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A METHOD FOR DETERMINING CONFIDENCE INTERVALS FOR TREND

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ABSTRACT

The method involves resampling (with replacement but without random numbers), numerical convolutions for sums and quotients, and estimation of confidence intervals for trend in average size claim. Starting with an original sample of comprehensive major medical claims (per claimant) for each of two calendar years, we use numerical convolutions for sums to generate distributions of average size claim (per claimant) for resamples of various sizes from each of the two calendar years. We also use numerical convolutions for quotients to generate distributions of trend (in average size claim per claimant) from the first to the second of the two calendar years, to note certain stabilities in standardized versions of these distributions, and to estimate confidence intervals for the underlying trends.

I. INTRODUCTION

Suppose for a given accident year, we have n claims with severities

$$x'_1, x'_2, x'_3, \ldots, x'_n;$$

and suppose for a later accident year, we have m claims with severities

$$y'_1, y'_2, y'_3, \ldots, y'_m.$$

If the coverage is a type for which inflationary trends are significant, we might want to estimate the trend from the earlier to the later of the given accident years.

An estimate $\hat{t}$ of the true trend ϑ in severity could be obtained from the ratio of the average claim severities in the later accident year to the average claim severities in the earlier accident year; namely,

$$\hat{t} = \frac{\frac{1}{m} \cdot \sum_{i=1}^{m} y'_i}{\frac{1}{n} \cdot \sum_{i=1}^{n} x'_i} - 1.$$

If the given accident years are s years apart, then the annual trend might be estimated by

$$(1 + \hat{t})^{1/s} - 1.$$

While useful, $\hat{t}$ is a single-point estimate for the true severity trend ϑ and gives no indication of the uncertainty involved in the estimate. To try to measure the degree of statistical uncertainty involved in this estimate, we begin by reinterpreting our data.

Instead of considering the set of values

$$(x_i')_{i=1,2,\ldots,n}$$

to be the experience for the earlier of the given accident years, we treat it as a sample[1] of n claims drawn from the population of all claims that could have occurred in that accident year.

We let the empirical distribution f_X of severities X for the earlier of the two given accident years be expressed as

$$f_X = (x_i, p_i)_{i=1,2,\ldots,n'},$$

where n' is the number of different severities in the set $(x_i')_{i=1,2,\ldots,n}$

$$x_j < x_k \text{ for } j < k$$

and p_i is the relative frequency of x_i for $i=1, 2, \ldots, n'$. Clearly, $n \geq n'$.

Similarly, the set of values

$$(y_i')_{i=1,2,\ldots,m}$$

can be treated as a sample[2] of m claims drawn from the population of all claims that could have occurred in the later of the two given accident years; and we let the empirical distribution f_Y of the severities Y for that accident year be expressed as

$$f_Y = (y_i, \bar{p}_i)_{i=1,2,\ldots,m'},$$

where m' is the number of different severities in the set $(y_i')_{i=1,2,\ldots,m}$

$$y_j < y_k \text{ for } j < k$$

and $\bar{p}_i$ is the frequency of y_i for $i=1, 2, \ldots, m'$. Clearly, $m \geq m'$.

[1]This sample is referred to as the original sample for this accident year.

[2]This sample is referred to as the original sample for the later of the two given accident years.

We can estimate the distribution f_T of resample point estimates $\hat{T}$ for the true severity trend ϑ as follows:

(1) Sample n times from the distribution f_X, summing the results and dividing by n, to obtain a possible average size claim, a, from the earlier of the two given accident years

(2) Sample m times from the distribution f_Y, summing the results and dividing by m, to obtain a possible average size claim, b, from the later of the two given accident years

(3) Calculate $\hat{t} = (b/a) - 1$, which is a trial resample point estimate of the true severity trend ϑ.

Repeating steps (1) through (3) many (say υ) times produces an approximation to the distribution $f_{\hat{T}}$ of possible sample point estimates $\hat{T}$ of the true severity trend ϑ.

We now describe this classical simulation process in more detail. Afterward, we offer a more efficient method (the generalized numerical convolution) for obtaining the distribution of resample point estimates.

II. BOOTSTRAPPING FOR TREND IN AVERAGE SIZE CLAIM

A. Resampling (with Replacement) Using Random Numbers

The cumulative empirical distributions for the two given accident years are

$$\left(x_i, \sum_{k=1}^{i} p_k\right)_{i=1,\,2,\,\ldots,\,n'} \quad \text{and} \quad \left(y_j, \sum_{k=1}^{j} \tilde{p}_k\right)_{j=1,\,2,\,\ldots,\,m'},$$

respectively. The resampling (with replacement) from the original samples involve the following steps:

(1) Generate a uniform [0,1] random number, r, and determine i such that $\Sigma_{k=1}^{i} p_k$ is the smallest cumulative probability greater than r. Look up x_i and add it to an accumulator.

(2) Repeat step (1) n times.

(3) Divide the resulting accumulation by n, to obtain the average size loss per claimant, and call the result a.

(4) Perform steps (1) through (3) again, but use $\Sigma_{k=1}^{j} \tilde{p}_k$ instead of $\Sigma_{k=1}^{i} p_k$ and y_j instead of x_i in step (1), and m instead of n in steps (2) and (3), and call the result b.

(5) Calculate $\hat{t}=(b/a)-1$, which is a resample point estimate $\hat{t}$ for the true trend ϑ.

(6) Repeat steps (1) through (5), say, υ times.

Let the frequency distribution of the resulting values of $1+\hat{T}$ be labelled as $f_{1+\hat{T}}$ and represented as

$$(1 + \hat{t}_k, r_k)_{k=1, 2, \ldots, \upsilon'},$$

where υ' is the number of different point estimates $\hat{t}$ obtained in step (5), and r_k is the frequency of $1+\hat{t}_k$ for $k=1, 2, \ldots, \upsilon'$. Now $f_{1+\hat{T}}$, once generated, can be used to estimate the standard error in trend or other such statistics. This procedure is referred to as bootstrapping.[3]

To use this approach, it is helpful to know how large υ should be to produce a reasonably good representation of the distribution of $1+\hat{T}$ if υ were chosen to be infinity. Table 1 shows results of this approach using $\upsilon=10^3$, 10^4 and 10^5 trial resample point estimates and $m=n=64$; the accident-year pair is 1983–84. The last column of Table 1 shows results from an almost exact representation of the distribution $f_{1+\hat{T}}$ of $1+\hat{T}$ if υ were chosen to be infinity.[4]

This resampling procedure is practical if $\upsilon=n$ and m are each small. However, as υ, n and/or m increase, this procedure becomes impractical. So we use the method below, which we call "Operational Bootstrapping."

B. Resampling (with Replacement) without Random Numbers

In contrast to classical bootstrapping, in which random numbers are used to do the resampling, operational bootstrapping uses numerical convolutions to generate the distributions without random numbers. For example, consider the distribution $f_{X_1+X_2}$ of X_1+X_2, where X_1 and X_2 are independent identically distributed random variables, each distributed as

$$(x_i, p_i)_{i=1, 2, \ldots, n}.$$

[3]For a detailed description of bootstrapping, see Efron and Tibshirani [1]. Efron coined the term "bootstrapping" in the late 1970s.

[4]For the method used to obtain this distribution, see Section II-B.

TABLE 1

BOOTSTRAP-TYPE DISTRIBUTIONS OF TREND FACTORS $(1+\hat{T})$
(RESAMPLING BY MONTE CARLO)

	$1+\hat{T}$			
Cumulative	$\upsilon = 10^3$	$\upsilon = 10^4$	$\upsilon = 10^5$	$\upsilon = \infty$
0.000001				0.109
0.00001			0.146	0.146
0.0001			0.206	0.179
0.001			0.264	0.247
0.01	0.378	0.392	0.389	0.375
0.025	0.472	0.467	0.462	0.457
0.05	0.544	0.535	0.532	0.526
0.1	0.636	0.622	0.622	0.618
0.2	0.748	0.737	0.742	0.737
0.3	0.838	0.834	0.840	0.837
0.4	0.938	0.925	0.930	0.932
0.5	1.022	1.017	1.025	1.028
0.6	1.125	1.123	1.130	1.135
0.7	1.236	1.243	1.256	1.261
0.8	1.418	1.409	1.424	1.431
0.9	1.723	1.687	1.706	1.723
0.95	2.037	1.968	2.000	2.027
0.975	2.398	2.323	2.331	2.370
0.99	2.964	2.871	2.833	2.940
0.999	4.068	4.433	4.274	4.844
0.9999		6.458	5.618	6.635
0.99999			7.299	8.360
0.999999				10.239
Mean	1.121	1.061	1.055	1.066
$Var \cdot 10^{+3}$	0.244	0.204	0.204	0.204
n	64	64	64	64
m	64	64	64	64

Symbolically, we can express $f_{X_1+X_2}$ in terms of the distributions of X_1 and X_2, as follows[5]:

$$f_{X_1+X_2} = f_{X_1} + f_{X_2}$$

$$= (x_i, p_i)_{i=1,2,\ldots,n'} + (x_j, p_j)_{j=1,2,\ldots,n'}$$

$$= \left\{ \begin{array}{l} (x_1 + x_1, p_1 \cdot p_1) \\ (x_1 + x_2, p_1 \cdot p_2) \\ \quad \vdots \\ (x_1 + x_{n'}, p_1 \cdot p_{n'}) \\ (x_2 + x_1, p_2 \cdot p_1) \\ (x_2 + x_2, p_2 \cdot p_2) \\ \quad \vdots \\ (x_2 + x_{n'}, p_2 \cdot p_{n'}) \\ \quad \vdots \\ (x_{n'} + x_1, p_{n'} \cdot p_1) \\ (x_{n'} + x_2, p_{n'} \cdot p_2) \\ \quad \vdots \\ (x_{n'} + x_{n'}, p_{n'} \cdot p_{n'}) \end{array} \right\}$$

which we might express as

$$(x_i + x_j, p_i \cdot p_j)_{i=1,2,\ldots,n' \text{ and } j=1,2,\ldots,n'}.$$ [6]

[5]The symbol + between two distributions means convolute for sums.

[6]This set of pairs is actually modified by replacing each of the pairs having identical values in the first position by one pair with that value in the first position and the sum of the corresponding probabilities in the second position. The resulting set of pairs then constitutes a distribution.

Using this distribution as a prototype and assuming that $X_1, X_2, \ldots, X_n$ are independent identically distributed random variables each distributed as

$$(x_i, p_i)_{i=1, 2, \ldots, n'},$$

we can proceed recursively to generate

$$f_{X_1+X_2+X_3+X_4},$$

where, since

$$f_{X_3+X_4} = f_{X_1+X_2},$$

we can write

$$f_{X_1+X_2+X_3+X_4} = f_{X_1+X_2} + f_{X_3+X_4} = f_{X_1+X_2} + f_{X_1+X_2};$$

and continue to perform convolutions between the results of other convolutions until we have obtained the desired result; namely,

$$f_{X_1+X_2+\ldots+X_n}.$$

If we proceed naively in this manner, the number of lines in the resulting distributions could become prohibitively large for both computer storage and computing time. The Appendix describes a method of overcoming this problem. This method (after dividing the amounts by n) produces a distribution having mean equal to the mean of the original sample and variance equal to the variance of the original sample.

