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AIDS AND THE CALCULATION OF LIFE INSURANCE FUNCTIONS

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ABSTRACT

We consider an HIV+ life that is subject to the various stages of the Walter Reed Staging Method with constant forces of mortality and progression in each stage. The traditional life insurance functions (such as the net single premium for an n-year term insurance or temporary annuity) are found. Net single premiums and net annual premiums per $1,000 whole life insurance are tabulated under various pre-AIDS mortality levels.

1. INTRODUCTION

Acquired Immune Deficiency Syndrome (AIDS) is one of the most devastating diseases to have afflicted humanity in recent times. Because of the nature of its transmission, there is a unique social stigma attached to this disease. As a result, there is great difficulty in assessing the spread of AIDS through the population.

The impact of AIDS on the financial health of the life and health insurance industry in the United States is now being assessed. The Society of Actuaries commissioned a task force [16] to investigate the various strategies available to insurance companies for dealing with the problems posed by AIDS. This task force noted that the outlook on AIDS mortality and morbidity was bleak, especially in the near term. In the opinion of some researchers, the earliest AIDS vaccine will not be available until the mid-1990s, while a number of other researchers feel that no vaccine will become available until the next century.

Because most of the reported AIDS cases are currently confined to the homosexual and IV-drug-using populations, one may question the existence of an insurable interest. However, since the current AIDS cases result from infections occurring 5 to 10 years ago, the currently infected heterosexual lives will not show up as AIDS cases until the mid-1990s. As a result, the number of reported cases of full-blown AIDS represents only the "tip of the iceberg." Of interest to insurers is the extent of the spread of the Human Immunodeficiency Virus (HIV) into the much larger (potentially insurable) heterosexual population. The Centers for Disease Control ([1] and [17]) estimate that the number of HIV+ cases is spreading in the heterosexual community, albeit at a stable rate. Some of these HIV+ heterosexuals may already have been insured.

Another problem facing insurance companies is the risk of antiselection. This problem can be dealt with, to a large extent, by more stringent underwriting standards such as using AIDS antibody tests and lowering AIDS testing limits. Unfortunately, AIDS antibody tests are not able to detect infections until enough antibodies have been produced. It may take 6–8 weeks for enough antibodies to be produced, thus allowing some HIV+ lives to be misclassified within this period (see [16], chapter 2). In a recent article, Imagawa et al. [7] found that it may take up to 36 months before antibodies to the HIV-1 virus can be detected.* Thus an insurable interest exists.

Clearly, as the disease spreads through the "insurable" heterosexual population, insurance companies will have HIV+ lives on their books and the proportion of deaths due to AIDS will increase. AIDS thus presents the valuation actuary with numerous problems, the major one being that the pricing and reserve standards at the time of issue of inforce business did not anticipate the AIDS risk. In order to deal with such problems, actuaries need to be able to compute the traditional life insurance functions in an AIDS environment. The objective of this paper is to provide expressions for some of these functions. Such expressions, to the best of the author's knowledge, are not available in the actuarial literature.

2. THE MODEL

Consider an HIV+ life that is otherwise healthy and asymptomatic. One method used to describe such a life's possible progression to AIDS is the Walter Reed Staging Method (WRSM) described by Redfield et al. [14]. The WRSM groups patients who have tested HIV+ into four stages along the route to full-blown AIDS, rather than grouping patients according to their complications. For completeness, we have added two more stages, "at-risk" and "death." These stages (labeled 0 to 5) are described below:

Stage 0 (At-risk) Healthy persons at risk for HIV+ infection, but testing negative

Stage 1 (HIV+) Otherwise asymptomatic persons testing HIV+

Stage 2 (LAS) Persons with HIV infection and lymphademopathy syndrome (LAS), together with moderate celluar immune deficiency

*This article was brought to my attention by Scott Holmes, Public Health Epidemiologist, Lincoln-Lancaster County Health Department, Lincoln, Neb.

Stage 3 (ARC)	Patients with HIV infection and LAS, plus severe cellular immune deficiency (AIDS-Related Complex, or ARC)
Stage 4 (AIDS)	Patients with AIDS
Stage 5 (Death)	Patients who died "of AIDS."
....	....
Stage 6 (Death)	Patients who died in stages 0, 1, 2 or 3. Patients who died in stage 4 of causes not related to AIDS.

Cowell and Hoskins [4] and Panjer [12] used the WRSM as the foundation of their respective models. They also assumed (1) that the progression of the disease through its various stages is sequential and irreversible and (2) that death prior to stage 4, that is, full-blown AIDS, is not permitted. Unfortunately, a life can die in stages 0 to 3 (never actually having developed AIDS) or in stage 4 of a cause unrelated to AIDS. The mortality in the earlier stages may be significantly worse than normal and must be included in any model. This fact has been recognized by researchers and is an established part of most mathematical models of HIV+ lives; see, for example, Dietz [5], Hyman and Stanley [6], Isham [8], May et al. [11], and Wilkie [18]. Another death stage (labeled as stage 6 above) has been added to accommodate this other type of death. Deaths prior to stage 4 are labeled as "immediate transitions" to stage 6; deaths in stage 4 that are not due to AIDS are also labeled as transitions to stage 6.

There are very few mathematical models of the *transition* dynamics of HIV using the WRSM. The only published mathematical models of which the author is aware that include these HIV transition stages are the above-mentioned papers by Cowell and Hoskins [4] and Panjer [12]. Most models deal with the *spread* of HIV in populations. In these models the assumption of constant intensities (or hazard rates) is often used to simplify the mathematics. In the few cases in which time-dependent hazard rates are assumed, proportional hazard functions are used for simplicity.

Like Panjer's model, we use a continuous time Markov process. For simplicity, assume a person in stage i is subject to a constant force of progression out of stage i into stage $i+1$ and to a constant force of mortality (out of stage i into stage 6). Once a life leaves a stage, it cannot return to that stage. This clearly yields a Markov process. These forces are operating simultaneously on the life in a "multiple decrement" environment. Actually, the situation is more akin to life tables with secondary decrements, in the sense of Jordan ([9], chapter 15), than to a multiple decrement table. This is because each stage i can be considered as producing its own "multiple decrement" table with two forces of decrement, mortality and progression.

In particular, for $i=0, 1, 2, 3, 4$, let μ_i be the force of progression from stage i to stage $i+1$, and μ_i' be the force of mortality while in stage i, that is, immediate transition to stage 6. Since both forces remain constant while in any stage, a "memoryless" property exists. This means that the length of time already spent in the current stage has no effect on the future length of time that the person will remain in this stage. This permits us to speak in terms of the future time spent in a stage without having to condition on the amount of time already spent in the stage. Let T_i be the (future) time spent on stage i before entering stage $i+1$, and T_i' be the (future) time spent in stage i before immediate transition to stage 6. Also let $T_i^{(d)}$ be the future lifetime until death (from any cause) for a life currently in stage i. The random variable $T_i^{(d)}$ is well defined and continuous and is given by

$$T_i^{(d)} = \begin{cases} T_i' & \text{if transition directly to stage 6} \\ T_i + T_{i+1}^{(d)} & \text{if progression to stage } i+1. \end{cases} \tag{1}$$

In the sequel we develop expressions for the traditional insurance functions, for example, net single premiums, actuarial present values of annuities, reserves, and distribution of loss functions. Throughout, it is assumed that insurance is issued to a life in stage i at the time of issue, that is, at $t=0$. This life is denoted by (i).

3. LIFE INSURANCE FUNCTIONS

The approach that is used to derive expressions for life insurance functions, probabilities, etc., is to derive a system of differential-difference equations called the Chapman-Kolmogorov backward equations; see Karlin and Taylor ([10], chapter 4, pp. 135–139). Smith [15], in his discussion of Panjer's paper, suggested the Chapman-Kolmogorov equations should be written in matrix form. The general matrix form of this system of equations for $t>0$ is

$$\frac{d}{dt}P(t) = MP(t) + B \tag{2}$$

where $P(t)$ is a column vector, M is the infinitesimal generator of the Markov transition process in the sense of Karlin and Taylor ([10], chapter 4.4, p.

132), and B is a column vector of constants. Smith suggested the solution to the homogeneous system, that is, with $B=0$, be written in series form as

$$P(t) = \left[I + \sum_{k=1}^{\infty} \frac{t^k}{k!} M^k \right] P(0)$$

where I is the identity matrix and $P(0)$ is the initial vector. As this summation is absolutely convergent, one may use as many terms as necessary to achieve the particular degree of accuracy desired in our computations. Smith pointed out that this series solution avoids the "near singularity" problems that may result whenever the eigenvalues of M are close. In this case the solution will necessarily be approximate.

If the eigenvalues of M are distinct, or some are equal and M can still be factorized as $M=U^{-1}DU$, where D is the diagonal matrix of eigenvalues of M and U is the matrix of independent eigenvectors of M, then $P(t)$ may be written in the form

$$P(t) = U^{-1}\Lambda(t)U$$

where $\Lambda(t)$ is a diagonal matrix. This solution does not require any approximations.

The solution to the nonhomogeneous equation (2) can be found by using well-developed techniques of differential equations; see Rainville and Bedient ([13], chapter 13). As noted above, these exact solutions require the inversion of matrices and the finding of eigenvectors. The approach used in this paper is straightforward and exact and requires no matrix inversions. The solutions fully exploit the constant intensities assumption. Since transitions to a previous stage are not permitted, we obtain simple recursive solutions to the differential equations. These solutions are also easily programmable.

Let $p_{ij}(t)$ be the probability that (i) will be alive t years from now and be in stage j. Consider the interval $(0, t+dt]$ as the union of $(0, dt]$ and $(dt, t+dt]$. For very small dt, we have

$$p_{ij}(t + dt) = 0 \times \mu_i' dt + \mu_i dt\, p_{i+1,j}(t) + (1 - \alpha_i dt)\, p_{ij}(t) + o(dt)$$

where $\alpha_i = \mu_i + \mu_i'$ and $o(x)$ satisfies

$$\lim_{x \to 0} \frac{o(x)}{x} = 0.$$

Transposing the term $p_{ij}(t)$ to the left-hand side and then dividing both sides by dt and letting $dt \downarrow 0$ gives us, for $j = i,\ i+1, \ldots, 4$,

$$\frac{d\,p_{ij}(t)}{dt} = \mu_i\, p_{i+1,j}(t) - \alpha_i\, p_{ij}(t). \tag{3}$$

It is clear that

$$p_{ij}(0) = \delta_{ij}$$
$$p_{jj}(t) = e^{-\alpha_j t}$$

where δ_{ij} is the Kronecker delta, that is,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The solution to Equation (3) is

$$p_{ij}(t) = e^{-\alpha_i t}\left[\delta_{ij} + \int_0^t \mu_i p_{i+1,j}(s) e^{\alpha_i s}\, ds\right]. \tag{4}$$

It can be written as the sum of exponential terms, that is,

$$p_{ij}(t) = \sum_{k=i}^{j} p_{ik}^{(j)}\, e^{-\alpha_k t}, \tag{5}$$

with

$$p_{ik}^{(j)} = 0 \text{ for } i > j \text{ or } k > j.$$

Substituting Equation (5) into Equation (4) and then comparing the coefficients of $e^{-\alpha_k t}$, on both sides of the resulting equation gives

$$\left.\begin{aligned} p_{jj}^{(j)} &= 1 && \text{for } j = 0, 1, \ldots, 4 \\ p_{ik}^{(j)} &= -(\mu_i\, p_{i+1,k}^{(j)})/(\alpha_k - \alpha_i) && \text{for } k = i+1, \ldots, j \\ p_{ii}^{(j)} &= -\sum_{k=i+1}^{j} p_{ik}^{(j)} && \text{for } i < j. \end{aligned}\right\} \tag{6}$$

The $p_{ij}(t)$'s are now known.

