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DISTRIBUTION OF AGGREGATE CLAIMS IN THE INDIVIDUAL RISK THEORY MODEL

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ABSTRACT

In this paper an algorithm is derived for computing the distribution of aggregate claims for the individual risk theory model. The procedure is an adaptation of the recursive method used in the solution of ordinary differential equations. Mathematical arguments are deliberately kept at an elementary level. The theory is intended to apply to a portfolio of individual life insurance policies with no attempt to generalize to other insurance situations.

Tentative remarks are made on the prospective role of risk theory in the individual life model office. These remarks are independent of the rest of the paper.

I. INTRODUCTION

In this paper a method is given for computing the aggregate claims distribution for the individual risk theory model. The computation is patterned after the recursive method recentl: articulated for the collective case.

The purpose is not mathematical originality or sophistication. Elementary combinatorial arguments, many no doubt familiar to the reader, are used whenever possible. In this regard it may also be of interest to note that the various recursive algorithms for computing aggregate claims amount to particular cases of the recursive method used in the power-series solution of ordinary differential equations [1]—a method that has been known for some time. In probability theory a recursive formula, in principle identical with the one used in the Poisson model, was used by T. N. Thiele to express sums of powers of observed values in terms of his "half invariants" [5]. This recursive formula was also used by T. N. E. Greville and Robert White in a problem involving multiple life contingencies [4].

The model pertains specifically to a portfolio of individual life insurance policies, although some connections with the collective case are described in Section IV. The simplifying assumption is made that insurance contracts expire with the payment of a single claim. On the other hand, one does not rely on a basic hypothesis of the collective model—namely, that a policy in claim status is immediately replaced by another in the same risk category. This hypothesis is much stronger than that of the stationary population and does not fit well even in many group cases. The hypothesis is clearly inappropriate for a portfolio of individual policies for which there is no necessary connection between the volume of new business and claims. A further distortion is caused by the fact that new individual business is subject to selection and is issued at the relatively younger ages, while the bulk of claims will occur at the older ages at ultimate durations. Such considerations are part of the problem of expanding one's short-term perspective into a long-range dynamic model—that is, a theory of surplus. Of course such a model is beyond the limited scope of this paper.

The algorithm is developed in Section II. A numerical example is presented in Section III. Analogies to the collective model are considered in Section IV, and some general observations are made in Section V.

II. THE ALGORITHM

In this section H is a portfolio of life insurance policies. Each policy is of some face amount n, where n is an integral multiple of some convenient unit such as \$1,000 or \$10,000. This face amount is payable during the coming year with probability q, where q is the mortality rate applicable to the cell to which the policy is assigned. In particular, it is assumed that this rate has already been adjusted for any differences between policy-year and calendar-year mortality. The contract expires with the payment of a claim. Interest is ignored. Claims arising from particular policies are assumed to be independent; in particular, each policyholder has a unique policy. Multiple policies belonging to a policyholder can be aggregated to achieve this latter condition, although practical problems, familiar from mortality studies, will arise.

We denote by

$$a_n = f(n) = f_H(n) \tag{1}$$

the probability that the aggregate claims arising from the portfolio H during the coming year will be precisely n units. The generating function of f_H is

$$R(z) = R_H(z) = \sum_{n=0}^{\infty} a_n z^n . \tag{2}$$

Now the maximum possible amount of claims, although very likely quite large, is nevertheless finite. Therefore, R_H is a polynomial in z—the coefficient of z^n indicating the probability of claims of exactly n units. We call R_H the probability-generating polynomial of the portfolio H.

A problem of some interest is that of determining the probability that the aggregate claims arising from H will not exceed some given number of units, such as N. Denoting this probability by $F(N)$, we clearly have

$$F(N) = \sum_{n=0}^{N} a_n . \tag{3}$$

The problem can therefore be restated as that of finding an "efficient" method of computing the coefficients of the probability-generating polynomial.

Suppose that K is another portfolio with claims arising independently of those of H, and let

$$R_K(z) = \sum_{n=0}^{\infty} b_n z^n . \tag{4}$$

Let c_r be the probability that total claims for the combined portfolio $H \cup K$ will be r units. This result occurs whenever claims for H and K are, respectively, m and $r - m$ units. Summing over all combinations, we have

$$c_r = \sum_{m=0}^{r} a_m b_{r-m} . \tag{5}$$

Note that c_r is just the coefficient of z^r in the (polynomial) product $R_H R_K$. Therefore,

$$R_{H \cup K} = R_H R_K ; \tag{6}$$

that is, the probability-generating polynomial of the combined portfolio is the product of the generating polynomials of its components.

In case H is a portfolio with just one policy, we have

$$R_H(z) = p + qz^n , \quad p = 1 - q , \tag{7}$$

where n is the face amount of the policy and q is the applicable mortality rate. For an arbitrary portfolio H it then follows from equation (6) that

$$R_H(z) = \prod_H (p + qz^n) , \tag{8}$$

where the product is taken over the entire portfolio H.

Let us now tabulate the portfolio by face amount, with H_n denoting those policies in H with a face amount of n units. We call these H_n's amount classes and allow for the possibility of amount classes with no members.

Now by properties of the logarithm,

$$\begin{aligned}
\log R_H(z) &= \sum_H \log(p + qz^n) \\
&= \sum_H \left[\log\left(1 + \frac{q}{p} z^n\right) - \log\left(1 + \frac{q}{p}\right)\right] \\
&= \sum_H \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\left(\frac{q}{p}\right)^k z^{nk} - \left(\frac{q}{p}\right)^k\right] \qquad (9) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\sum_n \sum_{H_n} \left(\frac{q}{p}\right)^k z^{nk} - \sum_H \left(\frac{q}{p}\right)^k\right] \\
&= S_1(z) - \tfrac{1}{2}S_2(z) + \tfrac{1}{3}S_3(z) - \ldots,
\end{aligned}$$

where

$$S_k(z) = -\sum_H \left(\frac{q}{p}\right)^k + \sum_n \sum_{H_n} \left(\frac{q}{p}\right)^k z^{nk}. \qquad (10)$$

In practice the polynomial $S_k(z)$ is computed by first computing, for each amount class H_n, the sum of kth powers

$$S_{n,k} = \sum_{H_n} \left(\frac{q}{p}\right)^k \qquad (11)$$

and then making a contribution $-S_{n,k} + S_{n,k}z^{nk}$ to $S_k(z)$.

To compute the probability-generating polynomial of H, we first prove the following:

LEMMA. *If the power series*

$$R(z) = \sum_{n=0}^{\infty} a_n z^n \quad \textit{and} \quad Q(z) = \sum_{n=0}^{\infty} b_n z^n$$

are related by

$$R(z) = e^{Q(z)} \qquad (12)$$

on some nontrivial (i.e., other than $z = 0$) interval of convergence, then the coefficients of Q and R are related by

$$a_0 = e^{b_0} \tag{13}$$

and

$$a_n = \frac{1}{n} \sum_{k=1}^{n} k a_{n-k} b_k , \quad n \geq 1 . \tag{14}$$

Proof. Equation (13) follows by the substitution of $z = 0$ in equation (12). Differentiating (12) with respect to z yields

$$R'(z) = R(z)Q'(z) . \tag{15}$$

We then obtain formula (14) by equating the coefficients of z^n in (15).

Our algorithm is based on the following *(see editor's note below)*:

THEOREM. *For $k = 1, 2, 3, \ldots$ let*

$$Q_k(z) = S_1(z) - \tfrac{1}{2}S_2(z) + \tfrac{1}{3}S_3(z) - \ldots + \frac{(-1)^{k+1}}{k} S_k(z) , \tag{16}$$

$$R_k(z) = e^{Q_k(z)} = \sum_{n=0}^{\infty} a_{n,k} z^n , \tag{17}$$

and let $F_k(N)$ be the sum of the first $N + 1$ coefficients of $R_k(z)$, that is,

$$F_k(N) = \sum_{n=0}^{N} a_{n,k} . \tag{18}$$

Then

(i) $\lim_{k \to \infty} F_k(N) = F(N)$,

(ii) $F_1(N) \leq F_3(N) \leq \ldots \leq F(N) \leq \ldots \leq F_4(N) \leq F_2(N)$.

Proof. By equation (9),

(A) $\lim_{k \to \infty} Q_k(z) = \log R(z)$,

so that

(B) $\lim_{k \to \infty} R_k(z) = \lim_{k \to \infty} e^{Q_k(z)} = e^{\log R(z)} = R(z)$.

EDITOR'S NOTE.—An error has been found in the proof of this theorem. The reader should disregard the remainder of this section and read the corrected version in the author's review of discussion.

