Fuzzy Regression Models

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Abstract

Recent articles, such as McCauley-Bell et al. (1999) and Sánchez and Gómez (2003a, 2003b, 2004), used fuzzy regression (FR) in their analysis. Following Tanaka et. al. (1982), their regression models included a fuzzy output, fuzzy coefficients and an non-fuzzy input vector. The fuzzy components were assumed to be triangular fuzzy numbers (TFNs). The basic idea was to minimize the fuzziness of the model by minimizing the total support of the fuzzy coefficients, subject to including all the given data.

The purpose of this article is to revisit the fuzzy regression portions of the foregoing studies and to discuss issues related to the Tanaka approach, including a consideration of fuzzy least-squares regression models.

Keywords: fuzzy linear regression, fuzzy least-squares regression, fuzzy coefficients, possibilistic regression, term structure of interest rates

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1 Introduction

Recent articles, such as McCauley-Bell et al. (1999) and Sánchez and Gómez (2003a, 2003b, 2004), used fuzzy regression (FR) in their analysis. The former use it to predict the relationship of known risk factors to the onset of occupational injury, while the latter used it to investigate the term structure of interest rates (TSIR). Following Tanaka et. al. (1982), their models took the general form:

$$\widetilde{Y} = \widetilde{A}_0 + \widetilde{A}_1 x_1 + \cdots + \widetilde{A}_n x_n \qquad (1)$$

where $\widetilde{Y}$ is the fuzzy output, $\tilde{A}_i$, j=1,2,..., n, is a fuzzy coefficient, and $\mathbf{x} = (x_1, ..., x_n)$ is an n-dimensional non-fuzzy input vector. The fuzzy components were assumed to be triangular fuzzy numbers (TFNs). Consequently, the coefficients, for example, can be characterized by a membership function (MF), $\mu_A(a)$, a representation of which is shown in Figure 1.

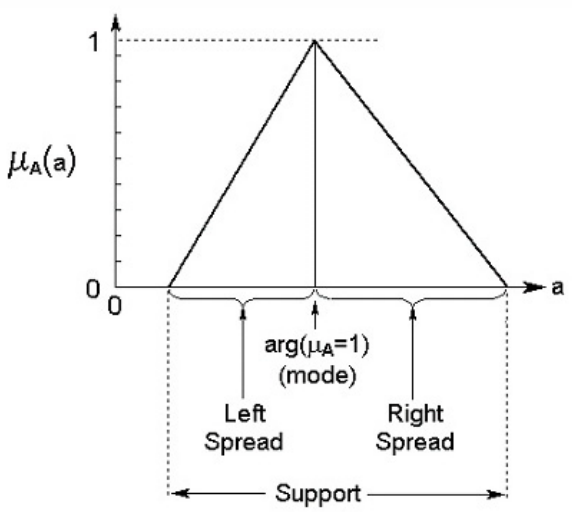


Figure 1: Fuzzy Coefficient

As indicated, the salient features of the TFN are its mode, its left and right spread, and its support. When the two spreads are equal, the TFN is known as a symmetrical TFN (STFN).

The basic idea of the Tanaka approach, often referred to as possibilistic regression, was to minimize the fuzziness of the model by minimizing the total spread of the fuzzy coefficients, subject to including all the given data.

The purpose of this article is to revisit the fuzzy regression portions of the foregoing studies, and to discuss issues related to the Tanaka (possibilistic) regression model. This

discussion is not meant to be exhaustive but, rather, is intended to point out some of the major considerations. The outline of the paper is as follows. We first define and conceptualize the general components of fuzzy regression. Next, the essence of the Tanaka model is explored, including a commentary on some of its potential limitations. Then, fuzzy least-squares regression models are discussed as an alternative to the Tanaka model. Throughout the paper, the same simple data set is used to show how the ideas are implemented. The paper ends with a summary of the conclusions of the study.

2 Fuzzy Linear Regression Basics

This section provides an introduction to fuzzy linear regression. The topics addressed include the motivation for FR, the components of FR, fuzzy coefficients, the h-certain factor, and fuzzy output.