The probability that (i) will die in stage j within the next t years is denoted by $q_{ij}(t)$, where

$$q_{ij}(t) = \int_0^t p_{ij}(s)\mu_j'\, ds.$$
$$= \sum_{k=i}^{j} \frac{\mu_j'}{\alpha_k} p_{ik}^{(j)} (1 - e^{-\alpha_k t}).$$

The probability that a life (i) will die within the next t years will be denoted by $q_i(t)$, where

$$q_i(t) = \Pr[T_i^{(d)} \le t]$$
$$= \sum_{j=i}^{4} q_{ij}(t).$$

Using the Chapman-Kolmogorov backward equation, a differential equation for $q_i(t)$ can be established as follows:

$$q_i(t + dt) = \mu_i' dt + q_{i+1}(t)\mu_i dt + (1 - \alpha_i dt)q_i(t) + o(dt),$$

which leads to

$$\frac{d\, q_i(t)}{dt} = \mu_i' + \mu_i q_{i+1}(t) - \alpha_i q_i(t), \tag{7}$$

with boundary conditions

$$q_i(0) = 0 \text{ and } q_6(t) \equiv q_5(t) \equiv 1,\ t \ge 0.$$

The solution to this differential equation is

$$q_i(t) = e^{-\alpha_i t} \int_0^t [\mu_i' + \mu_i q_{i+1}(s)] e^{\alpha_i s}\, ds \tag{8}$$

with

$$q_4(t) = 1 - e^{-\alpha_4 t}. \tag{9}$$

The general form of $q_i(t)$ is

$$q_i(t) = q_i + \sum_{j=i}^{4} \xi_{ij} e^{-\alpha_j t} \text{ for } 0 \le i \le 4, \tag{10}$$

where the q_i's and ξ_{ij}'s are constants. After substituting Equation (10) into Equation (8) and then comparing the coefficients of $e^{-\alpha_j t}$ on both sides of the resulting expression, we have:

$$\left.\begin{aligned} q_i &= 1 && \text{for } i = 0, 1, \ldots, 4; \\ \xi_{44} &= -1 && \\ \xi_{ij} &= -(\mu_i \xi_{i+1,j})/(\alpha_j - \alpha_i) && \text{for } j \geq i + 1; \\ \xi_{ii} &= -1 - \sum_{j=i+1}^{4} \xi_{ij} && \text{for } i = 0, 1, 2, 3. \end{aligned}\right\} \quad (11)$$

Of interest to actuaries is $E[T_i^{(d)}]$, the life expectancy for a life now in stage i. A recursive expression for this expectation can easily be derived as follows: let

$$\mathring{e}_i = E[T_i^{(d)}],$$
$$D_i = \{\text{the event that the life progresses to stage } i+1\},$$

and the complement of D_i be given by

$$D_i^c = \{\text{the event that the life dies in stage } i\}.$$

For $i = 0$, 1, 2, 3, and 4, it follows that

$$\begin{aligned} \mathring{e}_i &= E[T_i^{(d)}|D_i]\Pr[D_i] + E[T_i^{(d)}|D_i^c]\Pr[D_i^c] \\ &= E[T_i + T_{i+1}(d)|D_i]\Pr[D_i] + E[T_i'|D_i^c]\Pr[D_i^c] \\ &= \left(\frac{1}{\alpha_i} + \mathring{e}_{i+1}\right)\frac{\mu_i}{\alpha_i} + \frac{1}{\alpha_i}\left(\frac{\mu_i'}{\alpha_i}\right) \\ &= \frac{1}{\alpha_i} + \frac{\mu_i}{\alpha_i}\mathring{e}_{i+1}. \end{aligned} \quad (12)$$

This recursive expression can now be used because $\mathring{e}_5 = 0$.

Suppose the life (i) desires an n-year term insurance with a face value of \$1 issued on a fully continuous basis. By fully continuous we mean premiums are paid continuously until death or for n years, whichever is shorter, and the death benefit is paid immediately upon death if death occurs within the next n years. How can we determine the net premium and the net premium reserve for such a policy? Can we find the distribution of the prospective loss function? As we will see, the answers are similar to those one would expect from traditional life insurance arguments.

The following notation is used throughout the rest of this paper; the subscripts i refer to (i) at the present time:

$\bar{A}_i(t)$ = net single premium for t-year continuous term insurance
$\bar{a}_i(t)$ = actuarial present value for t-year continuous life annuity
$E_i(t)$ = net single premium for t-year pure endowment insurance
$\bar{P}_i(n)$ = net annual premium for fully continuous n-year term policy
$\bar{a}^*(t)$ = present value of t-year continuous annuity certain
δ = constant force of interest.

Recall the definition of $T_i^{(d)}$ in Equation (1). The prospective "net" loss random variable L_i at the time of issue of the policy is given by

$$L_i = \begin{cases} v^{T_i^{(d)}} - \bar{P}_i(n)\,\bar{a}^*(T_i^{(d)}) & \text{if } 0 \le T_i^{(d)} \le n; \\ 0 - \bar{P}_i(n)\,\bar{a}^*(n) & \text{if } T_i^{(d)} > n, \end{cases} \tag{13}$$

where

$$v^t = e^{-\delta t} \text{ and } \bar{a}^*(t) = \frac{1 - v^t}{\delta}.$$

Note that the term "net" loss is used because this loss is defined with respect to the net premium. For the sake of completeness, the random variables $Z_i(t)$ and $Y_i(t)$ are introduced and defined in the spirit of Bowers et al. ([2], chapters 4 and 5, respectively), as follows: $Z_i(t)$ is the present value of \$1 paid at the time of death of (i), providing death occurs within the next t years,

$$Z_i(t) = \begin{cases} v^{T_i^{(d)}} & \text{if } 0 \le T_i^{(d)} \le t; \\ 0 & \text{if } T_i^{(d)} > t. \end{cases} \tag{14}$$

Clearly,

$$E[Z_i(t)] = \bar{A}_i(t),$$

and

$$\text{Var}\,[Z_i(t)] = \bar{A}_i^{(2)}(t) - [\bar{A}_i(t)]^2$$

where $\bar{A}_i^{(2)}(t)$ is calculated as $\bar{A}_i(t)$ but at twice the force of interest. Similarly,

$$Y_i(t) = \begin{cases} \bar{a}^*(T_i^{(d)}) & \text{if } 0 \leq T_i^{(d)} \leq t; \\ \bar{a}^*(t) & \text{if } T_i^{(d)} > t, \end{cases} \tag{15}$$

with

$$E[Y_i(t)] = \bar{a}_i(t)$$

and

$$\text{Var}[Y_i(t)] = \frac{2}{\delta}\,[\bar{a}_i(t) - \bar{a}_i^{(2)}(t)] - [\bar{a}_i(t)]^2,$$

where $\bar{a}_i^{(2)}(t)$ is calculated as $\bar{a}_i(t)$ but at twice the force of interest. The prospective loss random variable can now be written as

$$L_i = Z_i(n) - \bar{P}_i(n)Y_i(n).$$

Using the equivalence principle, that is, $E[L_i] = 0$, gives us the standard expression

$$\bar{P}_i(n) = \frac{\bar{A}_i(n)}{\bar{a}_i(n)}. \tag{16}$$

We now proceed to develop differential equations for $\bar{A}_i(t)$, $\bar{a}_i(t)$, and $E_i(t)$.

The Chapman-Kolmogorov backward equation applied to $\bar{A}_i(t)$ yields

$$\bar{A}_i(t + dt) = v^{dt}[\mu_i' dt + \bar{A}_{i+1}(t)\mu_i dt + (1 - \alpha_i dt)\,\bar{A}_i(t) + o(dt)].$$

Since $v^{dt} = 1 - \delta dt + o(dt)$, then

$$\frac{d\,\bar{A}_i(t)}{dt} = \mu_i' + \mu_i \bar{A}_{i+1}(t) - (\delta + \alpha_i)\,\bar{A}_i(t), \tag{17}$$

with boundary conditions

$$\bar{A}_i(0) = 0 \text{ and } \bar{A}_5(t) \equiv \bar{A}_6(t) \equiv 1,\ t \geq 0.$$

The solution to this differential equation is

$$\bar{A}_i(t) = e^{-(\delta+\alpha_i)t} \int_0^t [\mu'_i + \mu_i \bar{A}_{i+1}(s)] e^{(\delta+\alpha_i)s} \, ds \tag{18}$$

with

$$\bar{A}_4(t) = \frac{\alpha_4}{\delta + \alpha_4} (1 - e^{-(\delta+\alpha_4)t}). \tag{19}$$

The general form of $\bar{A}_i(t)$ is

$$\bar{A}_i(t) = \bar{A}_i + \sum_{j=i}^{4} \omega_{ij} e^{-(\delta+\alpha_j)t} \text{ for } 0 \le i \le 4, \tag{20}$$

where the $\bar{A}_i$'s and ω_{ij}'s are constants. After substituting Equation (20) into Equation (18) and then comparing the coefficients of $e^{-(\delta+\alpha_j)t}$ on both sides of the resulting expression, we have

$$\left.\begin{aligned}
\bar{A}_4 &= \alpha_4/(\delta + \alpha_4) \\
\bar{A}_i &= (\mu_i' + \mu_i \bar{A}_{i+1})/(\delta + \alpha_i) \quad i = 0, 1, 2, 3; \\
\omega_{44} &= -\bar{A}_4 \\
\omega_{ij} &= -(\mu_i \omega_{i+1,j})/(\alpha_j - \alpha_i) \qquad j \ge i + 1; \\
\omega_{ii} &= -\bar{A}_i - \sum_{j=i+1}^{4} \omega_{ij} \qquad i = 0, 1, 2, 3.
\end{aligned}\right\} \tag{21}$$

The backward equation for $E_i(t)$ is

$$E_i(t + dt) = 0 \times \mu_i' dt + v^{dt} E_{i+1}(t) \mu_i dt + v^{dt} E_i(t)(1 - \alpha_i dt),$$

and the corresponding differential equation is

$$\frac{d\, E_i(t)}{dt} = \mu_i E_{i+1}(t) - (\delta + \alpha_i) E_i(t), \tag{22}$$

with boundary conditions

$$E_i(0) = 1 \text{ and } E_5(t) \equiv E_6(t) \equiv 0, t \geq 0.$$

The solution to this differential equation is

$$E_i(t) = e^{-(\delta+\alpha_i)t}[1 + \int_0^t \mu_i E_{i+1}(s) e^{(\delta+\alpha_i)s} ds] \quad (23)$$

with

$$E_4(t) = e^{-(\delta+\alpha_4)t}.$$

The general form of $E_i(t)$ is

$$E_i(t) = E_i + \sum_{j=i}^{4} \epsilon_{ij} e^{-(\delta+\alpha_j)t} \text{ for } 0 \leq i \leq 4, \quad (25)$$

where the E_i's and ω_{ij}'s are constants. Using the same technique as that used in deriving the constants given in Equation (21) leads us to

$$\begin{aligned} E_i &= 0 && \text{because } E_i(\infty) = 0; \\ \epsilon_{44} &= 1 && ; \\ \epsilon_{ij} &= -(\mu_i \epsilon_{i+1,j})/(\alpha_j - \alpha_i) && \text{for } j \geq i + 1; \\ \epsilon_{ii} &= 1 - \sum_{j=i+1}^{4} \epsilon_{ij} && \text{for } 1 = 0, 1, 2, 3. \end{aligned} \quad (26)$$

Finally, the Chapman-Kolmogorov backward equation applied to $\bar{a}_i(t)$ gives

$$\bar{a}_i(t + dt) = \bar{a}^*(dt) + v^{dt}[0 \times \mu_i' dt + \bar{a}_{i+1}(t)\mu_i dt + (1 - \alpha_i dt)\bar{a}_i(t) + o(dt)].$$

But $\bar{a}^*(dt) = dt + o(dt)$, therefore the following differential equation results

$$\frac{d\,\bar{a}_i(t)}{dt} = 1 + \mu_i \bar{a}_{i+1}(t) - (\delta + \alpha_i)\bar{a}_i(t), \quad (27)$$

with boundary conditions

$$\bar{a}_i(0) = 0 \text{ for } i = 0, 1, \ldots, 4 \text{ and } \bar{a}_5(t) \equiv \bar{a}_6(t) \equiv 0, t \geq 0.$$

The solution to this differential equation is

$$\bar{a}_i(t) = e^{-(\delta+\alpha_i)t} \int_0^t [1 + \mu_i \bar{a}_{i+1}(s)] e^{(\delta+\alpha_i)s}\, ds \quad (28)$$

with

$$\bar{a}_i(t) = \frac{1}{\delta + \alpha_4}(1 - e^{-(\delta+\alpha_4)t}). \qquad (29)$$

The general form of $\bar{a}_i(t)$ is

$$\bar{a}_i(t) = \bar{a}_i + \sum_{j=i}^{4} \gamma_{ij} e^{-(\delta+\alpha_j)t} \text{ for } 0 \le i \le 4, \qquad (30)$$

where the $\bar{a}_i$'s and γ_{ij}'s are constants. These constants are

$$\left.\begin{aligned}
\bar{a}_4 &= 1/(\delta + \alpha_4) & &; \\
\bar{a}_i &= (1 + \mu_i\bar{a}_{i+1})/(\delta + \alpha_i) & &\text{for } i = 0, 1, 2, 3; \\
\gamma_{44} &= -\,\bar{a}_4 & &; \\
\gamma_{ij} &= -\,(\mu_i\gamma_{i+1,j})/(\alpha_j - \alpha_i) & &\text{for } j \ge i + 1; \\
\gamma_{ii} &= -\,\bar{a}_i - \sum_{j=i+1}^{4} \gamma_{ij} & &\text{for } i = 0, 1, 2, 3.
\end{aligned}\right\} \qquad (31)$$

The net level premium can now be found. In fact, the net level premium for many different types of insurance plans can be arrived at by using the basic functions $\bar{A}_i(t)$, $E_i(t)$, and $\bar{a}_i(t)$. For example, the net level premium for an n-year endowment insurance, issued on a fully continuous basis, is

$$\bar{P}_i^e(n) = \frac{\bar{A}_i(n) + E_i(n)}{\bar{a}_i(n)}$$

Next we investigate the distribution and the moments of the loss random variable defined in Equation (13).