This establishes (i). Let $Q(z) = \log R(z)$.

If $A(z)$ and $B(z)$ are power series, we shall say that $A(z) \leqslant B(z)$ provided that, for any nonnegative integer N, the sum of the first $N + 1$ coefficients satisfies the condition

$$\text{(C)} \quad \sum_{n=0}^{N} a_n \leqslant \sum_{n=0}^{N} b_n \,.$$

We have

(D) If $A(z) \leqslant B(z)$ and $C(z) \leqslant D(z)$.

Then

(D1) $A(z) + C(z) \leqslant B(z) + D(z)$,
(D2) $A(z)C(z) \leqslant B(z)D(z)$.

Assertion (D1) is immediate. The sum of the first $N + 1$ coefficients of $A(z)C(z)$ is

$$\sum_{n=0}^{N} \sum_{k=0}^{n} a_{n-k}c_k = \sum_{k=0}^{N} c_k\left(\sum_{n=0}^{N-k} a_n\right)$$
$$\leqslant \sum_{k=0}^{N} c_k\left(\sum_{n=0}^{N-k} b_n\right) .$$

Therefore, $A(z)C(z) \leqslant B(z)C(z)$. Similarly, $B(z)C(z) = C(z)B(z) \leqslant D(z)B(z) = B(z)D(z)$, so that $A(z)C(z) \leqslant B(z)C(z) \leqslant B(z)D(z)$.

We also have the relation

(E) If $A(z) \leqslant B(z)$, then $e^{A(z)} \leqslant e^{B(z)}$.

This last statement can be established by applying (D) to the expansion

$$\text{(E1)} \quad e^{A(z)} = 1 + A(z) + \frac{A(z)^2}{2!} + \ldots ,$$

$$e^{B(z)} = 1 + B(z) + \frac{B(z)^2}{2!} + \ldots .$$

Note that

$$\text{(F)} \quad Q_1(z) \leqslant Q_3(z) \leqslant \ldots \leqslant Q(z) \leqslant \ldots \leqslant Q_4(z) \leqslant Q_2(z) .$$

This inequality can be established first for the case where the portfolio H consists of a single policy; we may then sum over the entire portfolio H, using (D1). It then follows from (E) that

$$\text{(G) } R_1(z) \leqslant R_3(z) \leqslant \ldots \leqslant R(z) \leqslant \ldots \leqslant R_4(z) \leqslant R_2(z) \, .$$

This establishes part (ii) of the theorem.

The algorithm now proceeds as follows:

A.1. Compute the polynomials

$$Q_k(z) = S_1(z) - \tfrac{1}{2}S_2(z) + \ldots + \frac{(-1)^{k+1}}{k} S_k(z); \tag{19}$$

and $Q_{k+1}(z)$ for an odd integer k.

A.2. Apply the recursive formula (13), (14) to compute the coefficients of $R_k(z)$ and $R_{k+1}(z)$.

A.3. By the theorem the probability $F(N)$ that claims will not exceed N units is in the interval

$$F_k(N) \leqslant F(N) \leqslant F_{k+1}(N) \, ,$$

where $F_j(N)$ is the sum of the first $N + 1$ coefficients of $R_j(z)$ for $j = k, k + 1$.

A.4. Increase the value of k, if desired, for greater accuracy. Since most values of q are low, a single application of steps A.1, A.2, A.3 for, say, $k = 5$ will generally yield results that are quite accurate (with respect to the assumptions). In any case, few applications of step A.4 are likely to be required.

III. AN EXAMPLE

In this section we apply the algorithm derived in Section II to calculate the aggregate claims distribution of a portfolio of 322 policies ranging in face amount from 1 to 5 units. The example is intended strictly as a conveniently verified simple illustration.

Table 1 shows the classification of the portfolio into amount and mortality classes. Table 2 lists the coefficients of the polynomials $Q_k(z)$ defined by equation (19). Table 3 indicates values of $F_k(N)$, the sum of the first $N + 1$ coefficients of $e^{Q_k(z)}$ calculated by equations (13) and (14). For early values of N we actually calculate $e^a F_k(N)$ for a convenient value of a.

This device circumvents arithmetical operations with minuscule quantities when the expected number of claims is large. Actual calculations were made to a greater number of significant figures than indicated.

After reading Section IV, the reader is invited to show that, assuming the Balducci hypothesis, the first column of Table 3 describes the aggregate

TABLE 1

CLASSIFICATION OF MODEL PORTFOLIO BY MORTALITY AND AMOUNT CLASSES

FACE AMOUNT IN UNITS	DEATHS PER THOUSAND				
	0.94	1.91	5.01	13.20	34.07
1.	12	23	2	14	20
2.	1	0	6	7	0
3.	0	3	13	31	0
4.	19	32	24	5	31
5.	6	14	1	36	22

Expected number of claims $=4.118$
Expected units of aggregate claims $=14.215$

TABLE 2

COEFFICIENTS OF THE POLYNOMIALS $Q_k(z)$

n	$Q_1(z)$	$Q_2(z)$	$Q_3(z)$	$Q_4(z)$	$Q_5(z)$
0	−4.2240129	−4.1695513	−4.1706954	−4.1706664	−4.1706672
1	0.9580813	0.9580813	0.9580813	0.9580813	0.9580813
2	0.1247882	0.1110220	0.1110220	0.1110220	0.1110220
3	0.4858726	0.4858726	0.4861765	0.4861765	0.4861765
4	1.3602650	1.3595622	1.3595622	1.3595544	1.3595544
5	1.2950058	1.2950058	1.2950058	1.2950058	1.2950060
6	0.0000000	−0.0029436	−0.0029378	−0.0029378	−0.0029378
7	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
8	0.0000000	−0.0201019	−0.0201019	−0.0201020	−0.0201020
9	0.0000000	0.0000000	0.0000253	0.0000253	0.0000253
10	0.0000000	−0.0169467	−0.0169467	−0.0169467	−0.0169467
11	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
12	0.0000000	0.0000000	0.0004585	0.0004583	0.0004583
13	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
14	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
15	0.0000000	0.0000000	0.0003506	0.0003506	0.0003506
16	0.0000000	0.0000000	0.0000000	−0.0000119	−0.0000119
17	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
18	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
19	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
20	0.0000000	0.0000000	0.0000000	−0.0000087	−0.0000084
21	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
22	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
23	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
24	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
25	0.0000000	0.0000000	0.0000000	0.0000000	0.0000002

TABLE 3

Distribution of Aggregate Claims from Model Portfolio

N	a	$e^aF_1(N)$	$e^aF_2(N)$	$e^aF_3(N)$	$e^aF_4(N)$	$e^aF_5(N)$
0	1	.039795	.042022	.041974	.041976	.041976
1	1	.077922	.082283	.082189	.082192	.082192
2	1	.101152	.106235	.106114	.106117	.106117
3	1	.131078	.137282	.137138	.137142	.137142
4	1	.207721	.217852	.217627	.217633	.217633
5	0	.119082	.125028	.124897	.124900	.124900
6	0	.152628	.159991	.159824	.159828	.159828
7	0	.181075	.189359	.189167	.189172	.189172
8	0	.221389	.231087	.230861	.230867	.230867
9	0	.277981	.290075	.289793	.289800	.289800
10	0	.334435	.348555	.348215	.348224	.348223
11	0	.379795	.395001	.394621	.394631	.394631
12	0	.425094	.441140	.440733	.440744	.440744
13	0	.479399	.496607	.496162	.496174	.496174
14	0	.536760	.555156	.554663	.554677	.554676
15	0	.586552	.605392	.604867	.604882	.604881
16	0	.629137	.647890	.647354	.647369	.647368
17	0	.671821	.690379	.689838	.689853	.689853
18	0	.715590	.733914	.733360	.733376	.733376
19	0	.755432	.773227	.772666	.772683	.772683
20	0	.788703	.805592	.805043	.805060	.805060
21	0	.817922	.833771	.833244	.833261	.833260
22	0	.845599	.860379	.859873	.859889	.859889
23	0	.870925	.884570	.884083	.884099	.884099
24	0	.892290	.904703	.904245	.904261	.904260
25	0	.909952	.921111	.920690	.920705	.920704
26	0	.925297	.935259	.934873	.934887	.934887
27	0	.938899	.947717	.947363	.947377	.947376
28	0	.950429	.958148	.957827	.957840	.957840
29	0	.959783	.966470	.966185	.966197	.966197
30	0	.967433	.973187	.972937	.972948	.972947
31	0	.973880	.978797	.978577	.978587	.978586
32	0	.979271	.983434	.983242	.983251	.983250
33	0	.983619	.987110	.986945	.986953	.986952
34	0	.987063	.989971	.989831	.989838	.989837
35	0	.989830	.992240	.992120	.992126	.992126
36	0	.992078	.994059	.993959	.993964	.993964
37	0	.993874	.995489	.995404	.995409	.995409
38	0	.995274	.996581	.996511	.996516	.996515
39	0	.996362	.997415	.997357	.997361	.997361
40	0	.997216	.998059	.998012	.998015	.998015
41	0	.997885	.998556	.998518	.998520	.998520
42	0	.998402	.998931	.998900	.998902	.998902
43	0	.998794	.999210	.999185	.999187	.999187
44	0	.999094	.999418	.999399	.999400	.999400
45	0	.999323	.999575	.999560	.999561	.999561
46	0	.999497	.999692	.999680	.999681	.999680
47	0	.999628	.999777	.999768	.999768	.999768
48	0	.999725	.999839	.999832	.999832	.999832
49	0	.999798	.999885	.999879	.999879	.999879

