and C-3 risks and their combination, with degree of adequacy expressed as a level of probability of ultimate ruin. Practical surrogates for such distributions are universes of deterministic cash flow scenarios of reasonable and plausible deviations from expected with bounding worst scenarios. These are related to ruin probabilities by stochastic generation of scenarios or by heuristic reasoning reflecting levels of comfort or intuitive feelings. The intent is to show the relationship of theoretical and practical procedures that differ only in degree of precision. Certain significant problems that arise in the absence of determination of risk surplus are discussed.

OBJECTIVES

This paper presents the probabilistic concepts basic to an understanding of the practical procedures in the valuation actuary effort. The emerging thrust of this effort is the measurement of the adequacy of total assets against the future contractual obligations and expenses on in-force business and the financial plans for new business, growth, and change. This is an ambitious undertaking for the actuarial profession and one that has not yet been discussed overall in the *Transactions*, although hundreds of pages on specific practical details have been published in the *Record*, Committee reports, seminar reports, the *Valuation Actuary Handbook* [6], and the *Transactions*. The author hopes that this paper, which deals only with some fundamental concepts, will stimulate publication of other comprehensive papers in the *Transactions* on the valuation actuary effort in all its ramifications.

At the outset, it is important to state what this paper is not. It is not a paper on mathematics, although it incorporates the language of mathematics. Indeed, the mathematics is generally familiar to most practicing actuaries and academically is almost trivial. Rigorous derivations of the mathematical statements can be found elsewhere [1], [2], and [5] and will not be presented here. Nor is this a paper surveying the practical procedures available to the practicing actuary. These procedures, to the extent already discovered, and the terms and definitions also can be found elsewhere [3], [4], [6], [7], and [8]. The paper assumes some familiarity with these lengthy references and makes no effort to reproduce them.

Rather, the paper is intended to fill a void in the literature. The literature has developed in piecemeal fashion, both theoretically and practically, without a cohesive theoretical structure. An intense discussion has arisen within the actuarial profession concerning not only the process of measuring the adequacy of assets, but also the professional responsibility to do so. No attempt is made here to consider such problems as the extent to which regulators rely on the objectivity of the valuation actuary, the reluctance of some managements to look to the valuation actuary for the effects of decisions on pricing, product, and financial planning, or the professional obligation of the valuation actuary to serve the public. These problems have been clarified in the United Kingdom and are being clarified in Canada, but in the United States progress has been slow, despite the mammoth valuation actuary effort, due to the complexity of the industry and the diversity of regulation.

The paper strives to provide a conceptual structure for understanding a number of basic questions, even though precise answers are not feasible:

- Can the actuary develop a workable measure of asset adequacy that is both understandable to management and regulators and acceptable theoretically?
- Can the actuary establish the adequacy of assets equal to reserves without studying assets equal to the sum of reserves and risk surplus needed on in-force business?
- Can the actuary report to management on the adequacy of total assets without becoming a "whistle-blower"?
- Can the actuary fulfill professional obligations without consideration of solvency, solidity, and vitality in reports to management?

BACKGROUND

The effort to determine the appropriate role of the valuation actuary is attracting widespread professional, regulatory, and management attention. Old theories are being modernized, and new theories are emerging. New structures of principles, practices, and standards are being researched. Investment and actuarial practices are being coordinated and merged. The once comfortable world of the valuation actuary, which entailed trust in tabular reserves based on static assumptions, has been destroyed. It has been replaced by a new world of volatile economic, investment, product, and claim environments calling for valuation based on professional judgment embracing pricing and surplus management as well.

Actuarial concepts and vocabulary have been extended to accommodate these developments, with old terms assuming new significance and new terms needed for new insights, for example, probability of ultimate ruin, asset and liability cash flows, universes of deterministic scenarios, asset adequacy, reserves requiring actuarial judgment, risk surplus, vitality surplus, C-1, C-2, and C-3 risks and their combination, and C-4 risk. The new world highlights uncertainty and deviations from expected. Actuarial judgment reflecting this uncertainty in a professional manner is required. A major intent of this paper is to sketch out a simple theoretical and practical structure of this uncertainty by utilizing the emerging nomenclature.

Cash flow from total assets of a life insurance company together with cash flow from premiums should be adequate to provide for the cash flow needed for contract obligations and expenses on in-force business and for cash flow demands under the financial plan for future business, growth, and change. Such asset and liability cash flows will unfold in a future that is unknown but that must be predicted within limits defined by levels of probability of ultimate ruin. Because return on invested capital and product prices must be competitive in the market, total assets, more particularly surplus, cannot be indefinitely large. Thus, a life insurance company, as a risk institution, must be managed explicitly or implicitly on the basis of an acceptable level of probability of ultimate ruin, given the existing assets and surplus. The actuary alone has the training and thus the professional responsibility for quantifying the effects of risks on the whole enterprise for guidance of management decisions. This responsibility overrides the valuation actuary's legal responsibility to provide the opinions and reports required by regulatory authorities.

For actuarial and planning purposes, total invested assets can be divided into (a) those equal to reserves on in-force business, (b) those equal to risk surplus need on in-force business against C-1, C-2, and C-3 risks (capacity utilized), and (c) the balance, the dynamics of which relate to financial plans for the future new business, growth, and change. By using terms now accepted in the literature, (a) relates to solvency and reasonable deviations from expected; the sum of (a) plus (b) relates to solidity and plausible deviations from expected; and the sum of (a) plus (b) plus (c) relates to vitality to prosper and grow profitably in the face of plausible deviations from expected.