2.1 Motivation

Classical statistical linear regressions takes the form

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \varepsilon_i, \quad i = 1,2,...,m \qquad (2)$$

where the dependent (response) variable, y_i , the independent (explanatory) variables, x_{ij}, and the coefficients (parameters), β_j, are crisp values, and ε_i is a crisp random error term with $E(\varepsilon_i)=0$, variance $\sigma^2(\varepsilon_i)=\sigma^2$, and covariance $\sigma(\varepsilon_i , \varepsilon_j) = 0$, $\forall i,j$, $i \neq j$.

Although statistical regression has many applications, problems can occur in the following situations:

- Number of observations is inadequate (Small data set)
- Difficulties verifying distribution assumptions
- Vagueness in the relationship between input and output variables
- Ambiguity of events or degree to which they occur
- Inaccuracy and distortion introduced by linearalization

Thus, statistical regression is problematic if the data set is too small, or there is difficulty verifying that the error is normally distributed, or if there is vagueness in the relationship between the independent and dependent variables, or if there is ambiguity associated with the event or if the linearity assumption is inappropriate. These are the very situations fuzzy regression was meant to address.

2.2 The Components of Fuzzy Regression

There are two general ways (not necessarily mutually exclusive) to develop a fuzzy regression model: (1) models where the relationship of the variables is fuzzy; and (2)

models where the variables themselves are fuzzy. Both of these models are explored in the rest of this article, but, for this conceptualization, we focus on models where the data is crisp and the relationship of the variables is fuzzy.

It is a simple matter to conceptualize fuzzy regression. Consider for this, and subsequent, examples the following simple Ishibuchi (1992) data:

Table 1: Data Pairs

i	1	2	3	4	5	6	7	8
x_i	2	4	6	8	10	12	14	16
y_i	14	16	14	18	18	22	18	22

Starting with this data, we fit a straight line through two or more data points in such a way that it bounds the data points from above. Here, these points are determined heuristically and OLS is used to compute the parameters of the line labeled Y^H, which takes the values $\hat{y} = 13 + .75x$, as shown in Figure 2(a).

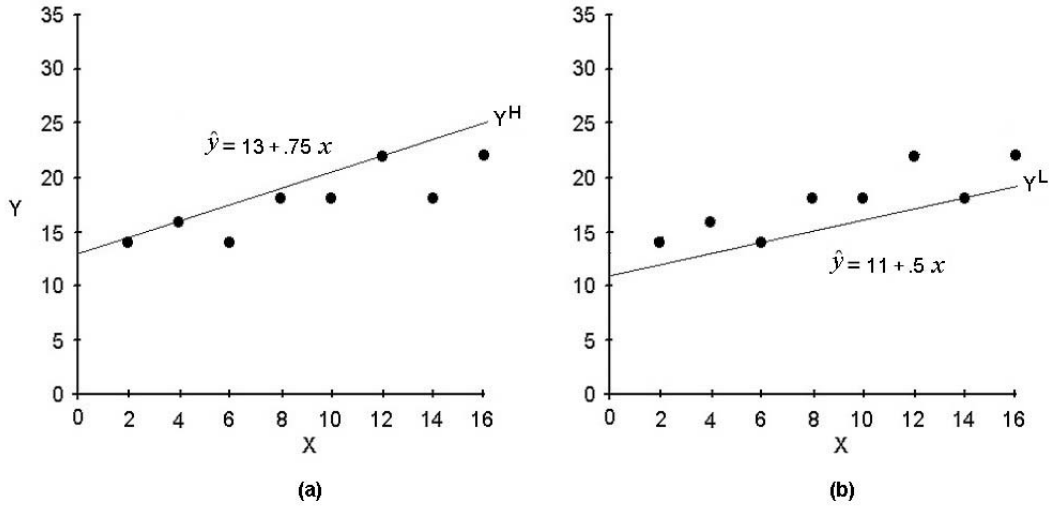


Figure 2: Conceptualizing the upper and lower bound

Similarly, we fit a second straight line through two or more data points in such a way that it bounds the data points from below. As shown in Figure 2(b), the fitted line in this case is labeled Y^L and takes the values $\hat{y} = 11 + .5x$.

Assuming, for the purpose of this example, that STFN are used for the MFs, the modes of the MFs fall midway between the boundary lines.[1]

[1] This approach to choosing the mode was discussed by Wang and Tsaur (2000) p. 357.