4. THE NET PREMIUM RESERVE

Recall Equation (13) defining the prospective net loss random variable. Let $F_i(t)$ be the cumulative distribution function (c.d.f.) of L_i at t years after the issue of the policy given that (i) is alive at time t, that is,

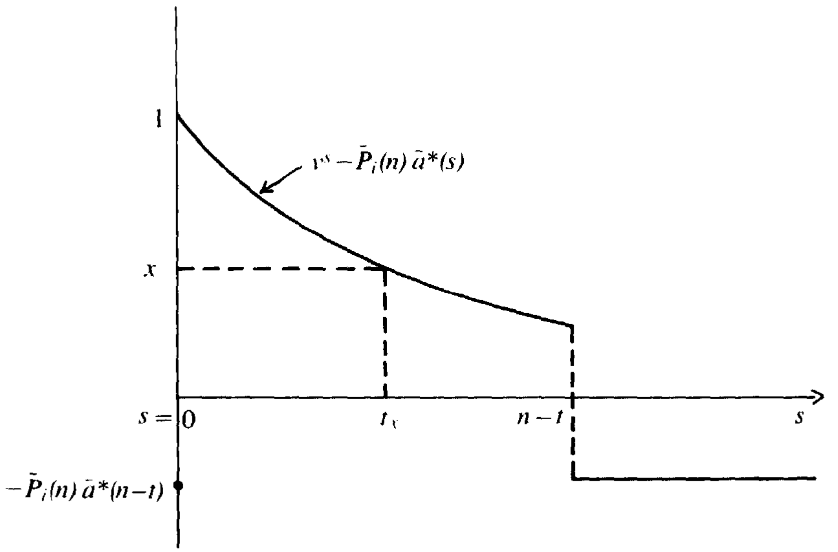
$$F_i(x,t) = \Pr[L_i \le x | T_i^{(d)} > t]. \qquad (32)$$

A graph of the loss over the remaining life of the policy t years after issue is shown in Figure 1. The prospective net loss at time t, given that the life is in stage j at time t, will be denoted $L_{ij}(t)$, where

$$L_{ij}(t) = \begin{cases} v^{T_j(d)} - \overline{P}_i(n)\bar{a}^*(T_j^{(d)}) \text{ if } 0 \le T_j^{(d)} \le n - t; \\ 0 - \overline{P}_i(n)\bar{a}^*(n - t) \text{ if } T_j^{(d)} > n - t. \end{cases} \tag{33}$$

FIGURE 1

PROSPECTIVE LOSS RANDOM VARIABLE, $L_{ii}(t)$



Denote its c.d.f. by

$$G_{ij}(x,t) = \Pr[L_{ij}(t) \le x]. \tag{34}$$

From Figure 1, one can see that

$$G_{ij}(x,t) = \begin{cases} 1 & \text{if } x > 1; \\ \Pr[T_j^{(d)} \geq t_x] & \text{if } v^{n-t} - \overline{P}_i(n)\overline{a}^*(n-t) \leq x \leq 1; \\ \Pr[T_j^{(d)} \geq n-t] & \text{if } -\overline{P}_i(n)\overline{a}^*(n-t) \leq x \leq \\ & \quad v^{n-t} - \overline{P}_i(n)\overline{a}^*(n-t); \\ 0 & \text{otherwise.} \end{cases}$$

where t_x is the solution to the equation (as a function of s)

$$v^s - \overline{P}_i(n)\overline{a}^*(s) = x$$

that is,

$$t_x = -\frac{1}{\delta}\log\left(\frac{\overline{P}_i(n) + x\delta}{\overline{P}_i(n) + \delta}\right) \tag{35}$$

Thus

$$G_{ij}(x,t) = \begin{cases} 1 & \text{if } x > 1; \\ 1 - q_j(t_x) & \text{if } v^{n-t} - \overline{P}_i(n)\overline{a}^*(n-t) \leq x \leq 1; \\ 1 - q_j(n-t) & \text{if } -\overline{P}_i(n)\overline{a}^*(n-t) \leq x \leq \\ & \quad v^{n-t} - \overline{P}_i(n)\overline{a}^*(n-t); \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

In our definition of $F_i(x,t)$, the actual stage that the life is in at time t is unknown, while in the definition of $G_{ij}(x,t)$ the stage at time t is known to be j. In order to connect these two probabilities, we must use Equation (37), which follows directly from the definition of conditional probability:

$$\Pr[(i) \text{ is in stage } j \text{ at } t|(i) \text{ alive at } t] = \frac{p_{ij}(t)}{1 - q_i(t)}. \tag{37}$$

This leads us to

$$F_i(x,t) = \sum_{j=i}^{4} \frac{p_{ij}(t)}{1 - q_i(t)} G_{ij}(x,t). \tag{38}$$

Using the standard arguments, as found in Bowers et al. ([2], chapter 7), the net premium reserve is

$$\bar{V}_{ij}(t) = E[L_{ij}(t)] = \bar{A}_j(n - t) - \bar{P}_i(n)\bar{a}_j(n - t) \tag{39}$$

where $\bar{V}_{ij}(t)$ is the net premium reserve given that the life is in stage j at time t. If the stage at time t is unknown, then we denote the net premium reserve as $\bar{V}_i(t)$, with

$$\bar{V}_i(t) = \sum_{j=i}^{4} \frac{p_{ij}(t)}{1 - q_i(t)} \bar{V}_{ij}(t). \tag{40}$$

The differential equation for $\bar{V}_{ij}(t)$ is

$$\frac{d\bar{V}_{ij}(t)}{dt} = \bar{P}_i(n) + (\delta + \alpha_j)\bar{V}_{ij}(t) - \mu_j\bar{V}_{i,j+1}(t) - \mu_j' \tag{41}$$

$$= \bar{P}_i(n) + \delta\bar{V}_{ij}(t) - [1 - \bar{V}_{ij}(t)]\mu_j' - [\bar{V}_{i,j+1}(t) - \bar{V}_{ij}(t)]\mu_j \tag{42}$$

with boundary conditions $\bar{V}_{ij}(0) = 0$ and $\bar{V}_{ij}(n) = 0$. As expected, Equations (41) and (42) are similar to Equations (7.11.3) and (7.11.5) in Bowers et al. ([2], chapter 7), and in an analogous manner, the last two terms in Equation (42) are the net amount at risk for dying in stage j and the net amount at risk for progressing to the next worse stage, respectively.

If we had an n-year endowment insurance instead, Equation (41) would still be valid, but the boundary conditions would become $\bar{V}_{ij}(0) = 0$ and $\bar{V}_{ij}(n) = 1$ and the premium would be different also.

The results derived thus far are valid for whole life insurances also. They can be adapted by simply letting n and $t \to \infty$. Thus, the following results will hold: The net premium is

$$\bar{P}_i = \frac{\bar{A}_i}{\bar{a}_i}, \tag{43}$$

with

$$1 = \bar{A}_i + \delta\bar{a}_i \text{ for } i = 0, 1, \ldots, 4. \tag{44}$$

Note that in stage 4, $\bar{P}_4 = \alpha_4$. The prospective net loss random variable at issue is

$$L_i = v^{T_i^{(d)}} - \bar{P}_i\bar{a}^*(T_i^{(d)}), \tag{45}$$

while at time t, given the life is in stage j, is

$$L_{ij}(t) = v^{T_j^{(d)}} - \bar{P}_i\bar{a}^*(T_j^{(d)}). \tag{46}$$

Clearly, for the whole life insurance,

$$\bar{V}_{ij}(t) = \bar{A}_j - \bar{P}_i\bar{a}_j \tag{47}$$

and

$$\mathrm{Var}[L_{ij}(t)] = \left(1 + \frac{\bar{P}_i}{\delta}\right)^2 [\bar{A}_j^{(2)} - (\bar{A}_j)^2] \tag{48}$$

independent of t because of the memoryless property of the constant forces assumption.

5. FULLY DISCRETE FUNCTIONS

For completeness, fully discrete functions are briefly examined. By fully discrete we mean that death benefits are paid at the end of the year of death and premiums are paid at the start of the year. As is traditional, the bar at the top of symbols is dropped to denote fully discrete functions. For annuities due, the bar is replaced with the double dots, for example, $\ddot{a}_i(t)$. The following results are easily derived from elementary probabilistic arguments: for $t=1, 2, 3, \ldots,$

$$A_i(t) = vq_i + v\sum_{j=i}^{4} p_{ij}\,A_j(t-1) \tag{49}$$

and

$$\ddot{a}_i(t) = 1 + v\sum_{j=i}^{4} p_{ij}\ddot{a}_j(t-1), \tag{50}$$

where $q_i=q_i(1)$ [see Equation (7)] and $p_{ij}=p_{ij}(1)$ [see Equation (4)].

Once $A_i(t)$ and $\ddot{a}_i(t)$ are known, the standard arguments similar to those in Section 4 above can be used to obtain expressions for such quantities as the net annual premium and the reserve. The recursive expression for the reserve of an n-year term insurance is

$$V_{ij}(t) + P_i(n) = vq_j + \sum_{k=j}^{4} vp_{jk}V_{ik}(t+1), \tag{51}$$

for $t=0, 1, 2, \ldots, n-1$ with $V_{ij}(n)=0$.

By letting $t\rightarrow\infty$, fully discrete whole life insurance functions will result. Applying this to Equation (49) yields

$$A_i = vq_i + v\sum_{j=i}^{4} p_{ij}A_j. \tag{52}$$

Note that

$$A_i + d\ddot{a}_i = 1 \tag{52}$$

will also hold.

6. AN EXAMPLE AND CONCLUSIONS

The estimates of μ_i given by Panjer ([12], Table 2) are used to calculate the net annual premium for a $1,000 whole life insurance on a fully continuous and fully discrete basis. They are $\mu_0=0.45$, $\mu_1=0.86$, $\mu_2=0.53$, $\mu_3=0.30$, and $\alpha_4=1.10$.

In order to proceed, assumptions about μ_i' must be made. It is assumed that μ_i' satisfies

$$\mu_i' = Bc^i, \text{ for } i = 0, 1, 2, 3. \tag{54}$$

Since α_4 is the force of death from all causes in stage 4, it is not unreasonable to assume that the level of the force of death in each of the stages 0 through 4 increases exponentially until it reaches 1.1 in stage 4. Remember that μ_i' remains level in each stage. Thus $Bc^4=1.1$ and

$$\mu_i' = \begin{cases} B\left(\dfrac{1.1}{B}\right)^{i/4} & \text{if } B > 0; \\ 0 & \text{if } B = 0. \end{cases} \tag{55}$$

In what follows, B is allowed to vary between 0 and 0.20 in steps of 0.005. The case $B=0$ implies that deaths can occur only after the life has entered stage 4, that is, has developed full-blown AIDS. This is the Walter Reed Staging Method and the model used by Panjer. Clearly Panjer's model ($B=0$) is a special case of the model described in this paper.

Net single premiums for whole life insurance are calculated by using Equation (23) for the fully continuous case and Equation (52) for the fully discrete case. Once the net single premiums are found, the corresponding net annual premiums can be found by using Equations (44) and (53), respectively. These premiums are displayed in Tables 1–8 and Figures 2–5.

