PORTFOLIO RISK MANAGEMENT WITH CVAR-LIKE CONSTRAINTS

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ABSTRACT. In his original monograph on portfolio selection, Markowitz (1952) discusses the tradeoff between the mean and variance of a portfolio. Since then, especially recently, much attention has been focused on asymmetric distributions to minimize risks with given return goal for investors who have special skewness preferences. To address this issue, we extend Krokhmal et al. (2002)'s approach by adding CVaR-like constraints to the traditional portfolio optimization problem. The CVaR optimization technique has the advantage of reshaping either the left or right tail of a distribution while not significantly affecting the other. Specifically, this approach is used to manage the skewness of asset-liability portfolios of financial institutions. In addition, we compare the CVaR-like constraints approach with traditional Markowitz method and some other alternatives such as, the CVaR approach (directly optimize CVaR), the Boyle-Ding approach as well as the mean-absolute deviation (MAD) approach. Our numerical analysis provides empirical support for the superiority of CVaR-like constraints approach in terms of skewness improvement of mean-variance portfolios.

1. INTRODUCTION

One of the fundamental roles of banks, insurance companies and other financial institutions is to invest in various financial assets. Correct assessment of their portfolio performance requires risk-return analysis. In his seminal work on modern portfolios, Markowitz (1952) quantifies the trade-off between the risk and expected return of a portfolio within a static context. However, more recently, higher moments of returns have become relevant to portfolio choice (Boyle and Emanuel, 1980). Markowitz (1952), Borch (1969) and Feldstein (1969) argue that introducing skewness of returns adds the dimension needed to improve the approximation provided by the mean and variance.

Early theories on portfolio choice including three moments were developed by Jean (1971), Arditti and Levy (1975), Ingersoll (1975), Kraus and Litzenberger (1976), Simkowitz and Beedles (1978), Conine and Tamarkin (1981) and others. Those theoretical framework on portfolio performance assessment has profound impact on portfolio risk management. Portfolio risk management, especially tail risk management, is crucial for financial institutions (Wright, 2007). Unfortunately, some commonly used tail risk measures nowadays, e.g. value-at-risk (VaR), do not capture all aspects of risk. For instance, the major shortcoming of the VaR-based risk management (VaR-RM)

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stems from its main focus on the probability rather than magnitude of a loss. Basak and Shapiro (2001) exhibit that, when a large loss occurs, the loss under VaR-RM is larger than that when not engaging in the VaR-RM. Moreover, Artzner et al. (1999) show that VaR has undesirable properties such as lack of sub-additivity, i.e., the VaR of a portfolio with two instruments may be greater than the sum of individual VaRs.

To overcome the limitations of the VaR-RM, Basak and Shapiro (2001) propose an alternative form of risk management that maintains a given level of conditional value at risk (CVaR) when losses occur. CVaR is also called mean excess loss, mean shortfall, or tail VaR. It is the conditional expected loss (or return) exceeding (or below) VaR. In contrast to the VaR-RM, losses in the CVaR-based risk management (CVaR-RM) are lower than those without. Moreover, CVaR is a more consistent risk measure than VaR because it is sub-additive and concave. It can also be optimized using linear programming (LP) and nonsmooth optimization algorithms.

Although the theories on portfolio two- or three-moment problems and tail risk management are rich, there are few studies explicitly examining the link between them. To fill this gap, this paper sheds light on the theoretical and empirical impact of tail risk management on the portfolio efficient frontier. We introduce a method to construct the portfolio efficient frontier by adding CVaR-like constraints to the traditional Markowitz (1952)'s mean-variance (MV) portfolio optimization problem. If portfolio managers disclose and monitor CVaR, their optimal behavior will not only reduce losses in the most adverse states (Basak and Shapiro, 2001) but also maximize the skewness given that, portfolios are not extremely positively skewed (Kane, 1982). Moreover, our approach extends the results of Rockafellar and Uryasev (2000). We show how to apply this method to the asset-liability management of a financial institution (e.g. an insurance company). Finally, we compare the CVaR-like constraints frontier with Markowitz (1952)'s MV and Boyle and Ding (2006)'s mean-variance-skewness (MVS) frontiers. Our study provides empirical support for the superiority of CVaR-like constraint approach over its alternatives.

Our paper is organized as follows: Section 2 lays the foundation of the analysis. We discuss the asset-liability portfolio and derive the optimization problems. Section 3 develops our CVaR-like constraint approach. Section 4 compares the CVaR-like constraint method with the Boyle-Ding approach theoretically. Section 5 presents the numerical illustrations with empirical data. Section 7 concludes the paper.

2. Portfolio and Efficient Frontier: Descriptions

2.1. **Asset-Liability Portfolio Problem.** Portfolio theory can be applied to the asset liability management (ALM) of financial institutions such as insurance companies. Insurers' ALM emphasizes the overall target profit earned on the asset side as well as the liability side. They collect premiums from several lines of business and invest the collected premiums and addition capitals in assets such as stocks, bonds, real estates, etc. Then they pay out losses and expenses. The margin is the net result, *i.e.*, the excess of written premiums over losses and expenses, divided by the written premiums,

$$\text{Margin} = \frac{\text{Written premiums} - \text{Losses Incurred} - \text{Expenses}}{\text{Written premiums}} = 1 - \text{Combined ratio}^1.$$

At the beginning of a year, the company writes a line of business i with premium Π_i for $i = 1, 2, \ldots, k_1$. The total premium is $\Pi = \Pi_1 + \cdots + \Pi_{k_1}$. The amount in the company's favor at the end of the year is $\Pi_i M_i$ for line i and the total for all lines is

$$\sum_{i=1}^{k_1} \Pi_i M_i = \Pi \sum_{i=1}^{k_1} a_i M_i,$$

where the weight of line i is $a_i = \Pi_i/\Pi$ and M_i is the margin of line i. Generally, a_i is given. However, we could allow the a_i to be decision variables in order to determine an optimal portfolio of lines of businesses.

From the view of investment, the company collects Π, and it has additional contingency capital $\lambda\Pi$ to be invested at the beginning of the year. For each line of business, $\Pi_i(1+\lambda_i)$ is invested.

$$\sum_{i=1}^{k_1} \Pi_i(1+\lambda_i) = \sum_{i=1}^{k_1} \Pi_i + \sum_{i=1}^{k_1} \Pi_i \lambda_i = \Pi(1+\lambda),$$

Assume the assets have returns R_j where $j = 1, \cdots, k_2$. Let b_j be the proportion invested in asset j. Let loss expense be included in the loss L_i where $i = 1, \ldots, k_1$. The total profits in the company's favor at the end of the year are written as follows:

$$\begin{aligned}
(1+\lambda)\Pi \sum_{j=1}^{k_2} b_j(1+R_j) - \sum_{i=1}^{k_1} L_i &= (1+\lambda)\Pi \sum_{j=1}^{k_2} b_j R_j + (1+\lambda)\Pi - \Pi \sum_{i=1}^{k_1} \frac{\Pi_i}{\Pi}\frac{L_i}{\Pi_i} \\
&= (1+\lambda)\Pi \sum_{j=1}^{k_2} b_j R_j + (1+\lambda)\Pi \sum_{i=1}^{k_1} a_i - \Pi \sum_{i=1}^{k_1} a_i \frac{L_i}{\Pi_i} \\
&= \Pi \sum_{i=1}^{k_1} a_i \left(1 - \frac{L_i}{\Pi_i}\right) + (1+\lambda)\Pi \sum_{j=1}^{k_2} b_j R_j + \lambda\Pi \\
&= \Pi \left(\sum_{i=1}^{k_1} a_i M_i + (1+\lambda) \sum_{j=1}^{k_2} b_j R_j + \lambda \right) \\
&= \Pi \left(\sum_{i=1}^{k_1} a_i M_i + \sum_{j=1}^{k_2} b_j R_j^* + \lambda \right),
\end{aligned} \tag{1}$$

[1]Combined ratio is generally defined as

$$\text{Combined ratio} = \frac{\text{Losses Incurred + Expenses}}{\text{Written premiums}} \quad \text{or} \quad \frac{\text{Losses Incurred}}{\text{Earned premiums}} + \frac{\text{Expenses}}{\text{Written premiums}}.$$

where L_i is the sum of claim payments and administrative expenses in year i, $M_i = 1 - \dfrac{L_i}{\Pi_i}$ and $R_j^* = (1+\lambda)R_j$. In addition, $a_1 + a_2 + \cdots + a_{k_1} = 1$ and $b_1 + b_2 + \cdots + b_{k_2} = 1$. Because λ is known at the beginning of the year, we can only consider part $\Pi\left(\sum_{i=1}^{k_1} a_i M_i + \sum_{j=1}^{k_2} b_j R_j^*\right)$. The λ above has no effect on the return maximization problem; nor does it contribute anything to the variance.