We can similarly generate the distribution

$$f_{Y_1+Y_2+\ldots+Y_m}$$

of $Y_1+Y_2+\ldots+Y_m$, where $Y_1, Y_2, \ldots, Y_m$ are independent identically distributed random variables, each distributed as

$$(y_i, \tilde{p}_i)_{i=1, 2, \ldots, m'}.$$

To generate the distribution

$$f_{1+\hat{T}} = f_{(1/m)(Y_1+Y_2+\ldots+Y_m)/f_{(1/n)(X_1+X_2+\ldots+X_n)}}$$

of

$$(1/m) \cdot (Y_1 + Y_2 + \ldots + Y_m)/[(1/n) \cdot (X_1 + X_2 + \ldots + X_n)],$$

we can first generate[7]

$$f_{Y_1+Y_2+\ldots+Y_m}/f_{X_1+X_2+\ldots+X_n}.$$

Letting

$$f_{X_1+X_2+\ldots+X_n}$$

be represented as

$$(u_i,\ p_i)_{i=1,\,2,\,\ldots,\,n^*}$$

and

$$f_{Y_1+Y_2+\ldots+Y_m}$$

be represented as

$$(v_i,\ \tilde{p}_i)_{i=1,\,2,\,\ldots,\,m^*},$$

we have

$$f_{(Y_1+Y_2+\ldots+Y_m)\,/\,(X_1+X_2+\ldots+X_n)} = (v_j,\ \tilde{p}_j)_{j=1,\,2,\,\ldots,\,m^*}/(u_i,\quad p_i)_{i=1,\,2,\,\ldots,\,n^*}$$

$$(v_j/u_i,\ p_i\cdot\tilde{p}_j)_{i=1,\,2,\,\ldots,\,n^*\text{ and }j=1,\,2,\,\ldots,\,m^*}.^6$$

Then

$$f_{1+\hat{T}} = f_{(1/m)(Y_1+Y_2+\ldots+Y_m)/[(1/n)(X_1+X_2+\ldots+X_n)]}$$

would be obtained by multiplying the amounts (not the probabilities) in the distribution

$$f_{(Y_1+Y_2+\ldots+Y_m)/(X_1+X_2+\ldots+X_n)}$$

by n/m.

The distributions $f_{1+\hat{T}}$ generated by the methods of this section are representations of the distribution $f_{1+\hat{T}}$, which would have been generated by the method of the previous section if we could have generated an infinite number of random numbers.[8] For this reason, we would expect the distributions shown in Table 1 in the columns headed $\upsilon=10^3$, $\upsilon=10^4$ and $\upsilon=10^5$ to approach the distribution shown in the column headed "υ = infinity" as υ increases.

[7]The symbol / between two distributions is being used to mean convolute for quotients, dividing the first random variable by the second.

[8]See the Appendix.

III. CONFIDENCE INTERVALS FOR TREND

A. Large Resamples

Lowrie and Lipsky [2] presented group major medical expense claims by claimant per accident year for each of the five years 1983 to 1987. Their distributions are shown separately for adult or child combined with either comprehensive or supplemental coverage. We focus on adult comprehensive coverage only, noting that the deductible is \$100 per calendar year and the coinsurance is 20 percent.

We considered the random variable

$$W = \frac{1 + \hat{T}}{E[1 + \hat{T}]} - 1.$$

Note that $E[1+W]$ is equal to unity. We were interested to find that f_{1+W} shows a remarkable degree of stability as the accident year pairs are varied. By using the operational bootstrapping approach described in the previous section, the distributions $f_{1+\hat{T}}$ and f_{1+W} were generated for each of the accident-year pairs 1983–84, 1984–85, 1985–86, and 1986–87 (see Table 2). In Table 2 the numbers of claims in the resamples varied from 66,260 to 111,263. We concluded that, provided the numbers of claims are of this order of magnitude, f_{1+W} can be used as a pivotal distribution; that is, for any true trend ϑ_0, the point estimates $1+\hat{T}|\vartheta_0$ can be considered to be distributed as $f_{(1+\vartheta_0)(1+W)}$. We note that f_{1+W} is approximately $N(1,\text{Var}[1+W])$ and $f_{1+\hat{T}}$ is approximately $N(E[1+\hat{T}],\text{Var}[1+\hat{T}])$.[9]

1. A Numerical Example of Determining a Confidence Interval for Trend Using Large Resamples

We now turn our attention to determining a confidence interval for the true trend ϑ. We wish to determine ϑ_1 such that $\Pr\{\vartheta_1 < \vartheta\} = 1 - \alpha/2$ and ϑ_2 such that $\Pr\{\vartheta < \vartheta_2\} = 1 - \alpha/2$, so that the random interval $(\vartheta_1, \vartheta_2)$ encloses the true trend ϑ at the desired level $(1-\alpha)$ of confidence.

From Table 2 we can select a value of W (say w_1) such that

$$1 - \alpha/2 = \Pr\{w_1 < W\};$$

[9]An expression such as $N(\mu,\sigma^2)$ is used, as is customary, to indicate a normal distribution with mean μ and variance σ^2.

TABLE 2

BOOTSTRAP DISTRIBUTIONS OF TREND FACTORS $(1+\hat{T})$ AND STANDARDIZED FACTORS $(1+W)$
(LARGE-SIZED RESAMPLING BY OPERATIONAL BOOTSTRAPPING)

	1983–84		1984–85		1985–86		1986–87	
Cumulative	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$
0.000001	0.965	0.934	0.982	0.936	0.979	0.938	1.001	0.938
0.00001	0.972	0.940	0.989	0.942	0.986	0.945	1.007	0.945
0.0001	0.979	0.948	0.996	0.949	0.993	0.951	1.015	0.951
0.001	0.988	0.956	1.005	0.958	1.001	0.959	1.023	0.959
0.01	0.999	0.967	1.016	0.968	1.011	0.969	1.034	0.969
0.025	1.004	0.972	1.021	0.973	1.016	0.974	1.039	0.974
0.05	1.009	0.976	1.026	0.977	1.021	0.978	1.043	0.978
0.1	1.014	0.982	1.031	0.982	1.026	0.983	1.048	0.983
0.2	1.021	0.988	1.037	0.988	1.032	0.989	1.054	0.989
0.3	1.025	0.992	1.042	0.993	1.036	0.993	1.059	0.993
0.4	1.029	0.996	1.046	0.996	1.040	0.997	1.063	0.997
0.5	1.033	1.000	1.049	1.000	1.043	1.000	1.066	1.000
0.6	1.037	1.004	1.053	1.003	1.047	1.003	1.070	1.003
0.7	1.041	1.007	1.057	1.007	1.051	1.007	1.074	1.007
0.8	1.046	1.012	1.062	1.012	1.055	1.011	1.078	1.011
0.9	1.053	1.019	1.068	1.018	1.062	1.017	1.085	1.017
0.95	1.058	1.024	1.073	1.023	1.067	1.022	1.090	1.022
0.975	1.063	1.029	1.078	1.027	1.071	1.027	1.095	1.027
0.99	1.069	1.034	1.084	1.033	1.077	1.032	1.100	1.032
0.999	1.081	1.046	1.095	1.043	1.088	1.042	1.112	1.042
0.9999	1.091	1.055	1.104	1.052	1.097	1.051	1.121	1.051
0.99999	1.099	1.064	1.113	1.060	1.105	1.059	1.129	1.059
0.999999	1.109	1.073	1.122	1.070	1.115	1.068	1.139	1.068
Mean	1.033	1.000	1.049	1.000	1.044	1.000	1.066	1.000
$\text{Var}\cdot 10^{+3}$	0.209	0.223	0.211	0.192	0.195	0.179	0.204	0.179
n	66260		76857		83457		88977	
m	76857		83457		88977		111263	

that is, such that

$$\begin{aligned} 1 - \alpha/2 &= \Pr\{1 + w_1 < 1 + W\} \\ &= \Pr\{1 + w_1 < (1 + \hat{T})/(1 + \vartheta)\} \\ &= \Pr\{(1 + w_1) \cdot (1 + \vartheta) < 1 + \hat{T}\} \\ &= \Pr\{1 + \vartheta < (1 + \hat{T})/(1 + w_1)\} \\ &= \Pr\{\vartheta < (1 + \hat{T})/(1 + w_1) - 1\} \end{aligned}$$

so we choose

$$\vartheta_1 = (1 + \hat{T})/(1 + w_1) - 1.$$

Similarly, from Table 2 we can select a value of W (say w_2) such that

$$1 - \alpha/2 = \Pr\{W < w_2\};$$

that is, such that

$$\begin{aligned} 1 - \alpha/2 &= \Pr\{1 + W < 1 + w_2\} \\ &= \Pr\{(1 + \hat{T})/(1 + \vartheta) < 1 + w_2\} \\ &= \Pr\{1 + \hat{T} < (1 + w_2) \cdot (1 + \vartheta)\} \\ &= \Pr\{(1 + \hat{T})/(1 + w_2) < 1 + \vartheta\} \\ &= \Pr\{(1 + \hat{T})/(1 + w_2) - 1 < \vartheta\} \end{aligned}$$

so we choose

$$\vartheta_2 = (1 + \hat{T})/(1 + w_2) - 1.$$

If $1-\alpha=95\%$, then referring to Table 2 (1983–84), we can let $1+w_1=0.972$ and $1+w_2=1.029$ and find that

$$\vartheta_1 = 1.033/1.029 - 1 = 0.004$$

and

$$\vartheta_2 = 1.033/0.972 - 1 = 0.063.$$

Therefore, the confidence interval for the true trend ϑ is

$$(\vartheta_2,\vartheta_1) = (0.4\%, 6.3\%);$$

3.3% was the corresponding point estimate. This result and the corresponding results for the other calendar-year pairs are shown in the following table:

STATISTICS AND CONFIDENCE INTERVALS FOR TREND FACTORS

Mean* and 50th Percentile	95% Confidence Interval	Calendar Year	n	m
1.033	(1.004,1.063)	1983–84	66,260	76,857
1.049	(1.021,1.078)	1984–85	76,857	83,457
1.044	(1.016,1.071)	1985–86	83,457	88,977
1.066	(1.039,1.095)	1986–87	88,977	111,263

*The mean and the median could turn out to be different, but here they happen to be identical to the number of decimal places shown.