TABLE 1

NET ANNUAL PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY CONTINUOUS BASIS

B	$1000 \cdot \bar{P}_0$	$1000 \cdot \bar{P}_1$	$1000 \cdot \bar{P}_2$	$1000 \cdot \bar{P}_3$	$1000 \cdot \bar{P}_4$
0.000	86.73	120.21	148.45	227.03	1100.00
0.005	129.89	200.36	268.10	453.70	1100.00
0.010	140.79	220.42	297.08	496.59	1100.00
0.015	149.21	235.39	318.15	525.35	1100.00
0.020	156.48	247.91	335.40	547.59	1100.00
0.025	163.07	258.95	350.32	565.98	1100.00
0.030	169.21	268.97	363.64	581.79	1100.00
0.035	175.01	278.24	375.77	595.73	1100.00
0.040	180.57	286.92	386.97	608.24	1100.00
0.045	185.93	295.14	397.43	619.63	1100.00
0.050	191.14	302.96	407.27	630.11	1100.00
0.055	196.23	310.46	416.60	639.83	1100.00
0.060	201.20	317.69	425.48	648.91	1100.00
0.065	206.09	324.67	433.98	657.44	1100.00
0.070	210.90	331.43	422.13	665.49	1100.00
0.075	215.64	338.01	449.98	673.11	1100.00
0.080	220.32	344.42	457.56	680.37	1100.00
0.085	224.96	350.68	464.89	687.29	1100.00
0.090	229.55	356.80	472.00	693.92	1100.00
0.095	234.10	362.79	478.91	700.27	1100.00
0.100	238.61	368.67	485.63	706.38	1100.00
0.105	243.10	374.44	492.17	712.26	1100.00
0.110	247.56	380.11	498.56	717.94	1100.00
0.115	251.99	385.70	504.80	723.42	1100.00
0.120	256.39	391.19	510.90	728.73	1100.00
0.125	260.78	396.61	516.87	733.86	1100.00
0.130	265.15	401.95	522.72	738.87	1100.00
0.135	269.50	407.23	528.46	743.73	1100.00
0.140	273.84	412.43	534.09	748.45	1100.00
0.145	278.17	417.58	539.62	753.04	1100.00
0.150	282.48	422.67	545.05	757.52	1100.00
0.155	286.77	427.70	550.38	761.88	1100.00
0.160	291.06	432.67	555.64	766.15	1100.00
0.165	295.34	437.60	560.81	770.31	1100.00
0.170	299.61	442.48	565.90	774.38	1100.00
0.175	303.87	447.31	570.91	778.36	1100.00
0.180	308.13	452.10	575.86	782.26	1100.00
0.185	312.38	456.85	580.73	786.07	1100.00
0.190	316.62	461.56	585.54	789.81	1100.00
0.195	320.86	466.23	590.28	793.48	1100.00
0.200	325.09	470.86	594.97	797.08	1100.00

TABLE 2

NET SINGLE PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY CONTINUOUS BASIS

B	$1000 \cdot \bar{A}_0$	$1000 \cdot \bar{A}_1$	$1000 \cdot \bar{A}_2$	$1000 \cdot \bar{A}_3$	$1000 \cdot \bar{A}_4$
0.000	618.29	691.86	734.93	809.17	953.59
0.005	708.12	789.13	833.54	894.45	953.59
0.010	724.49	804.57	847.30	902.68	953.59
0.015	735.93	814.69	855.95	907.51	953.59
0.020	745.07	822.39	862.34	910.93	953.59
0.025	752.83	828.67	867.43	913.58	953.59
0.030	759.63	833.99	871.66	915.73	953.59
0.035	765.74	838.63	875.29	917.54	953.59
0.040	771.30	842.74	878.46	919.10	953.59
0.045	776.42	846.45	881.28	920.47	953.59
0.050	781.19	849.82	883.81	921.68	953.59
0.055	785.64	852.91	886.12	922.78	953.59
0.060	789.82	855.77	888.23	923.78	953.59
0.065	793.78	858.44	890.18	924.69	953.59
0.070	797.53	860.92	891.98	925.54	953.59
0.075	801.10	863.26	893.67	926.32	953.59
0.080	804.50	865.46	895.24	927.05	953.59
0.085	807.75	867.55	896.73	927.73	953.59
0.090	810.87	869.52	898.12	928.37	953.59
0.095	813.86	871.40	899.44	928.97	953.59
0.100	816.74	873.19	900.70	929.54	953.59
0.105	819.51	874.90	901.89	930.09	953.59
0.110	822.18	876.54	903.02	930.60	953.59
0.115	824.76	878.11	904.11	931.09	953.59
0.120	827.25	879.61	905.14	931.56	953.59
0.125	829.66	881.06	906.14	932.01	953.59
0.130	832.00	882.46	907.09	932.43	953.59
0.135	834.26	883.80	908.01	932.84	953.59
0.140	836.46	885.10	908.89	933.24	953.59
0.145	838.59	886.35	909.74	933.62	953.59
0.150	840.66	887.57	910.55	933.99	953.59
0.155	842.67	888.74	911.35	934.34	953.59
0.160	844.63	889.88	912.11	934.68	953.59
0.165	846.54	890.99	912.85	935.01	953.59
0.170	848.39	892.06	913.57	935.33	953.59
0.175	850.20	893.10	914.26	935.64	953.59
0.180	851.96	894.11	914.93	935.94	953.59
0.185	853.68	895.10	915.59	936.23	953.59
0.190	855.36	896.06	916.22	936.51	953.59
0.195	857.00	896.99	916.84	936.79	953.59
0.200	858.59	897.90	917.44	937.06	953.59

TABLE 3

NET ANNUAL PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 7.0% INTEREST ON A FULLY CONTINUOUS BASIS

B	$1000 \cdot \bar{P}_0$	$1000 \cdot \bar{P}_1$	$1000 \cdot \bar{P}_2$	$1000 \cdot \bar{P}_3$	$1000 \cdot \bar{P}_4$
0.000	82.48	116.17	144.89	224.85	1100.00
0.005	125.64	196.44	264.97	452.08	1100.00
0.010	136.63	216.63	294.14	495.08	1100.00
0.015	145.13	231.69	315.33	523.91	1100.00
0.020	152.47	244.31	332.68	546.21	1100.00
0.025	159.13	255.42	347.69	564.65	1100.00
0.030	165.32	265.51	361.08	580.49	1100.00
0.035	171.19	274.84	373.27	594.47	1100.00
0.040	176.80	283.59	384.53	607.01	1100.00
0.045	182.22	291.86	395.05	618.43	1100.00
0.050	187.49	299.74	404.94	628.94	1100.00
0.055	192.62	307.29	414.32	638.68	1100.00
0.060	197.65	314.56	423.24	647.78	1100.00
0.065	202.58	321.58	431.78	656.33	1100.00
0.070	207.44	328.40	439.97	664.40	1100.00
0.075	212.22	335.02	447.86	672.05	1100.00
0.080	216.95	341.47	455.47	679.32	1100.00
0.085	221.63	347.77	462.84	686.26	1100.00
0.090	226.26	353.92	469.98	692.90	1100.00
0.095	230.86	359.96	476.92	699.27	1100.00
0.100	235.41	365.87	483.67	705.39	1100.00
0.105	239.94	371.68	490.25	711.29	1100.00
0.110	244.43	377.39	496.67	716.98	1100.00
0.115	248.90	383.00	502.93	722.48	1100.00
0.120	253.35	388.53	509.06	727.80	1100.00
0.125	257.77	393.98	515.06	732.96	1100.00
0.130	262.18	399.36	520.94	737.97	1100.00
0.135	266.57	404.66	526.70	742.83	1100.00
0.140	270.94	409.90	532.35	747.57	1100.00
0.145	275.30	415.07	537.90	752.17	1100.00
0.150	279.64	420.19	543.35	756.66	1100.00
0.155	283.98	425.25	548.71	761.04	1100.00
0.160	288.30	430.25	553.99	765.31	1100.00
0.165	292.61	435.21	559.18	769.48	1100.00
0.170	296.91	440.12	564.29	773.56	1100.00
0.175	301.20	444.98	569.32	777.55	1100.00
0.180	305.49	449.79	574.29	781.46	1100.00
0.185	309.77	454.57	579.18	785.29	1100.00
0.190	314.04	459.30	584.01	789.03	1100.00
0.195	318.31	463.99	588.77	792.71	1100.00
0.200	322.57	468.65	593.47	796.32	1100.00

TABLE 4

NET SINGLE PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 7.0% INTEREST ON A FULLY CONTINUOUS BASIS

B	$1000 \cdot \bar{A}_0$	$1000 \cdot \bar{A}_1$	$1000 \cdot \bar{A}_2$	$1000 \cdot \bar{A}_3$	$1000 \cdot \bar{A}_4$
0.000	549.36	631.96	681.67	768.69	942.06
0.005	649.98	743.81	796.59	869.82	942.06
0.010	668.81	762.00	812.99	879.77	942.06
0.015	682.04	773.98	823.34	885.63	942.06
0.020	692.64	783.12	831.00	889.78	942.06
0.025	701.66	790.58	837.10	893.00	942.06
0.030	709.60	796.93	842.19	895.61	942.06
0.035	716.73	802.46	846.55	897.82	942.06
0.040	723.24	807.38	850.38	899.72	942.06
0.045	729.24	811.81	853.78	901.39	942.06
0.050	734.82	815.84	856.84	902.87	942.06
0.055	740.05	819.55	859.62	904.21	942.06
0.060	744.98	822.98	862.18	905.43	942.06
0.065	749.63	826.18	864.53	906.55	942.06
0.070	754.05	829.17	866.72	907.58	942.06
0.075	758.26	831.98	868.76	908.53	942.06
0.080	762.28	834.63	870.67	909.42	942.06
0.085	766.12	837.13	872.46	910.26	942.06
0.090	769.81	839.51	874.16	911.04	942.06
0.095	773.35	841.78	875.76	911.78	942.06
0.100	776.76	843.94	877.28	912.48	942.06
0.105	780.04	846.00	878.73	913.14	942.06
0.110	783.21	847.97	880.11	913.77	942.06
0.115	786.27	849.87	881.42	914.37	942.06
0.120	789.23	851.69	882.68	914.94	942.06
0.125	792.10	853.44	883.89	915.49	942.06
0.130	794.87	855.13	885.05	916.02	942.06
0.135	797.57	856.75	886.16	916.52	942.06
0.140	800.18	858.32	887.24	917.01	942.06
0.145	802.72	859.84	888.27	917.47	942.06
0.150	805.19	861.31	889.27	917.92	942.06
0.155	807.59	862.74	890.23	918.36	942.06
0.160	809.92	864.12	891.16	918.77	942.06
0.165	812.20	865.45	892.06	919.18	942.06
0.170	814.42	866.75	892.94	919.57	942.06
0.175	816.58	868.02	893.78	919.95	942.06
0.180	818.68	869.25	894.60	920.32	942.06
0.185	820.74	870.44	895.40	920.68	942.06
0.190	822.74	871.61	896.18	921.02	942.06
0.195	824.70	872.74	896.93	921.36	942.06
0.200	826.62	873.84	897.66	921.69	942.06

TABLE 5

NET ANNUAL PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY DISCRETE BASIS

B	$1000 \cdot P_0$	$1000 \cdot P_1$	$1000 \cdot P_2$	$1000 \cdot P_3$	$1000 \cdot P_4$
0.000	78.86	107.62	131.18	193.61	632.35
0.005	115.75	172.86	224.13	347.81	632.35
0.010	124.83	188.41	245.02	373.20	632.35
0.015	131.77	199.81	259.81	389.60	632.35
0.020	137.71	209.22	271.70	401.96	632.35
0.025	143.06	217.42	281.82	411.97	632.35
0.030	148.02	224.78	290.71	420.42	632.35
0.035	152.67	231.52	298.71	427.76	632.35
0.040	157.10	237.78	306.01	434.26	632.35
0.045	161.36	243.65	312.75	440.10	632.35
0.050	165.47	249.20	319.03	445.41	632.35
0.055	169.46	254.48	324.92	450.29	632.35
0.060	173.35	259.52	330.47	454.80	632.35
0.065	177.15	264.36	335.74	459.00	632.35
0.070	180.86	269.02	340.75	462.92	632.35
0.075	184.52	273.52	345.54	466.62	632.35
0.080	188.11	277.88	350.12	470.10	632.35
0.085	191.64	282.10	354.53	473.40	632.35
0.090	195.12	286.21	358.76	476.54	632.35
0.095	198.56	290.20	362.85	479.53	632.35
0.100	201.96	294.10	366.80	482.38	632.35
0.105	205.31	297.90	370.62	485.11	632.35
0.110	208.64	301.61	374.32	487.73	632.35
0.115	211.92	305.25	377.91	490.24	632.35
0.120	215.18	308.81	381.41	492.67	632.35
0.125	218.41	312.30	384.80	495.00	632.35
0.130	221.61	315.72	388.11	497.25	632.35
0.135	224.78	319.07	391.33	499.43	632.35
0.140	227.92	322.38	394.47	501.54	632.35
0.145	231.01	325.65	397.54	503.58	632.35
0.150	234.16	328.81	400.54	505.56	632.35
0.155	237.23	331.94	403.47	507.49	632.35
0.160	240.29	335.03	406.34	509.35	632.35
0.165	243.33	338.06	409.15	511.17	632.35
0.170	246.35	341.07	411.90	512.94	632.35
0.175	249.35	344.01	414.59	514.66	632.35
0.180	252.34	346.93	417.23	516.34	632.35
0.185	255.31	349.80	419.83	517.98	632.35
0.190	258.26	352.63	422.37	519.58	632.35
0.195	261.19	355.42	424.87	521.14	632.35
0.200	264.10	358.19	427.33	522.66	632.35