Change the notation and write the margins and returns in one vector R with R_i the margin of line i if $0 \le i \le k_1$ and return of asset i if $k_1 < i \le n$ where $k_1 + k_2 = n$. The idea of portfolio theory is to determine the weights $X = [x_i]_{i=1}^n$ to maximize the expected return $\mathrm{E}(X^\top R)$ subject to variance and higher-moment constraints or equivalently, to minimize the variance $\mathrm{Var}(X^\top R)$ subject to return and higher-moment constraints. If we assume that the company cannot easily change its business, then the weights for the margins x_i with $i \le k_1$ are known and cannot be changed. In addition, the weights of asset x_i with $k_1 < i \le n$ may be subject to some conditions such as no short sales for some or all assets.

2.2. **Definition and Notation.** Consider the problem of selecting a portfolio with k_1 lines of business and k_2 assets ($k_1 + k_2 = n$). If $k_1 = 0$, we solve the general asset portfolio problem. Suppose we have observations of each assets (and/or lines of business) for m periods. For simplicity, we assume one period is a year. Define R_i the annual return for asset or line of business i for $i = 1, \ldots, n$. Note that the asset return considered in the portfolio problem is a return on assets including additional contingency capital reserves, *i.e.*, $R_i^* = (1+\lambda)R_i$ when $k_1 < i \le n$. In the following discussion, we remove the symbol "$*$" from R^*, again for simplicity. However, we should be aware that the new return notation R_i is still an asset return R_i adjusted by $(1+\lambda)$ for $k_1 < i \le n$. The first three moments of the model for asset return or margin R_i are as follows:

$$
\begin{aligned}
\mu_i &= \mathrm{E}[R_i], \quad i = 1, \ldots, n; \\
\sigma_{ij} &= \mathrm{E}[(R_i - \mu_i)(R_j - \mu_j)], \quad i, j = 1, \ldots, n; \\
\gamma_{ijk} &= \mathrm{E}[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)], \quad i, j, k = 1, \ldots, n.
\end{aligned} \tag{2}
$$

Let variable r_{il} represents the observed value of R_i in year l for $l = 1, \ldots, m$. Given the sample return $\{r_{il}\}$, we can write the empirical distribution moments (mean, covariance and co-skewness) as follows:

$$\begin{aligned}
\hat{\mu}_i &= \frac{1}{m}\sum_{l=1}^{m} r_{il}, \quad i = 1, \dots, n; \\
\hat{\sigma}_{ij} &= \frac{1}{m}\sum_{l=1}^{m} (r_{il} - \hat{\mu}_i)(r_{jl} - \hat{\mu}_j), \quad i, j = 1, \dots, n; \\
\hat{\gamma}_{ijk} &= \frac{1}{m}\sum_{l=1}^{m} (r_{il} - \hat{\mu}_i)(r_{jl} - \hat{\mu}_j)(r_{kl} - \hat{\mu}_k), \quad i, j, k = 1, \dots, n.
\end{aligned} \tag{3}$$

Next, we can calculate the portfolio empirical moments after we obtain the moments for each asset and/or line of business from equation (3). Let variable x_i be the proportion invested in asset or line of business i. The first three empirical moments of the portfolio are equal to:

$$\begin{aligned}
\hat{\mu}(x) &= \frac{1}{m}\sum_{l=1}^{m} \hat{\mu}(x)_l = \frac{1}{m}\sum_{l=1}^{m}\sum_{i=1}^{n} r_{il}x_i = \sum_{i=1}^{n} \hat{\mu}_i x_i, \\
\hat{\sigma}^2(x) &= \frac{1}{m}\sum_{l=1}^{m} [\hat{\mu}(x)_l - \hat{\mu}(x)]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} x_i x_j, \\
\frac{1}{m}\sum_{l=1}^{m} [\hat{\mu}(x)_l - \hat{\mu}(x)]^3 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \hat{\gamma}_{ijk} x_i x_j x_k,
\end{aligned} \tag{4}$$

where the portfolio empirical return in year l is

$$\hat{\mu}(x)_l = \sum_{i=1}^{n} r_{il} x_i \quad \forall\, l = 1, \dots, m.$$

2.3. **Optimization Problem Description.** The classical MV frontier is obtained by solving the following optimization problem, given moment information μ_i, μ_j and σ_{ij} of the return R_i and R_j. The traditional frontier consists of the points $(\sigma^2(x), \mu(x))$ where $\mu(x)$ varies over a range of values.

$$\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \\
\text{subject to} \quad & \sum_{i=1}^{n} x_i = 1 \\
& \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \\
& x_i \geq 0 \text{ for } i = 1, 2, \dots, n.
\end{aligned} \tag{5}$$

The constraint $x_i \geq 0$ can be eliminated to allow short sell of the i-th asset. Other inequality constraints can be added to reflect restrictions on proportions invested in the various assets.

Optimization problem (5) can also be applied to an ALM problem of an insurer. As for an asset-liability portfolio, the overall variance σ^2 is calculated as follows:

$$\begin{aligned}
\sigma^2(x) &= \sigma_L^2(a) + \sigma_V^2(b) + 2\sigma_{LV}(a,b) \\
&= \sum_{i=1}^{k_1}\sum_{j=1}^{k_1}\sigma_{ij}a_ia_j + \sum_{i=1}^{k_2}\sum_{j=1}^{k_2}\sigma_{ij}b_ib_j + 2\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sigma_{ij}a_ib_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j,
\end{aligned} \tag{6}$$

where σ_L^2 is the variance of the lines of business; σ_V^2 is the variance of the assets; and σ_{LV} is covariance of the lines of business and assets. Given a certain level of overall return $\mu_0(x)$, we can minimize the overall variance $\sigma^2(x)$ to obtain the optimal weights for assets and lines of business. Similar to problem (5), the ALM optimization problem is defined as follows:

$$\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j \\
\text{subject to} \quad & \sum_{i=1}^{k_1} x_i = 1 \\
& \sum_{i=k_1+1}^{n} x_i = 1 \\
& \sum_{i=1}^{n}\mu_i x_i = \mu_0(x), \quad \text{and} \quad n = k_1 + k_2.
\end{aligned} \tag{7}$$

If the portfolio only includes assets ($k_1 = 0$ and $k_2 = n$), we will return to the classical portfolio problem (5).

3. Improving Skewness of Mean-Variance Portfolio with CVaR-Like Constraints

In the MV analysis, the variance captures a portfolio's overall risk. A more recently introduced risk measure, VaR has been widely used for measuring downside risk and has become a part of the financial regulations in many countries (Jorion, 1997; Dowd, 1998; Saunders, 1999). It measures how the return of an asset or of a portfolio of assets (and liabilities) is likely to decrease over a certain time period. The β-level VaR is defined as follows:

$$\alpha(x,\beta) = \min\{\alpha \in \mathbb{R} : \mathbb{P}(R(x) \le \alpha) \ge \beta\}.$$

The variable $\alpha(x,\beta)$ is the β-*lower* quantile of the portfolio return distribution. Typically, the quantile β is set around 5%. Unfortunately, VaR is not the panacea of risk measurement methodologies. A major technical problem is that VaR is not sub-additive. For example, the variance of the sum of two variables $\mathrm{Var}(A + B)$ could be larger than the sum of these two variables' variances $\mathrm{Var}(A) + \mathrm{Var}(B)$. This imposes a problem for portfolio risk management because we hope portfolio diversification would reduce risk.

As an improved risk measure, the β-level CVaR, is the expected portfolio return, conditioned on the portfolio returns being lower than the β-level VaR over a given period. It is defined as

$$\mathrm{CVaR}(\mathrm{x}, \beta) = \mathbb{E}(\mathrm{R}(\mathrm{x})|\mathrm{R}(\mathrm{x}) \leq \alpha(\mathrm{x}, \beta)).$$

CVaR has some superior characteristics over variance and VaR (Rockafellar and Uryasev, 2000; Uryasev, 2000; D. Bertsimas and Samarovc, 2004; Wu et al., 2005). Variance is a symmetric measure and it does not differentiate between the desirable upside and the undesirable downside risks (Wu et al., 2005). In contrast, CVaR does not rely on the symmetric distribution assumption so we can use it to improve a portfolio's skewness. On the other hand, compared with VaR, CVaR not only takes into account probability but also the size of a return (or loss). Additionally, CVaR is a coherent risk measure that satisfies properties of monotonicity, sub-additivity, homogeneity, and translational invariance. Some of those desirable properties (e.g. sub-additivity) do not hold for VaR.

Some investors, especially institutional investors, may want to use CVaR to control downward risk and increase skewness but may not want to deviate too much from the Markowitz MV portfolios. To achieve this goal, Krokhmal et al. (2002) suggest using CVaR constraints to improve the skewness of MV portfolio. We extend it to a method which increases the skewness of Markowitz MV portfolios by adding one or more CVaR-like constraints. Imposing more than one CVaR-like constraints with several different β-levels can reshape the return distribution according to the customers' preferences. These preferences are specified directly in percentile terms. For instance, we may require that the mean values of the worst 1%, 5% and 10% losses are limited by some values.