In addition, there is an intangible asset, called "added value" in the literature, which cannot appear on balance sheets and net income statements. This intangible is the present value of net income expected in the future from the existing marketing organization. Although added value is obviously a component of vitality, it is not considered further here.

This paper examines only (a) and (b) and thus applies only to in-force business. Similar concepts, however, would be applicable to (c), with additional complications arising from the uncertainties and options inherent in financial planning for the future.

The paper specifically addresses only statutory financials, but the approach would be similar for addressing adequacy of invested assets in GAAP financials and internal management financials. The following relationships to statutory financials would apply:

- Invested assets would be essentially unchanged, equal to (a) + (b) + (c).
- The sum of (a) + (b) would be essentially unchanged.
- (a) would be equal to the benefit reserve, so that
- (b) would usually be increased, and
- (c) would be essentially unchanged, applying exclusively to future financial plans.
- Also, there would be an additional component of balance sheet assets, beyond invested assets, equal to the recoverable unamortized acquisition expenses on in-force business and similar items peculiar to such accounting systems.

CONCEPTUAL BASIS OF ASSET ADEQUACY

Definition of Adequate Assets

Assets are adequate, if, together with future premium and investment income, they provide for future contract obligations and expenses, based on a stated level of probability of ultimate ruin.

Ruin Probability Theory

Consider a single risk (for example, C-3 risk). As of the valuation date, define the following functions for in-force business on a specific product, a whole line, or a whole company:

A = invested assets being tested for adequacy, valued on the statutory basis.

A_o = invested assets exactly sufficient on expected experience on premiums; investment income; claims; expenses and charges for amortizing acquisition expenses; withdrawal, termination, and policy loan amounts paid; FIT; policyholder dividends and credits; stockholder dividends.

V = statutory reserves.

S = risk surplus needed, determined by the valuation actuary procedures.

A_v = invested assets equal to V.
A_{v+s} = invested assets equal to $V + S$.
U = $A - A_o$ = additional invested assets needed in excess of A_o to provide for a specific level of adverse deviations from expected.

Corresponding to A_v and A_{v+s}, U has the values U_v and U_{v+s}, respectively.

p = probability of ultimate ruin, that is, failure of assets to exceed reserves at some future duration.
$f(U)$ = probability density function of U being exactly sufficient to provide for a specific level of adverse deviations from expected.
$F(u)$ = probability distribution function

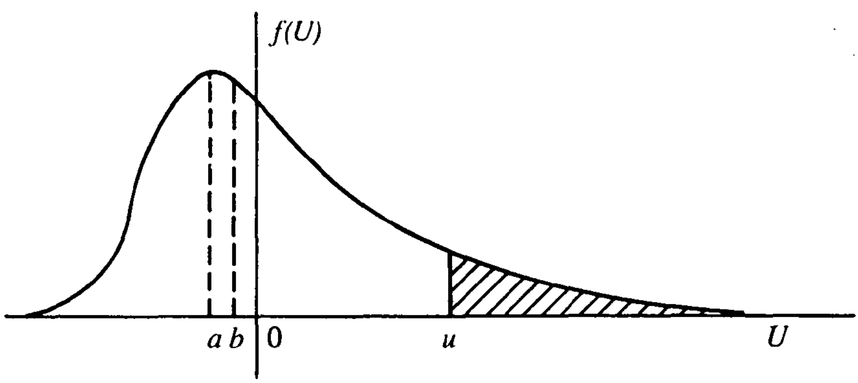
$$= \int_{-\infty}^{u} f(U)\, dU = \text{Prob}\ (U \leq u) = 1 - p$$

= probability that $A = A_o + u$ will be adequate at level of probability, p, of ultimate ruin.

Two values of u and p are significant:

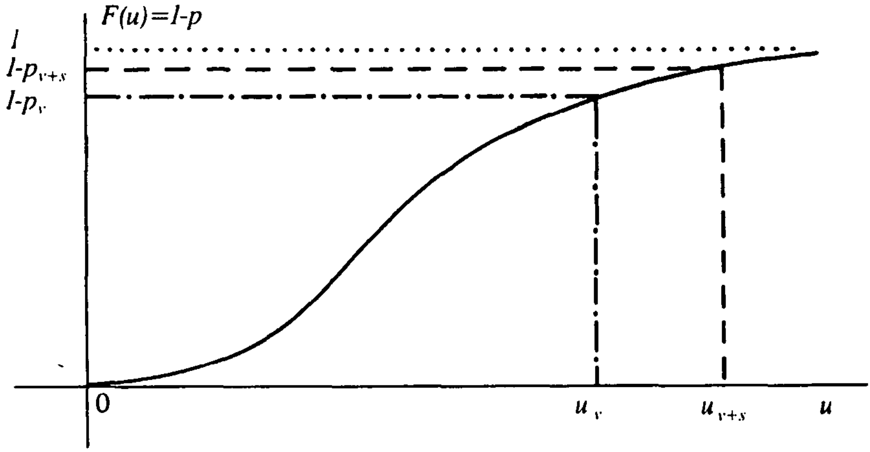
- For adequacy of assets equal to reserves at ruin probability level p_v where $A_v = V = A_o + u_v$.
- For adequacy of assets equal to reserves plus risk surplus at ruin probability level p_{v+s}, where $A_{v+s} = V + S = A_o + u_{v+s}$ and $S = u_{v+s} - u_v$.

Graphically, $f(U)$ is as follows, typically having a long tail to the right with the expected value of U equal to zero, modal value at a, and median value at b:



The area of the shaded space equals the ruin probability p, and the total area equals unity.