TABLE 6

NET SINGLE PREMIUM PER $1000 WHOLE LIFE INSURANCE
AT 5.5% INTEREST ON A FULLY DISCRETE BASIS

B	$1000 \cdot A_0$	$1000 \cdot A_1$	$1000 \cdot A_2$	$1000 \cdot A_3$	$1000 \cdot A_4$
0.000	602.03	673.66	715.61	787.86	923.84
0.005	689.48	768.30	811.29	869.65	923.84
0.010	705.40	783.27	824.56	877.43	923.84
0.015	716.52	793.08	832.88	881.98	923.84
0.020	725.39	800.53	839.01	885.19	923.84
0.025	732.92	806.59	843.89	887.67	923.84
0.030	739.53	811.74	847.94	889.68	923.84
0.035	745.45	816.21	851.41	891.37	923.84
0.040	750.84	820.18	854.44	892.82	923.84
0.045	755.81	823.75	857.12	894.09	923.84
0.050	760.43	826.99	859.54	895.22	923.84
0.055	764.74	829.97	861.74	896.24	923.84
0.060	768.79	832.72	863.74	897.16	923.84
0.065	772.62	835.28	865.59	898.00	923.84
0.070	776.25	837.67	867.31	898.78	923.84
0.075	779.70	839.91	868.91	899.50	923.84
0.080	783.00	842.03	870.40	900.17	923.84
0.085	786.14	844.02	871.80	900.80	923.84
0.090	789.15	845.92	873.12	901.39	923.84
0.095	792.05	847.71	874.37	901.94	923.84
0.100	794.83	849.43	875.56	902.47	923.84
0.105	797.50	851.06	876.68	902.96	923.84
0.110	800.08	852.63	877.75	903.43	923.84
0.115	802.57	854.13	878.77	903.88	923.84
0.120	804.98	855.56	879.75	904.31	923.84
0.125	807.30	856.95	880.69	904.72	923.84
0.130	809.55	858.28	881.58	905.11	923.84
0.135	811.74	859.56	882.44	905.48	923.84
0.140	813.85	860.80	883.27	905.84	923.84
0.145	815.88	862.01	884.07	906.19	923.84
0.150	817.90	863.15	884.83	906.52	923.84
0.155	819.84	864.27	885.57	906.84	923.84
0.160	821.72	865.35	886.29	907.15	923.84
0.165	823.56	866.39	886.98	907.45	923.84
0.170	825.34	867.41	887.65	907.74	923.84
0.175	827.08	868.40	888.30	908.02	923.84
0.180	828.78	869.36	888.93	908.29	923.84
0.185	830.43	870.29	889.54	908.56	923.84
0.190	832.04	871.20	890.13	908.81	923.84
0.195	833.61	872.08	890.71	909.06	923.84
0.200	835.14	872.95	891.27	909.30	923.84

TABLE 7

NET ANNUAL PREMIUM PER $1000 WHOLE LIFE INSURANCE
AT 7.0% INTEREST ON A FULLY DISCRETE BASIS

B	$1000 \cdot P_0$	$1000 \cdot P_1$	$1000 \cdot P_2$	$1000 \cdot P_3$	$1000 \cdot P_4$
0.000	74.12	102.78	126.51	189.35	623.48
0.005	110.65	167.48	218.82	342.06	623.48
0.010	119.72	182.97	239.60	367.19	623.48
0.015	126.65	194.33	254.32	383.42	623.48
0.020	132.60	203.71	266.14	395.66	623.48
0.025	137.95	211.88	276.20	405.56	623.48
0.030	142.90	219.22	285.04	413.92	623.48
0.035	147.56	225.93	292.99	421.18	623.48
0.040	151.99	232.17	300.25	427.61	623.48
0.045	156.25	238.02	306.94	433.39	623.48
0.050	160.36	243.55	313.18	438.65	623.48
0.055	164.35	248.80	319.03	443.47	623.48
0.060	168.23	253.83	324.55	447.93	623.48
0.065	172.03	258.64	329.78	452.09	623.48
0.070	175.74	263.28	334.76	455.97	623.48
0.075	179.39	267.76	339.51	459.62	623.48
0.080	182.98	272.10	344.06	463.07	623.48
0.085	186.50	276.30	348.43	466.34	623.48
0.090	189.98	280.39	352.63	469.44	623.48
0.095	193.42	284.36	356.69	472.39	623.48
0.100	196.81	288.24	360.60	475.22	623.48
0.105	200.16	292.02	364.40	477.92	623.48
0.110	203.47	295.72	368.07	480.51	623.48
0.115	206.75	299.33	371.63	483.00	623.48
0.120	210.00	302.87	375.10	485.39	623.48
0.125	213.22	306.34	378.47	487.70	623.48
0.130	216.41	309.74	381.74	489.93	623.48
0.135	219.57	313.08	384.94	492.08	623.48
0.140	222.70	316.37	388.06	494.17	623.48
0.145	225.78	319.62	391.10	496.19	623.48
0.150	228.92	322.75	394.07	498.15	623.48
0.155	231.98	325.87	396.98	500.05	623.48
0.160	235.03	328.94	399.82	501.90	623.48
0.165	238.06	331.96	402.61	503.69	623.48
0.170	241.07	334.94	405.33	505.44	623.48
0.175	244.06	337.86	408.00	507.15	623.48
0.180	247.03	340.77	410.62	508.81	623.48
0.185	249.99	343.62	413.19	510.42	623.48
0.190	252.92	346.43	415.72	512.01	623.48
0.195	255.84	349.20	418.19	513.55	623.48
0.200	258.74	351.95	420.63	515.06	623.48

TABLE 8

NET SINGLE PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 7.0% INTEREST ON A FULLY DISCRETE BASIS

B	$1000 \cdot A_0$	$1000 \cdot A_1$	$1000 \cdot A_2$	$1000 \cdot A_3$	$1000 \cdot A_4$
0.000	531.18	611.05	659.14	743.22	905.04
0.005	628.45	719.11	769.84	839.45	905.04
0.010	646.63	736.62	785.52	848.78	905.04
0.015	659.40	748.14	795.40	854.25	905.04
0.020	669.62	756.92	802.69	858.11	905.04
0.025	678.31	764.08	808.50	861.10	905.04
0.030	685.96	770.16	813.33	863.52	905.04
0.035	692.83	775.46	817.47	865.56	905.04
0.040	699.09	780.17	821.09	867.31	905.04
0.045	704.88	784.40	824.31	868.85	905.04
0.050	710.25	788.26	827.20	870.21	905.04
0.055	715.27	791.80	829.83	871.45	905.04
0.060	720.01	795.08	832.24	872.56	905.04
0.065	724.48	798.12	834.46	873.59	905.04
0.070	728.73	800.97	836.52	874.53	905.04
0.075	732.77	803.65	838.44	875.40	905.04
0.080	736.63	806.17	840.23	876.21	905.04
0.085	740.32	808.56	841.92	876.97	905.04
0.090	743.85	810.82	843.51	877.69	905.04
0.095	747.25	812.97	845.01	878.36	905.04
0.100	750.52	815.02	846.44	878.99	905.04
0.105	753.67	816.98	847.79	879.60	905.04
0.110	756.70	818.85	849.08	880.17	905.04
0.115	759.64	820.64	850.31	880.71	905.04
0.120	762.47	822.37	851.49	881.23	905.04
0.125	765.21	824.03	852.62	881.72	905.04
0.130	767.87	825.62	853.70	882.20	905.04
0.135	770.45	827.16	854.74	882.65	905.04
0.140	772.94	828.65	855.74	883.09	905.04
0.145	775.35	830.09	856.70	883.51	905.04
0.150	777.74	831.47	857.62	883.92	905.04
0.155	780.03	832.81	858.52	884.31	905.04
0.160	782.26	834.11	859.38	884.68	905.04
0.165	784.43	835.37	860.22	885.05	905.04
0.170	786.55	836.60	861.03	885.40	905.04
0.175	788.61	837.78	861.81	885.74	905.04
0.180	790.62	838.94	862.57	886.07	905.04
0.185	792.58	840.06	863.31	886.39	905.04
0.190	794.50	841.15	864.03	886.70	905.04
0.195	796.37	842.22	864.73	887.01	905.04
0.200	798.18	843.26	865.40	887.30	905.04

FIGURE 2

NET ANNUAL PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY CONTINUOUS CASE

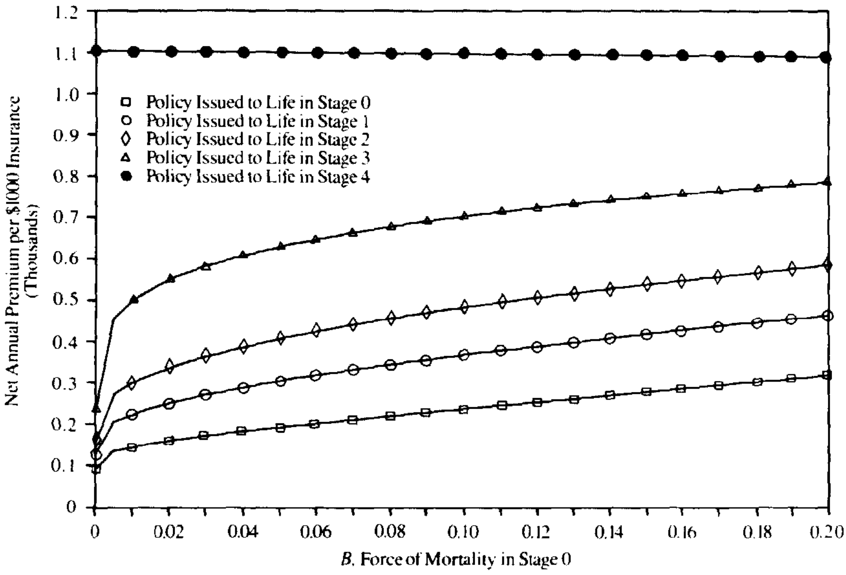


FIGURE 3

NET SINGLE PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY CONTINUOUS CASE