Given $\beta, w \in \mathbb{R}$ and a sample of asset returns (and/or lines of business), we write the sample version of the traditional Markowitz MV model (5) with a CVaR constraint as follows:

$$
\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\
&\text{subject to} && \text{CVaR}(x,\beta) \geq w \\
& && \sum_{i=1}^{k_1} x_i = 1 \\
& && \sum_{i=k_1+1}^{n} x_i = 1 \\
& && \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \\
& && x_i \geq 0, \qquad\qquad i = 1, 2, \ldots, n.
\end{aligned}
\tag{8}
$$

The above CVaR constraint ensures a lower tail expectation in an amount at least equal to w.

Based on Rockafellar and Uryasev (2000), β-level CVaR can be obtained by the following optimization:

$$
\mathrm{CVaR}(\mathrm{x},\beta) = \max_{\alpha}\ \alpha - \frac{1}{\beta}\mathbb{E}((\alpha - \mathrm{R}(\mathrm{x}))^+),
\tag{9}
$$

where $(a)^+$ is defined as $\max(a, 0)$.

Proof. See Appendix.

Based on the Equation (9), the model (8) can be written as:

$$
\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\
&\text{subject to} && \max_{\alpha}\ \ \alpha - \frac{1}{\beta}\frac{1}{m}\sum_{j=1}^{m}\left(\alpha - \sum_{i=1}^{n} r_{ij}x_i\right)^+ \geq w \\
& && \sum_{i=1}^{k_1} x_i = 1 \\
& && \sum_{i=k_1+1}^{n} x_i = 1 \\
& && \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \\
& && x_i \geq 0, \qquad\qquad i = 1, 2, \ldots, n.
\end{aligned}
\tag{10}
$$

Because obtaining a tractable formulation for the model (10) is difficult, Krokhmal et al. (2002) suggest dropping its maximization over α (see details in Theorem 2 in Krokhmal et al. (2002)). Therefore, we rewrite model (10) as follows:

$$
\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\
&\text{subject to} && \alpha - \frac{1}{\beta}\frac{1}{m}\sum_{j=1}^{m}\left(\alpha - \sum_{i=1}^{n} r_{ij}x_i\right)^{+} \geq w \\
& && \sum_{i=1}^{k_1} x_i = 1 \\
& && \sum_{i=k_1+1}^{n} x_i = 1 \\
& && \sum_{i=1}^{n}\mu_i x_i = \mu_0(x) \\
& && x_i \geq 0, \qquad i = 1, 2, \ldots, n.
\end{aligned} \tag{11}
$$

That is, the first constraint in the above model is not exactly a CVaR constraint, but a CVaR-like constraint. We call this method the “CVaR-like constraint approach” or “MV + CVaR approach”. After we linearize the CVaR-like constraint, model (11) is equivalent to:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j \\
\text{subject to} \quad & \alpha - \frac{1}{\beta}\frac{1}{m}\sum_{j=1}^{m} y_j \geq w \\
& y_j \geq \alpha - \sum_{i=1}^{n} r_{ij}x_i, \qquad j = 1,\dots,m \\
& \sum_{i=1}^{k_1} x_i = 1 \\
& \sum_{i=k_1+1}^{n} x_i = 1 \\
& \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \\
& x_i \geq 0, \qquad i = 1,2,\dots,n \\
& y_j \geq 0, \qquad j = 1,2,\dots,m.
\end{aligned} \tag{12}
$$

Notice that model (12) is a tractable problem. It has a quadratic convex objective (i.e. $\Sigma = \{\sigma_{ij}\}$ should be positive semidefinite) and linear constraints and thus can be solved as easy as the Markowitz MV problem.

As mentioned before, we can add more than one CVaR-like constraint with several different β-levels and reshape the return distribution according to the customers' preferences. For example, we can add p CVaR-like constraints by using various quantiles $\beta^1, \beta^2, \dots, \beta^p \in (0,1)$, and different $w^1, w^2, \dots, w^p \in \mathbb{R}$:

$$
\begin{aligned}
&\text{Minimize} && \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j \\
&\text{subject to} && \alpha^l - \frac{1}{\beta^l}\frac{1}{m}\sum_{j=1}^{m} y_j^l \geq w^l && l=1,2,\dots,p \\
& && y_j^l \geq \alpha^l - \sum_{i=1}^{n} r_{ij}x_i, \quad j=1,2,\dots,m; && l=1,2,\dots,p \\
& && \sum_{i=1}^{k_1} x_i = 1 \\
& && \sum_{i=k_1+1}^{n} x_i = 1 \\
& && \sum_{i=1}^{n}\mu_i x_i = \mu_0(x) \\
& && x_i \geq 0, && i=1,2,\dots,n \\
& && y_j^l \geq 0, \quad j=1,2,\dots,m; && l=1,2,\dots,p \\
& && \alpha^l \in \mathbb{R}, && l=1,2,\dots,p.
\end{aligned}
\tag{13}
$$

That is, we require that the mean values of the worst $\beta^1, \beta^2, \dots, \beta^p \in (0,1)$ losses are limited by different values of $w^1, w^2, \dots, w^p \in \mathbb{R}$ based on the customers' risk tolerance. Compared with the traditional approach, which specifies risk preferences in terms of utility functions, this approach provides a new efficient and flexible risk management tool and adds to the MVS literature. Furthermore, our proposed model (13) has an additional desirable feature: adding many CVaR-like constraints will not significantly increase computational costs while we can increase skewness and achieve portfolio optimization at the same time. Therefore, this approach provides a new efficient and flexible risk management tool so it contributes to the MVS literature.

4. Comparison between CVaR-like Constraint Approach and Boyle-Ding Approach

The MV frontier, as it is usually determined, has no explicit reference to skewness. Boyle and Ding (2006) give a method to increase the skewness of a given portfolio x^*, obtaining a new portfolio x for which the mean returns are equal and the variance of returns are almost equal. Moreover, the skewness of the new portfolio $R(x)$ should be greater than the skewness of the original $R(x^*)$. Investors should prefer x to x^* because a small increase in risk allows for a relatively large increase in skewness (and greater likelihood of a large return). These conditions can be written as follows:

$$\begin{aligned} \mu(x) &= \mu(x^*) \\ \sigma^2(x) &\geq \sigma^2(x^*) + \epsilon \\ \frac{1}{m}\sum_{j=1}^{m}[\mu(x)_j - \mu(x)]^3 &\geq \frac{1}{m}\sum_{j=1}^{m}[\mu(x^*)_j - \mu(x^*)]^3 + \delta, \end{aligned} \tag{14}$$

where both ϵ and δ are small positive numbers[2].

Now define

$$\begin{aligned} \alpha_j = \sum_{i=1}^{n}(r_{ij} - \mu_i)x_i^* = \mu(x^*)_j - \mu(x^*), \quad j = 1, \ldots, m \\ g(\alpha_j) = (\alpha_j - \epsilon)^2 + (\alpha_j - \epsilon)(\alpha_j + \epsilon) + (\alpha_j + \epsilon)^2 \\ c_i = \sum_{j=1}^{m} g(\alpha_j)(r_{ij} - \mu_i) \\ \beta = \sum_{i=1}^{n} c_i x_i^*. \end{aligned} \tag{15}$$

Next, we specify a condition as follows:

$$\alpha_j - \epsilon < \sum_{i=1}^{n}(r_{ij} - \mu_i)x_i < \alpha_j + \epsilon, \quad j = 1, \ldots, m. \tag{16}$$

With condition (16), we can linearize the third moment (or skewness) inequality in (14) as

$$\sum_{i=1}^{n} c_i x_i \geq \beta + \delta. \tag{17}$$

Proof. See Appendix.

[2]In most cases, $\epsilon < \delta$.

Boyle and Ding (2006) add a constant $\delta \geq 0$ to the right side to increase the likelihood that the resultant new portfolio has higher skewness. This is the statement of the new problem:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j \\
\text{subject to} & \sum_{i=1}^{k_1} x_i = 1 \\
& \sum_{i=k_1+1}^{n} x_i = 1 \\
& \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \\
& \sum_{i=1}^{n}(r_{ij}-\mu_i)x_i \leq \alpha_j + \epsilon \quad \forall j = 1,\dots m \\
& \sum_{i=1}^{n}(r_{ij}-\mu_i)x_i \geq \alpha_j - \epsilon \quad \forall j = 1,\dots m \\
& \sum_{i=1}^{n} c_i x_i \geq \beta + \delta \\
& x_i \geq 0 \quad \forall i = 1,2,\dots,n.
\end{aligned} \tag{18}
$$

They indicate that the problem should be solved iteratively, replacing x^* by the solution x, until no significant increase in skewness is obtained.