The corresponding function $F(u) = 1 - p$ is shown graphically as follows:



where u_v is the margin relative to expected inherent in assets equal to reserves, with probability p_v of ultimate ruin; u_{v+s} is the margin relative to expected inherent in assets equal to reserves plus risk surplus, with probability p_{v+s} of ultimate ruin; and $u_{v+s} - u_v$ equals risk surplus S.

The value of p_{v+s} is commonly chosen to be 1 percent. Indeed, in the current highly competitive market, it is unlikely that a lower value could be tolerated. The value of p_v might be, say 10 percent, but the level of p_v is conceptually less important than the level of p_{v+s}, for it is $A_o + u_{v+s}$ that establishes the solidity of the company with respect to its in-force business.

Ultimate ruin has been defined as failure of assets to exceed reserves at some future duration. If the valuation actuary in fact determines risk surplus at $p_{v+s} = 1$ percent on the basis of such definition of ultimate ruin (and management recognizes the determination in planning), a case can be made for redefining ultimate ruin for adequacy of assets equal to reserves less conservatively; the redefinition would be that ultimate ruin exists if the present value of future net cash flows is negative. The argument would be that $A_o + u_{v+s} = V + S$ is large enough to ensure availability of assets to cover reserves during those periods when $A_o + u_v$ provides insufficient assets to cover reserves, so that insolvency is not a problem. Indeed, $A_o + u_{v+s} = V + S$ itself becomes lower, because ultimate ruin for testing u_{v+s} is defined with respect to lower reserves. This implies more efficient use of capital.

A similar argument can be made for increasing the level of p_v within reason. An obvious corollary is that highly conservative reserves are a very inefficient use of capital. Margins in reserves and reserves themselves are released only by the dynamics of their specific contract class, while surplus is available not only for the particular risk under the specific contract class but also for all risks on all contract classes.

For multiple risks, the probability density function, similar to that presented above for an individual risk, is defined in multivariate probability space, reflecting the degree of correlation, if any, between pairs of individual risks. However, it is theoretically possible to reduce the aggregate $U = U_1 + U_2 + \ldots + U_N$ for all N risks to the above $f(U)$ and $F(u)$ functions, where the distribution of U is derived from the $U_1, U_2, \ldots, U_N$ multivariate distribution.

The above is the ideal theoretical structure for testing asset adequacy. However, most probability density functions are not available explicitly. One exception is the distribution of the sum of death claims, for which extensive literature exists. Another exception being developed is the application of the beta distribution to bond defaults. Fortunately, there are equivalent, though approximate, surrogate procedures that are entirely practicable. Described in the references, they involve universes of deterministic scenarios, which are, in effect, transformations of the above functions $f(U)$ and $F(u)$. Monte Carlo techniques have been used to generate the deterministic scenarios by some actuaries, leading directly to the distribution functions. In any event, in these universes of scenarios, classes of worst scenarios may be defined as attaching to p_v and p_{v+s}, allowing u_v and u_{v+s} to be determined. The next section elaborates this relationship.

PRACTICAL SOLUTION OF TESTS OF ASSET ADEQUACY

Universes of Deterministic Scenarios

Consider the individual risk situation (for example, C-3 risk). Corresponding to each U, there is a class of deterministic scenarios of the future (for example, C-3 risk scenarios of future yield curves) for which U will be exactly sufficient with probability $f(U)$. Corresponding to each u, there is a universe of these classes of scenarios, for which $U \leq u$ will be adequate with probability $1 - p$.

The universe of deterministic scenarios of the future constitutes all possible futures. For each such scenario, U can be calculated by cash flow analysis. Even in the case of the C-2 risk for deviations in the sum of death claims, where explicit distributions are classically available, scenarios of cash flows are implicit, although they are unnecessary for a solution.

The universe of classes of scenarios for which $U \leq u_\nu$ is identified in the references as consisting of scenarios of *reasonable* deviations from expected, with probability of $1 - p_\nu$. (The term "reasonable" has seemed readily acceptable to actuaries, apparently because it applies to the familiar reserve amounts.) The universe of classes of scenarios for which $U \leq u_{\nu+s}$ is identified in the references as consisting of scenarios of *plausible* deviations from expected with probability of $1 - p_{\nu+s}$. (The term "plausible" has seemed less acceptable; its genesis is the postulate that deviations from expected for which the level of probability of ultimate ruin is less than $p_{\nu+s}$ are "implausible" and should not be covered by assets equal to reserves plus risk surplus.) Thus, *reasonable* and *plausible* are words of art descriptive of these complex situations and have no other meanings.

Worst Scenarios

As can be seen from the graphic representations of $f(U)$ and $F(u)$, the monotonic shape of the tail of $f(U)$ and of the whole of $F(u)$ allows the classes of scenarios to be placed in order of adversity. Then, two classes of scenarios can be found that are the worst of all for $U \leqq u_\nu$ and for $U \leqq u_{\nu+s}$, respectively. Once these worst scenarios are identified, they can be used to calculate u_ν and $u_{\nu+s}$.

Theoretically, worst scenarios should be determined by first defining p_ν and $p_{\nu+s}$. Practically, it is usually necessary to define the worst scenarios as an initial step and then assign the values of p_ν and $p_{\nu+s}$ as magnitudes based on subjective feelings or expert opinion. The opinion or report could refer to "reasonable" and "plausible" scenarios no worse than those used in the testing. Of course, where the scenarios are stochastically generated, the ideal theoretical procedure is available.