distribution of claims for the portfolio under the collective Poisson model, with the Poisson parameter equal to the aggregate force of mortality. Thus the collective Poisson model can be regarded as the "first approximation" to the individual model.

IV. PARALLELS WITH THE COLLECTIVE CASE

A. It is now assumed that the reader is familiar with the Society's study note on risk theory [3]. It is straightforward to verify that the generating function of the frequency distribution,

$$f(x) = \sum_{n=0}^{\infty} p^{*n}(x) \frac{\lambda^n e^{-\lambda}}{n!} , \tag{20}$$

of a compound Poisson process is

$$R(z) = e^{Q(z)} , \tag{21}$$

where

$$Q(z) = \lambda P(z) - \lambda \tag{22}$$

and $P(z)$ is the generating function of $p(x)$. We define

$$\log f = Q(z) \tag{23}$$

and note that if g is the frequency function of another compound Poisson process, then so is $f * g$, and

$$\log (f * g) = \log f + \log g . \tag{24}$$

Intuitively, the convolution operation $*$ corresponds to combining portfolios. If $l = l(x)$ denotes the frequency function of the compound Poisson process for the *empty* portfolio,

$$\begin{aligned} l(0) &= 1 , \\ l(x) &= 0, \quad x > 0 , \end{aligned} \tag{25}$$

then

$$f * l = l * f = f , \tag{26}$$

so that $l(x)$ acts as unity if convolution is thought of as multiplication.

Define l/f to be the function whose generating function is

$$1/R(z) = e^{-Q(z)} , \tag{27}$$

where (27) is evaluated using the recursive relation given in equations (13) and (14). Then

$$f * (l/f) = l . \tag{28}$$

Letting

$$f/g = f * (l/g) , \tag{29}$$

we can define a "division" with respect to the convolution operation, with the caveat that l/f need not be a probability distribution. Calculations is this system can be performed by addition and subtraction of power series by virtue of formula (24). (The reader who is so inclined can formulate this discussion in terms of isomorphic Abelian groups.)

The algorithm is Section III can now be regarded as calculating successive values of the sequence f_k/g_k, where f_k and g_k are frequency functions of certain compound Poisson processes, the quotient is with respect to convolution, and

$$\lim_{k \to \infty} (f_k/g_k) = f/g . \tag{30}$$

The distribution of aggregate claims for the individual model is the quotient f/g; f and g are distributions of compound Poisson processes with parameters λ and $p(x)$, where

$$\lambda_x = \sum_D \sum_{H_y} \frac{y}{x} \left(\frac{q}{p}\right)^{x/y} , \tag{31}$$

$$\lambda = \sum_{x=1}^{\infty} \lambda_x , \tag{32}$$

and

$$p(x) = \lambda_x/\lambda . \tag{33}$$

Here, in the outer sum in (31), D indicates that the sum is taken over all divisors of x such that x/y is odd (*for* f) and even (*for* g).

B. Formula (20) may be generalized as

$$f(x) = \sum_{n=0}^{\infty} s(n)p^{*n}(x) , \tag{34}$$

where $s(n)$ is an arbitrary distribution on the collection of nonnegative integers—usually referred to as a counting distribution. If $S(z)$ is the generating function of $s(n)$, then it is easily verified that

$$R(z) = S(P(z)) . \tag{35}$$

Frequently, $s(n)$ is a standard a priori distribution that may make it possible to evaluate $f(x)$ in terms of (the experimentally determined) $p(x)$. For example, if $S(z)$ satisfies the first-order differential equation

$$A(z)\frac{dS}{dz} + B(z)S(z) + C(z) = 0 , \tag{36}$$

then

$$A(P(z))\frac{dR}{dz} + B(P(z))P'(z)R(z) + C(P(z))P'(z) = 0 , \tag{37}$$

which can be regarded as a first-degree differential equation in $R(z)$. If the functions $A(z)$, $B(z)$, and $C(z)$ are not "too complicated," then it may be practical to solve equation (37) by the method of undetermined coefficients, which, together with an initial value, will recursively produce the coefficients of $R(z)$. For example, if

$$A(z) = 1, \quad B(z) = -\lambda, \quad \text{and} \quad C(z) = 0 , \tag{38}$$

then we obtain the recursive relation (14) corresponding to the compound Poisson process.

As a second example, consider the relationship for certain constants a and b

$$s(n) = \left(a + \frac{b}{n}\right)s(n-1) , \tag{39}$$

satisfied by the Poisson, binomial, negative binomial, and geometric distributions, whose corresponding recursive formula was derived by Harry H. Panjer [2]. Given this relationship, then, equivalently,

$$A(z) = az - 1, \quad B(z) = a + b, \quad \text{and} \quad C(z) = 0 , \tag{40}$$

and solving equation (37) yields a recursive formula for the distribution of aggregate claims. Other variations on this theme, such as a recursive formula of the type in (39) but operative for only $n \geq N$, for a given N, can often be handled by suitable choices of $A(z)$, $B(z)$, and $C(z)$.

C. The essence of this section is not, of course, an attempt to derive all-inclusive general formulas. Rather, it is hoped that the reader is convinced that the devices employed in this discussion are not special "tricks" but represent general techniques. What has been done is to use to advantage, in a rather elementary way, to be sure, the structural similarity between different mathematical systems—a fundamental principle of mathematics.

V. REMARKS

In this section a number of admittedly subjective remarks are made on the possible role of risk theory in an individual life model office. A noteworthy fact is that the aggregate claims distribution in the individual model reflects *all* the mortality information about the underlying portfolio, in the sense that, starting with only the aggregate claims distribution, it is in theory possible to reconstruct the entire scheme of amount classes and mortality rates. This result, which can be verified by proving an appropriate unique factorization theorem for polynomials, points to a central role for the aggregate claims distribution.

Short-term models incorporating only the mortality risk can be constructed along the lines of Section III. Such a model could, for example, be of use in monitoring from year to year the ratio of actual to expected claims and would serve as a test of the appropriateness of the company's mortality tables. Short-term predictions of the probable fluctuations in claims and the resulting impact on cash flows could also be based on such a model.

A very difficult problem is the construction of long-term models within a risk-theoretic framework. A starting point could be a counterpart of ruin theory for the individual case; however, in the author's opinion the parallel should not be too finely drawn. From a technical point of view, the author believes that methods of finite combinatorics can be used to greater advantage than in collective risk theory. From a fundamental point of view a realistic long-range model will need to incorporate at least the traditional ingredients of the expense and investment risk and federal income taxes. It is at this point that one is truly in the middle of uncharted waters!

The point of view is frequently expressed that, for practical purposes, the mortality risk is negligible in proportion to the investment and expense risk and accounts in part, no doubt, for the meager use of risk theory in corporate modeling. This conclusion is based on the sound premise that the probability of a catastrophic mortality loss precipitating insolvency is small and is overshadowed by other, much more immediate, risks. The conclusion, however, contradicts the fact that, after all, the mortality risk is the fundamental risk that is covered by life insurance, and does not explain the considerable importance generally attached to reinsurance. The paradox is resolved by adopting a going-concern rather than a solvency perspective. Large swings in claims may not necessarily bring on insolvency but will wreak havoc with cash flows and the orderly operation of the company. Reinsurance costs are more appropriately viewed as the price to be paid for *smoothing* year-to-year claim costs rather than insurance against a remote contingency for which a stop-loss type of arrangement would be much more efficient. One must therefore look at the entire claims distribution curve rather than just the extreme right tail—a somewhat different emphasis from that of traditional risk theory.