2. *Test of Normality Assumptions*

To determine whether we could produce equally good confidence intervals making use of some normality assumptions, we assumed that f_X^{*n} and f_Y^{*m} could be approximated by the normal distributions $N(n \cdot E[X], n \cdot \text{Var}[X])$ and $N(m \cdot E[Y], m \cdot \text{Var}[Y])$, respectively. The distribution $f_{1+\hat{T}}$ was then obtained by generating

$$N(m \cdot E[Y], m \cdot \text{Var}[Y]) \,/\, N(n \cdot E[X], n \cdot \text{Var}[X])$$

and transforming the resulting distribution by multiplying the amounts (not the probabilities) by n/m. The resulting figures turned out to agree exactly with the figures shown in Table 2.[10]

In the following section we investigate the corresponding situation in which n and m are equal and medium-sized, say 64 to 16,384.

B. *Medium-Sized Resamples*

So far we have been dealing with resamples of size n or m from an original sample of size n or m, respectively, either using or not using random numbers. But even though the original samples are of size n or m, we can generate resamples of, say, size $\tilde{n}$ $(<n)$ and $\tilde{m}$ $(<m)$; in particular, we can choose $\tilde{n} = \tilde{m}$ $(<\min\{n,m\})$. The purpose of this would be to determine the confidence intervals for trend if the resamples were of medium (rather than large) size.

Consider

$$f_{1+\hat{T}} = f_{(1/\tilde{n})(Y_1+Y_2+\ldots+Y_{\tilde{n}})} \Big/ f_{(1/\tilde{n})(X_1+X_2+\ldots+X_{\tilde{n}})}$$

$$= \sum_{i=1}^{\tilde{n}} f_{(1/\tilde{n})\cdot Y_i} \Big/ \sum_{i=1}^{\tilde{n}} f_{(1/\tilde{n})\cdot X_i}{}^{*}$$

*$\Sigma_{i=1}^{n} f_{(1/n)Y_i} = f_{(1/n)Y_1} + f_{(1/n)Y_2} + \ldots + f_{(1/n)Y_m}$ is being used to mean convolute $f_{(1/m)Y_1}, f_{(1/m)Y_2}, \ldots,$ and $f_{(1/m)Y_m}$.

[10]A referee pointed out that if $\bar{X}$ and $\bar{Y}$ are asymptotically normal random variables and $\hat{T} = \bar{Y}/\bar{X} - 1$, then $\hat{T}$ is asymptotically $N(\mu,\sigma^2)$ with $\mu = \mu_Y/\mu_X - 1$ and $\sigma^2 = \mu_Y^2 \cdot \sigma_X^2/\mu_X^4 \cdot n + \sigma_Y^2/\mu_X^2 \cdot m$; and that these can be approximated by replacing the population quantities with the sample values.

If we had available (and used) the detailed data underlying the loss distributions presented by Lowrie and Lipsky [2], our confidence intervals would be slightly wider. Using the calendar-year pair 1987–1988 and the above formula for σ^2, we find that the ratio of σ^2 based on the detailed data to σ^2 based on the grouped data is 1.016, that is, a 1.6 percent deficiency in the variance. The data for 1988 were not shown in [2]; however, Lowrie was kind enough to furnish those data to me for this paragraph. Lowrie said that the "standard deviation" figures shown in [2] were calculated by an incorrect formula and should not be used.

where $\bar{n}$ takes on the value 64, 128, ..., or 1,024 and the X_i and Y_i are based on calendar years 1983 and 1984, 1984 and 1985, 1985 and 1986, or 1986 and 1987, respectively. The distributions $f_{1+\hat{T}}$ are shown in Table 3, along with the corresponding standardized distributions $f_{1+W} = f_{(1+\hat{T})/E[1+\hat{T}]}$.

For determining confidence intervals for trend where the resamples are of medium size, we wish to assume for given $\bar{n}$ that f_{1+W} does not differ significantly as we vary the calendar-year pairs. The reasonableness of making this assumption seems to be confirmed by the fact that for fixed $\bar{n} = \bar{m}$, the standardized distributions f_{1+W} in Table 3 vary as little as they do by calendar-year pair, at least in the portion of the distributions between cumulatives of 0.025 and 0.975.

1. A Numerical Example of Determining a Confidence Interval for Trend Using Medium-Sized Resamples

Suppose a trend factor of 1.15 has been observed from one year to another and the number of claims is 64 in each of the two accident years. We now determine a 95 percent confidence interval for the true severity trend ϑ, again using the formulas shown in the previous numerical example.

Referring to Table 3 (1983–84), we can let $1+w_1 = 2.108$ and $1+w_2 = 0.406$ if $1-\alpha = 0.95$; so $\vartheta_1 = 1.15/2.108 = 0.546$ and $\vartheta_2 = 1.15/0.406 = 2.83$. Thus the estimated 95 percent confidence interval for the underlying trend factor $1+\vartheta$ would be (0.546, 2.83). This result and the corresponding results for the other calendar-year pairs are shown in the following table:

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Mean	50th Percentile	95% Confidence Interval	Calendar Year	$\bar{n}$	$\bar{m}$
1.125	1.028	(0.546,2.83)	1983–84	64	64
1.147	1.050	(0.542,2.91)	1984–85	64	64
1.142	1.037	(0.533,2.96)	1985–86	64	64
1.171	1.059	(0.524,3.05)	1986–87	64	64

Table 3 includes distributions for $\bar{n} = \bar{m} = 64$, 128, 256, 512, and 1,024 for calendar-year pairs 1983–84, 1984–85, 1985–86, and 1986–87; and distributions for $\bar{n} = \bar{m} = 2{,}048$, 4,096, 8,192, and 16,384 for calendar-year pair 1983–84.

TABLE 3

BOOTSTRAP-TYPE DISTRIBUTIONS OF TREND FACTORS ($1+\hat{T}$) AND STANDARDIZED FACTORS ($1+W$)
(MEDIUM-SIZED RESAMPLING BY OPERATIONAL BOOTSTRAPPING)

	1983–84		1984–85		1985–86		1986–87	
Cumulative	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$
				$n=64$				
0.000001	0.109	0.097	0.101	0.088	0.123	0.107	0.101	0.087
0.00001	0.146	0.129	0.124	0.108	0.150	0.131	0.122	0.104
0.0001	0.179	0.159	0.161	0.140	0.193	0.169	0.159	0.135
0.001	0.247	0.220	0.220	0.191	0.253	0.221	0.225	0.192
0.01	0.375	0.334	0.366	0.319	0.369	0.323	0.360	0.308
0.025	0.457	0.406	0.453	0.395	0.444	0.389	0.441	0.377
0.05	0.526	0.468	0.531	0.463	0.518	0.453	0.520	0.444
0.1	0.618	0.550	0.623	0.543	0.610	0.535	0.614	0.524
0.2	0.737	0.655	0.749	0.652	0.737	0.646	0.749	0.639
0.3	0.837	0.744	0.854	0.744	0.839	0.735	0.853	0.728
0.4	0.932	0.829	0.949	0.827	0.937	0.821	0.951	0.812
0.5	1.028	0.914	1.050	0.915	1.037	0.909	1.059	0.904
0.6	1.135	1.009	1.159	1.010	1.151	1.008	1.175	1.003
0.7	1.261	1.121	1.294	1.127	1.283	1.124	1.316	1.123
0.8	1.431	1.272	1.474	1.284	1.464	1.282	1.502	1.283
0.9	1.723	1.532	1.775	1.547	1.770	1.551	1.826	1.559
0.95	2.027	1.802	2.094	1.825	2.099	1.839	2.172	1.854
0.975	2.370	2.108	2.435	2.122	2.465	2.159	2.570	2.194
0.99	2.940	2.614	2.924	2.548	3.011	2.637	3.218	2.747
0.999	4.844	4.308	4.237	3.693	4.714	4.129	5.435	4.640
0.9999	6.635	5.900	5.578	4.861	6.461	5.659	9.069	7.742
0.99999	8.360	7.434	6.972	6.076	8.138	7.128	12.108	10.336
0.999999	10.239	9.104	8.453	7.366	9.926	8.695	14.811	12.644
Mean	1.125	1.000	1.147	1.000	1.142	1.000	1.171	1.000
Var	0.267	0.211	0.264	0.201	0.283	0.217	0.338	0.246
$\bar{n}$	64		64		64		64	
$\bar{m}$	64		64		64		64	

TABLE 3—*Continued*

Cumulative	1983–84 $1+\hat{T}$	1983–84 $1+W$	1984–85 $1+\hat{T}$	1984–85 $1+W$	1985–86 $1+\hat{T}$	1985–86 $1+W$	1986–87 $1+\hat{T}$	1986–87 $1+W$
				$n=128$				
0.000001	0.209	0.193	0.187	0.170	0.221	0.202	0.192	0.171
0.00001	0.246	0.227	0.223	0.202	0.255	0.233	0.227	0.202
0.0001	0.295	0.272	0.269	0.244	0.305	0.278	0.274	0.244
0.001	0.366	0.338	0.339	0.307	0.376	0.343	0.347	0.309
0.01	0.489	0.452	0.474	0.429	0.489	0.446	0.475	0.423
0.025	0.560	0.518	0.555	0.503	0.556	0.507	0.551	0.490
0.05	0.624	0.576	0.626	0.568	0.618	0.564	0.619	0.551
0.1	0.702	0.649	0.710	0.643	0.697	0.636	0.704	0.627
0.2	0.803	0.742	0.816	0.740	0.804	0.737	0.813	0.724
0.3	0.881	0.814	0.899	0.814	0.886	0.808	0.902	0.802
0.4	0.954	0.881	0.975	0.883	0.963	0.878	0.980	0.872
0.5	1.028	0.949	1.051	0.953	1.039	0.947	1.060	0.943
0.6	1.107	1.023	1.135	1.028	1.122	1.024	1.147	1.021
0.7	1.200	1.108	1.231	1.116	1.220	1.112	1.249	1.111
0.8	1.319	1.219	1.356	1.229	1.347	1.228	1.383	1.231
0.9	1.515	1.399	1.559	1.413	1.556	1.419	1.602	1.426
0.95	1.717	1.586	1.758	1.593	1.766	1.610	1.832	1.630
0.975	1.936	1.788	1.958	1.774	1.983	1.809	2.081	1.851
0.99	2.271	2.098	2.224	2.015	2.291	2.089	2.452	2.181
0.999	3.168	2.927	2.887	2.616	3.129	2.854	3.689	3.282
0.9999	3.982	3.678	3.544	3.211	3.930	3.584	5.261	4.681
0.99999	4.813	4.446	4.211	3.817	4.733	4.316	6.456	5.743
0.999999	5.703	5.268	4.893	4.434	5.577	5.087	7.627	6.785
Mean	1.082	1.000	1.103	1.000	1.097	1.000	1.124	1.000
Var	0.126	0.108	0.127	0.104	0.135	0.112	0.160	0.127
$\bar{n}$	128		128		128		128	
$\bar{m}$	128		128		128		128	