Examples

The references contain numerous examples of practical procedures. For instance, with regard to the C-3 risk, a large number of deterministic scenarios of future yield curves may be hypothesized and worst scenarios selected. Or the scenarios may be generated by use of a stochastic Monte Carlo process with variability statistics from past experience or by opinions of management or experts. Then, U is calculated for each scenario (or for chosen worst scenarios) by cash flow procedures.

For future unforeseeable C-2 risk catastrophes, like epidemics (influenza, AIDS) or earthquakes, the level of losses U is first estimated, assuming the catastrophe has occurred (a worst scenario); then p_{v+s} is assumed to apply, where p_{v+s} equal to 1 percent seems appropriate. A similar approach has been used in the references for the C-1 risk in a major depression or stagflation of catastrophic proportions.

Combination of Risks for Risk Surplus

Risk surplus addresses future unforeseeable abnormal environments, and the list of identifiable risks is quite long, such as the following:

- C-1 risk: Lengthy serious deflationary or inflationary economic episode with widespread bankruptcies and very high unemployment; low-quality investments.
- C-2 risk: Uncontrolled AIDS or other epidemic; earthquake catastrophe; very high disability claims and inflated expenses correlated with the above C-1 risk episode; high death claims due to excessive retention limits; high health claims due to poor underwriting or concentration of risk; accidental catastrophes, etc.
- C-3 risk: Poor product/investment coordination and poor asset/liability cash flow management; C-3 risk correlated with the above C-1 risk episode.

Of course, once a catastrophe has occurred, the insurance company should set up reserves based on ruin probability p_v and consider additional risk surplus based on ruin probability p_{v+s} within corporate surplus.

It seems clear that straight addition of risk surpluses for individual risks will overstate the total risk surplus needed, because some pairs of risk are independent; for instance, death claims are independent of C-1 and C-3 risks and earthquakes are independent of all other risks. Also, some correlated risks are not 100 percent correlated, for example, C-1 and C-3 risks. And some pairs of risks are negatively correlated, an example being C-3 risk on single premium deferred annuities and C-3 risk on structured settlements. The general conditions for the existence of negative correlations have not yet been established, but such correlations are probably rare.

The references discuss the combination of risks at some length, although an exhaustive treatment must still be developed. There is a useful formula, however, that helps understanding and can easily be applied with caution, because of its approximate nature. Based on multivariate normal probability distribution theory, it is as follows:

$$u_T^2 = \sum_{i=1}^{N} u_i^2 + 2 \sum_{i,j=1}^{N} r_{ij}\, u_i\, u_j$$

where

u_i = u_{v+s} for individual risk i.
r_{ij} = correlation coefficient between u_i and u_j.
u_T = u_{v+s} for the combination of all risks.
u_i, u_j and u_T are all at the same level of probability p_{p+s} of ultimate ruin (e.g., $p_{p+s} = 1$ percent).
N = number of all risks.

If each negatively correlated pair of risks has previously been combined into a single risk, $0 \leq r_{ij} \leq 1$, and

$$\sum_1^N u_i^2 \leq u_T^2 \leq \left(\sum_1^N u_i \right)^2 .$$

Thus, u_T lies between a value based on the assumption that all risks are completely independent ($r_{ij} = 0$) and a value based on the assumption that all risks are 100 percent correlated ($r_{ij} = 1$).

Combination of Risks for Reserves

The list of risks for testing assets equal to reserves is much shorter than that for testing assets equal to reserves plus risk surplus, because abnormal and catastrophic happenings are not involved. Although the above formula should be considered for unusual situations, it is probably satisfactory generally just to set total margin equal to the sum of the margins for the individual risks.

IMPORTANCE OF DETERMINING RISK SURPLUS

At present, regulatory efforts in the United States are aimed primarily at the legally required actuarial opinion on adequacy of assets equal to reserves. However, in jurisdictions (such as New York) requiring or encouraging actuarial analysis to justify reserves on interest-sensitive products, it is likely that valuation actuaries will allude to surplus available when establishing the level of worst scenarios of C-1 and C-3 risks contemplated in testing adequacy of assets equal to reserves.

At the professional level, the *1987 Report of the Joint Committee on the Role of the Valuation Actuary in the U.S.* [8] contains a recommendation that, as a goal, the valuation actuary should eventually report to management on the adequacy of total invested assets equal to reserves, risk surplus, and the amounts needed for financial plans for new business, growth, and change.

This is already a legal requirement in the United Kingdom and is being approached rapidly in Canada.

Measurement of assets equal to reserves plus risk surplus is an important professional desideratum today and should be a professional requirement in standards as soon as practical procedures are available (if, indeed, they are not already available) and actuaries are educated to apply them. Not only do actuaries owe such advice to management and clients, but determination of risk surplus is central to a number of actuarial interests:

1. If assets equal to reserves plus risk surplus are determined to be adequate, it is acceptable to establish statutory reserves less conservatively, thereby utilizing capital more efficiently. This was discussed above. A corollary is that, if risk surplus determination is not included in reports to management available on request to regulators, statutory reserves are bound to be more conservative.
2. The pricing/product actuary should reflect risk surplus (target surplus) S in pricing. Risk surplus (target surplus) is capital advanced from corporate entity surplus, commensurate with the risk on each product. It has a cost. The cost is $(r - i_s)\ S$, where r is the target return on equity providing for enhancement of corporate entity surplus and i_s is investment earnings rate on S, all after federal income tax (FIT). (Acquisition expenses are also advanced from corporate entity surplus and are amortized on the basis of persistency and interest rate r.) Because FIT in i_s is without offset and in mutuals contains a surplus tax, i_s is small, probably of the order of 5 percent. The charge r is commonly of the order of 15 percent. Hence, the cost of risk surplus advanced is of the order of $(0.15 - 0.05)$ S, or $0.1\ (S/V)V$. This is appreciable where C-1 and C-3 risks are large. If we regard this as a priority charge in pricing and valuation (and it should be), it should be deducted from interest earned.
3. The corporate actuary involved in internal management financials and financial planning should recognize (i) invested assets required for in-force business equal to statutory reserves plus statutory statement risk surplus and (ii) the charge for advance of risk surplus as discussed above for the pricing actuary.

