

PARAMETRIC MODELS FOR LIFE TABLES

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ABSTRACT

This paper presents a general law of mortality that is equal to a mixture of Gompertz, Weibull, Inverse-Gompertz, and Inverse-Weibull survival functions. We demonstrate that a special case of our model fits the pattern of mortality of a U.S. life table up to age 90. We investigate several loss criteria for parameter estimation including some information theoretic criteria. We also present parsimonious special cases of our general law that fit the male and female 1980 CSO mortality tables to age 90 and the male and female 1983 Table *a* to age 100. Plots of our estimates of the valuation mortality rates are almost indistinguishable from the valuation rates themselves, and so we conclude that future valuation tables can be defined with reasonable mathematical formulas. Finally, we demonstrate that modeling a law of mortality as a mixture of survival functions is equivalent to using a multiple decrement table.

1. INTRODUCTION

Research into a law of mortality has been conducted ever since Abraham De Moivre proposed the model $l_x = l_0(1 - x/\omega)$, where l_0 is the radix of the life table and ω is the terminal age of the population. Jordan [6] demonstrates that many actuarial functions are readily derived from the survival function $s(x) = l_x/l_0$, and so a parametric mathematical law of $s(x)$ can be quite useful, as Tenenbein and Vanderhoof [11] suggest. Probably the most useful parametric mathematical law was proposed by Benjamin Gompertz in which the force of mortality

$$\mu_x = Bc^x \tag{1.1}$$

was modeled as an exponential function. Gompertz's law fits observed mortality rates very well at the adult ages, and it is a good tool for comparing mortality tables, as Wetterstrand [12] demonstrated. Brillinger [3] argued that if the human body is considered as a series system of independent components, then the force of mortality may follow Gompertz's law. In fact, Brillinger argued that the distribution of the time of death can be approximated with one of the three extreme-value distribution functions. We also use extreme-value distributions for modeling the pattern of mortality.

Recently, Heligman and Pollard [5] proposed a formula that fits Australian mortality rates fairly well at all ages. This formula is

$$q_x/p_x = A^{(x+B)^c} + D \exp\{-E (\log x - \log F)^2\} + GH^x \quad (1.2)$$

where $p_x = 1 - q_x$ and q_x is the probability that a life aged x will die within a year. The Heligman-Pollard model is an eight-parameter model containing three terms, each representing a distinct component of mortality. The last term reflects the exponential pattern of mortality at the adult ages, while the first term reflects the fall in mortality during childhood. The middle term models the hump at age 23 that is found in many mortality tables. Tenenbein and Vanderhoof [11] showed that this pattern of mortality is evident in tables other than the Australian tables. Following the example of Heligman and Pollard, we present a special case of our general law that reflects the exponential pattern at the adult ages, the fall of mortality at the childhood years, and the hump at age 23. Heligman and Pollard also state that (1.2) has relatively few parameters and that all the parameters have a demographic interpretation. However, they caution that interpreting the parameters is difficult when (1.2) is generalized. We also strive for parsimonious parametric models. Moreover, all our model parameters have an easy demographic and statistical interpretation regardless of the number of parameters.

Both Brillinger [3] and Heligman and Pollard [5] give general formulas for laws of mortality. Brillinger does not support any of the possible models with empirical evidence, while Heligman and Pollard only support (1.2) with empirical evidence. Often mortality tables do not exhibit the classical pattern suggested by (1.2). A typical example is the male 1983 Table *a*. We present a general parametric law of mortality that is equal to a mixture of extreme-value survival functions. Moreover, we support various special cases of our model by fitting them to population and valuation tables. We show that one version of our general model fits the pattern of mortality in a life table for the U.S. population up to age 90. In this case we also show that our model has a smaller loss than the Heligman-Pollard model. We also fit our formulas to the male and female 1980 CSO tables and to the male and female 1983 Table *a*. Based on the performance of our formulas, we conclude that valuation mortality rates in the future can be defined with a reasonable mathematical formula. Finally, we demonstrate that modeling a law of mortality as a mixture of survival functions is equivalent to using a multiple decrement model.

2. MIXTURES OF PARAMETRIC SURVIVAL FUNCTIONS

In this section we present a general law of mortality that is equal to a mixture of Gompertz, Weibull, Inverse-Gompertz, and Inverse-Weibull survival functions, and we give a heuristic justification of this model. Let ψ_k for $k = 1, \dots, m$ be the probability that a new life must die from cause k and let $s_k(x)$ be the probability of surviving to age x given that a life must die from cause k . Then the survival function $s(x)$ can be expressed as a mixture of $s_1(x), \dots, s_m(x)$. That is, the probability of living to age $x > 0$ is

$$s(x) = \sum_{k=1}^m \psi_k s_k(x). \quad (2.1)$$

In the Appendix we demonstrate that a mixture of survival functions such as (2.1) is equivalent to a multiple decrement model. This implies that the survival function associated with (1.2) can be represented as (2.1). Moreover, the survival function associated with the general form of μ_x as given in Brillinger [3] can be represented as (2.1).