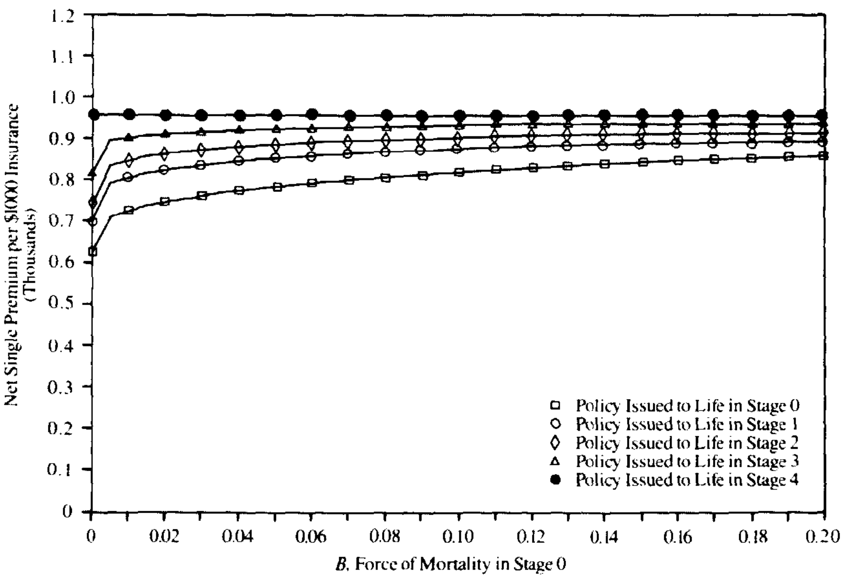


FIGURE 4

NET ANNUAL PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY DISCRETE CASE

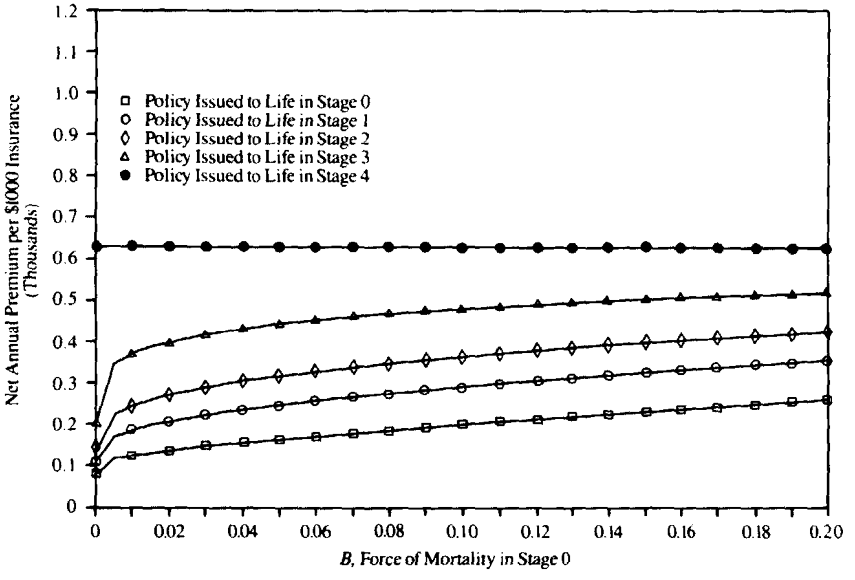
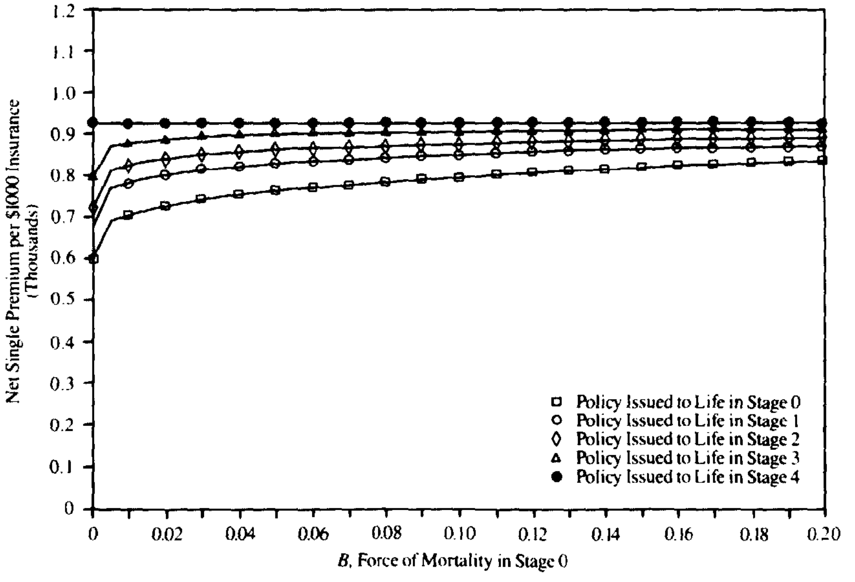


FIGURE 5

NET SINGLE PREMIUM PER $1000 WHOLE LIFE INSURANCE AT 5.5% INTEREST ON A FULLY DISCRETE CASE



It is instructive to compare these results with those given in Chapter 2, page 894 of the Society of Actuaries AIDS Task Force Report [16]. That report lists some estimates of the net single premium per 1,000 for whole life insurance issued to a life in stage 0 and one in stage 4.

	5.5% Interest	7.0% Interest
Stage 0	573 (602*)	502 (531*)
Stage 4	876 (924)	849 (905)

*Denotes minimum value of net single premium, that is, at $B=0$.

The results in parentheses are taken from Tables 1–8; for example, 602 was taken from Table 2 column $1000*A_0$ at $B=0$. Notice these results are consistently higher than those given by the AIDS Task Force. From a marketing perspective, they are prohibitively high; that is, HIV+ lives are practically uninsurable. Comparing the values in Table 1 to those given by the Commissioners 1980 Standard Ordinary Mortality Table Male [3] at 5.5 percent under column $1000\,A_x$, one sees that an HIV+ life is equivalent to a "normal" life 80 years old and an AIDS life is equivalent to a "normal" life 97 years old!

It is hoped that this paper will assist actuaries in calculating premiums and reserves for lives in the so-called high-risk groups and provide tools for analyzing models of the impact of AIDS in an insurance environment. The material presented here can be a valuable addition to the Course 150 syllabus in the Society of Actuaries Associateship Examination.

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DISCUSSION OF PRECEDING PAPER

E.S. SEAH AND ELIAS S.W. SHIU:

Dr. Ramsay is to be complimented for this interesting and thought-provoking paper. We agree with him that it would be instructive for Course 150 students to learn about the material presented in the paper. Indeed, the first sentence of Dr. Jan Hoem's [3] introductory lecture on Subject 3 at the 23rd International Congress of Actuaries was: "The mathematics of Markov chains can be used to develop a complete theory of the common life contingencies and their extensions." Below are some alternative derivations for several of the results in Section 3.

Consider Equation (2) in the paper without B:

$$\frac{d}{dt}P(t) = MP(t), \tag{D.1}$$

where M is a square matrix and $P(t)$ is a column vector or a matrix. The solution to (D.1) is given by

$$P(t) = e^{tM}P(0) \tag{D.2}$$

$$= \left(\sum_{k=0}^{\infty} \frac{t^kM^k}{k!}\right)P(0). \tag{D.3}$$

For the system of differential equations described by Equation (3) of the paper,

$$M = \begin{pmatrix} -\alpha_0 & \mu_0 & 0 & 0 & 0 \\ 0 & -\alpha_1 & \mu_1 & 0 & 0 \\ 0 & 0 & -\alpha_2 & \mu_2 & 0 \\ 0 & 0 & 0 & -\alpha_3 & \mu_3 \\ 0 & 0 & 0 & 0 & -\alpha_4 \end{pmatrix} \tag{D.4}$$

We give two methods to compute e^{tM}. The first method is by means of a generalization of the Cauchy integral formula.

Let C be a positively oriented Jordan curve in the complex plane, and let f be a function that is analytic everywhere within and on C. The Cauchy integral formula states that, for each z interior to C,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw. \tag{D.5}$$

Formula (D.5) can be generalized to the case of finite dimensional matrices [2, Theorem VII.1.10]. Let T be an n by n matrix and f be a function analytic in a domain containing the closure of an open set O. Suppose that all the eigenvalues of T are in O and that the boundary ∂O and O consists of a finite number of closed rectifiable Jordan curves, oriented in the positive sense. Then $f(T)$ may be expressed as a Riemann contour integral over ∂O by the formula:

$$f(T) = \frac{1}{2\pi i} \int_{\partial O} f(w)(wI - T)^{-1} dw, \tag{D.6}$$

where I is the identity matrix. Formula (D.6), which is also valid for bounded linear operators on Banach spaces, is usually attributed to Riesz, Dunford and/or Taylor. However, Bellman [1, p. 104, #43] pointed out that the formula can be found in an 1899 paper by H. Poincaré.

If the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of T are distinct, then

$$f(T) = \sum_{k=1}^{n} f(\lambda_k)\, E(\lambda_k), \tag{D.7}$$

where $E(\lambda_k)$ is the spectral projection corresponding to the eigenvalue λ_k ([2, Theorem VII.1.8], [4, p. 272, Equation (VI.6.10)]). The spectral projection can be evaluated by the contour integral

$$E(\lambda_k) = \frac{1}{2\pi i} \int_{C_k} (wI - T)^{-1} dw, \tag{D.8}$$

where C_k is a closed contour enclosing exactly one eigenvalue of T, λ_k, in its interior, or by the Lagrange interpolation polynomial formula [2, p. 562, #6]

$$E(\lambda_k) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{T - \lambda_j I}{\lambda_k - \lambda_j}. \tag{D.9}$$

(The formula

$$f(T) = \sum_{k=1}^{n} f(\lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{T - \lambda_j I}{\lambda_k - \lambda_j}$$

is known as the Sylvester interpolation formula [1, p. 102, #33].) We shall apply formulas (D.7) and (D.8) to evaluate $f(M)=e^{tM}$. We note that, as M is an upper triangular matrix, the eigenvalues of M are its diagonal entries, and we assume them to be distinct.

To apply Formula (D.7), we need to first evaluate $(wI-M)^{-1}$. Let

$$D = \begin{pmatrix} -\alpha_0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_3 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_4 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & \mu_0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $M=D+N$ and

$$\begin{aligned}(wI - M)^{-1} &= [(wI - D) - N]^{-1} \\ &= (wI - D)^{-1}[I - N(wI - D)^{-1}]^{-1}. \qquad \text{(D.10)}\end{aligned}$$

As the matrix $N(wI-D)^{-1}$ is nilpotent,

$$\begin{aligned}[I - N(wI - D)^{-1}]^{-1} &= \sum_{m=0}^{\infty}[N(wI - D)^{-1}]^m \\ &= \sum_{m=0}^{4}[N(wI - D)^{-1}]^m. \qquad \text{(D.11)}\end{aligned}$$

We note that, if

$$A = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then

$$A^2 = \begin{pmatrix} 0 & 0 & ab & 0 & 0 \\ 0 & 0 & 0 & bc & 0 \\ 0 & 0 & 0 & 0 & cd \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & abc & 0 \\ 0 & 0 & 0 & 0 & bcd \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$A^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & abcd \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the matrices $(wI-D)^{-1}$, $(wI-D)^{-1}N(wI-D)^{-1}$, ... , and $(wI-D)^{-1}[N(wI-D)^{-1}]^4$ are

$$\begin{pmatrix} \frac{1}{w+\alpha_0} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{w+\alpha_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{w+\alpha_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{w+\alpha_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{w+\alpha_4} \end{pmatrix},$$

$$\begin{pmatrix} 0 & \frac{\mu_0}{(w+\alpha_0)(w+\alpha_1)} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu_1}{(w+\alpha_1)(w+\alpha_2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu_2}{(w+\alpha_2)(w+\alpha_3)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\mu_3}{(w+\alpha_3)(w+\alpha_4)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

... , and

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\mu_0\mu_1\mu_2\mu_3}{(w+\alpha_0)(w+\alpha_1)(w+\alpha_2)(w+\alpha_3)(w+\alpha_4)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

respectively.

Now, let C_k be a small circle in the complex plane centered at $-\alpha_k$ and enclosing none of the other eigenvalues of M; then

$$\frac{1}{2\pi i}\int_{C_k}\left(\prod_{m=i}^{j}\frac{1}{w+\alpha_m}\right)dw = \begin{cases}\displaystyle\prod_{\substack{m=i\\ m\neq k}}^{j}\frac{1}{\alpha_m-\alpha_k} & \text{if } i\le k\le j\\ 0 & \text{otherwise.}\end{cases}$$

Thus the matrices $E(-\alpha_0)$, $E(-\alpha_1)$, ... , and $E(-\alpha_4)$ are

$$\begin{pmatrix} 1 & \frac{\mu_0}{\alpha_1-\alpha_0} & \frac{\mu_0\mu_1}{(\alpha_1-\alpha_0)(\alpha_2-\alpha_0)} & \frac{\mu_0\mu_1\mu_2}{(\alpha_1-\alpha_0)(\alpha_2-\alpha_0)(\alpha_3-\alpha_0)} & \frac{\mu_0\mu_1\mu_2\mu_3}{(\alpha_1-\alpha_0)(\alpha_2-\alpha_0)(\alpha_3-\alpha_0)(\alpha_4-\alpha_0)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \frac{\mu_0}{\alpha_0-\alpha_1} & \frac{\mu_0\mu_1}{(\alpha_0-\alpha_1)(\alpha_2-\alpha_1)} & \frac{\mu_0\mu_1\mu_2}{(\alpha_0-\alpha_1)(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)} & \frac{\mu_0\mu_1\mu_2\mu_3}{(\alpha_0-\alpha_1)(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)} \\ 0 & 1 & \frac{\mu_1}{\alpha_2-\alpha_1} & \frac{\mu_1\mu_2}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)} & \frac{\mu_1\mu_2\mu_3}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

..., and

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dfrac{\mu_0\mu_1\mu_2\mu_3}{(\alpha_0-\alpha_4)(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)} \\ 0 & 0 & 0 & 0 & \dfrac{\mu_1\mu_2\mu_3}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)} \\ 0 & 0 & 0 & 0 & \dfrac{\mu_2\mu_3}{(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)} \\ 0 & 0 & 0 & 0 & \dfrac{\mu_3}{\alpha_3-\alpha_4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Since $P(0)$ is the identity matrix,

$$P(t) = e^{tM}P(0) = e^{tM} = \sum_{k=0}^{4} e^{-t\alpha_k}E(-\alpha_k). \tag{D.12}$$

Formulas (5) and (6) of the paper are equivalent to (D.12).