TABLE 3—*Continued*

Cumulative	1983–84		1984–85		1985–86		1986–87	
	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$	$1+\hat{T}$	$1+W$
				$n=256$				
0.000001	0.327	0.309	0.301	0.279	0.338	0.315	0.309	0.281
0.00001	0.369	0.349	0.342	0.317	0.378	0.353	0.350	0.319
0.0001	0.423	0.399	0.397	0.369	0.430	0.402	0.405	0.369
0.001	0.495	0.468	0.472	0.438	0.501	0.467	0.480	0.437
0.01	0.602	0.569	0.589	0.546	0.603	0.563	0.594	0.542
0.025	0.660	0.623	0.656	0.608	0.660	0.616	0.656	0.598
0.05	0.713	0.673	0.715	0.663	0.711	0.664	0.714	0.651
0.1	0.776	0.732	0.784	0.727	0.776	0.724	0.783	0.714
0.2	0.857	0.809	0.871	0.808	0.859	0.802	0.871	0.794
0.3	0.919	0.867	0.936	0.868	0.923	0.862	0.939	0.856
0.4	0.974	0.920	0.995	0.923	0.982	0.916	1.001	0.913
0.5	1.029	0.971	1.052	0.975	1.040	0.970	1.061	0.967
0.6	1.088	1.027	1.113	1.032	1.102	1.028	1.126	1.026
0.7	1.154	1.089	1.183	1.097	1.172	1.094	1.201	1.094
0.8	1.239	1.170	1.270	1.178	1.262	1.178	1.296	1.181
0.9	1.375	1.298	1.406	1.303	1.403	1.309	1.448	1.319
0.95	1.508	1.423	1.531	1.420	1.537	1.435	1.597	1.456
0.975	1.644	1.552	1.650	1.430	1.670	1.559	1.750	1.595
0.99	1.829	1.726	1.803	1.671	1.846	1.723	1.963	1.789
0.999	2.278	2.151	2.167	2.010	2.279	2.127	2.654	2.419
0.9999	2.271	2.555	2.518	2.335	2.696	2.516	3.324	3.029
0.99999	3.137	2.962	2.864	2.655	3.111	2.904	3.904	3.558
0.999999	3.570	3.370	3.206	2.973	3.528	3.293	4.522	4.121
Mean	1.059	1.000	1.078	1.000	1.097	1.000	1.097	1.000
Var	0.062	0.055	0.063	0.055	0.135	0.058	0.079	0.066
$\bar{n}$	256		256		256		256	
$\bar{m}$	256		256		256		256	

TABLE 3—*Continued*

Cumulative	1983–84 $1+\hat{T}$	1983–84 $1+W$	1984–85 $1+\hat{T}$	1984–85 $1+W$	1985–86 $1+\hat{T}$	1985–86 $1+W$	1986–87 $1+\hat{T}$	1986–87 $1+W$
				$n=512$				
0.000001	0.457	0.436	0.432	0.405	0.465	0.440	0.440	0.406
0.00001	0.498	0.476	0.475	0.446	0.505	0.478	0.483	0.446
0.0001	0.549	0.525	0.529	0.497	0.555	0.525	0.536	0.495
0.001	0.614	0.587	0.599	0.563	0.618	0.585	0.606	0.560
0.01	0.702	0.671	0.696	0.654	0.705	0.666	0.701	0.648
0.025	0.748	0.715	0.748	0.702	0.750	0.709	0.751	0.694
0.05	0.789	0.754	0.793	0.745	0.791	0.748	0.796	0.735
0.1	0.838	0.801	0.847	0.796	0.841	0.795	0.850	0.785
0.2	0.901	0.860	0.914	0.859	0.906	0.856	0.919	0.849
0.3	0.947	0.905	0.965	0.906	0.954	0.902	0.971	0.897
0.4	0.989	0.945	1.009	0.948	0.998	0.943	1.017	0.940
0.5	1.030	0.984	1.052	0.988	1.041	0.984	1.063	0.982
0.6	1.073	1.025	1.097	1.030	1.086	1.027	1.110	1.026
0.7	1.121	1.071	1.146	1.076	1.137	1.075	1.164	1.075
0.8	1.182	1.129	1.207	1.134	1.200	1.134	1.232	1.138
0.9	1.274	1.217	1.298	1.219	1.295	1.224	1.336	1.234
0.95	1.361	1.300	1.379	1.295	1.381	1.306	1.433	1.324
0.975	1.443	1.379	1.454	1.365	1.463	1.323	1.529	1.413
0.99	1.549	1.480	1.546	1.452	1.566	1.481	1.660	1.533
0.999	1.798	1.718	1.760	1.653	1.811	1.712	2.017	1.863
0.9999	2.035	1.944	1.959	1.840	2.042	1.930	2.350	2.171
0.99999	2.266	2.165	2.149	2.019	2.266	2.142	2.667	2.464
0.999999	2.492	2.381	2.334	2.192	2.485	2.349	3.000	2.767
Mean	1.047	1.000	1.065	1.000	1.058	1.000	1.082	1.000
Var	0.031	0.028	0.032	0.028	0.033	0.029	0.039	0.034
$\bar{n}$	512		512		512		512	
$\bar{m}$	512		512		512		512	

TABLE 3—*Continued*

Cumulative	1983–84 $1+\hat{T}$	1983–84 $1+W$	1984–85 $1+\hat{T}$	1984–85 $1+W$	1985–86 $1+\hat{T}$	1985–86 $1+W$	1986–87 $1+\hat{T}$	1986–87 $1+W$
				$n = 1024$				
0.000001	0.549	0.559	0.565	0.532	0.549	0.559	0.524	0.531
0.00001	0.592	0.594	0.607	0.570	0.593	0.593	0.573	0.567
0.0001	0.641	0.636	0.655	0.615	0.643	0.634	0.627	0.611
0.001	0.699	0.687	0.713	0.670	0.702	0.684	0.693	0.665
0.01	0.774	0.754	0.787	0.743	0.778	0.751	0.776	0.736
0.025	0.812	0.788	0.824	0.780	0.816	0.784	0.817	0.772
0.05	0.844	0.819	0.857	0.812	0.850	0.814	0.854	0.804
0.1	0.884	0.854	0.897	0.851	0.890	0.851	0.898	0.842
0.2	0.933	0.899	0.947	0.898	0.940	0.897	0.953	0.890
0.3	0.970	0.933	0.984	0.934	0.978	0.931	0.995	0.927
0.4	1.002	0.963	1.017	0.965	1.011	0.962	1.031	0.959
0.5	1.033	0.992	1.049	0.994	1.043	0.992	1.066	0.990
0.6	1.065	1.022	1.082	1.024	1.076	1.022	1.103	1.022
0.7	1.101	1.055	1.119	1.058	1.113	1.057	1.143	1.058
0.8	1.144	1.096	1.164	1.098	1.157	1.098	1.192	1.103
0.9	1.206	1.156	1.229	1.157	1.222	1.160	1.263	1.169
0.95	1.261	1.210	1.287	1.207	1.278	1.214	1.325	1.229
0.975	1.311	1.260	1.339	1.253	1.330	1.263	1.382	1.286
0.99	1.371	1.322	1.404	1.209	1.392	1.324	1.453	1.360
0.999	1.510	1.464	1.555	1.434	1.537	1.464	1.615	1.540
0.9999	1.639	1.595	1.697	1.547	1.673	1.591	1.769	1.710
0.99999	1.766	1.720	1.839	1.652	1.806	1.713	1.921	1.874
0.999999	1.892	1.841	1.984	1.752	1.941	1.829	2.076	2.036
Mean	1.040	1.000	1.058	1.000	1.051	1.000	1.075	1.000
Var	0.016	0.014	0.017	0.015	0.017	0.015	0.021	0.017
$\bar{n}$	1024		1024		1024		1024	
$\bar{m}$	1024		1024		1024		1024	

TABLE 3—*Continued*

Cumulative	1983–84 $1+\hat{T}$	1983–84 $1+W$	1983–84 $1+\hat{T}$	1983–84 $1+W$	1983–84 $1+\hat{T}$	1983–84 $1+W$	1983–84 $1+\hat{T}$	1983–84 $1+W$
				$n=\bar{m},\bar{n}$				
0.000001	0.674	0.650	0.769	0.743	0.841	0.813	0.894	0.865
0.00001	0.707	0.682	0.794	0.767	0.859	0.831	0.907	0.878
0.0001	0.744	0.718	0.822	0.794	0.880	0.851	0.923	0.893
0.001	0.788	0.760	0.855	0.826	0.904	0.875	0.941	0.910
0.01	0.844	0.814	0.897	0.866	0.935	0.904	0.963	0.932
0.025	0.872	0.841	0.917	0.886	0.950	0.919	0.974	0.942
0.05	0.897	0.865	0.935	0.903	0.963	0.931	0.983	0.951
0.1	0.925	0.893	0.956	0.924	0.978	0.946	0.994	0.962
0.2	0.961	0.927	0.982	0.949	0.997	0.964	1.007	0.974
0.3	0.988	0.953	1.001	0.967	1.010	0.977	1.017	0.984
0.4	1.011	0.975	1.018	0.983	1.022	0.988	1.025	0.992
0.5	1.033	0.997	1.033	0.998	1.033	0.999	1.033	1.000
0.6	1.056	1.018	1.049	1.014	1.045	1.010	1.041	1.007
0.7	1.080	1.042	1.066	1.030	1.057	1.022	1.050	1.016
0.8	1.110	1.071	1.087	1.050	1.071	1.036	1.060	1.025
0.9	1.153	1.112	1.116	1.079	1.091	1.055	1.074	1.039
0.95	1.189	1.147	1.141	1.102	1.108	1.072	1.086	1.050
0.975	1.222	1.178	1.163	1.124	1.123	1.086	1.096	1.060
0.99	1.261	1.216	1.189	1.149	1.141	1.103	1.108	1.072
0.999	1.347	1.299	1.246	1.203	1.179	1.140	1.134	1.097
0.9999	1.424	1.374	1.294	1.251	1.211	1.171	1.156	1.118
0.99999	1.496	1.443	1.339	1.293	1.240	1.199	1.175	1.137
0.999999	1.564	1.509	1.380	1.333	1.266	1.224	1.193	1.154
Mean	1.037	1.000	1.035	1.000	1.034	1.000	1.034	1.000
Var	0.008	0.007	0.004	0.004	0.002	0.002	0.001	0.001
$\bar{n}$	2048		4096		8192		16384	
$\bar{m}$	2048		4096		8192		16384	