Since Boyle-Ding approach needs to set the constants ϵ and δ beforehand, one should do several try-and-error experiments to make feasible decision. Boyle and Ding (2006) suggests iterately performing the optimization process to obtain the "best" optimum. This also depends on some try-and-error tests and cannot be done automatically. In contrast, CVaR-like constraints approach is more easily to implement. In addition, CVaR-like constraints approach can accurately reshape the distribution more effectively by adding specific quantile constraint with (β, w) according to the individual's preferences.

Moreover, as long as the portfolio distribution is not skewed extremely positively (Kane, 1982), the CVaR-like constraint approach offers much higher skewness than the MVS approach with only slight deviation from the MV efficient frontier. The experiment in Section 5 also shows that the Boyle-Ding approach can only increase skewness for low-variance portfolios. So it loses its power when customers prefer relatively higher risks. In this case, the MV + CVaR approach is a better choice when management of high-risk portfolios is at stake.

5. Empirical Illustration: Multiple Assets and Lines of business

We first compute the optimal portfolios of five assets ($k_1 = 0$ and $k_2 = 5$) based on the CVaR-like constraint (also called "MV + CVaR") and Boyle-Ding MVS approaches, respectively, using yearly data ranging from 1980 to 2005 ($m = 26$). Then we extend our comparison to 20 assets ($k_1 = 0$ and $k_2 = 20$). As stated before, our analysis can also be applied to an asset-liability portfolio. To illustrate, we selecte fourteen lines of business ($k_1 = 14$) and five assets ($k_2 = 5$). When evaluating the MV + CVaR and Boyle-Ding MVS approaches, we plot their efficient frontiers and compare them with the traditional MV frontier. In addition, we compare their skewness-variance graphs and asset mix plots. All of our examples assume no borrowing is allowed.

Table 1 summarizes statistics of the annual returns of 20 assets and the annual margins of 14 lines of business in our examples. Most of the insurance lines of business have negative skewness. This negative skewness suggests that the margins are pulled down by rare catastrophic events. Among all 20 assets, the S&P 500 has the highest average rate of return and, the lowest skewness. This is consistent with the observations made by David (1997). He concludes that stock market returns exhibit negative skewness and that large negative returns are more common than large positive ones. Moreover, mortgage-backed securities have the highest skewness.

Example 1. We first examine a portfolio with five assets ($k_1 = 0$ and $k_2 = 5$). These five assets include a short-term US Treasury bill, a long-term US Treasury bond, a mortgage-backed security, a crude oil future and the S&P 500. Our observation period is from 1980 to 2005 ($m = 26$). There are five optimal portfolios whose weights are to be determined, x_i for $i = 1, \ldots, 5$. The portfolio return is

$$\sum_{i=1}^{5} \mu_i x_i = 0.0794x_1 + 0.0983x_2 + 0.0976x_3 + 0.0766x_4 + 0.1431x_5.$$

We solved the model (12) to obtain MV + CVaR optimal portfolios. We set w equal to

$$\text{CVaR}_{0.05}^{\text{MV}}(r) + 0.05|\text{CVaR}_{0.05}^{\text{MV}}(r)|,$$

where $\text{CVaR}_{0.05}^{\text{MV}}(r)$ is the empirical 5%-level CVaR obtained from Markowitz MV optimization. The construct of w is reasonable because the empirical 5%-level CVaR obtained by the MV + CVaR approach should be close to and, should be a little larger than its Markowitz MV counterpart. With the Boyle-Ding MVS approach, we solve equation (18) to find another portfolio based on MV that had the same return, approximately the same variance, and had increased skewness with parameters $\epsilon = 0.2$ and $\delta = 0.0001$.[3]

After obtaining the optimal weights of x_j^* for three methods, respectively, we plot their efficient frontiers and skewness-variance graphs in Figures 1 and 2. Figure 1 shows the Markowitz MV, the Boyle-Ding MVS and the MV + 5%-level CVaR frontiers. The frontiers of these three approaches

[3]The problem is sensitive to the values of ϵ and δ. For example, with $\epsilon = 0.3$ and $\delta = 0$, the solutions to the new problem are essentially identical to the original MV solutions.

TABLE 1. Descriptive Statistics of assets and lines of business from 1980 to 2005

Assets	Mean	Variance	Skewness	Lines	Mean	Variance	Skewness
TSY: 1-3	0.0794	0.0022	0.6774	Cml Prop	0.0030	0.0101	0.2743
TSY: 7-10	0.0983	0.0090	0.5775	Allied	-0.0732	0.0319	-0.5677
MBS	0.0976	0.0075	1.9130	Hm/Fr	-0.0978	0.0143	-2.2626
Crude	0.0766	0.0757	0.2840	CMP	-0.1146	0.0114	0.0493
S&P 500	0.1431	0.0259	-0.5031	Comp	-0.1225	0.0062	0.1934
Agcy 1-3	0.0817	0.0023	0.7848	GL	-0.1572	0.0720	3.1982
Agcy 7-10	0.0999	0.0080	0.8760	Med/Prof	-0.2607	0.0441	-0.2115
Corp AAA 3-5	0.0897	0.0035	0.9103	PPAuto	-0.0402	0.0018	0.5091
Corp AA 3-5	0.0931	0.0037	0.9236	Cauto	-0.0830	0.0086	-0.4159
Corp A 3-5	0.0950	0.0038	1.0309	FSB	0.1106	0.0168	-0.2171
Corp BBB 3-5	0.0954	0.0038	1.0475	BC/BS	0.0060	0.0007	-0.6929
Corp HYld	0.1003	0.0102	0.9256	PCHlth	-0.0551	0.0042	-0.8627
Sovrgn: Inter	0.1057	0.0042	0.6630	Reins	-0.1607	0.0173	-1.9770
Yanky	0.0991	0.0068	0.8512	Other	-0.0790	0.0098	0.0760
ABS	0.0801	0.0018	0.6451				
Muni 3-5 Yrs	0.0580	0.0012	0.4472				
Commodities	0.0269	0.0007	1.7360				
Lumber	0.0254	0.0027	1.4153				
Currency	0.0114	0.0109	0.4489				
ML Convert	0.1228	0.0190	0.0225				

The sample includes the annual returns of 20 assets and the annual margins of 14 lines of business from 1980 to 2005. Data are offered by the General Re Company. The asset “TSY: 1-3” stands for the short-term US Treasury bill; “TSY: 7-10” is the long-term US Treasury bond; “MBS” is the mortgage-backed security; “Crude” is the crude oil future; “S&P 500” is the S&P 500 Index; “Agcy 1-3” is the short-term agency bond; “Agcy 7-10” is the long-term agency bond; “Corp AAA 3-5” is the middle-term AAA corporate bond; “Corp AA 3-5” is the middle-term AA corporate bond; “Corp AAA 3-5” is the middle-term A corporate bond; “Corp BBB 3-5” is the middle-term BBB corporate bond; “Corp HYld” is the corporate high yield bond; “Sovrgn: Inter” is the international sovereign bond; “Yanky” is the Yankee bond; “ABS” is the asset-backed security; “Muni 3-5 Yrs” is the middle-term municipal bond, “Commodities” is the commodity future; “Lumber” is the lumber future; “Currency” is the currency future and “ML Convert” is the convertible bond. The lines of business include Commercial Property (Cml Prop), Allied Lines (Allied), Farmowners/Farmers Multiple Peril (Hm/Fr), Commercial Multiple Peril (CMP), Workers’ Compensation (Comp), General Liability (GL), Medical Professional Liability (Med/Prof), Private Passenger Auto Liability (PPAuto), Commercial Auto/Truck Liability (Cauto), Fidelity/Surety (FSB), Blue Cross Blue Shield (BC/BS), Public and Commercial Health Insurance (PCHlth), Reinsurance (Reins) and Other Insurance (Other).

are similar. As we expect, the frontier of the MV + CVaR approach is almost the same as that of the Markowitz MV because it is derived from the traditional MV by adding more constraints to the MV problem.

The desirability of the MV + CVaR approach is shown in Figure 2. Figure 2 compares the 5-asset skewness-variance graphs of the three approaches. With a reasonable sacrifice of the return variance, the MV + CVaR approach has a higher skewness than the Markowitz MV. The skewness is increased not much for the Boyle-Ding MVS. Figure 2 suggests that the MV + CVaR approach

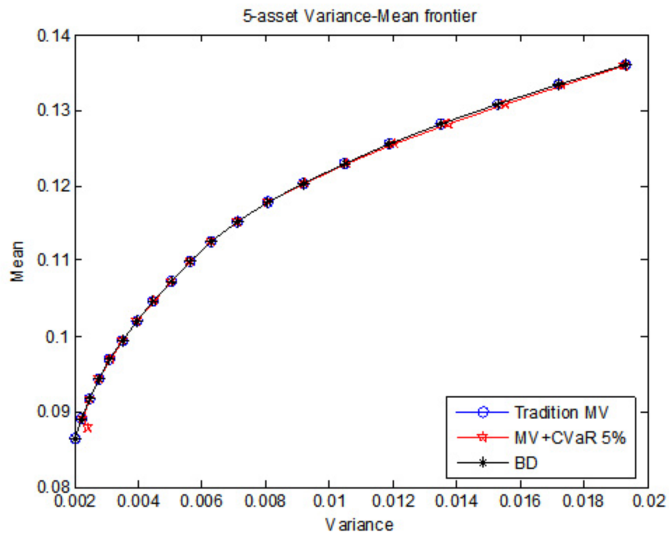

FIGURE 1. The efficient frontiers of 5-asset portfolios based on the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD"). The vertical axis stands for the expected returns of portfolios, and the horizontal axis is for variances.