The preceding remarks are only tentative opinions. Their intent is to stimulate interest in the risk theory aspects of model offices for individual life companies. The development of such a model is proposed as a research problem, with the frank admission that one cannot at this time even ask the right questions with sufficient precision.

VI. ACKNOWLEDGMENT

The author is indebted to John Beekman for his valuable comments on an earlier version of this paper.

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DISCUSSION OF PRECEDING PAPER

DAVID C. MC INTOSH:

Mr. Kornya's succinct and elegant paper appears likely to become set reading for students preparing to take the Society's examination in risk theory. It is therefore particularly unfortunate that the paper's central theorem should contain a serious mathematical error.

The error occurs in the purported proof of statement (D2). Given

$$\sum_{n=0}^{N-k} a_n \leq \sum_{n=0}^{N-k} b_n \quad \text{for all } k \in \{0, 1, \ldots, N\},$$

it does not follow that

$$\sum_{k=0}^{N} c_k \left(\sum_{n=0}^{N-k} a_n\right) \leq \sum_{k=0}^{N} c_k \left(\sum_{n=0}^{N-k} b_n\right)$$

for arbitrarily chosen real numbers $c_0, c_1, \ldots, c_N$. Mr. Kornya seems to be assuming that $c_0, c_1, \ldots, c_N$ are all nonnegative, a condition which does not apply to the coefficients of the polynomials $Q_k(z)$.

In fact, if we consider a portfolio with only one member, it is evidently false that

$$[Q_1(z)]^2 \leq [Q_2(z)]^2,$$

even though it is true that

$$Q_1(z) \leq Q_2(z).$$

Consequently, statement (D2) is invalid in the situation where Mr. Kornya wishes to apply it. (The partial ordering of real power series is that defined by Mr. Kornya for use in his proof.)

Of course, statement (D2) is merely an incidental step in the purported proof of statement (E). Statement (E) is not true for arbitrarily chosen real polynomials $A(z)$ and $B(z)$, and *a fortiori* it is not true for real power series in general. In order to see this, consider

$$A(z) = \log_e 2.7 - \frac{1}{2} z + \frac{7}{16} z^2,$$
$$B(z) = 1.$$

It is evident that $A(z) \leqslant B(z)$, since $2.7 < e$ and $7/16 < 1/2$. However, the third partial sums of coefficients of $e^{A(z)}$ and $e^{B(z)}$ are, respectively, 2.86875 and e. Therefore, since $2.86875 > e$, it is not true that $e^{A(z)} \leqslant e^{B(z)}$.

In order to determine whether part (ii) of Mr. Kornya's theorem is or is not true, it is necessary to establish whether or not statement (E) is true in the case where $A(z) = Q_k(z)$ and $B(z) = Q_j(z)$ for some positive integers k and j. This would require analysis of the particular polynomials $Q_k(z)$, the series $R_k(z)$, and the sums $F_k(N)$. Mr. Kornya's reasoning, involving general power series, simply is not valid.

In fact, it is not difficult to show that, for a given N, the sequence $F_k(N)$ falls into the desired pattern of alternately monotonic convergence toward $F(N)$ when $k \geqslant N$ (see below for a proof). Unfortunately, this result is too weak for the application that Mr. Kornya wishes to make. For that application it is necessary that the alternating pattern should be followed for all $k \geqslant 1$, even when N is a large amount. Mr. Kornya's numerical example suggests that the stronger result may indeed be true, but it remains unproved.

To prove the relatively weak result, let us adopt the following notation for a portfolio H: for each $h \in H$, write

n_h = Face amount of policy h (a positive integer);
q_h = Probability of claim under policy h;
$p_h = 1 - q_h$;
$r_h = q_h/p_h$.

For each positive integer m,

$$S_m(z) = -\sum_{h\in H} r_h^m + \sum_{h\in H} r_h^m z^{mn_h}.$$

$$R(z) = \sum_{t=0}^{\infty} a_t z^t = \exp\left[\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} S_m(z)\right]$$

$$= R_k(z) \quad \exp\left[\sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} S_m(z)\right] \qquad \text{for all } k \geqslant 1.$$

Therefore, for all positive integers k,

$$R_k(z) = R(z) \exp\left[- \sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} S_m(z) \right]$$
$$= \exp\left[\sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{h \epsilon H} r_h^m \right] R(z) \times$$
$$\exp\left[- \sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{h \epsilon H} r_h^m z^{mnh} \right].$$

Now, $R(z) = a_0 + a_1 z + \ldots + a_k z^k +$ terms of greater order than z^k. Using the methods described on pages 179–81 of Ahlfors [1], it can be shown that

$$\exp\left[- \sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{h \epsilon H} r_h^m z^{mnh} \right] = 1 +$$
terms of greater order than z^k,

and hence that

$$R_k(z) = \exp\left[\sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{h \epsilon H} r_h^m \right] \times$$

$$(a_0 + a_1 z + \ldots + a_k z^k + \text{terms of greater order than } z^k).$$

Therefore, for all $N \geqslant 1$ and for all $k \geqslant N$,

$$F_k(N) = F(N) \exp\left[\sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{h \epsilon H} r_h^m \right].$$

It follows, by considering the alternately monotonic convergence to zero of the sequence

$$t_k = \sum_{m=k+1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{h \epsilon H} r_h^m,$$

that the sequence $F_k(N)$ follows the desired alternating pattern of convergence to $F(N)$ when $k \geqslant N$.

As a final comment, it is worth noting that the various series in Mr.

Kornya's paper are divergent if $q_h > 1/2$ for some policy $h \in H$. Extremely high risks must therefore be excluded from the portfolio before applying the algorithm described in the paper.

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DONALD P. MINASSIAN:

If $R(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ is the generating function for the number of unit claims, then $F(N) \equiv \sum_{n=0}^{N} a_n$ is the probability that the number of unit claims does not exceed N. Further, if $R_k(z) \equiv e^{Qk(z)} \equiv \sum_{n=0}^{\infty} a_{n,k} z^n$ and if $F_k(N) \equiv \sum_{n=0}^{N} a_{n,k}$, the question has arisen whether we can have monotonic convergence, that is, specifically whether

$$F_1(N) \leqslant F_3(N) \leqslant \ldots \leqslant F(N) \leqslant \ldots \leqslant F_4(N) \leqslant F_2(N), \tag{1}$$

since $F_k(N)$ can readily be calculated for small k (see Kornya's paper). For future reference, this result would follow from

$$R_1(z) \leqslant R_3(z) \leqslant \ldots \leqslant R(z) \leqslant \ldots \leqslant R_4(z) \leqslant R_2(z), \tag{1a}$$

where $\leqslant$ is as defined below.

In this discussion we show (A) that the general answer is no (counterexamples are provided) and (B) that there *is* a monotonic convergence theorem but the algorithm is somewhat more complicated, and convergence slower, than is the case with the original Kornya algorithm. We omit empirical studies of the latter result, but hope that the algorithm will be so tested by others.

A. *Counterexamples to Formula (1)*

We present the development of this result, rather than simply the result itself, because it was the inspiration for Part B of this discussion.

We repeat a definition used by Kornya.

Definition. If $A(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ and $B(z) \equiv \sum_{n=0}^{\infty} b_n z^n$, then we say $A(z) \leq B(z)$ if, for every nonnegative integer N, we have $\sum_{n=0}^{N} a_n \leq \sum_{n=0}^{N} b_n$. (The motivation is that in the series for $R(z) \equiv \sum_{n=0}^{N} a_n z^n$ the probability $F(N)$ that the number of unit claims does not exceed N is $\sum_{n=0}^{N} a_n$.)

PROPOSITION 1. *Let $A(z)$, $B(z)$, and $\leq$ be as in the definition. Then $A(z) \leq B(z)$ if and only if, as power series, $A(z)/(1 - z) \leq_s B(z)/(1 - z)$, where $\leq_s$ (s for "strong") means that* each *coefficient of $A(z)/(1 - z)$ does not exceed the corresponding coefficient of $B(z)/(1 - z)$. In fact, a much stronger result holds: for* any fixed *nonnegative integer N, $\sum_{n=0}^{N} a_n$ does not exceed $\sum_{n=0}^{N} b_n$ in the original series $A(z)$ and $B(z)$ if and only if the Nth coefficient of $A(z)/(1 - z)$ does not exceed the Nth coefficient of $B(z)/(1 - z)$.*

Proof. The Nth coefficient of $A(z)/(1 - z)$ is precisely $\sum_{n=0}^{N} a_n$; similarly for $B(z)/(1 - z)$.