2. Test of Normality Assumptions

If $\bar{n}$ and $\bar{m}$ are sufficiently large, we can avoid performing the convolutions to produce

$$f_{Y_1+Y_2+\ldots+Y_{\bar{m}}} \text{ and } f_{X_1+X_2+\ldots+X_{\bar{n}}}.$$

That is, if $f_{Y_1+Y_2+\ldots+Y_{\bar{m}}}$ and $f_{X_1+X_2+\ldots+X_{\bar{n}}}$ are close to being normal distributions, we can assume that $f_{Y_1+Y_2+\ldots+Y_{\bar{m}}}$ is[11]

$$N[E(Y_1 + Y_2 + \ldots + Y_{\bar{m}}), \mathrm{Var}(Y_1 + Y_2 + \ldots + Y_{\bar{m}})]$$

and $f_{X_1+X_2+\ldots+X_{\bar{n}}}$ is

$$N[E(X_1 + X_2 + \ldots + X_{\bar{n}}), \mathrm{Var}(X_1 + X_2 + \ldots + X_{\bar{n}})]$$

and do only a single convolution for quotients; namely,

$$N\{E[(1/\bar{m}) \cdot (Y_1 + Y_2 + \ldots + Y_{\bar{m}})], \mathrm{Var}[(1/\bar{m}) \cdot (Y_1 + Y_2 + \ldots + Y_{\bar{m}})]\}/$$

$$N\{E[(1/\bar{n}) \cdot (X_1 + X_2 + \ldots + X_{\bar{n}})], \mathrm{Var}[(1/\bar{n}) \cdot (X_1 + X_2 + \ldots + X_{\bar{n}})]\}.$$

Based on the underlying adult comprehensive major medical claim samples and the generated distributions, we can draw the following conclusions for these data:

1. For resample sizes of 256 or less, the assumption of normality for distributions of average size claims may not be particularly useful; this is because such an assumption produces negative average size claim per claimant with appreciable probability.
2. From Table 4 it can be ascertained how well the assumption of normality for distributions of average size claim per claimant generates distributions of point estimates of trend for resamples of size $\bar{n} = \bar{m} = 512$.
3. Recalculating Table 4 for $\bar{m} = \bar{n} = 1{,}024$ (not shown) demonstrates that the assumption of normality for distributions of average size claim distributions for resamples of size 1,024 produces point estimates of trend distributions shown in Table 3 (for $\bar{n} = \bar{m} = 1{,}024$), to an accuracy of at least three decimal places in $1 + \hat{T}$. This does not imply that the point estimates of trend distributions themselves are normal.

[11]A good discretized version of a normal distribution can be obtained by generating a binomial distribution $b(n;p)$, where n is large and $p = 0.5$; and then a discretized version of $n(\mu,\sigma^2)$ can be obtained by performing the usual type of transformation $z = \mu + \sigma(x - np)/\sqrt{npq}$.

TABLE 4*

TREND FACTORS $(1+T')$ OBTAINED BY CONVOLUTING TWO NORMAL DISTRIBUTIONS FOR QUOTIENTS

	1983–84		1984–85		1985–86		1986–87	
Cumulative	$1+\hat{T}$	$1+\hat{T}'$	$1+\hat{T}$	$1+\hat{T}'$	$1+\hat{T}$	$1+\hat{T}'$	$1+\hat{T}$	$1+\hat{T}'$
0.000001	0.457	0.384	0.432	0.405	0.465	0.381	0.440	0.338
0.00001	0.498	0.440	0.475	0.460	0.505	0.439	0.483	0.403
0.0001	0.549	0.506	0.529	0.523	0.555	0.504	0.536	0.475
0.001	0.614	0.582	0.599	0.597	0.618	0.583	0.606	0.563
0.01	0.702	0.680	0.696	0.694	0.705	0.683	0.701	0.671
0.025	0.748	0.730	0.748	0.743	0.750	0.734	0.751	0.726
0.05	0.789	0.774	0.793	0.787	0.791	0.778	0.796	0.776
0.1	0.838	0.826	0.847	0.840	0.841	0.832	0.850	0.835
0.2	0.901	0.894	0.914	0.907	0.906	0.900	0.919	0.909
0.3	0.947	0.944	0.965	0.959	0.954	0.953	0.971	0.966
0.4	0.989	0.990	1.009	1.004	0.998	0.998	1.017	1.017
0.5	1.030	1.033	1.052	1.050	1.041	1.043	1.063	1.066
0.6	1.073	1.079	1.097	1.096	1.096	1.091	1.110	1.117
0.7	1.121	1.130	1.146	1.149	1.137	1.143	1.164	1.176
0.8	1.182	1.193	1.207	1.214	1.200	1.208	1.232	1.248
0.9	1.274	1.286	1.298	1.315	1.295	1.305	1.336	1.355
0.95	1.361	1.371	1.379	1.404	1.381	1.393	1.433	1.452
0.975	1.443	1.450	1.454	1.490	1.463	1.475	1.529	1.545
0.99	1.549	1.550	1.546	1.598	1.566	1.579	1.660	1.662
0.999	1.798	1.792	1.760	1.869	1.811	1.834	2.017	1.953
0.9999	2.035	2.038	1.959	2.155	2.042	2.097	2.350	2.259
0.99999	2.266	2.303	2.149	2.472	2.266	2.382	2.667	2.599
0.999999	2.492	2.598	2.334	2.840	2.485	2.703	3.000	2.367
Mean	1.047	1.048	1.065	1.066	1.058	1.059	1.082	1.084
Var	0.031	0.034	0.032	0.036	0.033	0.036	0.039	0.044
$\tilde{n}$	512		512		512		512	
$\tilde{m}$	512		512		512		512	

*The columns headed $1+\hat{T}$ in this table are taken from Table 3.

4. Table 3 can be used almost directly to determine the size of the resamples such that the trend distributions themselves are essentially normal; that is, whether $f_{1+\hat{T}}$ is approximately $N(E[1+\hat{T}],\text{Var}[1+\hat{T}])$ or f_{1+W} is approximately $N(E[1+W],\text{Var}[1+W])$. Of course, such normality is lacking if the median is not equal to the mean or if symmetry is lacking. If the median is close to the mean and a fair degree of symmetry exists, then we may want to compare $N(E[1+\hat{T}],\text{Var}[1+\hat{T}])$ with $f_{1+\hat{T}}$ or $N(E[1+W],\text{Var}[1+W])$ with f_{1+W} at selected cumulative probabilities, for example, 0.025, 0.05, 0.95, and 0.975.

IV. CONCLUSIONS

We started with original samples of comprehensive major medical claims per claimant, one sample for each of two calendar years. By resampling with replacement (using numerical convolutions) from the corresponding empirical distributions, we generated distributions of average size claim per claimant, in which the number of resamples was a power of 2 from 6 to 15 (that is, 64, 128, 256, 512, 1,024, 2,048, 4,096, 8,192, or 16,384). Assuming an equal number of resamples in each of two calendar years, we convoluted these latter distributions for quotients to obtain distributions of point estimates for trend in average size claim per claimant from one calendar year to the other. The results are shown in Table 3.

Table 2 presents similar distributions of resample point estimates for trend in average size claim per claimant in which the numbers of resamples in adjacent calendar years are those of the original experience during the observation period (1983 to 1987, inclusive). The distributions in Table 2 are close to normal, which is perhaps not unexpected because the numbers of claims lie in the range from 66,260 to 111,263. Standardizing these trend distributions by dividing the amounts (not the probabilities) by their respective mean values, we find a high degree of stability as we move from one pair of calendar years to another. Thus we can use the distributions in Table 2 for determining confidence intervals for trend in average size claim per claimant, when we are dealing with such large resample sizes.

We show how we might use Table 3 to estimate 95 percent confidence intervals for trend when medium-sized samples of comprehensive major medical losses per claimant are available. Of course, because the underlying experience data involve $100 deductible/20 percent coinsurance and essentially no maximum, Table 3 should be used with caution if the major medical plan deviates significantly from this. Table 3 shows considerable stability[12] in the standardized distributions of resample point estimates for trend, from one pair of calendar years to another. Thus, the distributions in Table 3 can be used for determining 95 percent confidence intervals for trend in average size claims per claimant, when resamples are medium-sized.

The numerical convolutions (for sums and quotients) used in producing the figures in Tables 2, 3, and 4 were generated using the methods described in the Appendix using $\varepsilon = 10^{-15}$ and $nax = 1000$. For any one convolution, the total of the discarded probability products did not exceed $5 \cdot 10^{-7}$; choosing a smaller value for ε would make this figure even smaller.

[12]At least where the cumulative is in the range from 0.025 to 0.975.

ACKNOWLEDGMENTS

I thank Robert Bender of Kemper National Insurance Company for his truly helpful editorial assistance. Josh Zirin, formerly of Kemper Group, designed a practical way of implementing the Von Mises calculations. Larry Hickey of PolySystems, Inc. called my attention to the Von Mises Theorem. Anne Hoban, also of Kemper National, has been helpful with the selection of notation.

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APPENDIX

UNIVARIATE GENERALIZED NUMERICAL CONVOLUTIONS

If f_X and f_Y are independent distributions of the discrete finite univariate random variables X and Y, respectively, then the distribution f_{X+Y} of the sum $W=X+Y$ is the convolution f_X+f_Y of f_X and f_Y for sums.[13]

Let f_x be expressed in element notation as

$$\begin{Bmatrix} (x1_1, p1_1) \\ \vdots \\ (x1_{n1}, p1_{n1}) \end{Bmatrix}$$

which is also expressed as

$$(x1_i, p1_i)_{i=1, 2, \ldots, n1}$$

Similarly, let f_Y be

$$(x2_j, p2_j)_{j=1, 2, \ldots, n2}.$$

[13]We are using the operation + instead of * between two distributions to indicate convolution for sums; that is, f_X+f_Y instead of f_X*f_Y. We use the notation f_X/f_Y for the convolution of f_X and f_Y for quotients X/Y.