SUMMARY OBSERVATION

The valuation actuary must be satisfied with less than precise application of ruin probability theory. We are dealing in magnitudes, for example, level of probability of ultimate ruin of "about 1 percent" or "about 10 percent."

Precision cannot be sought. Whatever process is used, the process itself will give the actuary, management, and regulators an understanding of risk-sharing, risk control, and the extent of avoidance of unreasonable risk-taking.

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DISCUSSION OF PRECEDING PAPER

DAVID L. CRESWELL:

The Society once again owes Mr. Cody a debt of gratitude, this time for providing us with an overall theoretical framework for the work of the valuation actuary. My comments on the paper relate specifically to the C-2 risk.

I was concerned by AIDS being referred to as a C-2 risk catastrophe and the later statement that, when testing assets equal to reserves, catastrophic happenings are not involved. I'm sure the author would agree that, using $p_v = 0.10$, the estimation of u_v, on most blocks of business, would properly involve considerable attention to AIDS, both its expected impact and the more adverse side of the probability density function of such impact. I make this point because I believe we actuaries are still somewhat remiss in not fully recognizing the uncertainty of the AIDS epidemic in our pricing, reserving and surplus analysis.

The author explicitly recognizes that his formula for the combination of risks should be applied with caution. The remainder of my discussion is an expansion of this cautionary note as it relates specifically to C-2 risk, with some related questions at the end.

The use of a normal curve for the AIDS risk produces values that I believe most informed observers would consider significant distortions. Our estimate of a worst 1 percent case scenario for AIDS involves eventual infections of several times the expected value. Intermediate values between this and the expected case using the normal curve seem too pessimistic. Because the AIDS risk is quite significant for our company, a more accurate representation is necessary. We use a step function for $f(u)$ for AIDS and numerical methods to combine this with other risks that we model as normal.

The use of a simple formula for combining u_i's in an operation dominated by various term coverages (that is, group, credit, property and casualty) appears to suffer from lack of recognition of the time element of losses. In this situation, the most significant risk is often that of future losses, not of the inadequacy of the claim reserves and liabilities (which may be negligible).

As an example, let us assume the following:

u_i = beginning surplus sufficient for line of business i, as a stand-alone operation, so that the probability of its eventual reduction to zero is p_{v+s}.

u_j = defined similarly for line of business j.

Further, let us assume that line i is very profitable, with no AIDS risk but enough other immediate risks to require u_i of initial surplus and that line j is less profitable, with little immediate risk but significant AIDS risk ten years in the future.

In this case, line i's likelihood of failure occurs predominately in early years, before the high profits cause an essentially impregnable defense of accumulated surplus. The likelihood of j's failure, however, is centered out in the zone of maximum AIDS impact, in which the combined accumulation of u_j and the more modest profits must suffice for a cushion against such impact. Using more sophisticated techniques, I find that in many such cases, u_{i+j} is actually significantly less than u_j in spite of a positive correlation between the lines. This occurs because of the considerable help generated by high earnings in line i as a cushion against AIDS claims in line j. I believe the problem with the simple formula can be significant in any situation in which projected future profits interact with multiple years in which failure can occur.

Thus, in lines of business in which C-2 risk predominates and in which the main concerns are year-by-year fluctuations, underwriting cycles, uncertainties to long-term average profitability, and a superimposed AIDS risk, I found no practical substitute for a stochastic model measuring year-by-year probabilities of ruin given survival to the beginning of such year. This is accomplished without Monte Carlo sampling, although the current model measures only C-2 risk.

This also leads to interesting questions on evolving valuation requirements. If a line of business has very little in reserves and claim liabilities, it may still require backing of considerable surplus because of the risk of future losses. If reserves are designed for $p_v = 0.10$ or 0.25, does this not necessitate similar modeling of future losses? Is the difference between u_v and u_{v+s} not merely quantitative, depending on p_v and p_{v+s}? Would it not then be appropriate to have a separate reserve item for lines of business where A_v may be several times A_o (that is, a statutorily required contingency reserve)?

DONALD R. SONDERGELD:

We are indebted to Don Cody, who has condensed a lot of important information into just 12 pages of the *Transactions*.

The purpose of this discussion is to point out that although a company should determine how much statutory surplus it needs for its evaluation of the risk, it is possible that additional surplus may be needed if an A+ A.M.

Best rating is desired. The same comment applies to other rating agencies such as Moody's and Standard and Poor's.

In my 1982 TSA paper,* I described one of the guidelines A.M. Best used at that time for its financial rating of life insurance companies. A.M. Best still calculates a ratio of total exposure (Z) to net capital and surplus (S) $= Z \div S$. In the 1982 paper, the reciprocals embedded in that ratio were displayed to show the surplus needed as a percentage of reserves to maintain a good rating.