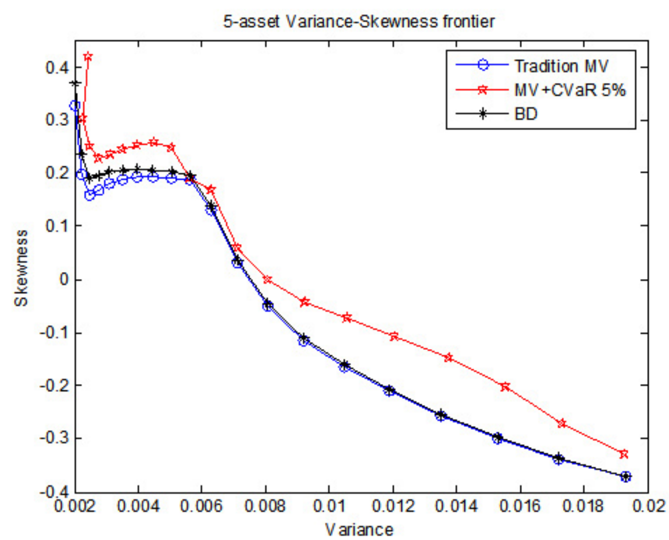

FIGURE 2. The 5-asset portfolios skewness-variance graphs of the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD"). The vertical axis stands for the skewness of portfolios, and the horizontal axis is for variances.

not only achieves left-tail risk management but also has higher skewness. That is, this method would let the financial institutions enjoy more potential for higher returns.

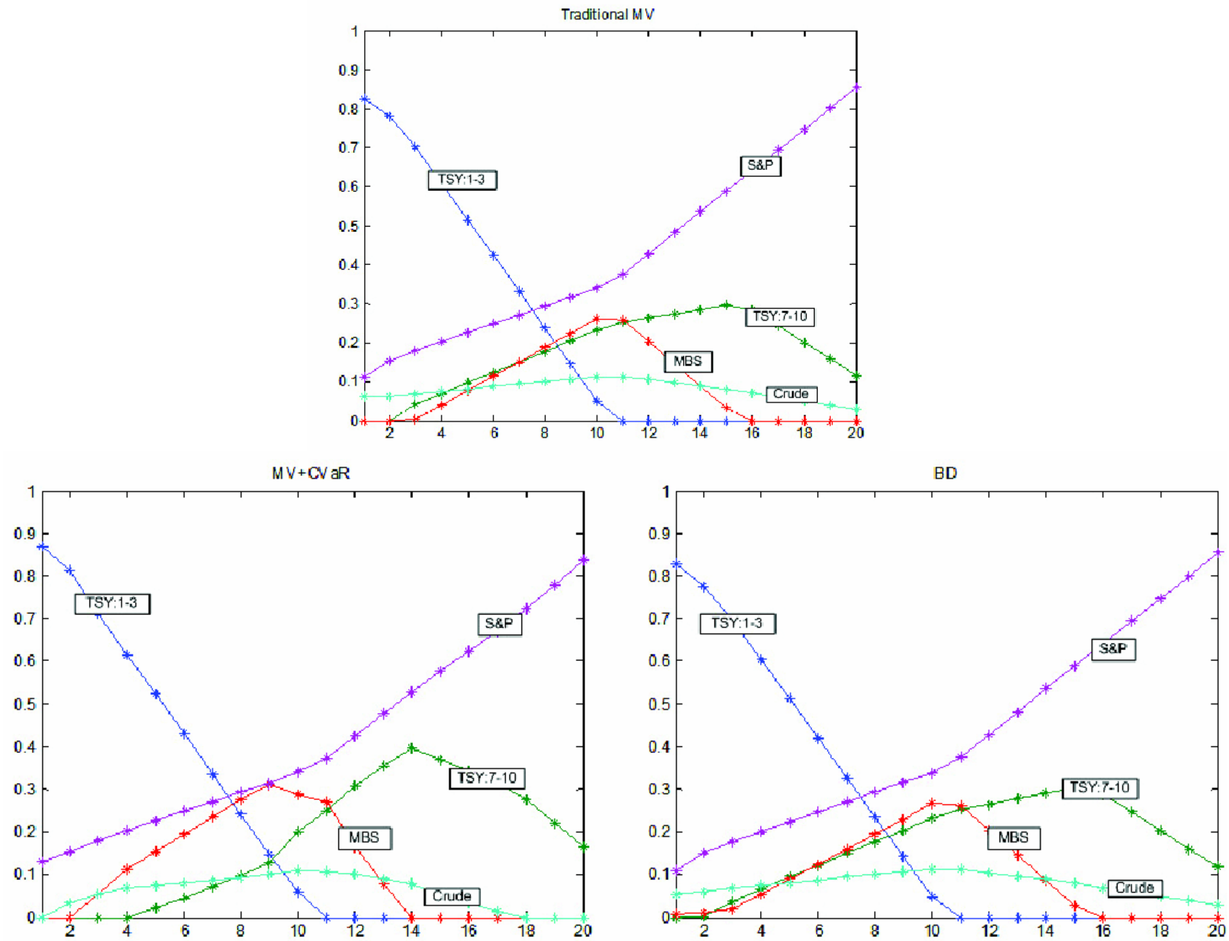

FIGURE 3. The 5-asset mix for the efficient portfolios of the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD"). The vertical axis stands for the weight of each asset, and the horizontal axis is the solution number for the efficient portfolios in Figure 1. The asset "TSY: 1-3" stands for the short-term US Treasury bill; "TSY: 7-10" is the long-term US Treasury bond; "MBS" is the mortgage-backed security; "Crude" is the crude oil future; and "S&P" is the S&P 500 Index.

We also plot in Figure 3 the asset mix for the 20 efficient portfolios of these three methods. The horizontal axis shows only the solution number; return and variance increase as the solution number increases. We can think of the horizontal axis as representing either the return or the variance. As the required return increases, the mix shifts from bonds to equity as the weight of MBS first rises and then falls. All three methods requires all five assets to form the efficient frontier. None of the portfolios in these three approaches contain a lot of crude oil future. Figure 3 also shows the source of skewness. Although the three methods have similar holdings in the S&P 500, the MV + CVaR approach invests relatively more in the long-term US Treasury bond and less

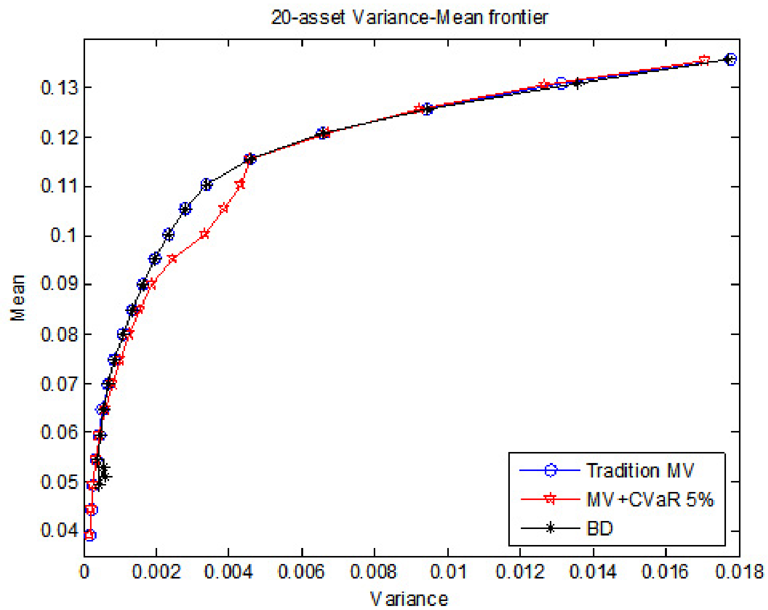

FIGURE 4. The efficient frontiers of 20-asset portfolios based on the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD") . The vertical axis stands for the expected returns of portfolios, and the horizontal axis is for variances.

in the crude oil future. Since the skewness of the long-term US Treasury bond is higher than that of the crude oil future, this confirms our result shown in Figure 2.