Then $f_W = f_{X+Y} = f_X + f_Y =$

$$\left\{\begin{array}{ll} (x1_1 + x2_1, & p1_1 \cdot p2_1) \\ (x1_1 + x2_2, & p1_1 \cdot p2_2) \\ \cdot & \\ \cdot & \\ \cdot & \\ (x1_1 + x2_{n_2}, & p1_1 \cdot p2_{n_2}) \\ (x1_2 + x2_1, & p1_2 \cdot p2_1) \\ (x1_2 + x2_2, & p1_2 \cdot p2_2) \\ \cdot & \\ \cdot & \\ \cdot & \\ (x1_2 + x2_{n_2}, & p1_2 \cdot p2_{n_2}) \\ (x1_3 + x2_1, & p1_3 \cdot p2_1) \\ (x1_3 + x2_2, & p1_3 \cdot p2_2) \\ \cdot & \\ \cdot & \\ \cdot & \\ (x1_3 + x2_{n_2}, & p1_3 \cdot p2_{n_2}) \\ \cdot & \\ \cdot & \\ \cdot & \\ (x1_{n_1} + x2_1, & p1_{n_1} \cdot p2_1) \\ (x1_{n_1} + x2_2, & p1_{n_1} \cdot p2_2) \\ \cdot & \\ \cdot & \\ \cdot & \\ (x1_{n_1} + x2_{n_2}, & p1_{n_1} \cdot p2_{n_2}) \end{array}\right\} \quad \text{Set}(1)$$

which can also be expressed as[14]

$$(x1_i + x2_j, p1_i \cdot p2_j)_{i=1,2,\ldots,n \text{ and } j=1,2,\ldots,n} \, .$$

[14]For a generalized convolution of f_{X1} and f_{X2} to generate the distribution $f_{X1/X2}$ of the random variable X_1/X_2, this set would be replaced by

$$(x1_i/x2_j, p1_i \cdot p2_j)_{i=1,2,\ldots,n_1 \text{ and } j=1,2,\ldots,n_2} .$$

If n_1 and n_2 are (say) 1000, then generating this matrix would involve 10^6 lines.[15] This would be practical if we do not intend to use f_W in further convolutions. But if (for example) we want to generate the distribution $f_U = f_W + f_Z$ of $U = (X+Y) + Z$ where

$$f_Z = (x3_k, p3_k)_{k=1,2,\ldots,1000},$$

then we would be dealing with 10^9 lines. And further convolutions would become impractical, because of the amount of both computer storage and computing time required. The following algorithm has been designed to overcome these problems.

The Univariate Generalized Numerical Convolution Algorithm

Choose $\varepsilon > 0$. Typically ε is chosen to be 10^{-10} or 10^{-15}.

Loop 1:

Perform the calculations indicated in Matrix (1) above, discarding any lines for which the resulting probability is less than ε; that is, discard lines for which $p1_i \cdot p2_j < \varepsilon$. The purpose of this is to avoid underflow problems and to increase the fineness of the partitions (meshes) to be imposed.

Calculate[16,17]

$$low_x = \min\{x1_i + x2_j \neq 0 | p1_i \cdot p2_j > \varepsilon\} \text{ for } \begin{matrix} i = 1, 2, \ldots, n_1 \\ j = 1, 2, \ldots, n_2 \end{matrix}$$

[15]There may be some collapsing due to identical amounts on different lines. The number of lines produced is reduced by representing on a single line all lines with identical amounts; on that line is the amount and the sum of the original probabilities.

[16]In many applications we replace $x1_i + x2_j$ by $\log(x1_i + x2_j)$, which will allow finer subintervals at the low end of the range. Of course, to be able to use logs the range of $X+Y$ should not include values less than 1 (to avoid theoretical and numerical problems).

[17]For a generalized convolution of f_{X1} and f_{X2} to generate the distribution $f_{X1/X2}$ of the random variable X_1/X_2, these expressions would be replaced by

$$low_x = \min\{x1_i/x2_j \neq 0 \mid p1_i \cdot p2_j > \varepsilon\} \text{ for } \begin{matrix} i = 1,2,\ldots,n_1 \\ j = 1,2,\ldots,n_2 \end{matrix}$$

and

$$high_x = \max\{x1_i/x2_j \neq 0 \mid p1_i \cdot p2_j > \varepsilon\} \text{ for } \begin{matrix} i = 1,2,\ldots,n_1 \\ j = 1,2,\ldots,n_2 \end{matrix}$$

and

$$high_x = \max\{x1_i + x2_j \neq 0 | p1_i \cdot p2_j > \varepsilon\} \text{ for } {i = 1, 2, \ldots, n_1 \atop j = 1, 2, \ldots, n_2}.$$

Let *nax* be a positive integer selected for the purpose of creating the following partition: let

$$\text{delta} = \frac{high_x - low_x}{nax/2 - 1};$$

partition the interval (low_x − delta, $high_x$ + delta) into $nax/2 + 1$ subintervals: let

$$\text{delta} = \frac{high_x - low_x}{nax/2 - 1};$$

partition the interval (low_x − delta, $high_x$ + delta) into $nax/2 + 1$ subintervals:

r	Subinterval $I(r)$
1	$[0,0]$
2	$(low_x - \text{delta}, low_x)$
3	$[low_x, low_x + 1 \cdot \text{delta})$
4	$[low_x + 1 \cdot \text{delta}, low_x + 2 \cdot \text{delta})$
.	.
.	.
.	.
$nax/2 - 1$	$[low_x + (nax/2 - 3) \cdot \text{delta}, high_x - \text{delta})$
$nax/2$	$[high_x - \text{delta}, high_x)$
$nax/2 + 1$	$[high_x, high_x + \text{delta})$

Subinterval I_1 is the degenerate interval consisting of 0 alone. If for some $r_0 > 1$, $0 \in I_{r_0}$, then 0 is deleted from I_{r_0}; that is, that particular subinterval has a hole at 0.

Loop 2:

For each r ($r = 1, 2, \ldots, nax/2 + 1$) set to zero, the initial value of each of the accumulators $m_0[I(r)]$, $m_1[I(r)]$, $m_2[I(r)]$, and $m_3[I(r)]$.

For each i $(1=1, 2, \ldots, n_1)$ and j $(j=1, 2, \ldots, n_2)$ for which $x1_i+x2_j>\varepsilon$, determine the positive integer r for which $x1_i+x2_j \epsilon I(r)$ and perform the accumulations

$$\begin{aligned} m_0[I(r)] &= m_0[I(r)] + p1_i \cdot p2_j \\ m_1[I(r)] &= m_1[I(r)] + (x1_i + x2_j)^1 \cdot p1_i \cdot p2_j \\ m_2[I(r)] &= m_2[I(r)] + (x1_i + x2_j)^2 \cdot p1_i \cdot p2_j \\ m_3[I(r)] &= m_3[I(r)] + (x1_i + x2_j)^3 \cdot p1_i \cdot p2_j \end{aligned}$$

That is, we generate the probability and the 1st through 3rd moments for each mesh interval $I(r)$ $(r=1, 2, \ldots, nax/2+1)$.

Loop 3:

The Von Mises Theorem and algorithm [3] guarantee that for each r $(r=1, 2, \ldots, nax/2+1)$, there exist and we can find two pairs of real numbers[17] $[x_1(r), p_1(r)]$ and $[x_2(r), p_2(r)]$ such that $x_1(r) \epsilon I(r)$ and $x_2(r) \epsilon I(r)$ and such that the following relationships hold:

Moment	Relationship
0	$\sum_{i=1}^{2} p_i(r) = m_0[I(r)]$
1	$\sum_{i=1}^{2} x_i(r)^1 \cdot p_i(r) = m_1[I(r)]$
2	$\sum_{i=1}^{2} x_i(r)^2 \cdot p_i(r) = m_2[I(r)]$
3	$\sum_{i=1}^{2} x_i(r)^3 \cdot p_i(r) = m_3[I(r)]$

We accept the 0-th through 3rd moments and produce two points[18] and associated probabilities, with the feature that these moments are accurately retained.

Having kept accurately the 0-th through 3rd moments of $X+Y$ within each mesh interval, we have automatically kept accurately the corresponding global moments.

[18]In some cases $x_1=x_2$ and what would otherwise be two pairs $[x_1(r), p_1(r)]$ and $[x_2(r), p_2(r)]$ collapse into one pair $[x_1(r), p_1(r)+p_2(r)]$. This would happen, for example, when the values of $x1_i+x2_j$ that fall into $I(r)$ are all identical.

We can then express the full distribution f_{X+Y} of the univariate random variable $X+Y$ as

$$\left(x_k(r),\, p_k(r)\right)_{r=1,2,\,\ldots,\,nax/2+1 \text{ and } k=1,2}$$

We now describe how we actually obtain the number pairs $[x_1(r), p_1(r)]$ and $[x_2(r), p_2(r)]$ for any given value of r ($r=1, 2, \ldots, nax/2+1$). To simplify the notation somewhat in this description, we replace the symbols $m_0[I(r)]$, $m_1[I(r)]$, $m_2[I(r)]$, and $m_3[I(r)]$ by m_0, m_1, m_2, and m_3, respectively. If $m_1=0$ and $m_0 \neq 0$, then we let

$$\begin{aligned} x_1(r) &= 0 \qquad & p_1(r) &= m_0 \\ x_2(r) &= 0 \qquad & p_2(r) &= 0; \end{aligned}$$

otherwise, if $m_0 \cdot m_2 - m_1 \cdot m_1 < 10^{-10} \cdot |m_1|$, we let

$$\begin{aligned} x_1(r) &= m_1/m_0 \qquad & p_1(r) &= m_0 \\ x_2(r) &= 0 \qquad & p_2(r) &= 0; \end{aligned}$$

that is, in effect, use a single-number pair rather than two-number pairs if the variance in $I(r)$ is close to zero[19]; otherwise, perform the following calculations:

$$c_0 = \frac{m_1 \cdot m_3 - m_2 \cdot m_2}{m_0 \cdot m_2 - m_1 \cdot m_1}$$

$$c_1 = \frac{m_1 \cdot m_2 - m_0 \cdot m_3}{m_0 \cdot m_2 - m_1 \cdot m_1}$$

$$a_1 = \frac{1}{2} \cdot (-c_1 - |c_1 \cdot c_1 - 4 \cdot c_0|^{0.5})$$

$$a_2 = \frac{1}{2} \cdot (-c_1 + |c_1 \cdot c_1 - 4 \cdot c_0|^{0.5})$$

$$s_1 = \frac{m_0 \cdot a_2 - m_1}{a_2 - a_1}$$

$$s_2 = \frac{m_1 - m_0 \cdot a_1}{a_2 - a_1}$$

[19]We treat this situation differently to avoid exceeding the limits of precision of the numbers being held by the computer.

$$x_1(r) = a_1 \qquad p_1(r) = s_1$$
$$x_2(r) = a_2 \qquad p_2(r) = s_2.$$

We check that $x_1(r)$ and $x_2(r)$ both lie in $I(r)$; and if not, then if $I(r)$ is a degenerate interval (that is, consists of a single point), then we let

$$x_1(r) = m_1/m_0 \qquad p_1(r) = m_0$$
$$x_2(r) = 0 \qquad p_2(r) = 0;$$

otherwise,[20] we let

$$\sigma = |(-m_1/m_0) \cdot (m_1/m_0) + (m_2/m_0)|^{0.5}$$

$$\tau = \left| \frac{m_1/m_0 - \text{left endpoint of } I(r)}{\text{right endpoint of } I(r) - m_1/m_0} \right|$$

$$x_1(r) = -\sigma \cdot \tau^{0.5} + m_1/m_0 \qquad p_1(r) = m_0/(1 + \tau)$$

$$x_2(r) = \sigma/\tau^{0.5} + m_1/m_0 \qquad p_2(r) = p_1(r) \cdot \tau.$$

It is desirable to use double precision floating point numbers in performing these calculations; otherwise, numerical difficulties could occur.