However, A.M. Best now also calculates something called "gross leverage," which is used with other items to determine the rating. All other things being equal, the lower the gross leverage, the better the rating.

$$\text{Gross Leverage} = \left\{ [V^2 + W^2 + X^2 + Y^2] \frac{[Z]}{[S]} + (-5W + 7X + 8Y) \right\} [\text{Size Factor}]$$

where V = ratio of life exposure to total exposure
W = ratio of annuity exposure to total exposure
X = ratio of health exposure to total exposure
Y = ratio of other exposure to total exposure.

As the "total exposure" (Z) increases, the "size factor" reduces. This formula resembles the multivariate normal probability distribution formula referred to by Mr. Cody.

I recently received the following comments from A.M. Best:

"The development and application of our gross leverage calculation is an attempt to assess the degree to which a company maintains a cushion or ability to withstand unusual adverse circumstances relative to carriers writing similar lines of business and the life/health industry as a whole. In general, the lower the gross leverage value the greater a company's ability to withstand occasional unfavorable operating experience. A gross leverage value which is substantially higher than the industry norm may indicate a company which is highly leveraged, that is, it may be writing more business than its capital and surplus can support.

"Important components of the gross leverage formula are the peer and size (spread of risk) factors. The peer weighting factors, which are based on peer company (life, annuity, accident and health and other) median values, enable us to fairly evaluate all companies

*SONDERGELD, D.R., "Profitability As a Return on Total Capital," *TSA* XXXIV (1982):415-33.

on a similar leverage scale for rating purposes regardless of a specific company's marketing emphasis. The size or spread of risk factor recognizes the fact that as a company's exposure increases, its spread of risk also normally increases. In other words a company's capital and surplus base becomes less susceptible to being adversely affected by one single event or risk as the concentration of exposure is dispersed to a greater degree.

"It is important to note that in most instances gross leverage is only a starting point for analytical purposes. Appropriate adjustments to capital and surplus, exposure and the peer and spread of risk factors are manually made in many cases to reflect conservative reserving methods, non-recurring transactions, balance sheet quality, profitability, reinsurance activities and/or operations which do not facilitate accurate peer company analysis. The results of any applicable manual adjustments is the net leverage value which represents our major leverage calculation for analytical and rating purposes."

ERIC S. SEAH AND ELIAS S.W. SHIU:

Mr. Cody is to be thanked for this exposition on a very important subject. The paper has certainly filled a long-felt void in the actuarial literature. We wish to take this opportunity to present a practical formula for the computation of the probability of ultimate ruin in the classical collective risk model.

In the classical collective risk model, it is assumed that the number of insurance claims up to time t, $t \geq 0$, is a Poisson process $N(t)$ and the individual claim amounts X_1, X_2, X_3, . . . are mutually independent and identically distributed random variables. The Poisson process $N(t)$ is independent $P(0) = 0$. Premiums are received continuously at a constant rate c. The $p_1 < \infty$ and $E[N(t)] = \lambda t$. The claim amounts are always positive, that is, $P(0) = 0$. Premiums are received continuously at a constant rate c. The positive number

$$\theta = c(p_1\lambda)^{-1} - 1$$

is called the relative security loading.

For $k = 1, 2, 3, \ldots$, define

$$S_k = X_1 + X_2 + \ldots + X_k. \tag{1}$$

Put $S_0 = 0$. The probability of ultimate ruin $\psi(u)$ is the probability that the risk reserve

$$u + ct - S_{N(t)} \tag{2}$$

is ever negative. Note that the argument of the ruin function ψ is the amount of risk reserve at time 0.

Let a denote the Lundberg security factor:

$$a = \lambda c^{-1} = [(1 + \theta)p_1]^{-1}. \tag{3}$$

For $\alpha \geq 0$, define

$$x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0 & x < 0 \end{cases}. \tag{4}$$

We shall prove that, for $u \geq 0$,

$$\psi(u) = 1 - \frac{\theta e^{au}}{1 + \theta} \sum_{j=0}^{\infty} \frac{(-a)^j E[(u - S_j)_+^j \, e^{-aS_j}]}{j!}. \tag{5}$$

It is somewhat easier to work with the nonruin function

$$\phi(u) = 1 - \psi(u).$$

The function ϕ is monotone increasing; it takes the value 0 on the negative axis and $\phi(+\infty) = 1$. In Section 12.6 of *Actuarial Mathematics* [2], ϕ is denoted as F_L, where L is the maximal aggregate loss random variable.

Consider a small time interval $(0, s)$. By the Poisson assumption, the probability that a claim will occur in the interval is $\lambda s + o(s)$. Hence, for $u \geq 0$, the nonruin function $\phi(u)$ satisfies the relation:

$$\phi(u) = \lambda s E[\phi(u + cs - X)] + (1 - \lambda s)\phi(u + cs) + o(s). \tag{6}$$

Dividing (6) by s, rearranging and letting s tend to $0+$, we obtain the integro-differential equation

$$0 = \lambda E[\phi(u - X)] + c\phi'(u) - \lambda\phi(u), \quad u > 0,$$

or

$$\phi'(u) = a\{\phi(u) - E[\phi(u - X)]\}, \quad u > 0. \tag{7}$$

The convolution of two functions g and h is defined by

$$(g * h)(x) = \int_{-\infty}^{\infty} g(x - t)\, h(t)\, dt. \tag{8}$$

Let $p(x)$ denote the derivative of the probability distribution function $P(x)$. If $P(x)$ is not differentiable, $p(x)$ is a generalized function [4; 8; 14; 15].