Example 2. More assets are included in the portfolio this time. In addition to 5 assets in Example 1, we include another 15 assets. That is, we expand the sample to 20 assets ($k_1 = 0$ and $k_2 = 20$). These 15 new assets include a short-term agency bond, a long-term agency bond, a middle-term AAA corporate bond, a middle-term AA corporate bond, a middle-term A corporate bond, a middle-term BBB corporate bond, a corporate high yield bond, an international sovereign bond, a Yankee bond, an asset-backed security, a middle-term municipal bond, a commodity future, a lumber future, a currency future and a convertible bond. Their mean-variance frontiers and skewness-variance graphs based on the three approaches analyzed are shown in Figures 4 and 5. These graphs are similar to those in Example 1. Specifically, skewness with the MV + CVaR approach is higher than those of the MV and Boyle-Ding approaches although its portfolios are relatively less efficient.

Example 3. In this example, we study the ALM problem by maximizing the overall profits of assets and lines of business. We use 14 lines of business ($k_1 = 14$) and the same five assets as in Example 1 ($k_2 = 5$). The data are from the General Re Company from 1980 to 2005. These 14 lines of business are Commercial Property, Allied Lines, Farmowners/Farmers Multiple Peril,

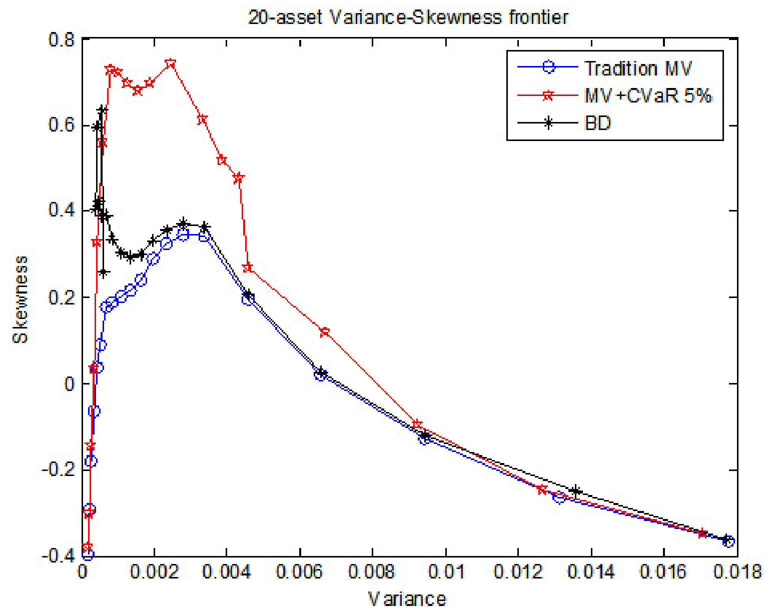

FIGURE 5. The 20-asset portfolios skewness-variance graphs of the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD"). The vertical axis stands for the skewness of portfolios, and the horizontal axis is for variances.

Commercial Multiple Peril, Workers' Compensation, General Liability, Medical Professional Liability, Private Passenger Auto Liability, Commercial Auto/Truck Liability, Fidelity/Surety, Blue Cross Blue Shield, Public and Commercial Health Insurance, Reinsurance and Other Insurance.

As for the asset-liability portfolio containing 14 lines of business and 5 assets, since both the weights of lines and the weights of assets are required to sum to one separately, the mean-variance frontier becomes more "sparse". The "sparse" here means that there are more available portfolios that can satisfy a specific combination of (σ^2, μ). Therefore, it is more likely to increase skewness without sacrificing variance (increase variance). The following experiments confirm this inference.

According to equation (18), ϵ constrains the biggest increase of variance and δ denotes the highest possible improvement of skewness. In general, a big increase of skewness is accompanied with a large sacrifice of variance. Therefore, these two tolerance parameters change in the same direction. In the figures 6 and 7, the Boyle-Ding curves are obtained by setting $\epsilon = 0.03$ and $\delta = 0.0003$. While, we set $\epsilon = 0.01$ and $\delta = 0.00005$ for the Boyle-Ding approach in figures 8 and 9.

In both cases, mean-variance frontiers got from skewness-improving methods match the Markowitz frontier very well, no matter the CVaR-like constraints or Boyle-Ding approach is considered. When δ is high (Figure 7), Boyle-Ding approach outperforms CVaR-like constraints approach

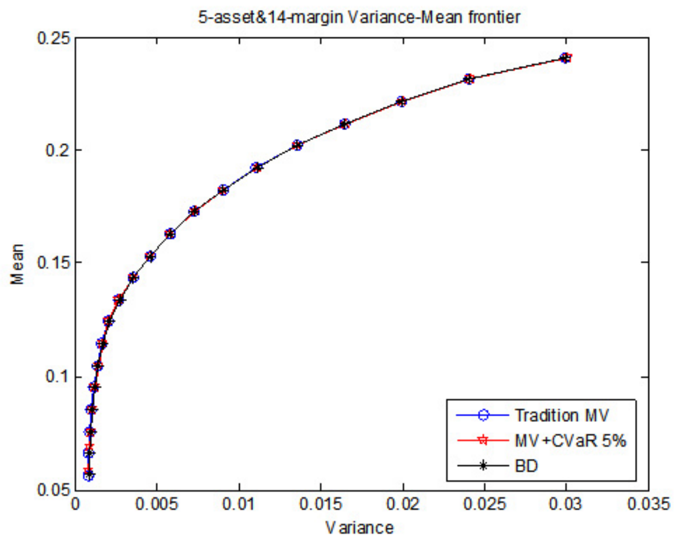

FIGURE 6. The efficient frontiers of 14-line and 5-asset portfolios based on the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD") with $\epsilon = 0.03$ and $\delta = 0.0003$. The vertical axis stands for the expected returns of portfolios, and the horizontal axis is for variances.

in the low-variance interval (0, 0.013) and CVaR-like constraints approach is better in the high-variance regime. Theoretically, one should increase δ as high as possible conditional on no sacrifice of the mean-variance frontier. In our example, $\delta = 0.0003$ is preferred.

Notice that for both cases, in a small mediate variance interval, CVaR-like constraints approach obtains lower skewness than classical Markowitz method does. This phenomenon can also be found in the 20-asset portfolio example. In Figure 4, the "MV+CVaR" frontier deviates from the Markowitz mean-variance frontier in the variance interval (0.002, 0.005). In figure 10, we use maximum-entropy distribution to show effects of CVaR-like constraints on portfolio distribution.

The maximum-entropy distribution is the representative distribution which is most likely to realize with given moments and support. It only considers moment information and therefore, can be used to check effects of our portfolio optimization approaches on distributions. In Figure 10, three moments are given and the support is set at $[\mu - 4\sigma, \mu + 4\sigma]$. Since this method is not the focus of our paper, we will not discuss it in details. Below is a simplified "mathematical definition" of maximum-entropy approach we used in our paper.

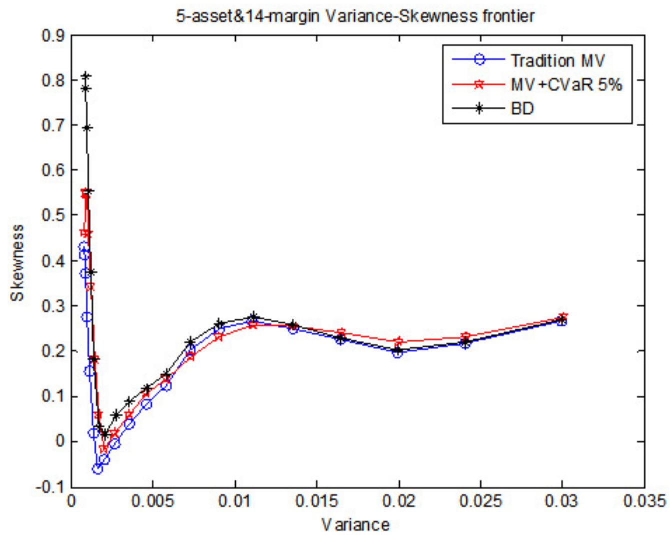

FIGURE 7. The 14-line and 5-asset portfolios skewness-variance graphs of the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD") with $\epsilon = 0.03$ and $\delta = 0.0003$. The vertical axis stands for the skewness of portfolios, and the horizontal axis is for variances.