[20]This situation occurs only when the accuracy of the numbers being held by the computer is being impaired by the fact that the computer can hold numbers to only a limited degree of precision. Because this situation occurs only when the associated probability is extremely small, the fact that not all of the first three moments are being retained in this situation is not of practical significance.

DISCUSSION OF PRECEDING PAPER

ROY GOLDMAN:

I found the author's paper informative from a theoretical point of view. It is especially helpful to know that the convolutions can be approximated by normal distributions.

Although the author applied his methodology to group major medical claims, I think that other forms of casualty coverages may be more suited to this methodology than group health coverages.

I see several problems in applying this methodology to a large block of group health business. First, how is a claim defined? Presumably, each claim transaction is not a claim. The number of transactions on traditional health business has been increasing rapidly for reasons unrelated to frequency: for example, physician claim unbundling, prescription drug submissions, hospital billings, and the like. The average severity would be distorted if transactions were used.

Therefore in order to apply the methodology, all transactions during a calendar year must be aggregated for each individual (or employee unit).

Individuals then need to be grouped by age (or adult/child), coverage, deductible, and so on. I venture to say that most claim data bases are not constructed that way for group insurance, so this type of aggregation would be expensive on a regular basis.

Even if one could obtain the trend for severity in this manner, trend must then be derived for frequency. In group insurance, what is important is the overall trend on a case (or pool) basis.

Overall trend can be obtained by comparing the yearly increase, on a case-by-case basis, of the ratio of incurred claims to employee units exposed. This methodology captures both frequency and severity and uses data that are readily available to group insurers. Means, variances, and confidence intervals can be calculated directly. Groups can, of course, be broken down by case size, industry, or any other category the actuary may want to study.

CHARLES S. FUHRER:

Mr. Bailey is to be congratulated for giving us a new way to calculate how accurate our trend estimates are. The traditional method is nonetheless still adequate.

I. The Traditional Method

Modern statistics was established during the first part of this century. Sometimes early authors in this field needed to devise methods that involved only relatively short computations. Today, we have data processing equipment, so some new methods are now available. However, these methods may not provide a worthwhile improvement in all cases.

According to the central limit theorem [6, p. 260, 9.3.1a] (the author mentions this in footnote 9), the distribution of the ratio of two sample means is asymptotically normal if the random variables have finite variances. Consequently, for large samples, the normal approximation gives reasonably accurate confidence limits. Of course, as pointed out in author's reference [2] there may be some good reasons to suspect that the variance of medical care claim data may not be finite. The author's bootstrap methodology also makes the asumption that the variance is finite. In fact, it assumes that the population's distribution is exactly the same as the sample (empirical) distribution. Thus, the author's method assumes that all moments of the distribution are finite. This is certainly possible, but is not a property of many distributions that might be used to model claim size distributions, such as the Pareto.

II. The Estimator

The calculated confidence intervals are remarkably wide. Even for samples of more than 80,000 claims per year (Table 2, 1986–1987), the 95 percent confidence interval for an estimate of 6.6 percent had a width of 5.6 percent (3.9 percent to 9.5 percent). This is so wide that the estimate becomes almost useless. To get the 95 percent confidence interval down to a more reasonable 1 percent width, about 3,000,000 claims per year are needed. Most health insurance actuaries would not have access to this volume of claim data.

Fortunately, there is a modified method for estimating the severity trend that does greatly reduce the width of the confidence intervals. This method involves ignoring claim amounts above a fixed limit point, which I will call p. This method is practiced by most health actuaries, who usually do not calculate the confidence intervals. It is actually a robust statistical technique. In [1, p. 105] it is called the Huber estimator [3] and is one of the M estimators. The paper might have been more useful if the author had calculated the confidence intervals for this modification. Most health actuaries select the value p arbitrarily. Another useful result would have been to analyze how to select the optimum value of p. There are other estimators that can be used from robust statistics. One method would smoothly give less weight to large claim amounts, instead of completely ignoring the amount over p.

Robust statistical methods were discussed at the 14th annual Actuarial Research Conference held at the University of Iowa, September 6–8, 1979. The proceedings of this conference appear in *ARCH* 1979.3; in particular, see [2] and also [4].

III. The Convolution Technique

The Appendix to the paper presents a method for calculating convolutions of discrete random variables. It is known (for example, [5]) that convolutions of many random variables can be highly inaccurate due to round-off errors. The author's method replaces an intermediate calculated distribution with a distribution that has fewer points. He does this in a somewhat clever way that ensures that the first three moments are unchanged. Unfortunately, the author does not present any evidence as to whether this method actually helps reduce the round-off error. Furthermore, it can be shown that in replacing the distribution, the fourth moments are lowered. The effect of understating the fourth moment is to underestimate the tails of a distribution. Thus, his method may underestimate the width of the confidence intervals.

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(AUTHOR'S REVIEW OF DISCUSSIONS)

WILLIAM A. BAILEY:

I thank Mr. Fuhrer and Mr. Goldman for their comments.

With reference to Mr. Goldman's comments, my methodology has proved suitable for various lines of business, including group medical and workers' compensation among others. He is correct in inferring that the claim dollars need to be aggregated for each unit (employee or dependent) or for each life on which there was a claim. We have not found it difficult or expensive to do this aggregation.

I did not build frequency of claim into my analysis, because group medical frequency data were not available to me. However, it may be better to examine frequency separately; otherwise, the distribution of number of claims may be incorporated into the analysis implicitly as a binomial distribution. The ratio of variance to mean of an empirical distribution of number of claims is likely to be on the order of a multiple of the mean. This rules out the Poisson (ratio = 1) as well as the binomial (ratio = $q<1$). Otherwise, I have no objection to looking at trends in claim costs rather than trends in severities.

With reference to Mr. Fuhrer's comments, the purposes of the paper were to:

(1) Describe an algorithm for performing univariate generalized numerical convolutions
(2) Show how such convolutions could be used to determine confidence intervals for trend
(3) Determine how large the samples need to be before the normality assumptions can safely be used
(4) Demonstrate that the confidence intervals are likely to be rather large in the absence of huge volumes of data.

Purpose (3) was achieved in Section III.A.2 for large resamples and in Section III.B.2 for medium-size resamples. These subsections help to quantify Mr. Fuhrer's "reasonably large samples" needed before "the normal approximation gives reasonably accurate confidence limits." Table 3 helps us to see how wide the intervals may be for trend estimates in which the sample sizes are smaller than this.

I leave to other investigators the question of whether the underlying distributions have infinite moments.

Whether we use classical bootstrapping (that is, resampling with replacement and with random numbers) or operational bootstrapping (that is, resampling with replacement and without random numbers) via numerical generalized convolutions, we are estimating the bootstrap distribution. In most cases, especially where we are interested in the tails of the distributions, the operational bootstrapping approach produces the better results. However, the question of the power of the bootstrap distribution itself in the estimation process has been the source of a vast statistical literature during the past decade and a half, and the discussions continue.

The bootstrap distribution also can be viewed from an entirely different perspective. The "jackknife" is a statistical method developed early in the century. It involved eliminating from the sample one observation at a time and recalculating the statistic of interest. Thus was obtained a distribution of the statistic of interest. The approach of forming resamples by excluding some of the original observations is a common approach. The bootstrap distribution expands on this idea by considering every possible resample that might be obtained from the original sample by resampling with replacement. From this point of view, the bootstrap distribution (operational or classical) might be viewed as a part of the field of descriptive statistics, valuable in its own right.

Purpose (4) was the original motivation for writing the paper. I believed that too much credibility was being given to trends observed on various blocks of medical business. I agree with Mr. Fuhrer that the confidence intervals generated without limiting the size of the claims are likely to be too wide to be useful. I also know from my own experience that underwriters and actuaries (in group health insurance and in workers' compensation) like to limit the size of the claims to give greater credibility to the observed experience (the excess over the claim limit is charged for on an expected value basis). Table A-1 shows operational bootstrap distributions comparable to those in Table 3 in the paper, but the claims are limited to $10,000 and $5,000. The new results seem to indicate that simply limiting the size of the claims may not enable us to achieve an objective such as trying to be within 0.01 of the real trend 90 percent of the time, at least where the number of claims is 16,384 or fewer.

Finally, we come to Mr. Fuhrer's questions about the convolution technique itself.

His first point concerns "round-off" errors. And I confess that, if I tried to convolute $2^{10^{15}}$ distributions together, there would be a round-off problem.

TABLE A–1

BOOTSTRAP-TYPE DISTRIBUTIONS OF TREND FACTORS $(1+\hat{T})$ FOR 1983-84
(MEDIUM-SIZED RESAMPLING BY OPERATIONAL BOOTSTRAPPING)

Percentile	$1+\hat{T}$	$1+\hat{T}$	$1+\hat{T}$	$1+\hat{T}$	$1+\hat{T}$
	Percentiles for Which Claims in Excess of $10,000 Have Been Excluded				
0.000001	0.319	0.450	0.574	0.680	0.767
0.000010	0.360	0.490	0.609	0.709	0.790
0.000100	0.411	0.539	0.651	0.744	0.816
0.001000	0.480	0.601	0.703	0.785	0.848
0.010000	0.579	0.686	0.771	0.838	0.888
0.025000	0.634	0.730	0.806	0.864	0.908
0.050000	0.685	0.771	0.837	0.888	0.925
0.100000	0.748	0.820	0.875	0.916	0.946
0.200000	0.833	0.885	0.923	0.951	0.971
0.300000	0.900	0.934	0.959	0.977	0.990
0.400000	0.961	0.978	0.991	1.000	1.006
0.500000	1.022	1.022	1.022	1.022	1.022
0.600000	1.087	1.067	1.053	1.044	1.037
0.700000	1.161	1.117	1.088	1.068	1.054
0.900000	1.396	1.272	1.193	1.140	1.104
0.900000	1.396	1.272	1.193	1.140	1.104
0.950000	1.526	1.354	1.246	1.176	1.128
0.975000	1.648	1.430	1.295	1.208	1.150
0.990000	1.804	1.523	1.354	1.246	1.175
0.995000	1.919	1.590	1.396	1.273	1.193
0.999900	2.545	1.939	1.604	1.404	1.279
0.999990	2.912	2.132	1.715	1.472	1.322
0.999999	3.283	2.323	1.820	1.535	1.362
Mean	1.053	1.037	1.029	1.025	1.023
Var	0.068	0.032	0.016	0.008	0.004
$\tilde{n}$	64	128	256	512	1024
$\tilde{m}$	64	128	256	512	1024