Let $\delta(x)$ denote the Dirac delta function. Then Equation (7) becomes

$$\begin{aligned}\phi'(u) &= a[\phi(u) - (\phi * p)(u)] \\ &= a\{[\delta(u) - p(u)] * \phi(u)\}, \ u > 0. \end{aligned} \tag{9}$$

With the definition

$$f(u) = a[\delta(u) - p(u)], \tag{10}$$

Equation (9) can be written as

$$\phi'(u) = (f * \phi)(u), \ u > 0. \tag{11}$$

We are to seek a function ϕ, which is zero on the negative axis and satisfies (11) on the positive axis.

Put $f^{*0}(x) = \delta(x)$; for $n = 1, 2, 3, \ldots$, define

$$f^{*n}(x) = f(x) * f^{*(n-1)}(x). \tag{12}$$

As

$$(g * h)' = g * (h'), \tag{13}$$

one can show that the function

$$\sum_{n=0}^{\infty} \frac{f^{*n}(u) * u_+^n}{n!} \tag{14}$$

or any scalar multiple of it satisfies Equation (11) and is zero on the negative axis. Hence, we have the formula

$$\phi(u) = \phi(0) \sum_{n=0}^{\infty} \frac{f^{*n}(u) * u_+^n}{n!}. \tag{15}$$

We shall determine the value of $\phi(0)$ later.

Substituting

$$\begin{aligned} f^{*n}(u) &= a^n[\delta(u) - p(u)]^{*n} \\ &= a^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} p^{*j}(u) \end{aligned}$$

into (15) and interchanging the order of summation yields

$$\begin{aligned}
\phi(u) &= \phi(0) \sum_{j=0}^{\infty} \frac{(-1)^j p^{*j}(u)}{j!} * \left(\sum_{n=j}^{\infty} \frac{a^n\, u_+^n}{(n-j)!}\right) \\
&= \phi(0) \sum_{j=0}^{\infty} \frac{(-a)^j\, p^{*j}(u) * (u_+^j\, e^{au_+})}{j!} \\
&= \phi(0) \sum_{j=0}^{\infty} \frac{(-a)^j\, E[(u - S_j)_+^j\, e^{a(u-S_j)_+}]}{j!} \\
&= \phi(0)\, e^{au} \sum_{j=0}^{\infty} \frac{(-a)^j\, E[(u - S_j)_+^j\, e^{-aS_j}]}{j!} \qquad (16)
\end{aligned}$$

Formula (16) is (5), if we can prove that

$$\phi(0) = \theta(1 + \theta)^{-1}. \qquad (17)$$

To this end we integrate (9) with respect to u from 0 to w:

$$\phi(w) - \phi(0) = a \int_0^w \phi(w - x)\, [1 - P(x)]\, dx. \qquad (18)$$

(Formula (18) is Equation (XI.7.2) of [5].) Letting w tend to $+\infty$ in (18) and applying the Lebesgue dominated convergence theorem, we obtain

$$1 - \phi(0) = a \int_0^{\infty} [1 - P(x)]\, dx = ap_1 = \frac{1}{1 + \theta},$$

which gives Formula (17).

Let $\eta = \sup\{x \mid P(x) = 0\}$. By assumption, $\eta \geq 0$. If $\eta > 0$, then (5) is a finite sum, with the index j ranging from 0 to $[u/\eta]$. For a real number r, we let $[r]$ denote the greatest integer less than or equal to r.

Formula (5) is valid for continuous or discrete X. Willmot [13] has used (5) to evaluate $\psi(u)$ when X is Gamma (with arbitrary nonscale parameter) or continuous uniform. In [10], it is assumed that X takes values on positive integers only: If

$$c_k^{*j} = Pr\left(\sum_{i=1}^{j} X_i = k\right) = Pr(S_j = k),$$

then, for $u \geq 1$,

$$\phi(u) = \frac{\theta e^{au}}{1 + \theta} \left\{ 1 + \sum_{k=1}^{[u]} e^{-ak} \sum_{j=1}^{k} \frac{c_k^{*j}[a(k - u)]^j}{j!} \right\}. \tag{19}$$

The coefficients $\{c_k^{*j}\}$ can be evaluated recursively by the formula

$$c_k^{*(m+n)} = \sum_i c_i^{*m} c_{k-i}^{*n}. \tag{20}$$

Formula (19) can be easily programmed. Below is a listing of *APL* codes. The program is efficient unless u is large. However, for large values of u, one can use Lundberg's asymptotic formula [1, p. 52; 3, (119); 5, p. 378; 6, (5.27); 9, (6.64)]. For more discussion on the probability of ultimate ruin, see [11] or [12].

```
    ∇ X RUIN C;THETA;U;A;P1;P;CSTAR
   ⍝ X IS THE VECTOR OF CLAIM AMOUNTS AND C IS THE ASSOCIATED
   ⍝ PROBABILITIES.  THE ENTRIES OF X ARE POSITIVE INTEGERS AND
   ⍝ THE ENTRIES OF C SUM TO 1.
   LP:'INITIAL RISK RESERVE? (TO QUIT THE PROGRAM, ENTER A NEGATIVE NUMBER)'
    →(0>U←⎕)/0
    'ENTER RELATIVE SECURITY LOADING'
    THETA←⎕
    A←÷(1+THETA)×P1←+/X×C
    CREATEP
    CREATECSTAR
    CALCPROB
    3 1 ⍴' '
    →LP
  ∇
```

```
    ∇ CREATEP
    P←(1+⌈/X)⍴0
    P[1+X]←C
  ∇
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```
    ∇ CREATECSTAR;I;R;T
   ⍝ ADAPTED FROM GRENANDER ([7], PAGE 402)
    R←1+⌊U
    CSTAR←(R,1)⍴1,(¯1+R)⍴0
    T←(R,R)↑(¯1+⍳R)⌽((R,R)⍴⌽R↑P),(R,R)⍴0
    I←2
   LP:CSTAR←CSTAR,⌽CSTAR[;(⍴CSTAR)[2]]+.×T
    →(R≥I←I+1)/LP
  ∇
```

```
    ∇ CALCPROB;K;SUM;T
    SUM←0
    K←0
   LP:→((⌊U)<K←K+1)/END
    T←+/(K↑1↓CSTAR[K+1;])××\(K⍴A×K-U)÷⍳K
    SUM←SUM+T×*-A×K
    →LP
   END:'INITIAL RISK RESERVE ',⍕U
    'RELATIVE SECURITY LOADING ',⍕THETA
    'PROBABILITY OF RUIN ',⍕1-(*A×U)×(1+SUM)×THETA÷1+THETA
  ∇
```