COROLLARY 1. *If $Q_i(z)$ and $Q_j(z)$ are any two power series, then as power series $e^{Q_i(z)} \leq e^{Q_j(z)}$ if and only if*

$$\exp\,[Q_i(z) + z + (z^2/2) + (z^3/3) + \ldots] \leq \exp\,[Q_j(z) + z + (z^2/2) + (z^3/3) + \ldots] \tag{2}$$

(where in general $e^{C(z)} \equiv \sum_{n=0}^{\infty} [C(z)]^n/n!$; we assume we are within the radius of convergence for $C(z)$, so order of terms is immaterial).

Proof. The expression on the left is

$$e^{Q_i(z) - \log(1 - z)} = e^{Q_i(z)}/e^{\log(1 - z)} = e^{Q_i(z)}/(1 - z). \tag{2'}$$

Similarly this is for the expression on the right.

Now consider $Q_1(z)$ and $Q_3(z)$, with definitions as in Kornya's paper. While it is true that $Q_1(z) \leq Q_3(z)$, the crucial question respecting formula (1) above is whether $e^{Q_1(z)} \leq e^{Q_3(z)}$. By Corollary 1, this holds if and only if formula (2) holds where we take $Q_i(z) \equiv Q_1(z)$ and $Q_j(z) \equiv Q_3(z)$. We further note (calculational details omitted):

LEMMA. *If* $C(z) \equiv \sum_{i=0}^{\infty} c_i z^i$ *is any power series inside its radius of convergence and if* $e^{C(z)} \equiv \sum_{i=0}^{\infty} e_i z^i$, *then*

$$\begin{aligned} e_0 &= e^{c_0} \\ e_1 &= (1 + c_1)e^{c_0} \\ e_2 &= (c_1^2 + 2!c_2)e^{c_0}/2! \\ e_3 &= (c_1^3 + 6c_1c_2 + 3!c_3)e^{c_0}/3! \\ e_4 &= (c_1^4 + 12c_1^2c_2 + 24c_1c_3 + 12c_2^2 + 4!c^4)e^{c_0}/4!\ . \end{aligned} \tag{3}$$

(Note: at this point we require only the expression for e_2; we include the rest for future reference.)

Now let $A(z) \equiv \sum_{i=0}^{\infty} a_i z^i$ represent the exponent on the left-hand side of formula (2), where $Q_1(z)$ replaces $Q_i(z)$, and let $B(z) \equiv \sum_{i=0}^{\infty} b_i z^i$ represent the exponent on the right, where $Q_3(z)$ replaces $Q_j(z)$. Were $e^{Q_1(z)} \leq e^{Q_3(z)}$, then formula (2) would hold, where $Q_1(z)$, $Q_3(z)$ replace $Q_i(z)$, $Q_j(z)$. So, by the lemma,

$$\text{“}e_2 \text{ for } e^{Q_1(z)}\text{” does not exceed “}e_2 \text{ for } e^{Q_3(z)}\text{”; i.e.,} \tag{4}$$

$$\left(\frac{a_1^2 + 2a_2}{2}\right) e^{a_0} \leq \left(\frac{b_1^2 + 2b_2}{2}\right) e^{b_0}.$$

Now assume a portfolio of insureds where each insured has precisely one unit of insurance. Then $Q_1(z)$ lacks the quadratic term. Hence (compare Kornya) a_2 (which is the coefficient from adding $(z^2/2)$ to the quadratic term—cf. formula (2)) is ½. Also, in this case $b_1 = a_1 = 1 + \Sigma\,(q/p)$; $b_2 = ½ - ½\,\Sigma\,(q/p)^2$; and $b_0 - a_0 = ½\,\Sigma\,(q/p)^2 - ⅓\,\Sigma\,(q/p)^3$. Multiplying both sides of formula (4) by $2e^{-a_0}$ and using the expressions above, we see that (4) is equivalent to

$$[1 + \Sigma\,(q/p)]^2 + 1 \leqslant \left\{[1 + \Sigma\,(q/p)]^2 + 1 - \Sigma\,(q/p)^2\right\} \times \qquad (4')$$

$$\exp\,[{}^1\!/_2\,\Sigma\,(q/p)^2 - {}^1\!/_3\,\Sigma\,(q/p)^3]\ .$$

Now assume one insured (so each Σ above reduces to one term), and let $q/p = 1$—perhaps inadmissible in view of convergence-of-series problems, but we remedy this below.

We obtain

$$5 \leqslant 4e^{1/6}, \qquad (4'')$$

which is *not* true, since $5 > 4e^{1/6}$. Hence, by continuity arguments, if we take q/p slightly less than 1, which *is* admissible, formula (4″) is still violated and our counterexample is established. In fact, checking (4′) above, we can have any number of insureds in our counterexample as long as precisely one insured has q/p sufficiently close to (but less than) 1, and all remaining insureds have q/p sufficiently close to zero (still assuming that each insured is insured for one unit).

In fact, a second—and perhaps more serious—counterexample is provided under the same assumptions (each insured is insured for one unit; one insured has q/p close to 1, and all remaining q/p's are close to zero). In this case it will be seen that e_4 (see formula (3) above) with respect to $R_1(z) \equiv e^{Q1(z)}$ exceeds e_4 for $R_2(z) \equiv e^{Q2(z)}$; so $R_1(z) \leqslant R_2(z)$ does not hold—a serious contradiction of the desired formula (1a) above. I shall not provide computational details here; the reader can recreate them (or I will mail them to interested parties). In fact, in this case Kornya has found that $R(z)$ is not even caught between $R_1(z)$ and $R_2(z)$.

B. *A Monotonic Convergence Algorithm*

First we point out that if, as power series, $A(z) \leqslant B(z)$ and $C(z) \leqslant D(z)$, then $A(z) + C(z) \leqslant B(z) + D(z)$ (proof is immediate), and *if all coefficients in all series are positive,* then $A(z)C(z) \leqslant B(z)D(z)$. The proof is on page 828 of the Kornya paper, where the author also shows the following:

LEMMA. *If $A(z)$ and $B(z)$ are power series with positive coefficients and if $A(z) \leqslant B(z)$, then $e^{A(z)} \leqslant e^{B(z)}$ (assuming that both $A(z)$ and $B(z)$ are inside their radius of convergence).*

Now assume for the moment that all coefficients in the exponents of (2) are positive except for the constant term (see Kornya's definition of $Q_i(z)$). We discuss the validity of this "positivity" assumption below. Let $A_i(z)$ and $A_j(z)$ represent the power series for the exponents of (2). Now, for any two power series, $C(z) \leqslant D(z)$ if and only if $kC(z) \leqslant kD(z)$ for any positive constant k. Thus $e^{A_i(z)} \leqslant e^{A_j(z)}$ if and only if, for any real r, $e^{r + A_i(z)} \leqslant e^{r + A_j(z)}$. Hence, by taking r large enough, we may assume that the constant terms of $r + A_i(z)$ and $r + A_j(z)$ are positive. Hence, by our assumption that all other coefficients of $A_i(z)$ and $A_j(z)$ are positive, we may assume that all coefficients of $A_i(z)$ and $A_j(z)$ below are positive. Now, following Kornya we note that

$$Q_1(z) \leqslant Q_3(z) \leqslant \ldots \leqslant Q(z) \leqslant \ldots \leqslant Q_4(z) \leqslant Q_2(z), \tag{5}$$

and hence

$$A_1(z) \leqslant A_3(z) \leqslant \ldots \leqslant A(z) \leqslant \ldots \leqslant A_4(z) \leqslant A_2(z) \tag{5'}$$

in view of the additive property of $\leqslant$; (recall that each $A_i(z) = Q_i(z) - \log(1 - z)$). Thus, since we assume that each coefficient of each $A_i(z)$ is positive, we have, from the Lemma,

$$e^{A_1(z)} \leqslant e^{A_3(z)} \leqslant \ldots \leqslant e^{A(z)} \leqslant \ldots \leqslant e^{A_4(z)} \leqslant e^{A_2(z)}, \tag{6}$$

or, equivalently, using the Kornya notation ($R_i(z) \equiv e^{Q_i(z)}$),

$$\frac{R_1(z)}{1-z} \leqslant \frac{R_3(z)}{1-z} \leqslant \ldots \leqslant \frac{R(z)}{1-z} \leqslant \ldots \leqslant \frac{R_4(z)}{1-z} \leqslant \frac{R_2(z)}{1-z}. \tag{6'}$$

This yields our algorithm. For, expanding upon the steps in the Kornya paper (p. 829), we proceed as follows:

A.1. Compute the polynomials $Q_k(z)$ and $Q_{k+1}(z)$ for odd integer k.