TABLE A-1—*Continued*

Percentile	$1 + \hat{t}$	$1 + \hat{t}$	$1 + \hat{t}$	$1 + \hat{t}$
Percentiles for Which Claims in Excess of $10,000 Have Been Excluded				
0.000001	0.834	0.885	0.923	0.951
0.000010	0.852	0.898	0.933	0.958
0.000100	0.872	0.913	0.944	0.966
0.001000	0.895	0.931	0.956	0.975
0.010000	0.925	0.952	0.972	0.986
0.025000	0.940	0.963	0.980	0.992
0.500000	0.952	0.972	0.986	0.996
0.100000	0.967	0.983	0.994	1.002
0.200000	0.986	0.996	1.003	1.009
0.300000	0.999	1.005	1.010	1.013
0.400000	1.011	1.014	1.016	1.018
0.500000	1.022	1.022	1.022	1.022
0.600000	1.033	1.029	1.027	1.025
0.700000	1.045	1.038	1.033	1.030
0.900000	1.079	1.062	1.050	1.041
0.900000	1.079	1.062	1.050	1.041
0.950000	1.096	1.073	1.058	1.047
0.975000	1.111	1.084	1.065	1.052
0.990000	1.128	1.096	1.073	1.058
0.995000	1.140	1.104	1.079	1.062
0.999900	1.197	1.143	1.106	1.080
0.999990	1.226	1.162	1.119	1.089
0.999999	1.251	1.179	1.131	1.097
Mean	1.022	1.022	1.022	1.022
Var	0.002	0.001	0.000	0.000
$\tilde{n}$	2048	4096	8192	16384
$\tilde{m}$	2048	4096	8192	16384

TABLE A-1—*Continued*

Percentile	$1 + \hat{t}$	$1 + \hat{t}$	$1 + \hat{t}$	$1 + \hat{t}$	$1 + \hat{t}$
Percentiles for Which Claims in Excess of $5,000 Have Been Excluded					
0.000001	0.386	0.516	0.632	0.728	0.804
0.000010	0.427	0.554	0.664	0.753	0.824
0.000100	0.478	0.599	0.702	0.783	0.847
0.001000	0.544	0.656	0.748	0.819	0.874
0.010000	0.637	0.732	0.808	0.865	0.908
0.025000	0.686	0.772	0.838	0.888	0.925
0.050000	0.732	0.807	0.865	0.908	0.940
0.100000	0.788	0.850	0.897	0.932	0.957
0.200000	0.861	0.905	0.938	0.962	0.978
0.300000	0.918	0.947	0.968	0.983	0.994
0.400000	0.970	0.985	0.995	1.003	1.008
0.500000	1.021	1.021	1.021	1.021	1.021
0.600000	1.074	1.058	1.047	1.039	1.034
0.700000	1.135	1.100	1.076	1.059	1.048
0.900000	1.323	1.225	1.161	1.118	1.089
0.900000	1.323	1.225	1.161	1.118	1.089
0.950000	1.424	1.291	1.205	1.147	1.109
0.975000	1.520	1.350	1.243	1.173	1.126
0.990000	1.638	1.423	1.290	1.204	1.147
0.995000	1.725	1.475	1.323	1.226	1.162
0.999900	2.186	1.741	1.486	1.330	1.231
0.999990	2.451	1.886	1.571	1.383	1.265
0.999999	2.715	2.025	1.652	1.433	1.297
Mean	1.042	1.031	1.026	1.023	1.022
Var	0.046	0.022	0.011	0.005	0.003
$\bar{n}$	64	128	256	512	1024
$\bar{m}$	64	128	256	512	1024

TABLE A-1—*Continued*

Percentile	$1+\hat{T}$	$1+\hat{T}$	$1+\hat{T}$	$1+\hat{T}$
Percentiles for Which Claims in Excess of $5,000 Have Been Excluded				
0.000001	0.862	0.906	0.938	0.962
0.000010	0.877	0.917	0.946	0.968
0.000100	0.894	0.930	0.955	0.974
0.001000	0.915	0.945	0.966	0.982
0.010000	0.940	0.963	0.979	0.991
0.025000	0.952	0.972	0.986	0.996
0.050000	0.963	0.979	0.991	1.000
0.100000	0.975	0.988	0.998	1.004
0.200000	0.991	0.999	1.006	1.010
0.300000	1.002	1.007	1.011	1.014
0.400000	1.012	1.014	1.016	1.017
0.500000	1.021	1.021	1.021	1.021
0.600000	1.030	1.027	1.025	1.024
0.700000	1.040	1.034	1.030	1.027
0.900000	1.068	1.054	1.044	1.037
0.900000	1.068	1.054	1.044	1.037
0.950000	1.082	1.064	1.051	1.042
0.975000	1.094	1.072	1.057	1.046
0.990000	1.109	1.082	1.064	1.051
0.995000	1.118	1.089	1.068	1.054
0.999900	1.165	1.121	1.090	1.069
0.999990	1.188	1.136	1.101	1.077
0.999999	1.209	1.150	1.111	1.084
Mean	1.021	1.021	1.021	1.021
Var	0.001	0.001	0.000	0.000
$\bar{n}$	2048	4096	8192	16384
$\bar{m}$	2048	4096	8192	16384

It may not be the precise type of round-off problem that Mr. Fuhrer had in mind, but if I could do that many convolutions, I expect that the probability products discarded in Loop 1 on page 35 would have accumulated to unity by the end of that series of convolutions. That is, the resulting distribution would be null, because the lines in the resulting distributions would have disappeared. Fortunately, I observe the total of the discarded probability products (at the end, but also during the series of convolutions), so I would be aware if this ever became a problem. In practice, I have done thousands upon thousands of convolutions without this ever having been a problem.

His second point concerns the possible deterioration in the fourth moment and the possibility of thereby understating the tails of the distribution. For confirmation that this is not a problem, I suggest turning to Ormsby et al. [2], specifically, the discussion of pages 1321–1327 (the comparisons are between univariate distributions calculated by (a) Monte Carlo vs. (b) numerical convolutions, using an earlier convolution algorithm) and to Bailey [1], specifically, the numerical example for the fifth and sixth bridges (the comparison is between a probability of ruin calculated (a) more or less precisely by an analytic method vs. (b) one of either of two series of univariate convolutions, using the current convolution algorithm).

Perhaps Mr. Fuhrer could use some analytical methods or the Monte Carlo method to confirm or reject the operational bootstrap distributions shown in the current paper.

Although it may be difficult or impossible to prove mathematically just how accurate are the results produced by numerical generalized convolutions, my experience has been that the univariate results are very accurate. Bivariate results, although acceptable in practice, are not likely to be as accurate as univariate results. For the bivariate case, see the numerical example for the fourth bridge in the above paper in *ARCH* 1993.1.

Tables A-2 and A-3 were used to calculate Table A-4, which shows the effect of excluding claims in excess of the upper end of the range in the X column. These latter figures are shown for comparison with the means shown in Table A-1.

REFERENCES

1. BAILEY, WILLIAM A. "Six Bridges to ψ's," *ARCH* 1993.1: 143-227.
2. ORMSBY, CHARLES A., SILLETTO, C. DAVID, SIBIGROTH, JOSEPH C., AND NICOL, WILLIAM K. "New Valuation Mortality Tables for Individual Life Insurance," *RSA* 5, no. 4 (1979): 1301–1335.

TABLE A-2

COMPREHENSIVE ADULT 1983

(1) Amount of Claim	(2) Number of Claims	(3) Frequency	(4) (1) × (3)	(5) Cumulative of (4)
127	6,500	0.098098	12.458	46.893
197	11,582	0.174796	34.435	98.912
317	10,873	0.164096	52.018	129.654
447	4,557	0.068775	30.742	156.367
549	3,224	0.048657	26.713	245.321
773	7,625	0.115077	88.954	297.758
1,119	3,105	0.046861	52.437	345.990
1,371	2,331	0.035180	48.231	441.498
1,740	3,637	0.054890	95.508	528.461
2,236	2,577	0.038892	86.963	604.135
2,734	1,834	0.027679	75.674	738.612
3,455	2,579	0.038922	134.477	794.487
4,236	874	0.013190	55.875	845.256
4,738	710	0.010715	50.769	1,026.560
6,052	1,985	0.029958	181.304	1,142.066
8,609	889	0.013417	115.506	1,263.154
12,047	666	0.010051	121.088	1,382.301
18,932	417	0.006293	119.146	1,446.068
28,743	147	0.002219	63.767	1,500.177
41,210	87	0.001313	54.109	1,529.674
57,485	34	0.000513	29.497	1,546.193
84,194	13	0.000196	16.519	1,560.395
117,628	8	0.000121	14.202	1,571.105
177,414	4	0.000060	10.710	1,577.840
223,126	2	0.000030	6.735	1,577.840
	66,260	1.000000		

TABLE A–3

COMPREHENSIVE ADULT 1984

(1) Amount of Claim	(2) Number of Claims	(3) Frequency	(4) (1) × (3)	(5) Cumulative of (4)
126	6,934	0.090219	11.368	43.920
197	12,700	0.165242	32.553	95.942
318	12,573	0.163590	52.021	127.757
448	5,458	0.071015	31.815	156.113
548	3,977	0.051745	28.357	251.490
773	9,483	0.123385	95.377	305.054
1,119	3,679	0.047868	53.564	353.610
1,368	2,728	0.035494	48.556	446.820
1,735	4,129	0.053723	93.210	532.403
2,235	2,943	0.038292	85.582	606.843
2,744	2,085	0.027128	74.440	745.107
3,475	3,058	0.039788	138.264	804.990
4,238	1,086	0.014130	59.884	860.419
4,744	898	0.011684	55.429	1,052.898
6,016	2,459	0.031994	192.479	1,165.696
8,592	1,009	0.013128	112.798	1,295.187
12,078	824	0.010721	129.491	1,413.653
19,048	478	0.006219	118.466	1,480.693
28,785	179	0.002329	67.040	1,530.986
41,563	93	0.001210	50.293	1,570.165
60,224	50	0.000651	39.179	1,583.202
83,499	12	0.000156	13.037	1,598.344
116,377	10	0.000130	15.142	1,616.034
169,947	8	0.000104	17.690	1,630.334
274,771	4	0.000052	14.300	1,630.334
	76,857	1.000000		

TABLE A-4

TREND FROM 1983 TO 1984

X	Trend from 1983 to 1984*
0–150	0.937
150–250	0.97
250–400	0.985
400–500	0.998
500–600	1.025
600–1,000	1.025
1,000–1,250	1.022
1,250–1,500	1.012
1,500–2,000	1.007
2,000–2,500	1.004
2,500–3,000	1.009
3,000–4,000	1.013
4,000–4,500	1.018
4,500–5,000	1.026
5,000–7,500	1.021
7,500–10,000	1.025
10,000–15,000	1.023
15,000–25,000	1.024
25,000–35,000	1.021
35,000–50,000	1.026
50,000–75,000	1.024
75,000–100,000	1.024
100,000–150,000	1.029
150,000–200,000	1.033
200,000–400,000	1.033

*Excluding claim amounts above the upper end of the range shown on the same line in the *X* column.