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(AUTHOR'S REVIEW OF DISCUSSION)

DONALD D. CODY:

I am appreciative of the discussions contributed by Mr. Creswell, Professors Seah and Shiu, and Mr. Sondergeld and for their kind words.

Mr. Creswell discussed the financial effects of the AIDS epidemic and raised important questions on other matters. His analysis has increased the value of the paper, and I am pleased to add further views.

Among the future unforeseeable catastrophes to be covered by risk surplus, I listed epidemics, citing influenza and AIDS as examples. It was not my intention to imply that, once an epidemic has occurred, it would continue to be a surplus matter. (The final text will clarify this.) The financial effects of the World War I influenza epidemic were increased cash outflows from death claims lasting for some months. AIDS is different because it involves some ten years from HIV infection to death of the last survivor and new HIV infections over an indeterminate period. Thus, the AIDS occurrence must involve both reserves and risk surplus. Let me suggest how the concepts of the paper would apply:

One approach is suggested in two papers by David M. Holland [2]. The approach involves an aggregate reserve accumulated over time like a pension fund. It would be funded by annual contributions from net operating income and would pay AIDS life insurance and health insurance claims as they occur (less any available credits, such as reductions in policyholder dividends). The target level of the reserve would be adjusted as the experience of the company and the characteristics of the epidemic emerge. The concepts of the paper imply that the reserve should be established at the ruin probability level of p_v. Because the level of the reserve is to be adjusted as experience and characteristics emerge, it seems reasonable to base it on expected, implying a p_v of somewhat less than 50 percent. In addition, the company should consider the availability of risk surplus at ruin probability level of p_{v+s} within corporate surplus, looking forward to future AIDS claim strains not contemplated in the reserves. In my text, I suggest that p_{v+s} be at the 1 percent level, which probably is too strong here for the next few years while AIDS epidemiology and company experience are so uncertain.

Mr. Creswell is rightfully concerned about use of the normal distribution. Indeed, a normal distribution would almost never be appropriate in determining the u_i's. Almost every C-1, C-2, and C-3 risk has an underlying distribution with considerable skewness to the right. I used the multivariate normal distribution only in the combinatorial formula; it is necessary there to derive the formula. The u_{v+s}'s for each risk on the right-hand side are *not* based on the normal distribution; it is for this reason that I believe that the combinatorial formula involves only higher-order errors.

He offers an interesting example of negative C-2 risk correlation between his line i and line j. The paper refers to a similar negative correlation between C-3 risks for single premium deferred annuities and structured settlements. This has been published by Peter B. Deakins in the December 1988 *The Financial Reporter* [1]. Let me enlarge on Mr. Creswell's text: For negatively correlated risks i and j,

$$u_{i+j}^2 = u_i^2 + u_j^2 - 2 \mid r_{ij} \mid u_i u_j \qquad -1 \leqq r_{ij} \leqq 0$$

This equation can be reduced to the following:

$$u_{i+j} - u_i = -\frac{2\, u_i\, u_j}{u_{i+j} + u_i} \left\{ \mid r_{ij} \mid - \frac{1}{2} \frac{u_j}{u_i} \right\}.$$

Thus, $u_{i+j} < u_i$ provided .

$$\frac{u_j}{u_i} < 2 \mid r_{ij} \mid.$$

For instance, if $r_{ij} = -1$, then $u_{i+j} < u_i$, provided $u_i < u_j < 2\, u_i$. However, there is no way to determine the value of r_{ij} exactly. For this reason, the paper suggests that risks i and j be tested as one combined risk.

I agree completely with the implications of Mr. Creswell's final paragraph. Unfortunately, it is not feasible to make risk surplus (designated surplus) a statutory requirement in the U.S., even though the Canadian Institute of Actuaries and the regulatory authorities are designing such a structure in Canada. Nevertheless, my personal feeling is that the valuation actuary in the U.S. should advise his management about risk surplus and vitality surplus, that is, the adequacy of total assets.

Professor Seah and Shiu have increased the value of the paper by reminding us that the C-2 risk of variation in the sum of death claims can be solved explicity by the classical collective risk model. Probability distributions based on this model intrinsically contain a complete universe of cash flow scenarios. There is no need to resort to cash-flow scenarios generated by Monte Carlo techniques.

Their discussion sets forth an elegant solution of the ultimate ruin probability based on methods of the operational calculus and adds a valuable bibliography of the application of the method to C-2 risks amenable to the collective risk model.

Mr. Sondergeld has enhanced our knowledge of the A.M. Best criteria for company ratings. Whether the new leverage rating bears more than a superficial relationship to the multivariate normal is not disclosed by the A. M. Best commentary.

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