We observe that, with the definition

$$\mathbf{x} \otimes \mathbf{y} = \mathbf{x}\mathbf{y}^T = (x_i y_j),$$

we have

$$E(-\alpha_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \dfrac{\mu_0}{\alpha_1 - \alpha_0} \\ \dfrac{\mu_0\mu_1}{(\alpha_1 - \alpha_0)(\alpha_2 - \alpha_0)} \\ \dfrac{\mu_0\mu_1\mu_2}{(\alpha_1 - \alpha_0)(\alpha_2 - \alpha_0)(\alpha_3 - \alpha_0)} \\ \dfrac{\mu_0\mu_1\mu_2\mu_3}{(\alpha_1 - \alpha_0)(\alpha_2 - \alpha_0)(\alpha_3 - \alpha_0)(\alpha_4 - \alpha_0)} \end{pmatrix},$$

$$E(-\alpha_1) = \begin{pmatrix} \dfrac{\mu_0}{\alpha_0 - \alpha_1} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ \dfrac{\mu_1}{\alpha_2 - \alpha_1} \\ \dfrac{\mu_1\mu_2}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)} \\ \dfrac{\mu_1\mu_2\mu_3}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_4 - \alpha_1)} \end{pmatrix},$$

$$E(-\alpha_2) = \begin{pmatrix} \dfrac{\mu_0\mu_1}{(\alpha_0 - \alpha_2)(\alpha_1 - \alpha_2)} \\ \dfrac{\mu_1}{\alpha_1 - \alpha_2} \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \\ \dfrac{\mu_2}{\alpha_3 - \alpha_2} \\ \dfrac{\mu_2\mu_3}{(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)} \end{pmatrix},$$

and so on. Write

$$E(-\alpha_k) = \mathbf{u}_k \otimes \mathbf{v}_k, \qquad k = 0, 1, 2, 3, 4. \tag{D.13}$$

It is not difficult to check that the sequences $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ are bi-orthogonal, that is,

$$\mathbf{u}_i^T\mathbf{v}_j = \mathbf{v}_j^T\mathbf{u}_i = \delta_{ij}. \tag{D.14}$$

The vector $\mathbf{u}_k$ is an eigenvector of M and $\mathbf{v}_k$ is an eigenvector of M^T; the corresponding eigenvalue for either case is $-\alpha_k$.

Here is an *APL* program that computes the spectral projection matrices.

```
    ∇A PROJECTIONS B;U;V;Z

     ⍝ This function calculates the spectral projections of matrix M.
     ⍝ A:  Main diagonal of M with positive signs.
     ⍝ B:  Diagonal immediately above the main diagonal of M.
     ⍝ U:  Vector u defined in equation (D.13).
     ⍝ V:  Vector v defined in equation (D.13).
     ⍝ Z:  Spectral projection.

     ' '
     'Alpha: ',10 6⍕A
     'Mu:    ',10 6⍕B
     I←1
     LOOP: U←(-ρA)↑(⌽(×\⌽(I-1)↑B)÷×\(⌽(I-1)↑A)-A[I]),1,((ρA)-I)ρ0
     V←(ρA)ρ((I-1)ρ0),1,(×\(I-1)↓B)÷×\(I↓A)-A[I]
     ' '
     'The spectral projection corresponding to the eigenvalue',(10 6⍕-A[I]),' is'
     12 6⍕Z←U∘.×V
     →((ρA)≥I←I+1)/LOOP
    ∇
```

```
Alpha:    0.650000  1.166281  0.999042  1.018294  2.200000
Mu:       0.450000  0.860000  0.530000  0.300000

The spectral projection corresponding to the eigenvalue ¯0.650000 is
    1.000000    0.871618    2.147570    3.090502    0.598162
    0.000000    0.000000    0.000000    0.000000    0.000000
    0.000000    0.000000    0.000000    0.000000    0.000000
    0.000000    0.000000    0.000000    0.000000    0.000000
    0.000000    0.000000    0.000000    0.000000    0.000000

The spectral projection corresponding to the eigenvalue ¯1.166281 is
    0.000000   ¯0.871618    4.482132  ¯16.052202   ¯4.658580
    0.000000    1.000000   ¯5.142315   18.416564    5.344752
    0.000000    0.000000    0.000000    0.000000    0.000000
    0.000000    0.000000    0.000000    0.000000    0.000000
    0.000000    0.000000    0.000000    0.000000    0.000000

The spectral projection corresponding to the eigenvalue ¯0.999042 is
    0.000000    0.000000   ¯6.629702 ¯182.512625  ¯45.591743
    0.000000    0.000000    5.142315  141.565543   35.363142
    0.000000    0.000000    1.000000   27.529536    6.876891
    0.000000    0.000000    0.000000    0.000000    0.000000
    0.000000    0.000000    0.000000    0.000000    0.000000
```

The spectral projection corresponding to the eigenvalue ¯1.018294 is

0.000000	0.000000	0.000000	195.474325	49.625100
0.000000	0.000000	0.000000	¯159.982107	¯40.614685
0.000000	0.000000	0.000000	¯27.529536	¯6.988928
0.000000	0.000000	0.000000	1.000000	0.253870
0.000000	0.000000	0.000000	0.000000	0.000000

The spectral projection corresponding to the eigenvalue ¯2.200000 is

0.000000	0.000000	0.000000	0.000000	0.027061
0.000000	0.000000	0.000000	0.000000	¯0.093209
0.000000	0.000000	0.000000	0.000000	0.112037
0.000000	0.000000	0.000000	0.000000	¯0.253870
0.000000	0.000000	0.000000	0.000000	1.000000

Another way to evaluate e^{tM} is to diagonalize the matrix M. As pointed out in the paper, if $M=UDU^{-1}$, then $e^{tM}=Ue^{tD}U^{-1}$. The columns of U are eigenvectors of M. As M is upper triangular, U can also be chosen to be upper triangular. Since M is quite simple, it is not too difficult to find such a matrix U. For example, put

$$U = (\boldsymbol{u}_0\ \boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3\ \boldsymbol{u}_4);$$

by (D.14),

$$U^{-1} = (\boldsymbol{v}_0\ \boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4)^T.$$

Next, we consider the system of differential equations defined by (7) of the paper. Put

$$\boldsymbol{q}(t) = \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{pmatrix}$$

and

$$\boldsymbol{b} = \begin{pmatrix} \mu_0' \\ \mu_1' \\ \mu_2' \\ \mu_3' \\ \alpha_4 \end{pmatrix}$$

Then (7) is equivalent to

$$\frac{d}{dt}\boldsymbol{q}(t) = M\boldsymbol{q}(t) + \boldsymbol{b}, \tag{D.15}$$

which, in turn, is the same as

$$\frac{d}{dt}[e^{-tM}\,\boldsymbol{q}(t)] = e^{-tM}\boldsymbol{b}. \tag{D.16}$$

Integrating (D.16) yields

$$\begin{aligned} e^{-tM}\,\boldsymbol{q}(t) - \boldsymbol{q}(0) &= \int_0^t e^{-sM}\boldsymbol{b}\,ds \\ &= (I - e^{-tM})M^{-1}\boldsymbol{b}. \end{aligned}$$

Since $\boldsymbol{q}(0) = \boldsymbol{0}$, we have

$$\boldsymbol{q}(t) = (e^{tM} - I)M^{-1}\boldsymbol{b}. \tag{D.17}$$

Applying formula (D.7), we obtain

$$\boldsymbol{q}(t) = \left[\sum_{k=0}^{4}\frac{1 - e^{-t\alpha_k}}{\alpha_k}E(-\alpha_k)\right]\boldsymbol{b}.$$

Write $\mathbf{1} = (1, 1, 1, 1, 1)^T$ and note that $-M\mathbf{1} = \boldsymbol{b}$. Hence,

$$\begin{aligned} \boldsymbol{q}(t) &= \mathbf{1} - e^{tM}\mathbf{1} \\ &= \mathbf{1} - P(t)\mathbf{1}. \end{aligned}$$

To derive the results on term insurance, temporary life annuities and pure endowments, consider the differential equation

$$\frac{d}{dt}\boldsymbol{f}(t) = (M - \delta I)\boldsymbol{f}(t) + \boldsymbol{c}. \tag{D.18}$$

If $\boldsymbol{f}(0) = \boldsymbol{0}$ and $\boldsymbol{c} = \boldsymbol{b} = -M\mathbf{1}$, then

$$\boldsymbol{f}(t) = (\bar{A}_0(t), \ldots, \bar{A}_4(t))^T.$$

If $\boldsymbol{f}(0) = \mathbf{1}$ and $\boldsymbol{c} = \boldsymbol{0}$, then

$$\boldsymbol{f}(t) = (E_0(t), \ldots, E_4(t))^T.$$

(Here E denotes endowment, not the expectation operator or spectral projection.) If $\boldsymbol{f}(0)=\mathbf{0}$ and $\boldsymbol{c}=\mathbf{1}$, then

$$\boldsymbol{f}(t) = (\bar{a}_0(t), \ldots, \bar{a}_4(t))^T.$$

The eigenvalues of the matrix $M-\delta I$ are $-\alpha_0-\delta$, $-\alpha_1-\delta$, $-\alpha_2-\delta$, $-\alpha_3-\delta$, and $-\alpha_4-\delta$. The spectral projections with respect to $M-\delta I$ are identical to the spectral projections with respect to M. Thus these vector functions can easily be evaluated by applying (D.7).

We note that, if

$$\left.\frac{d}{dt}\boldsymbol{f}(t)\right|_{t=\tau} = \mathbf{0},$$

then (D.18) implies that $\boldsymbol{f}(\tau)=(\delta I-M)^{-1}\boldsymbol{c}$. Consequently, with $\tau=+\infty$, we have

$$\begin{aligned}(\bar{A}_0, \ldots, \bar{A}_4)^T &= (\delta I - M)^{-1}\boldsymbol{b} \\ &= (I - \delta M^{-1})^{-1}\,\mathbf{1}\end{aligned}$$

and

$$(\bar{a}_0, \ldots, \bar{a}_4)^T = (\delta I - M)^{-1}\,\mathbf{1}.$$

It is easy to check that Formula (44) of the paper follows from these two formulas.

The formulas corresponding to

$$\bar{A}_{x:\overline{t}|} + \delta\bar{a}_{x:\overline{t}|} - 1 = 0$$

can also be obtained. Let

$$\boldsymbol{h}(t) = (h_0(t), \ldots, h_4(t))^T,$$

where

$$h_i(t) = \bar{A}_i(t) + E_i(t) + \delta\bar{a}_i(t) - 1.$$

Then

$$\frac{d}{dt}\boldsymbol{h}(t) = (M - \delta I)\boldsymbol{h}(t).$$

Since $h(0) = \mathbf{0}$, we have $h(t) = \mathbf{0}$ for all t. Consequently, we can also derive formulas such as

$$\bar{P}^c_i(n) = \frac{\bar{A}_i(n) + E_i(n)}{\bar{a}_i(n)} = \frac{1}{\bar{a}_i(n)} - \delta$$

and

$$\bar{V}^c_{ij}(t) = \bar{A}_j(n - t) + E_j(n - t) - \bar{P}^c_i(n)\bar{a}_j(n - t) = 1 - \frac{\bar{a}_j(n - t)}{\bar{a}_i(n)}.$$

We have two final comments. For an insurance policy designed for potential AIDS patients, should the insurance premium be waived when the policyholder reaches stage 4 (or stage 3)? Indeed, some companies would even pay out much of the death benefit to a policyholder in stage 4. Our last comment concerns μ'_0, which is the force of mortality of a person in stage 0. It does not seem realistic that the force of mortality of a healthy person is constant.