We now argue that extreme-value survival functions are reasonable models for $s_k(x)$. In the following discussion we assume that the causes of death can be categorized according to childhood causes, teenage causes and adult causes. The discussion also draws extensively from the theory of reliability as presented in Barlow and Proschan [1].

At the adult ages the law of mortality follows a Gompertz law. Using reliability theory, we can argue that this is a reasonable model. Suppose that a human body can be represented as a series system of independent and identically distributed components. In this system, the first failure of a component results in death, and the time of death is approximately distributed as an extreme-value distribution that can take three different forms. One of these extreme-value distributions is the Gumbel distribution for minima, which is approximately a Gompertz distribution. Another of these extreme-value distributions for minima is the Weibull distribution. For certain parameter values the Weibull has a decreasing force of mortality, and so it seems that this may be a plausible model for early childhood where mortality rates are decreasing. Barlow and Proschan [1] point out that the third extreme-value distribution for minima is not very useful for modeling lifetimes, and we concur with this opinion.

Some other extreme-value distributions are the Inverse-Weibull distribution and the Gumbel distribution for maxima, which is approximately an Inverse-Gompertz distribution. Later we give empirical evidence that these

distributions are plausible models for the teenage component of our model. These distributions arise in a parallel system of independent and identically distributed components. This system fails when all components fail and the approximate distribution of the time of failure can take three forms, including the Gumbel distribution for maxima and the Inverse-Weibull distribution. The third extreme-value distribution is not appropriate for modeling lifetimes.

We now present the survival functions for the Gompertz, Weibull, Inverse-Compertz, and Inverse-Weibull models. We also give a parametrization of these models that reveals the location and dispersion of the distribution.

A. The Gompertz Model

Consider the Gompertz law shown in (1.1). Instead of using the standard parametrization $\mu_x = Bc^x$, we prefer to use the informative parametrization

$$\mu_x = \frac{1}{\sigma} \exp\left\{\frac{x - m}{\sigma}\right\} \quad (2.2)$$

where $c = \exp(1/\sigma)$ and $B = \exp(-m/\sigma)/\sigma$. Tenenbein and Vanderhoof [11] state that in many human populations $B > 0$ will increase whenever $c > 0$ decreases. This property is readily explained with our parametrization whenever $m > \sigma$. In this parametrization $m > 0$ is a measure of location because it is the mode of the density, while $\sigma > 0$ represents the dispersion of the density about the mode. We demonstrate both facts. The Gompertz survival function is

$$s(x) = \exp\{e^{-m/\sigma} - e^{(x-m)/\sigma}\}, \quad (2.3)$$

and the density is

$$f(x) = \frac{1}{\sigma} \exp\left\{\frac{x-m}{\sigma} + e^{-m/\sigma} - e^{(x-m)/\sigma}\right\}. \quad (2.4)$$

Using the inequality $e^y \geq 1 + y$, we deduce that

$$e^{-m/\sigma} - 1 \geq e^{-m/\sigma} - e^{(x-m)/\sigma} + (x-m)/\sigma$$

and that

$$f(m) \geq f(x) \quad \forall x > 0.$$

This demonstrates that m is the mode. The parameter σ is a measure of dispersion about m because

$$\lim_{\sigma \rightarrow 0} s(m - \epsilon) - s(m + \epsilon) = 1 \quad (2.5)$$

for an arbitrary $\epsilon > 0$. This also demonstrates that m is a measure of location because all the mass concentrates about m when σ is small.

B. The Inverse-Gompertz Model

A model that is closely associated with Gompertz's law is the Inverse-Gompertz, where the survival function is

$$s(x) = (1 - \exp\{-e^{-(x-m)/\sigma}\}) / (1 - \exp\{-e^{m/\sigma}\}), \quad (2.6)$$