For any given data pair, (x_i, y_i), the foregoing conceptualizations can be summarized by the fuzzy regression interval $[Y_i^L, Y_i^U]$ shown in Figure 3.[2]

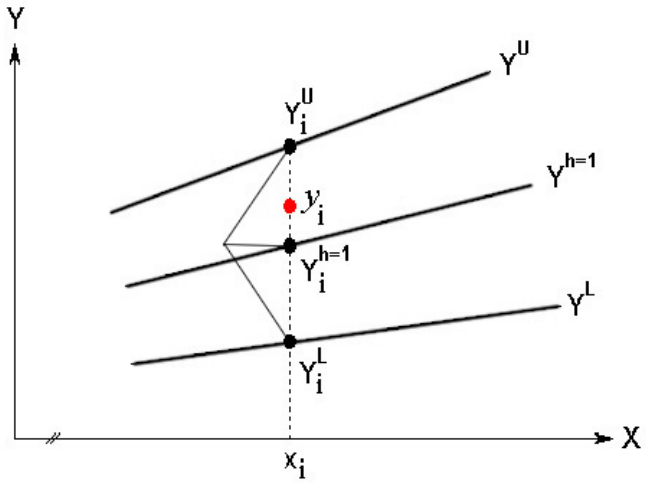


Figure 3: Fuzzy Regression Interval

$Y_i^{h=1}$ is the mode of the MF and if a SFTN is assumed, $Y_i^{h=1} = \overline{Y}_i = (Y_i^U + Y_i^L)/2$. Given the parameters, $(Y^U, Y^L, Y^{h=1})$, which characterize the fuzzy regression model, the i-th data pair (x_i, y_i), is associated with the model parameters $(Y_i^U, Y_i^L, Y_i^{h=1})$. From a regression perspective, we can view $Y_i^U - y_i$ and $y_i - Y_i^L$ as components of the SST, $y_i - Y_i^{h=1}$ as a component of SSE, and $Y_i^U - Y_i^{h=1}$ and $Y_i^{h=1} - Y_i^L$ as components of the SSR, as discussed by Wang and Tsaur (2000).

In possibilistic regression based on STFN, only the data points involved in determining the upper and lower bounds determine the structure of the model, as depicted in Figure 2. The rest of the data points have no impact on the structure. This problem is resolved by using asymmetric TFNs.

2.3 The Fuzzy Coefficients

Combining Equation (1) and Figure 1, and, for the present, restricting the discussion to STFNs, the MF of the j-th coefficient, may be defined as:

$$\mu_{A_j}(a) = \max\left\{1 - \frac{|a - a_j|}{c_j}, 0\right\} \qquad (3)$$

where a_j is the mode and c_j is the spread, and represented as shown in Figure 4.

[2] Adapted from Wang and Tsaur (2000), Figure 1.

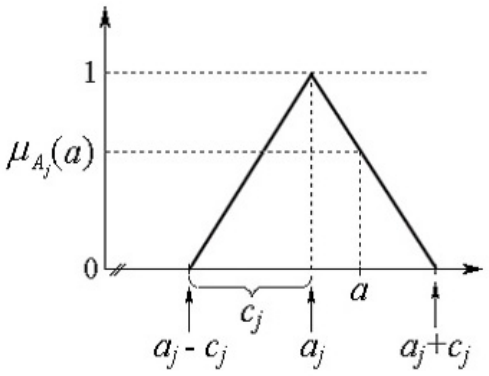


Figure 4: Symmetrical fuzzy parameters

Defining

$$\widetilde{A}_j = \{a_j, c_j\}_L = \{\widetilde{A}_j : a_j - c_j \le \widetilde{A}_j \le a_j + c_j\}_L, \quad j = 0,1,\cdots,n \tag{4}$$

and restricting consideration to the case where only the coefficients are fuzzy, we can write

$$\begin{aligned} \widetilde{Y}_i &= \widetilde{A}_0 + \sum_{j=1}^{n} \widetilde{A}_1 x_{ij} \\ &= (a_0, c_0)_L + \sum_{j=1}^{n} (a_j, c_j)_L x_{ij} \end{aligned} \tag{5}$$

This is a useful formulation because it explicitly portrays the mode and spreads of the fuzzy parameters. In a subsequent section, we explore fuzzy independent variables.