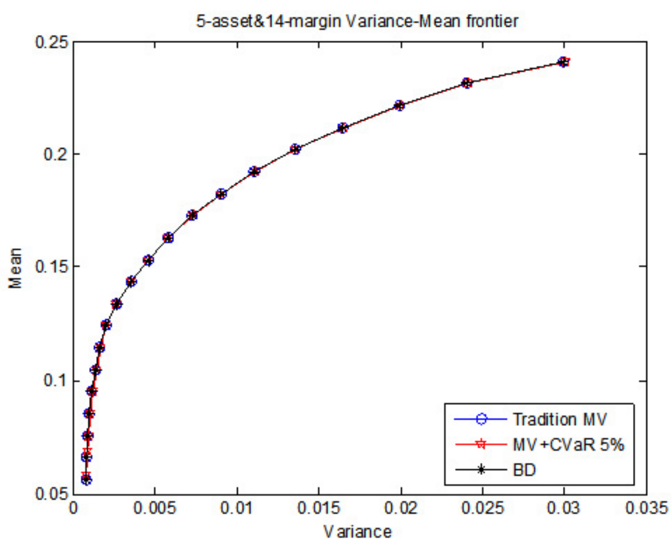

FIGURE 8. The efficient frontiers of 14-line and 5-asset portfolios based on the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD") with $\epsilon = 0.01$ and $\delta = 0.00005$. The vertical axis stands for the expected returns of portfolios, and the horizontal axis is for variances.

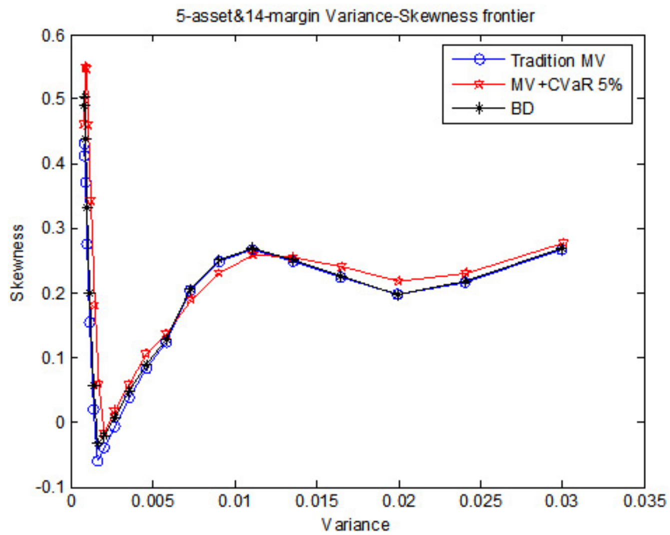

FIGURE 9. The 14-line and 5-asset portfolios skewness-variance graphs of the Markowitz Mean-Variance approach ("Traditional MV"), the Markowitz Mean-Variance approach with 5%-level CVaR constraint ("MV + CVaR") and the Boyle-Ding Mean-Variance-Skewness approach ("BD") with $\epsilon = 0.01$ and $\delta = 0.00005$. The vertical axis stands for the skewness of portfolios, and the horizontal axis is for variances.

The maximum-entropy distribution has a density function $f^*(x)$ that solves the following optimization problem:

$$\begin{aligned} &\max_{f(x)} \quad -\int_a^b f(x) \log f(x) \, \mathrm{d}x \\ &\text{subject to} \int_a^b x^i f(x) \, \mathrm{d}x = \mu_i \text{ for } i = 0, 1, \ldots, n \\ &\text{and } f(x) \geq 0 \end{aligned} \tag{19}$$

Where $\mu_0, \mu_1, \ldots, \mu_n$ are the given sequence of moments.

According to the left plot in Figure 10, 5% CVaR-like constraint makes the left tail shift to the right, putting more mass on the right tail. However, the part around the mean decreases a little bit. This partly explains why CVaR-like constraints approach deteriorates skewness in a small mediate variance interval.

6. ROBUSTNESS CHECK

In this section, we compare the CVaR-like constraints approach with two more alternatives: the CVaR optimization approach and the Mean-absolute Deviation (MAD) approach.

The CVaR optimization approach chooses CVaR as the objective function. It is proposed by Krokhmal et al. (2002). They suggest minimizing the CVaR of loss portfolios. Therefore, for return portfolios, we maximize CVaR to control risks and increase the likely of getting higher

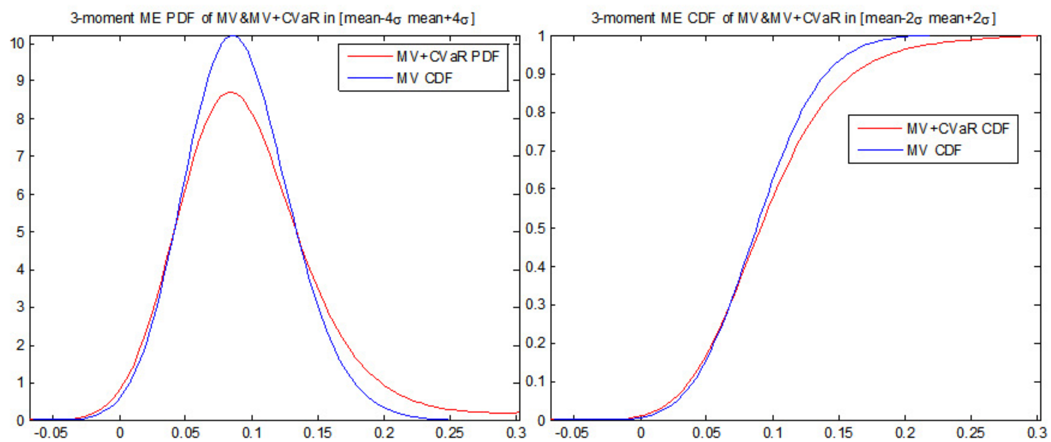

FIGURE 10. Maximum-entropy distribution (red curve) and the corresponding normal distribution (blue curve) with the same mean and variance of the portfolios obtains by adding 5% CVaR-like constraint (Figure 6). The maximum-entropy distribution can be considered as the portfolio return distribution with 5% CVaR-like constraint and the corresponding normal represents the distribution of traditional Markowitz return. The left plot shows the probability density function (pdf) and the right one shows the cumulative distribution function (CDF).

returns. The optimization problem (We call it "CVaR approach") is:

$$
\begin{aligned}
&\text{Maximize} && \mathrm{CVaR}(\mathrm{x}, \beta) \\
&\text{subject to} && \sum_{i=1}^{k_1} x_i = 1 \\
& && \sum_{i=k_1+1}^{n} x_i = 1 \\
& && \sum_{i=1}^{n} \mu_i x_i = \mu_0(x) \\
& && x_i \geq 0, \qquad\qquad i = 1, \ldots, n.
\end{aligned} \tag{20}
$$

If the portfolio returns are normally distributed, the Markowitz MV and CVaR will generate the same efficient frontier. However, the solutions of CVaR approach may be far away from the traditional MV frontier. In the case of non-normal, and especially non-symmetric distributions, CVaR and MV portfolio optimization approaches may reveal significant differences (Rockafellar and Uryasev, 2000).

For the mean-absolute Deviation (MAD) approach, instead of using variance, which is in the quadratic form, one uses absolute value of the dispersion of the portfolio returns to measure risks:

$$
\mathrm{MAD}(R(x)) = \mathbb{E}(|R(x) - \mathbb{E}(R(x))|) = \mathbb{E}\left(\left|\sum_{i=1}^{n} R_i x_i - \mathbb{E}\left(\sum_{i=1}^{n} R_i x_i\right)\right|\right).
$$

The sample version obtained from the historical data r_{ij} is

$$\mathrm{MAD}(R(x)) = \frac{1}{m}\sum_{j=1}^{m}\left|\sum_{i=1}^{n} r_{ij}x_i - \sum_{i=1}^{n}\hat{\mu}_i x_i\right| = \frac{1}{m}\sum_{j=1}^{m}\left|\sum_{i=1}^{n}(r_{ij}-\hat{\mu}_i)x_i\right|,$$

where

$$\hat{\mu}_i = \frac{1}{m}\sum_{j=1}^{m} r_{ij}.$$

Replacing the variance of the portfolio returns with this measure in the MV model, the portfolio optimization problem based on the MAD approach is defined as follows:

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{m}\sum_{j=1}^{m}\left|\sum_{i=1}^{n}(r_{ij}-\hat{\mu}_i)x_i\right| \\
\text{subject to} \quad & \sum_{i=1}^{k_1} x_i = 1 \\
& \sum_{i=k_1+1}^{n} x_i = 1 \\
& \sum_{i=1}^{n}\hat{\mu}_i x_i = \mu_0(x) \\
& x_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}$$

We first solve the same portfolio optimization problems as those in Examples 1, 2 and 3 based on the CVaR and MAD approaches respectively and plot their MV frontier and skewness-variance graphs (which are not shown here).[4] Then we compare the CVaR-like constraints method with these two approaches.

Our results indicate that the CVaR approach is the least efficient one among these three in terms of the mean-variance tradeoff especially in the low level variance range, but it offers the highest skewness. Intuitively, as a risk management measure, CVaR tends to maximize the expected return just below a given level of VaR, but not on the whole distribution. Therefore, it has more room to reshape the tail to increase the skewness of portfolios but at some time it sacrifices more portfolio efficiency. Whether it is an acceptable technique depends on the extent to which investors are willing to deviate from the traditional MV frontier.