A.2. Apply the recursive formula (given by Kornya) to compute the coefficients of $R_k(z)$ and $R_{k+1}(z)$.

A.3. Calculate the coefficients of $R_k(z)/(1 - z)$ and $R_{k+1}(z)/(1 - z)$, a straightforward computation if we recall that the Nth coefficient of $A(z)/(1 - z)$ for any $A(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ is $\sum_{n=0}^{N} a_n$. Note that the sum of the first N coefficients of $R(z)/(1 - z)$ is, by definition of $\leq$, caught between the sum of the first N coefficients of $R_k(z)/(1 - z)$ and $R_{k+1}(z)/(1 - z)$ for each nonnegative integer N.

A.3.′ Calculate the *sum* of the first i coefficients of $R_k(z)/(1 - z)$ and $R_{k+1}(z)/(1 - z)$ for *all* $i = 0, 1, 2, \ldots, N$ (recall that the subscript for the constant term is zero). Thus, if $R(z) \equiv \sum_{n=0}^{\infty} r_n z^n$, and recalling that the ith coefficient of $R(z)/(1 - z)$ is $\sum_{n=0}^{i} r_n$, we have bounded respectively $r_0, r_0 + (r_0 + r_1) \equiv 2r_0 + r_1, (2r_0 + r_1) + (r_0 + r_1 + r_2) \equiv 3r_0 + 2r_1 + r_2, \ldots, (N + 1)r_0 + Nr_1 + \ldots + r_N$. From this we can calculate the desired bounds, that is, for r_0, $r_0 + r_1$, $r_0 + r_1 + r_2$, etc., where $\sum_{n=0}^{N} r_n$ is the probability that the number of unit claims does not exceed N. We calculate these bounds by adding (actually subtracting) the inequality bounds for $R(z)/(1 - z)$. For example, if $a \leqslant r_0 \leqslant b$ and $c \leqslant 2r_0 + r_1 \leqslant d$, then $c - b \leqslant r_0 + r_1 \leqslant d - a$, etc. Similarly, we may calculate bounds for $r_0 + r_1 + r_2$, $r_0 + r_1 + r_2 + r_3, \ldots, r_0 + r_1 + r_2 \ldots + r_N$.

A.4. Increase the value of k, if desired, for greater accuracy.

Of course, subtracting the above inequalities diminishes accuracy; i.e., convergence is slowed. At least, however, the accuracy loss accumulates "arithmetically" instead of "geometrically." Thus, if our original bounds on the sums of the coefficients of $R(z)/(1 - z)$ are very tight (as the computer study in the Kornya paper suggests is possible), we may get reasonable accuracy—particularly when the number of insureds is not too great and mortalities are reasonably low. Elementary hand computation shows, for example, that six-decimal-place accuracy in the bounds for $R(z)/(1 - z)$ might achieve two- or three-decimal-place accuracy in the bounds for $R(z)$—accuracy sufficient for many applications.

Finally, how reasonable is the supposition that all coefficients in the power series for each $Q_i(z) - \log(1 - z)$ are positive (except the constant term)? Recall that we assumed this in our algorithm.

First, if this is *not* so, we can continue dividing by $1 - z$ until we find an n such that all coefficients (first degree and higher) for

$$\frac{e^{Q_i(z)}}{(1 - z)^n} \equiv e^{Q_i(z) - n \log(1 - z)} \equiv e^{Q_i(z) + nz + (nz^2/2) + \ldots} \tag{7}$$

are positive. Indeed, we can find a single n that works for *all* $Q_i(z)$ and for

$Q(z)$, since the $Q_i(z)$ converge uniformly to $Q(z)$ and since the coefficients of $Q(z)$ are bounded (the series for $Q(z)$ converges for $z = 1$) if—as assumed tacitly by Kornya—each $q/p < 1$; thus, with only a finite number of q/p's, the q/p's are bounded away from 1.

Having found such an n, we find, in order, bounds for $R(z)/(1 - z)^n$, $R(z)/(1 - z)^{n-1}, \ldots, R(z)/(1 - z), R(z)$, using the methodology described above. Now clearly the speed of convergence lessens "geometrically" as n increases, so we do not wish n to be very large!

Second, returning to the desirable case $n = 1$, we can point to some fairly realistic situations where all coefficients (except the constant term) for each $Q_i(z) - \log(1 - z)$ and for $Q(z) - \log(1 - z)$ are positive.

Except for the constant term, the largest (in absolute value) possible negative coefficient in *any* $Q_i(z)$ is

$$-\frac{1}{2}\sum\left(\frac{q}{p}\right)^2 - \frac{1}{4}\sum\left(\frac{q}{p}\right)^4 - \frac{1}{6}\sum\left(\frac{q}{p}\right)^6 - \ldots, \tag{8}$$

whose absolute value is less than that of

$$-\frac{1}{2}\left[\sum\left(\frac{q}{p}\right)^2 + \sum\left(\frac{q}{p}\right)^4 + \sum\left(\frac{q}{p}\right)^6 + \ldots\right]. \tag{8$'$}$$

Now let L be the largest q/p, and, using the formula for summing a geometric series, the absolute value of (8′) does not exceed that of

$$-\frac{1}{2}(ML^2)/(1 - L^4) \doteq -\frac{1}{2}ML^2, \tag{8$''$}$$

where M is the total number of policyholders. (We assume that L is reasonably small, so L^4 is relatively insignificant.) Now suppose that $L \leq 0.001$ (as would be the case at most younger ages for annual q), and suppose that the maximum number of units of insurance is 100 (the unit might be \$10,000, so that an insured can buy up to \$1,000,000 face value). The worst that can happen for any $Q_i(z)$ is that all the negatives accumulate in the coefficient for z^{200}. This is "worst" because the $(q/p)^2$ terms—by far the worst culprits—must be included among coefficients for z^{2c}, where $c = 1, 2, \ldots, 100$ under our assumption of a maximal 100 units of insurance; further, the coefficients of the $-\log(1 - z)$ series (which, we recall, is added to the series for $Q_i(z)$) hardly decrease beyond this point. They are $\frac{1}{200}$, $\frac{1}{201}$, $\frac{1}{202}$, etc., whereas the coefficients of the $Q_i(z)$ series decrease "geometrically."

Now when we add the log series to $Q_i(z)$, the coefficient of z^{200} is in the worst case $(\frac{1}{200}) - \frac{1}{2}ML^2$ under our assumptions. Thus, under the assumption $L \leq 0.001$, we can have 10,000 policyholders (M) and still all coefficients in the series expansion of $Q_i(z) - \log(1 - z)$, save the constant, will be positive. Obviously, other "mixes" can be had. For example, if we increase the unit claim size, we can increase the number of policyholders. Projected declines in mortalities help our case: for the typical pools of insureds with ages in the twenties or thirties, we could have 100,000 or perhaps 1,000,000 policyholders without violating the required positivity. This is particularly true if we note that the above derivation was a worst-case scenario in that all the negativity went to the "highest" power of z, here z^{200}. This will probably never arise in practice. Indeed, one could, in a computer study for a particular pool of insureds, "spread" the $(q/p)^i$'s to their respective coefficients. It would probably suffice to so assign only the $(q/p)^2$'s and $(q/p)^1$'s, the dominant terms in the coefficients of $Q_i(z)$, and then add in the $-\log(1 - z)$ series to see whether any term (save the constant term) had a negative coefficient. My guess is that for most pools of insureds we would have satisfactory results at all but the advanced ages, particularly if unit claim size were large.

ELIAS S.W. SHIU:

Mr. Kornya is to be complimented for an elegant paper. I would like to make several comments.

The algorithm in Section II works very well when the values of q are small. However, if there is a q with value greater than $\frac{1}{2}$, then the algorithm breaks down, since the logarithmic series

$$\sum (-1)^{k+1}x^k/k$$

diverges for $x > 1$. As the logarithmic series converges rapidly only for small x (cf. [4], sec. 6.5), it might be advisable to evaluate $R_H(z)$ as

$$\left[\prod_{q \text{ large}} (p + qz^n)\right]\left[\prod_{q \text{ small}} (p + qz^n)\right],$$

where the first product contains relatively few factors and the second product can be computed efficiently using the algorithm.