2.4 The "h-certain" Factor

If, as in Figure 3, the supports[3] are just sufficient to include all the data points of the sample, there would be only limited confidence in out-of-sample projection using the estimated FR model. This is resolved for FR, just as it is with statistical regression, by extending the supports.

Consider the MF associated with the j-th fuzzy coefficient, a representation of which is shown in Figure 5.

[3] Support functions are discussed in Diamond (1988: 143) and Wünsche and Näther (2002: 47).

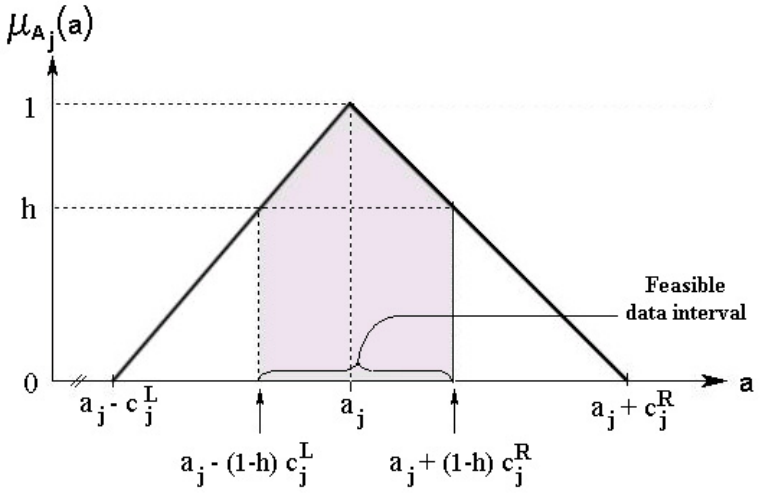


Figure 5: Estimating A_j using an "h-certain" factor

For illustrative purposes, a non-symmetric TFN is shown, where c_j^L and c_j^R represent the left and right spread respectively. Beyond that, what makes this MF materially different from the one shown in Figure 4, is that it contains a point "h" on the y-axis, called an "h-certain factor," which, by controlling the size of the feasible data interval (the base of the shaded area), extends the support of the MF.[4] In particular, as the h-factor increases for a given data set, so increases the spreads, c_j^L and c_j^R .

2.5 Observed Fuzzy Output

An h-certain factor also can be applied to the observed output. Thus, the i-th output data might be represented by the STFN, $\widetilde{Y}_i = (y_i, e_i)$, where y_i is the mode and e_i is the spread, as shown in Figure 6. Here, the actual data points fall within the interval $y_i \pm (1\text{-}h)\, e_i$, the base of the shaded portion of the graph.

[4] Note that the h-factor has the opposite purpose of an α-cut, in that the former is used to extend the support, while the latter is used to reduce the support.

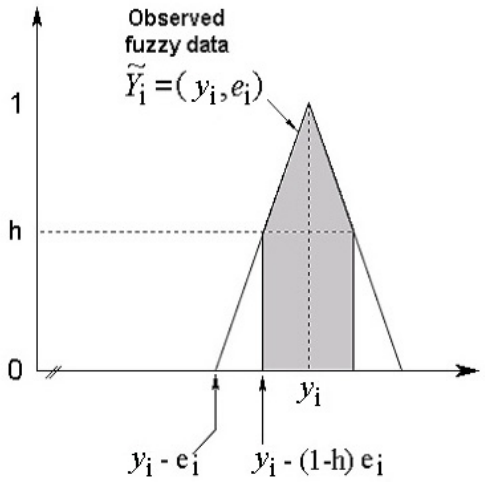


Figure 6: Observed Fuzzy Output

2.6 Fitting the Fuzzy Regression Model

Given the foregoing, two general approaches are used to fit the fuzzy regression model:

The possibilistic model. Minimize the fuzziness of the model by minimizing the total spreads of its fuzzy coefficients (see Figure 1), subject to including the data points of each sample within a specified feasible data interval.

The least-squares model. Minimize the distance between the output of the model and the observed output, based on their modes and spreads.