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J.C. MCKENZIE SMITH:

Dr. Ramsay is to be congratulated for his success in applying mathematical theory to real problems.

The purpose of this discussion is to demonstrate the value of matrix methods in performing the calculations. Whether the matrix approach is better will depend on circumstances.

Matrix Methods

The author's continuous models, (3), (7), (17), (22), (27), and (41), can be formulated using Equation (D) (2 in the paper) with appropriate choices M and B:

$$\frac{dP(t)}{dt} = MP(t) + B \tag{D}$$

where B and P are column vectors and M is a square matrix. The general solution is [1, p. 40]

$$P(t) = F(t,s)P(s) + G(t,s)B$$

where $F(t,s) = F(t-s, 0) = \exp[(t-s)M]$

$G(t,s) = G(t-s, 0) =$ integral of $F(t,r)$ from $r = s$ to $r = t$.

In particular, if $s = t-1$, the result is a recursive equation (R):

$$P(t) = F(t,t-1)P(t-1) + G(t,t-1)B = F\,P(t-1) + GB \tag{R}$$

into which form (49) and (51) may be cast directly, without recourse to differential equations. This recursive formulation is discussed by Smith [3]. In general, the matrix formulation is simpler to express (and, in some languages, like APL, simpler to program) but it does necessitate matrix algebra.

There are two difficulties with solving the differential equation: integrating $F(t,r)$ and the possible slow convergence of the power series solution. The slow convergence problem is easily handled by solving for $F(1,0)$, using the power series, and then using $F(t,s) = F(1,0)$ times itself $(t-s)$ times; or, better still, by using the recursive formulation directly.

The integration of $F(t,r)$ can be accomplished by reformulating (D) as a homogeneous differential equation as follows:

$$\begin{pmatrix} d/dt\ P(t) \\ d/dt\ B \end{pmatrix} = \begin{pmatrix} M & I \\ O & O \end{pmatrix} \begin{pmatrix} P(t) \\ B \end{pmatrix}$$

It is then simply a matter of applying the power series solution [1], [4] to get

$$\exp \begin{pmatrix} M & I \\ O & O \end{pmatrix} = \begin{pmatrix} F & G \\ O & I \end{pmatrix}$$

which supplies F and G for the recursive equation, which can then be used to generate the required values.

Simplification for Whole Life

Things can be simplified greatly for the whole life case. Under continuous assumptions, the reserve model (41) for 5.5 percent interest is expressed by (D) with $B = GU$ where

$$G = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

and U = {net premium} and with

$$P(t) = \begin{Bmatrix} \text{reserve in state 0} & \text{at time } t \\ \text{reserve in state 1} & \text{at time } t \\ \text{reserve in state 2} & \text{at time } t \\ \text{reserve in state 3} & \text{at time } t \\ \text{reserve in state 4} & \text{at time } t \\ \text{death benefit} & \text{at time } t \end{Bmatrix}$$

and

$$M = \begin{Bmatrix} 0.50854 & -0.45000 & 0.00000 & 0.00000 & 0.00000 & -0.00500 \\ 0.00000 & 0.93280 & -0.86000 & 0.00000 & 0.00000 & -0.01926 \\ 0.00000 & 0.00000 & 0.65770 & -0.53000 & 0.00000 & -0.07416 \\ 0.00000 & 0.00000 & 0.00000 & 0.63916 & -0.30000 & -0.28562 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.15354 & -1.10000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{Bmatrix} .$$

Each diagonal element of M is the sum of the two forces of decrement plus the force of interest. Each term just to the right of the diagonal is the negative of the force of progression and each term in the sixth column is the negative of the force of mortality.

State i	Force of Interest	Force of Progression	Force of Death	$M_{i+1,i+1}$	$M_{i+1,i+2}$	$M_{i+1,6}$
0	0.05354	0.45000	0.00500	0.50854	−0.45000	−0.00500
1	0.05354	0.86000	0.01926	0.93280	−0.86000	−0.01926
2	0.05354	0.53000	0.07416	0.65770	−0.53000	−0.07416
3	0.05354	0.30000	0.28562	0.63916	−0.30000	−0.28562
4	0.05354	0.00000	1.10000	1.15354	−1.10000	−1.10000

Denote by M' the 7-by-7 matrix formed by adding a row of zeros and then a column of ones in the first five positions and zeros in the rest:

$$M' = \begin{Bmatrix} 0.50854 & -0.45000 & 0.00000 & 0.00000 & 0.00000 & -0.00500 & 1.00000 \\ 0.00000 & 0.93280 & -0.86000 & 0.00000 & 0.00000 & -0.01926 & 1.00000 \\ 0.00000 & 0.00000 & 0.65770 & -0.53000 & 0.00000 & -0.07416 & 1.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.63916 & -0.30000 & -0.28562 & 1.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.15354 & -1.10000 & 1.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{Bmatrix}$$

Denote by P' the column vector formed by incorporating the premium into the state vector:

$$P' = \begin{Bmatrix} \text{reserve in state 0} & \text{at time } t \\ \text{reserve in state 1} & \text{at time } t \\ \text{reserve in state 2} & \text{at time } t \\ \text{reserve in state 3} & \text{at time } t \\ \text{reserve in state 4} & \text{at time } t \\ \text{death benefit} & \text{at time } t \\ \text{net premium} & \end{Bmatrix}$$

Equation 2 can now be expressed as

$$\frac{dP'(t)}{dt} = M'P'$$

Since the author's model is ageless, it follows that, for whole life, $d\mathrm{P}(t)/dt = MP(t) + GU = 0$, which implies that $dP'(t)/dt = M'P'(t) = 0$, which, in turn, implies that $P'(t) = P'$ lies in the null space of M'.

To characterize the null space of M', it is only necessary to reduce it to row reduced echelon form M'' [2, p. 11]:

$$M'' = \begin{Bmatrix} 1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & -0.70812 & 5.45154 \\ 0.00000 & 1.00000 & 0.00000 & 0.00000 & 0.00000 & -0.78913 & 3.93851 \\ 0.00000 & 0.00000 & 1.00000 & 0.00000 & 0.00000 & -0.83354 & 3.10910 \\ 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.00000 & -0.89445 & 1.97145 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & -0.95359 & 0.86690 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{Bmatrix}$$

The first thing to note is that column 6 contains the net single premium (NSP) factors in Table 2 for $B=0.005$, per \$1 rather than \$1,000 of face. The second thing to notice is that the numbers in the last column are annuity factors and, for each row from 1 to 5, (44) holds:

$$-(\text{column } 6) + \ln(0.055) \times (\text{column } 7) = 1$$

Also, each row is equivalent to the following:

$$(\text{reserve}) - (\text{NSP}) \times (\text{death benefit}) + (\text{net prem.}) \times (\text{annuity factor}) = 0$$

which is merely a statement of the prospective reserve formula. To obtain the annual premiums for $B=0.005$ in Table 1, note that the author has defined the net premium for each state as the respective ratio of 1000 × (net single premium) to the annuity factor, which will result in a reserve of nil for each state. This is simply −1000 times the ratio of the column 6 number to the column 7 number:

1000 ×	0.70812 divided by	5.45154	equals	\$129.89
1000 ×	0.78913 divided by	3.93851	equals	\$200.36
1000 ×	0.83354 divided by	3.10910	equals	\$268.10
1000 ×	0.89445 divided by	1.97145	equals	\$453.70
1000 ×	0.95359 divided by	0.86690	equals	\$1,100.00

in agreement with Table 1, line 2.

For a real policy, the net premium would not change as the life insured progressed through the stages of the disease, and the net premium would be determined by setting the stage 1 reserve to 0, causing positive net level premium reserves in the other stages.

The Discrete Case

The discrete formulas, (51), can be expressed by means of the recursive formulation where F and G are defined by

$$G(t, t-1) = \left\{\begin{matrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \end{matrix}\right\} = G \text{ and } U = \{\text{net premium}\}$$

and

$$F(t, t-1) = \left\{\begin{matrix} 0.60137 & 0.22054 & 0.09649 & 0.01728 & 0.00118 & 0.01101 \\ 0.00000 & 0.39345 & 0.38949 & 0.10863 & 0.00983 & 0.04648 \\ 0.00000 & 0.00000 & 0.51804 & 0.27712 & 0.03537 & 0.11733 \\ 0.00000 & 0.00000 & 0.00000 & 0.52774 & 0.12377 & 0.29636 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.31552 & 0.63235 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 \end{matrix}\right\} = F.$$

where $F(t)$ contains reserves for t years remaining, rather than t years from issue, as in (51), and where the elements of all rows of F but the last are equal to the transition probabilities divided by 1.055 and the last row has 1 in its diagonal element. The transition probabilities themselves can be determined by calculating $\exp(-M)$ where M is as above except that the force of interest is zero [4].

Equation (R), for whole life, takes the form

$$P = F\,P + G\,B$$

which is equivalent to

$$(F - I)P + GB = 0$$

(where I denotes the identity matrix), which is equivalent to

$$F'P' = 0$$

where

$$F' = (F - IB) = \begin{Bmatrix} -0.39863 & 0.22054 & 0.09649 & 0.01728 & 0.00118 & 0.01101 & -1.00000 \\ 0.00000 & -0.60655 & 0.38949 & 0.10863 & 0.00983 & 0.04648 & -1.00000 \\ 0.00000 & 0.00000 & -0.48196 & 0.27712 & 0.03537 & 0.11733 & -1.00000 \\ 0.00000 & 0.00000 & 0.00000 & -0.47226 & 0.12377 & 0.29636 & -1.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & -0.68448 & 0.63235 & -1.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{Bmatrix}$$

and

$$P' = \begin{pmatrix} P \\ U \end{pmatrix}$$

The row reduced echelon form of F' is F'':

$$F'' = \begin{Bmatrix} 1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & -0.68948 & 5.95639 \\ 0.00000 & 1.00000 & 0.00000 & 0.00000 & 0.00000 & -0.76830 & 4.44451 \\ 0.00000 & 0.00000 & 1.00000 & 0.00000 & 0.00000 & -0.81129 & 3.61976 \\ 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.00000 & -0.86965 & 2.50035 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & -0.92384 & 1.46096 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{Bmatrix}$$

Since $F''P' = 0$ for a whole life, the negatives of the fully discrete single premiums are, as before, displayed per unit of face amount in the sixth column in agreement with Table 6, line 2. The first five elements in the seventh column are annuity due factors satisfying Equation (52) in the paper:

$$-(\text{column } 6) + (\text{column } 7) \times 0.055/1.055 = 1$$

As before, each of the first five rows expresses the prospective reserve formula:

$$(\text{reserve}) - (\text{NSP}) \times (\text{death benefit}) + (\text{net prem.}) \times (\text{annuity factor}) = 0$$

and the ratio of the sixth element to the seventh element provides the net premium for the respective stage of the disease:

1000 × 0.68948 divided by 5.95639 equals $115.75
1000 × 0.76830 divided by 4.44451 equals $172.86
1000 × 0.81129 divided by 3.61976 equals $224.13
1000 × 0.86965 divided by 2.50035 equals $347.81
1000 × 0.92384 divided by 1.46096 equals $632.35

in agreement with Table 5, line 2.

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(AUTHOR'S REVIEW OF DISCUSSION)

COLIN M. RAMSAY:

I would like to thank Messrs. Seah and Shiu and Mr. Smith for participating in the discussion of my paper.

Seah and Shiu's use of complex analysis would appear to be an "overkill" if viewed only in the context of my paper. However, they have provided us with techniques that can be used in more general settings. I was not aware of the existence of the matrix analogue of the Lagrange interpolation formula as given in their Equation (D.9).

As far as designing insurance policies with the possibility of the life developing AIDS in mind, it does seem reasonable to waive premiums when the policyholder reaches stage 4. Such a policy provision will require the insurer to charge a larger premium and carry a larger reserve per policy.

I agree that the assumption of constant forces of the mortality and/or progression is somewhat unrealistic. However, this paper was intended to develop the mathematics of life contingencies in as simple an HIV+/AIDS environment as possible.

Mr. Smith's comments are well taken. I agree that, in general, the matrix formation is simpler to program in matrix-based languages such as APL and GAUSS. With these languages, the matrix approach becomes more attractive as the size of the matrix increases.