There is another modification I would like to suggest. Since

$$\Pi\,(p + qz^n) = \left(\Pi\, p\right) \Pi \left(1 + \frac{q}{p} z^n\right),$$

one needs only to apply the algorithm to find

$$\Pi \left(1 + \frac{q}{p} z^n\right).$$

In this case one can state a stronger theorem. Instead of the result

$$\lim_{k \to \infty} F_k(N) = F(N),$$

one has

$$F_k(N) = F(N)$$

for each $k \geqslant N/\min\{n\}$.

To understand this paper, I found it instructive to consider the special case where each policy has identical face amount, that is, $H = H_n$ for some n. The problem then becomes the evaluation of the coefficients of the polynomial

$$g(t) = \prod_H \left(1 + \frac{q}{p} t\right).$$

Interesting algorithms for this problem can be found in [8]. Following the method in this paper, consider

$$g(t) = e^{\log g(t)}.$$

Equation (9) of the paper becomes

$$\begin{aligned} \log g(t) &= \sum_H \log \left(1 + \frac{q}{p} t\right) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_H \left(\frac{q}{p}\right)^k t^k. \end{aligned}$$

If one defines

$$S_k = \sum_H \left(\frac{q}{p}\right)^k ,$$

then (cf. [5], p. 92, eq. [34])

$$\begin{aligned} g(t) &= \exp(S_1 t) \exp(-S_2 t^2/2) \exp(S_3 t^3/3) \ldots \\ &= (1 + S_1 t + S_1^2 t^2/2! + \ldots)(1 - S_2 t^2/2 \\ &\quad + S_2^2 t^4/2^2 2! - \ldots) \ldots \\ &= \sum_{m \geq 0} a_m t^m , \end{aligned}$$

where

$$a_m = \sum_{\substack{k_1, k_2, \ldots, k_m \geq 0 \\ k_1 + 2k_2 + \ldots + mk_m = m}} (-1)^{k_2 + k_4 + \ldots} \frac{S_1^{k_1}}{1^{k_1} k_1!} \frac{S_2^{k_2}}{2^{k_2} k_2!} \cdots \frac{S_m^{k_m}}{m^{k_m} k_m!}$$

It is pointed out in Section IV that if a function S satisfies a simple differential equation, then the coefficients of the power series of $S(P(z))$ can be computed recursively. If $S(x) = x^{\alpha}$, α arbitrary, then the resulting recursive formula is called the *J. C. P. Miller formula* in the computer science literature ([2]; [3], Theorem 1.6c; [6], pp. 445–46). Formulas such as equations (19) and (23) of [7] and equations (8) and (9) of [1] are particular cases of the J. C. P. Miller formula. The lemma in Section II presents the recursive formula for the coefficients of the power series of $S(P(z))$, where $S(x) = e^x$; this result also appears as problem 6 on page 43 of [3] and exercise 4 on page 450 of [6], with the answer given on page 561. Several interesting recursive formulas are given in [1].

This paper has provided an interesting introduction to the manipulation of generating functions and power series. For the readers who wish to take a *second course* on this topic, I would recommend section 4.7 of [6] and chapter 1 of [3].

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BEDA CHAN AND PROMOD K. SHARMA*:

Mr. Kornya presented a new, interesting, and useful algorithm for the distribution of aggregate claims. In this discussion we review two traditional algorithms.

The traditional way of computing the aggregate claims distribution is by convolution [1]. We use the example in Setion III of Mr. Kornya's paper for illustration.

For Table 1 of the paper, let

n_{ij} = Number of policies in row i, column j;
q_j = Mortality rate for policies in column j;
N_{ij} = Number of claims arising from n_{ij};
S = Total amount of claims.

Then

$$N_{ij} \sim \text{Binomial } (n_{ij}, q_j),$$

and

$$S = \sum_{i=1}^{5} \sum_{j=1}^{5} iN_{ij} .$$

The aggregate claims distribution computed by convolution is given in the second column of Table 1 of this discussion. In our computation, we modify

* Mr. Sharma, not a member of the Society, is an honor actuarial science student at the University of Western Ontario.

the APL programs supplied in [3]. Our figures agree completely with Mr. Kornya's Table 3, last column.

In convoluting two probability vectors $\{p_i | \sum_{i=0}^{m} p_i = 1, p_i \geq 0\}$ and $\{q_j | \sum_{j=0}^{n} q_j = 1, q_j \geq 0\}$ using APL, an $(m + 1) \times (n + 1)$ matrix is formed. In a small-size example such as the one illustrated here, the matrix has grown too large to fit into the workspace after a few convolutions. We solve this problem by truncating the probability vectors. The truncation error is held under control by the following propositions. Proposition 1 states that the error of a truncation is bounded by the sum of the truncated tails. Proposition 2 gives a bound on the size of the tail of a binomial distribution.

Proposition 1. *Let* $\boldsymbol{p} = \{p_i | \sum_{i=0}^{m} p_i = 1, p_i \geq 0\}$, $\boldsymbol{q} = \{q_j | \sum_{j=0}^{n} q_j = 1, q_j \geq 0\}$, $\tilde{\boldsymbol{p}} = \{p_i | i = 0, \ldots, \tilde{m}$ *with* $\tilde{m} \leq m\}$, *and* $\tilde{\boldsymbol{q}} = \{q_j | j = 0, \ldots, \tilde{n}$ with $\tilde{n} \leq n\}$.
Then

$$(p*q)_k \geq (\tilde{p}*\tilde{q})_k$$

and

$$\sum_{k=0}^{m+n} [(\boldsymbol{p}*\boldsymbol{q})_k - (\tilde{\boldsymbol{p}}*\tilde{\boldsymbol{q}})_k] \leq \sum_{i=\tilde{m}+1}^{m} p_i + \sum_{j=\tilde{n}+1}^{n} q_j .$$

Proof. Straightforward, by using the definition of convolution and the fact that $\boldsymbol{p}$ and $\boldsymbol{q}$ are probability vectors.

Proposition 2.

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1 - p)^{n-k} = \int_0^p \frac{\Gamma(n + 1)}{\Gamma(r)\Gamma(n - r + 1)} x^{r-1}(1 - x)^{n-r}dx < \binom{n}{r} p^r .$$

Proof. The equality is from [2; p. 173, problem 45]. The inequality requires using $(1 - x)^{n-r} \leq 1$ for $0 \leq x \leq p$, and integration.

One traditional way of approximating the aggregate claims distribution is by fitting a normal distribution. Note that

$$E(S) = \sum_{i=1}^{5}\sum_{j=1}^{5} iE(N_{ij}) = \sum_{i=1}^{5}\sum_{j=1}^{5} in_{ij}q_j = 14.21462,$$

$$\begin{aligned}\text{Var}(S) &= \sum_{i=1}^{5}\sum_{j=1}^{5} i^2 \,\text{Var}(N_{ij}) \\ &= \sum_{i=1}^{5}\sum_{j=1}^{5} i^2\, n_{ij}q_j(1-q_j) = 56.9594007622.\end{aligned}$$

The approximation by a normal distribution with matching mean and variance is displayed in column 3 of Table 1 of this discussion.

We share Mr. Kornya's view that risk theory computation should be applied more often. One reason for its infrequent use is probably the volume of computation needed for convolution. Indeed, in a small-size example such as the one considered here, FORTRAN requires a long time for looping and APL requires very large workspace for huge matrices. Our solution here is to truncate the probability vector. Mr. Kornya's elegant solution is to construct a sequence F_k squeezing the true distribution F. If a rough, quick answer is desired, one may use the normal approximation. Compare columns 2 and 3 in Table 1.

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TABLE 1

Aggregate Claims	Cumulative Distribution by Convolution	Cumulative Distribution by Normal Aproximation	Aggregate Claims	Cumulative Distribution by Convolution	Cumulative Distribution by Normal Approximation
0	.015442	.345938D − 01	35	.992126	.997601D 00
1	.0302366	.460239D − 01	36	.993964	.998426D 00
2	.0390382	.603081D − 01	37	.995409	.998983D 00
3	.0504517	.778487D − 01	38	.996515	.999354D 00
4	.0800628	.990138D − 01	39	.997361	.999596D 00
5	.1249	.124109D 00	40	.998015	.999752D 00
6	.159828	.153345D 00	41	.99852	.999850D 00
7	.189172	.186816D 00	42	.998902	.999911D 00
8	.230867	.224468D 00	43	.999187	.999948D 00
9	.2898	.266088D 00	44	.9994	.999970D 00
10	.348223	.311293D 00	45	.999561	.999983D 00
11	.394631	.359540D 00	46	.99968	.999991D 00
12	.440744	.410139D 00	47	.999768	.999995D 00
13	.496174	.462282D 00	48	.999832	.999997D 00
14	.554676	.515082D 00	49	.999879	.999999D 00
15	.604881	.567618D 00	50	.999914	.999999D 00
16	.647368	.618984D 00	51	.999939	.100000D 01
17	.689853	.668333D 00	52	.999956	.100000D 01
18	.733376	.714920D 00	53	.999969	.100000D 01
19	.772683	.758135D 00	54	.999978	.100000D 01
20	.80506	.797526D 00	55	.999985	.100000D 01
21	.83326	.832807D 00	56	.999989	.100000D 01
22	.859889	.863858D 00	57	.999993	.100000D 01
23	.884099	.890711D 00	58	.999995	.100000D 01
24	.90426	.913530D 00	59	.999996	.100000D 01
25	.920704	.932584D 00	60	.999997	.100000D 01
26	.934887	.948218D 00	61	.999998	.100000D 01
27	.947376	.960823D 00	62	.999999	.100000D 01
28	.95784	.970809D 00			
29	.966197	.978583D 00			
30	.972947	.984529D 00			
31	.978586	.988999D 00			
32	.98325	.992300D 00			
33	.986952	.994696D 00			
34	.989837	.996404D 00			

(AUTHOR'S REVIEW OF DISCUSSION)

PETER S. KORNYA:

I would like to thank the discussants for their stimulating comments.