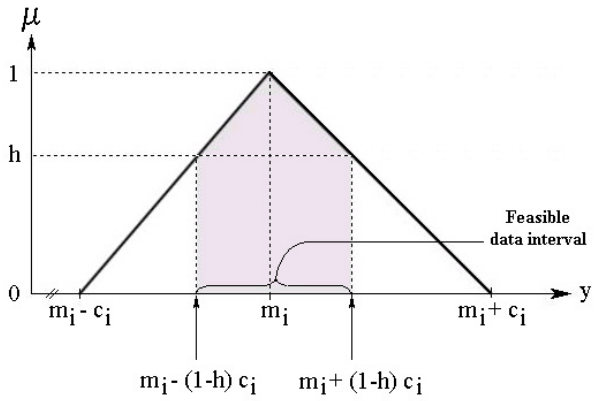
The details of these approaches are addressed in the next two sections of this paper.

3 The Possibilistic Regression Model

The possibilistic regression model is optimized by minimizing the spread, subject to adequate containment of the data. The spread is minimized

$$\min\left[c_0 + \sum_{j=1}^{n} c_j \, | x_{ij} | \right], \; c_j \geq 0 \qquad (6)$$

Figure 7 shows the first step in the containment requirement, by showing how Figure 5 can be easily extended to portray the fuzzy output of the model.



$$m_i = a_0 + \sum_{j=1}^{n} a_j x_{ij} \qquad c_i = c_0 + \sum_{j=1}^{n} c_j \mid x_{ij} \mid$$

Figure 7: Fuzzy output of the model

Putting this together with the observed fuzzy output, Figure 6, results in Figure 8, which shows a representation of how the estimated fuzzy output may be fitted to the observed fuzzy data.

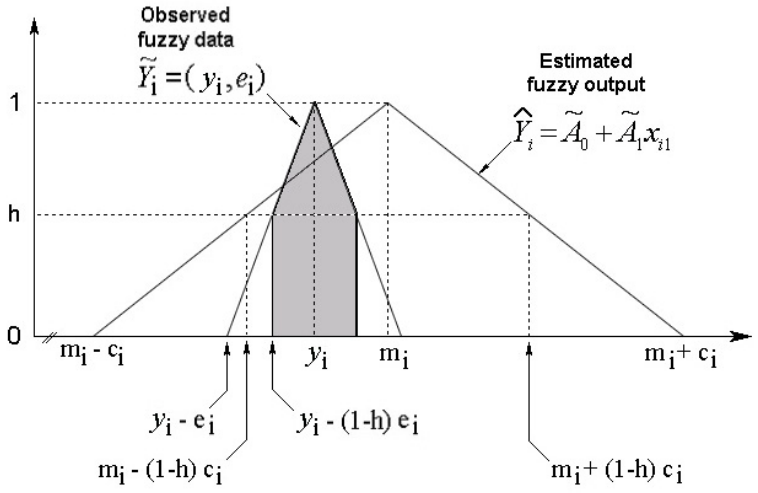


Figure 8: Fitting the estimated output to the observed output

The key is that the observed fuzzy data, adjusted for the h-certain factor, is contained within the estimated fuzzy output, adjusted for the h-certain factor. Formally,

$$a_0 + \sum_{j=1}^{n} a_j x_{ij} + (1-h)\left[c_0 + \sum_{j=1}^{n} c_j \mid x_{ij} \mid \right] > y_i + (1-h)\, e_i \qquad (7)$$

$$a_0 + \sum_{j=1}^{n} a_j x_{ij} - (1-h)\left[c_0 + \sum_{j=1}^{n} c_j \mid x_{ij} \mid \right] < y_i - (1-h) e_i$$

$$c_j \geq 0,\ i = 0, 1, ..., m,\ j = 0, 1, ..., n$$

Figure 9[5] shows the impact of the h-factor on the sample data, given h=0 and h=.7.