As for the MAD approach, it is always the least desirable in terms of securing higher skewness, especially with higher variance portfolios. In all three examples, the MAD skewness-variance line jumps up and down from the Markowitz line. These erratic results suggest that the MAD approach is not a good method, at least with our examples. It implies that the MAD approach may be subject to functional form bias and not a good risk surrogate (Lee, 1977).

After we compare the CVaR-like constraint approach with two more methods, we further confirm that it is a promising method for portfolio optimization and risk management, especially for

[4]The results are available upon request.

investors who are interested in increasing their portfolio's skewness while not deviating far from the traditional MV frontier.

7. Conclusion

In this paper, we develop a new effective way, the CVaR-like constraints approach, to improve the skewness of a MV portfolio. Specifically, we add one or more CVaR-like constraints to the traditional portfolio optimization problem. This method is compared with the Boyle-Ding approach. Numerical analysis shows that the CVaR-like constraint approach is a more effective way to improve the skewness given it does not deviate too much from the traditional MV frontier. Our robustness check also shows its superiority over two other methods: the CVaR approach and the MAD approach. Moreover, we have demonstrated that the CVaR-like constraints approach can be used to successfully manage asset-liability portfolios of financial institutions such as insurance companies.

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Appendix

Proof of CVaR Expression Transformation: Equation (9).

Call $F(x,\alpha,\beta) = \alpha - \frac{1}{\beta}\mathbb{E}((\alpha - R(x))^+)$. If we fix x, for $\lambda \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}(((\lambda\alpha_1 + (1-\lambda)\alpha_2) - R(x))^+) = \\
&\mathbb{E}(((\lambda(\alpha_1 - R(x)) + (1-\lambda)(\alpha_2 - R(x)))^+) \leq \\
&\mathbb{E}((\lambda(\alpha_1 - R(x))^+ + (1-\lambda)(\alpha_2 - R(x))^+) = \\
&\lambda\mathbb{E}((\alpha_1 - R(x))^+) + (1-\lambda)\mathbb{E}((\alpha_2 - R(x))^+).
\end{aligned}$$

So $\mathbb{E}((\alpha - R(x))^+)$ is convex on α. The inequality above follows

$$\max\{a+b,0\} \leq \max\{a,0\} + \max\{b,0\}.$$

Since $-\frac{1}{\beta} \leq 0$ and the first term in $F(x,\alpha,\beta)$ is linear, the function $F(x,\alpha,\beta)$ is concave. Thus the maximum can be found by differentiating $F(x,\alpha,\beta)$ with respect to α and then setting differentiated function equal to zero.

$$\frac{\delta}{\delta\alpha}F(x,\alpha,\beta) = 1 - \frac{1}{\beta}\mathbb{E}(\mathbb{I}(R(x) \leq \alpha) = 1 - \frac{1}{\beta}\mathbb{P}(R(x) \leq \alpha).$$

So the maximizer α^* satisfies

$$1 - \frac{1}{\beta}\mathbb{P}(R(x) \leq \alpha^*) = 0,$$

or

$$\mathbb{P}(R(x) \leq \alpha^*) = \beta.$$

That is, α^* is the β-level VaR or $\alpha^* = \alpha(x,\beta)$. So

$$\max_{\alpha}\ \alpha - \frac{1}{\beta}\mathbb{E}((\alpha - R(x))^+) = \alpha(x,\beta) - \frac{1}{\beta}\mathbb{E}((\alpha(x,\beta) - R(x))^+).$$

To finish, we notice

$$\begin{aligned}
\mathbb{E}((\alpha(x,\beta) - R(x))^+) = \ & \mathbb{E}((\alpha(x,\beta) - R(x))^+ | R(x) \geq \alpha(x,\beta))\mathbb{P}(R(x) \geq \alpha(x,\beta)) \\
&+\mathbb{E}((\alpha(x,\beta) - R(x))^+ | R(x) \leq \alpha(x,\beta))\mathbb{P}(R(x) \leq \alpha(x,\beta)).
\end{aligned}$$

The first term on the right of the above equation is zero and the second term becomes

$$\begin{aligned}
&\mathbb{E}((\alpha(x,\beta) - R(x))^+ | R(x) \leq \alpha(x,\beta))P(R(x) \leq \alpha(x,\beta)) \\
&= \mathbb{E}((\alpha(x,\beta) - R(x)) | R(x) \leq \alpha(x,\beta))\beta \\
&= \beta\alpha(x,\beta) - \beta\mathbb{E}((R(x)) | R(x) \leq \alpha(x,\beta)) \\
&= \beta\alpha(x,\beta) - \beta\mathrm{CVaR}(\mathrm{x},\beta).
\end{aligned}$$

Replacing it back, we get

$$\max_{\alpha}\ \alpha - \frac{1}{\beta}\mathbb{E}((\alpha - R(x))^+) = \alpha(x,\beta) - \frac{1}{\beta}(\beta\alpha(x,\beta) - \beta\mathrm{CVaR}(\mathrm{x},\beta)) = \mathrm{CVaR}(\mathrm{x},\beta).$$

Proof of Skewness Condition of Equation (17).

The skewness condition is difficult to handle. The usual techniques of portfolio optimization will not handle such a non-linear constraint. Boyle and Ding (2006) replace this constraint by a set of m linear inequalities. In order to increase the skewness, it is sufficient that

$$[\mu(x)_j - \mu(x)]^3 \geq [\mu(x^*)_j - \mu(x^*)]^3 \text{ for each period } j = 1, 2, \ldots, m. \tag{21}$$

Each of these cubic constraints is replaced by a linear constraint. The linear constraint is based on the approximation to t^3 obtained joining the points (a, a^3) and (b, b^3) with a line. In the notation of the paper $a = t_0 - \epsilon$ and $b = t_0 + \epsilon$, where ϵ is a small positive number, and

$$\begin{aligned} t_0 &= \mu(x^*)_j - \mu(x^*) = \alpha_j, \\ t &= \mu(x)_j - \mu(x) = \sum_{i=1}^{n} (r_{ij} - \mu_i) x_i. \end{aligned}$$

This gives us

$$\begin{aligned} t^3 &\approx a^3 + \frac{b^3 - a^3}{b - a}(t - a) = a^3 + [a^2 + ab + b^2](t - a) \\ &= (t_0 - \epsilon)^3 + \left[(t_0 - \epsilon)^2 + (t_0 - \epsilon)(t_0 + \epsilon) + (t_0 + \epsilon)^2 (r_{ij} - \mu_j)\right](t - t_0 + \epsilon) \\ &= (t_0 - \epsilon)^3 + g(t_0)(t - t_0 + \epsilon). \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned} \left(\sum_{i=1}^{n} (r_{ij} - \mu_i) x_i\right)^3 &\approx (\alpha_j - \epsilon)^3 + g(\alpha_j) \left(\sum_{i=1}^{n} (r_{ij} - \mu_i) x_i - \alpha_j + \epsilon\right) \\ &= (\alpha_j - \epsilon)^3 - (\alpha_j - \epsilon) g(\alpha_j) + g(\alpha_j) \sum_{i=1}^{n} (r_{ij} - \mu_i) x_i, \end{aligned} \tag{23}$$

where $g(t_0) = (t_0 - \epsilon)^2 + (t_0 - \epsilon)(t_0 + \epsilon) + (t_0 + \epsilon)^2$. This is a good approximation when $a < t < b$ and $|b - a|$ is small, *i.e.*, when x satisfies the following inequalities:

$$\alpha_j - \epsilon < \sum_{i=1}^{n} (r_{ij} - \mu_i) x_i < \alpha_j + \epsilon, \quad j = 1, \ldots, m. \tag{24}$$

The constraints (16) are used in Boyle and Ding (2006). This implies that the mean of the new portfolio cannot change more than $\pm\epsilon$ from the initial mean for each observation period j.

Provided the inequalities (16) hold for each $j = 1, \cdots, m$, then from (22) we have

$$\begin{aligned} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} (r_{ij} - \mu_i) x_i\right)^3 &\approx \sum_{j=1}^{m} \left[(\alpha_j - \epsilon)^3 - (\alpha_j - \epsilon) g(\alpha_j)\right] + \sum_{j=1}^{m} g(\alpha_j) \sum_{i=1}^{n} (r_{ij} - \mu_i) x_i \\ &= C + \sum_{i=1}^{n} c_i x_i, \end{aligned} \tag{25}$$

where

$$C = \sum_{j=1}^{m} \left[(\alpha_j - \epsilon)^3 - (\alpha_j - \epsilon) g(\alpha_j) \right] \text{ and } c_i = \sum_{j=1}^{m} g(\alpha_j)(r_{ij} - \mu_i).$$

The same analysis applies to the original portfolio x^*:

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} (r_{ij} - \mu_i) x_i^* \right)^3 \approx C + \sum_{i=1}^{n} c_i x_i^*.$$

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