I am indebted to Mr. McIntosh for pointing out the error in the proof of the theorem and for giving me the opportunity to correct it. The reader should *disregard* the theorem in Section II and replace it with the following:

THEOREM. *Suppose that* $q \leq 1/3$ *for each policy in the portfolio H and for* $k = 1, 2, 3, \ldots$ let

$$Q_k(z) = S_1(z) - \frac{1}{2} S_2(z) + \ldots + \frac{(-1)^{k+1}}{k} S_k(z), \tag{16 a}$$

$$Q(z) = \lim_{k \to \infty} Q_k(z) = \log R(z), \tag{16b}$$

$$R_k(z) = e^{Q_k(z)} = \sum_{n=0}^{\infty} a_{n,k} z^n, \tag{17}$$

$$F_k(N) = \sum_{n=0}^{N} |a_{n,k}|. \tag{18}$$

and

$$\epsilon = \frac{3}{k+1} \sum_H \left(\frac{q}{p}\right)^{k+1}; \tag{19}$$

then

$$|F(N) - F_k(N)| \leq e^{\epsilon} - 1 \tag{i}$$

and

$$\lim_{k \to \infty} F_k(N) = F(N). \tag{ii}$$

Proof. If $A(z)$ and $B(z)$ are power series, say that $A(z) \leq B(z)$ provided that, for any nonnegative integer N, the sum of the first $N + 1$ coefficients satisfies

$$\sum_{N=0}^{N} a_n \leq \sum_{n=0}^{N} b_n. \tag{A}$$

Also define $|A(z)|$ to be the power series

$$|A(z)| = \sum_{n=0}^{\infty} |a_n| z^n. \tag{B}$$

Then it is easily shown that

$$A(z) \leqslant |A(z)|, \tag{C1}$$

$$|A(z) + B(z)| \leq |A(z)| + |B(z)|, \tag{C2}$$

$$|A(z)\, B(z)| \leqslant |A(z)|\, |B(z)|. \tag{C3}$$

$$\text{If } |A(z)| \leqslant |B(z)| \text{ and } |C(z)| \leqslant |D(z)|, \tag{C4}$$

$$\text{then } |A(z)|\, |C(z)| \leqslant |B(z)|\, |D(z)|,$$

and, using equations (13) and (14), we have the following:

$$|e^{A(z)}| \leqslant e^{|A(z)|}. \tag{C5}$$

Note that an arbitrary constant can be regarded as a power series with all but the first term equal to zero. Then, using (C1), (C2), (C3), (C4), and (C5), we have the following:

$$\begin{aligned} &|Q(z) - Q_k(z)| \\ &= \left| \sum_H \sum_{j=1}^{\infty} \frac{(-1)^{k+j}}{k+j}\left(\frac{q}{p}\right)^{k+j} + \frac{(-1)^{k+j+1}}{k+j}\left(\frac{q}{p}\right)^{k+j} z^{n(k+j)} \right| \\ &\leqslant \sum_H \left| \sum_{j=1}^{\infty} \frac{(-1)^{k+j}}{k+j}\left(\frac{q}{p}\right)^{k+j} \right| + \sum_H \sum_{j=1}^{\infty} \frac{1}{k+j}\left(\frac{q}{p}\right)^{k+j} z^{n(k+j)} \\ &\leqslant \frac{1}{k+1} \sum_H \left(\frac{q}{p}\right)^{k+1} + \frac{1}{k+1} \sum_H \sum_{j=1}^{\infty} \left(\frac{q}{p}\right)^{k+j} \end{aligned} \tag{D}$$

$$= \frac{1}{k+1} \sum_H \left(\frac{q}{p}\right)^{k+1} + \frac{1}{k+1} \sum_H \left(\frac{q}{p}\right)^{k+1} \left(\frac{1}{1-q/p}\right)$$

$$\leqq \epsilon.$$

$q \leqslant 1/3$ is used in the last step. Now note that

$$|R(z)| = |R_k(z)\, e^{Q(z) - Q_k(z)}| \leqslant |R_k(z)|\, e^{|Q(z)-Q_k(z)|}, \tag{E}$$

so that

$$F(N) \leqslant F_k(N) e^{\epsilon}. \tag{F}$$

Similarly,

$$F_k(N) \leqslant F(N)\, e^{\epsilon}; \tag{G}$$

therefore,

$$F_k(N) - F(N) \leqslant (e^{\epsilon} - 1)\, F(N) \leqslant e^{\epsilon} - 1 \tag{H}$$

and

$$F(N) - F_k(N) \leqslant (1 - e^{-\epsilon})\, F(N) \leqslant 1 - e^{-\epsilon} \leqslant e^{\epsilon} - 1, \tag{I}$$

from which (i) follows. Statement (ii) is an immediate consequence of (i).

The algorithm now proceeds as follows:

A.1. Choose a value of k for which the magnitude of error

$$e^{\epsilon} - 1$$

is sufficiently small. A value of $k = 5$ will, for a typical insurance portfolio of several hundred thousand policies, generally yield results accurate to the fifth decimal place.

A.2. Compute the polynomial $Q_k(z)$ and apply the recursive formulas (13) and (14) to compute the coefficients of $R_k(z)$. By the theorem, a conservative estimate for the distribution of aggregate claims is

$$F(N) \leq F_k(N) + e^{\epsilon} - 1.$$

Dr. Minassian has pointed out that it is not necessarily true that

$$F_1(N) \leq F_3(N) \leq \ldots \leq F(N) \leq \ldots \leq F_4(N) \leq F_2(N). \qquad \text{(J)}$$

His counterexamples saved me from heroic attempts to prove the original version of the theorem. As far as the *algorithm* is concerned, however, the inequality (J) is not really necessary—all that is needed is an efficient *bound* on the error $|F(N) - F_k(N)|$. There is no need to modify the algorithm described in Section II.

Dr. Shiu and others point out that values of q must be small for the algorithm to work well. Although $q < ½$ is the absolute requirement for convergence, the values of q should be considerably smaller for *efficient* convergence. The algorithm is intended to apply to a typical insurance portfolio with a one year horizon where the q's are generally quite small. Taking $k = 5$, for example, the algorithm evaluates the claims distribution to within six decimal places for a portfolio of 322,000 policies distributed according to the example in Section III. His suggestion to partition the portfolio into subportfolios can be extended to partitioning by other categories, such as male/female, standard/substandard, etc. The suggested modification for which

$$F(N) = F_k(N) \qquad \text{(K)}$$

is, I believe, of limited practical importance because of the large value of k generally required. In the example just cited, one needs $k = 14{,}215$ just to ensure equality up to the *expected* amount of claims. The suggested algorithm is in fact tantamount to an *exact* evaluation of the coefficients of the polynomial $R(z)$.

Dr. Shiu lists many excellent references, to which I would like to add one more. The algorithm described in Section II is really nothing but Newton's formula for the relationship between the coefficients and the sums of powers of the roots of a polynomial.

Dr. Chan and Mr. Sharma give an alternative method of computation by direct convolutions, with truncation of the tail end of the distribution to cut down on the volume of computation. This method, I suspect, still involves a considerable amount of computation in order to suitably approximate portfolios with a substantial number of policies. In such a case, their suggestion of using the normal distribution may work quite well in view of the convergence of the binomial to the normal distribution. A useful result would be a practical bound on the *degree of error* in the normal approximation to the aggregate claims distribution.