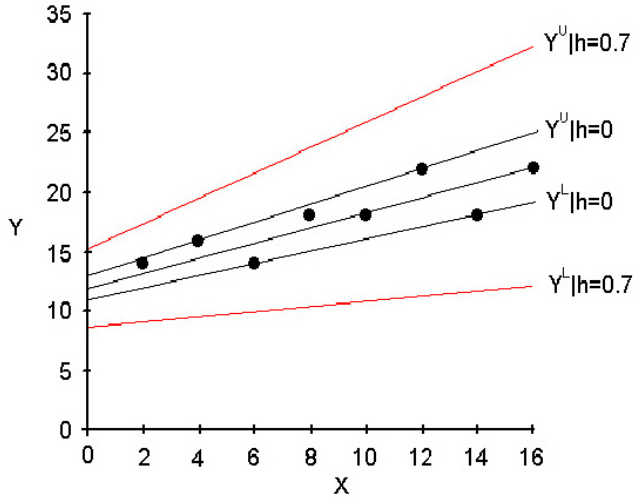


Figure 9: FLR and h-certain model

The result is what one would expect. Increasing the h-factor expands the confidence interval and, thus, increases the probability that out-of-sample values will fall within the model. This is comparable to increasing the confidence in statistical regression by increasing the confidence interval.

The possibilistic linear regression model, as depicted by equations (6) and (7), is essentially the fuzzy regression model used by Sánchez and Gómez (2003a, 2003b, 2004) to investigate the TSIR.[6]

[5] Adapted from Chang and Ayyub (2001), Figure 4.

[6] Key components of the Sánchez and Gómez methodology included constructing a discount function from a linear combination of quadratic or cubic splines, the coefficients of which were assumed to be TFNs or STFNs, and using the minimum and maximum negotiated price of fixed income assets to obtain the spreads of the dependent variable observations. Given the fuzzy discount functions, the authors provided TFN approximations for the corresponding spot rates and forward rates. It was necessary to approximate the spot rates and forward rates since they are nonlinear functions of the discount function, and hence are not TFNs even though the discount function is a TFN.

3.1 Criticisms of the Possibilistic Regression Model

There are a number of criticisms of the possibilistic regression model. Some of the major ones are the following:

- Tanaka et al "used linear programming techniques to develop a model superficially resembling linear regression, but it is unclear what the relation is to a least-squares concept, or that any measure of best fit by residuals is present." [Diamond (1988: 141-2)]
- The original Tanaka model was extremely sensitive to the outliers. [Peters (1994)].
- There is no proper interpretation about the fuzzy regression interval [Wang and Tsaur (2000)]
- Issue of forecasting have to be addressed [Savic and Pedrycz (1991)]
- The fuzzy linear regression may tend to become multicollinear as more independent variables are collected [Kim et al (1996)].
- The solution is x_j point-of-reference dependent, in the sense that the predicted function will be very different if we first subtract the mean of the independent variables, using (x_j - $\bar{x}_i$) instead of x_j. [Hojati (2004), Bardossy (1990) and Bardossy et al (1990)]

4 The Fuzzy Least-Squares Regression (FLSR) Model

An obvious way to bring the FR more in line with statistical regression is to model the fuzzy regression along the same lines. In the case of a single explanatory variable, we start with the standard linear regression model: [Kao and Chyu (2003)]

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1,2,..., m \tag{8}$$

which in a comparable fuzzy model might take the form:

$$\widetilde{Y}_i = \beta_0 + \beta_1 \widetilde{X}_i + \widetilde{\varepsilon}_i, \quad i = 1,2,..., m \tag{9}$$

Conceptually, the relationship between the fuzzy i-th response and explanatory variables in (9) can be represented as shown in Figure 10.

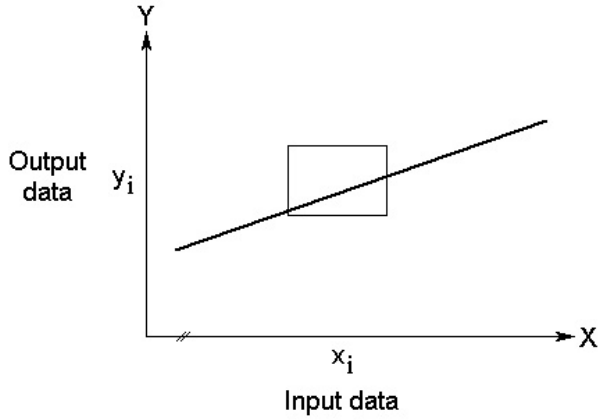


Figure 10: Fuzzy i-th response and explanatory variables

Rearranging the terms in (9),

$$\tilde{\varepsilon}_i = \tilde{Y}_i - \beta_0 - \beta_1 \tilde{X}_i, \quad i = 1,2,...,m \tag{10}$$

From a least-squares perspective, the problem then becomes

$$\min \sum_{i=1}^{n} (\tilde{Y}_i - b_0 - b_1 \tilde{X}_i)^2 \tag{11}$$

There are a number of ways to implement FLSR, but the two basic approaches are FLSR using distance measures and FLSR using compatibility measures. A description of these methods follows.

4.1 FLSR using Distance Measures (Diamond's Approach)

Diamond (1988) was the first to implement the FLSR using distance measures and his methodology is the most commonly used. Essentially, he defined an L^2- metric $d(.,.)^2$ between two TFNs by [Diamond (1988: 143) equation (2)]

$$d\left(\langle m_1, l_1, r_1\rangle, \langle m_2, l_2, r_2\rangle\right)^2 = (m_1 - m_2)^2 + \left((m_1 - l_1) - (m_2 - l_2)\right)^2 + \left((m_1 + r_1) - (m_2 + r_2)\right)^2 \tag{12}$$

Given TFNs, it provides a measure of the distance between two fuzzy numbers based on their modes, left spread and right spread.[7]

[7] The methods of Diamond's paper are rigorously justified by a projection-type theorem for cones on a Banach space containing the cone of triangular fuzzy numbers, where a Banach space is a normed vector space that is complete as a metric space under the metric d(x, y) = ||x-y|| induced by the norm.

The case most similar to the Sánchez and Gómez model takes the form

$$\widetilde{Y}_i = \widetilde{\beta}_0 + \widetilde{\beta}_1 x_i + \widetilde{\varepsilon}_i, \quad i = 1,2,...,m \tag{13}$$

and requires the optimization of

$$\min_{A,B} \sum d(\widetilde{A} + \widetilde{B}x_i, \widetilde{Y}_i)^2 \tag{14}$$

The solution follows from (12), and if $\widetilde{B}$ is positive, it takes the form:

$$\begin{aligned} d(\widetilde{A} + x_i\widetilde{B}, \widetilde{Y}_i)^2 = (a + bx_i - y_i)^2 + (a + bx_i - c_A^L - c_B^L x_i - y_i + c_{Y_i}^L)^2 \\ + (a + bx_i + c_A^R + c_B^R x_i - y_i + c_{Y_i}^R)^2 \end{aligned} \tag{15}$$

A similar expression holds when $\widetilde{B}$ is negative. If the solutions exist, the parameters of $\widetilde{A}$ and $\widetilde{B}$ satisfy a system of six equations in the same number of unknowns, these equations arising from the derivatives associated with (15) being set equal to zero. Of course, this fitted model has the same general characteristics as previously shown, but now we can use the residual sum of d-squares to gauge the effectiveness of model.

In the case most reminiscent of statistical regression, the coefficients are crisp and the task becomes the least-squares optimization problem

$$\min_{a,b} \sum d(a + b\widetilde{X}_i, \widetilde{Y}_i)^2 \tag{16}$$

Once again, the solution is given by (12), adjusted to take into account the sign of b.

Finally, an interesting problem when implementing the Diamond approach is associated with models of the form

$$\widetilde{Y}_i = \widetilde{\beta}_0 + \widetilde{\beta}_1 \widetilde{X}_i + \widetilde{\varepsilon}_i, \quad i = 1,2,...,m \tag{17}$$

for which there is no general solution, since the LHS, $\widetilde{Y}_i$, is a TFN while the RHS involves the fuzzy product $\widetilde{\beta}_1 \widetilde{X}_i$, whose sides are drumlike.

One approach to this problem (Hong et al (2001)) is to replace the t-norm min(a,b) with the t-norm $T_w(a,b)$ = a, if b=1; b, if a=1; 0, otherwise. Since $T_w(a,b)$ is a shape preserving operation under multiplication, it resolves the problem. This approach is used in Koissi and Shapiro (2005).

Another approach is to use approximate TFNs. This was done by Sánchez and Gómez (2003a), albeit in another context.

4.2 FLSR using compatibility measures

An alternate least-squares approach is based on the Celmiņš (1987) compatibility measure

$$\gamma(\widetilde{A},\widetilde{B}) = \max_{x} \min\{\mu_A(X),\mu_B(X)\} \qquad (18)$$

representative examples of which are shown in Figure 11.[8]

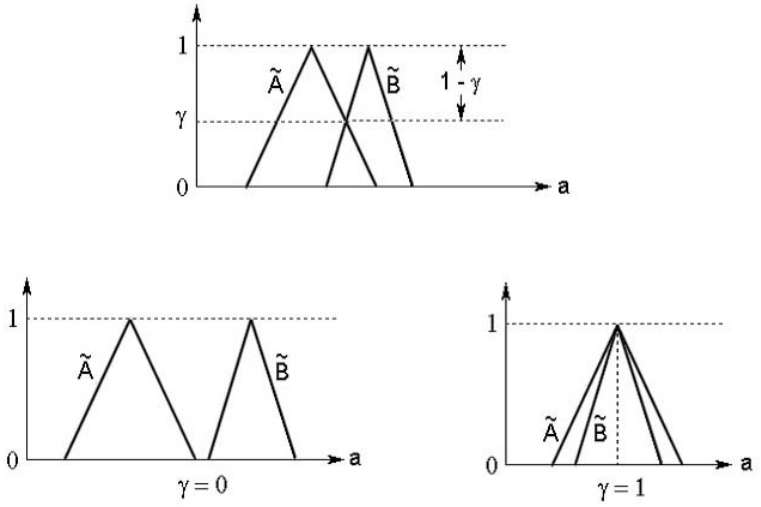


Figure 11: Celmiņš Compatibility Measure

As indicated, γ ranges from 0, when the MFs are mutually exclusive, to 1, when the modes of the MFs coincide.

Celmiņš compatibility model, which involved maximizing the compatibility between the data and the fitted model, follows from this measure. The objective function is

$$\sum_{i=1}^{m}(1-\gamma_i)^2 \qquad (19)$$

Thus, for example, when there is a single crisp explanatory variable, [Chang and Ayyub (2001: 190)]

$$\begin{aligned} \hat{Y} &= \widetilde{A}_0 + \widetilde{A}_1 x \qquad (20) \\ &= m_0 + m_1 x \pm \sqrt{c_0 + 2c_{01}x + c_1^2 x^2} \end{aligned}$$

where m_0 and m_1 are determined using weighted LS regression, and c_0, c_1, and c_{01} are determined using iteration and the desired compatibility measure.

[8] Adapted from Chang and Ayyub (2001), Figure 2.

An example of the use of the Celmiņš compatibility model applied to our sample data is shown in Figure 12.[9]

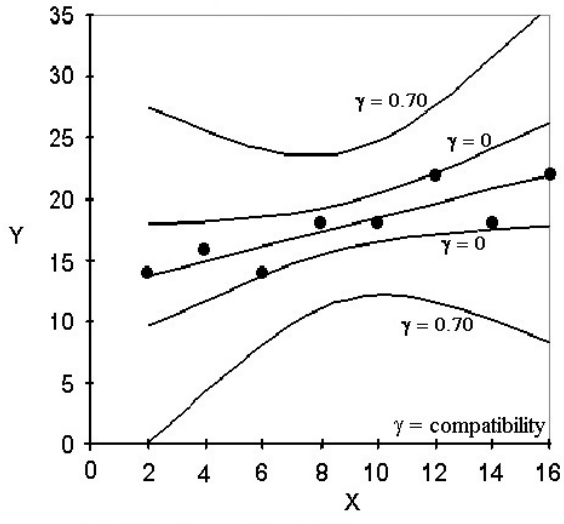


Figure 12: FLS using maximum compatibility criterion

The essential characteristics of the model in this case are the parabolic curves for the upper and lower bounds and that the higher the compatibility level, the broader the width of the bounds.

5 Comment

The studies of McCauley-Bell et al. (1999) and Sánchez and Gómez (2003a, 2003b, 2004) provide some interesting insights into the use of fuzzy regression. However, their methodology relies on possibilitic regression, which has the potential limitations mentioned in section 3.1. Since some of these limitations can be circumvented by using FLSR techniques, it is important that researchers are familiar with these techniques as well. If this article helps in this regard, it will have served its purpose.

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