the density is

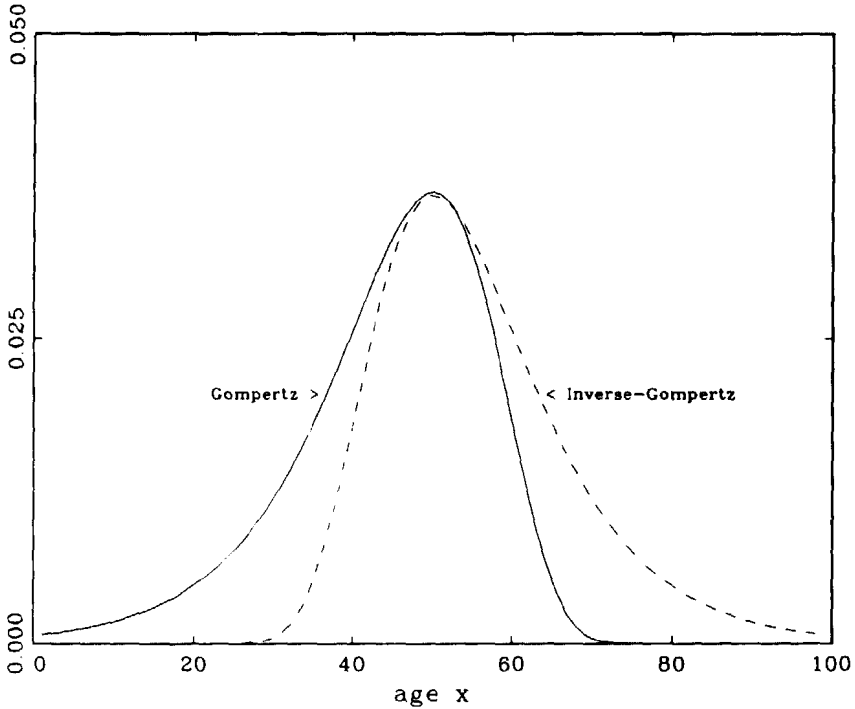
$$f(x) = \frac{1}{\sigma} \exp\left\{-\frac{x-m}{\sigma} - e^{-(x-m)/\sigma}\right\} / \left(1 - \exp\{-e^{m/\sigma}\}\right), \quad (2.7)$$

and the force of mortality is

$$\mu_x = \frac{1}{\sigma} \exp\left\{-\frac{x-m}{\sigma}\right\} / \left(\exp\{e^{-(x-m)/\sigma}\} - 1\right). \quad (2.8)$$

In this parametrization $m > 0$ is a measure of location because it is the mode of the density, while $\sigma > 0$ represents the dispersion of the density about the mode because (2.5) holds. To understand the difference between the Gompertz and Inverse-Gompertz models, we present Figure 1, in which the densities are plotted with the parameter values $m = 50$ and $\sigma = 10$. In both cases the densities are unimodal, and they peak at age 50. Note that the Gompertz is skewed to the left, while the Inverse-Gompertz is skewed to the right. Also, note that the Inverse-Gompertz density is simply equal to the Gompertz density reflected around m . All the plots in this paper were created with the computer language GAUSS.

FIGURE 1
A PLOT OF SOME GOMPERTZ AND INVERSE-GOMPERTZ DENSITIES



C. The Weibull Model

Another survival function is the Weibull, which can be parametrized as follows

$$s(x) = \exp \left\{ - \left(\frac{x}{m} \right)^{m/\sigma} \right\}. \quad (2.9)$$

In this parametrization, $m > 0$ is a location parameter and $\sigma > 0$ is a dispersion parameter because (2.5) holds. The density function is

$$f(x) = \frac{1}{\sigma} \left(\frac{x}{m} \right)^{\frac{m}{\sigma}-1} \exp \left\{ - \left(\frac{x}{m} \right)^{m/\sigma} \right\} \quad (2.10)$$

and the force of mortality is

$$\mu_x = \frac{1}{\sigma} \left(\frac{x}{m} \right)^{\frac{m}{\sigma}-1}. \quad (2.11)$$

Note that if $\sigma \geq m$, then the mode of the density is 0 and μ_x is a non-increasing function of x , while if $\sigma < m$, then the mode is greater than 0 and μ_x is an increasing function.

D. The Inverse-Weibull Model

A model that is closely associated with Weibull's law is the Inverse-Weibull, in which the survival function is

$$s(x) = 1 - \exp \left\{ - \left(\frac{x}{m} \right)^{-m/\sigma} \right\}, \quad (2.12)$$

the density function is

$$f(x) = \frac{1}{\sigma} \left(\frac{x}{m} \right)^{-\frac{m}{\sigma}-1} \exp \left\{ - \left(\frac{x}{m} \right)^{-m/\sigma} \right\}, \quad (2.13)$$

and the force of mortality is

$$\mu_x = \frac{1}{\sigma} \left(\frac{x}{m} \right)^{-\frac{m}{\sigma}-1} / \left(\exp \left\{ \left(\frac{x}{m} \right)^{-m/\sigma} \right\} - 1 \right). \quad (2.14)$$

In this model $m > 0$ is a measure of location and $\sigma > 0$ is a measure of dispersion because (2.5) holds. The Inverse-Weibull proves useful for modeling the teenage years, because the logarithm of (2.14) is a very concave function. Another model that has a similar logarithmic force is the Transformed Normal, in which the force of mortality is equal to

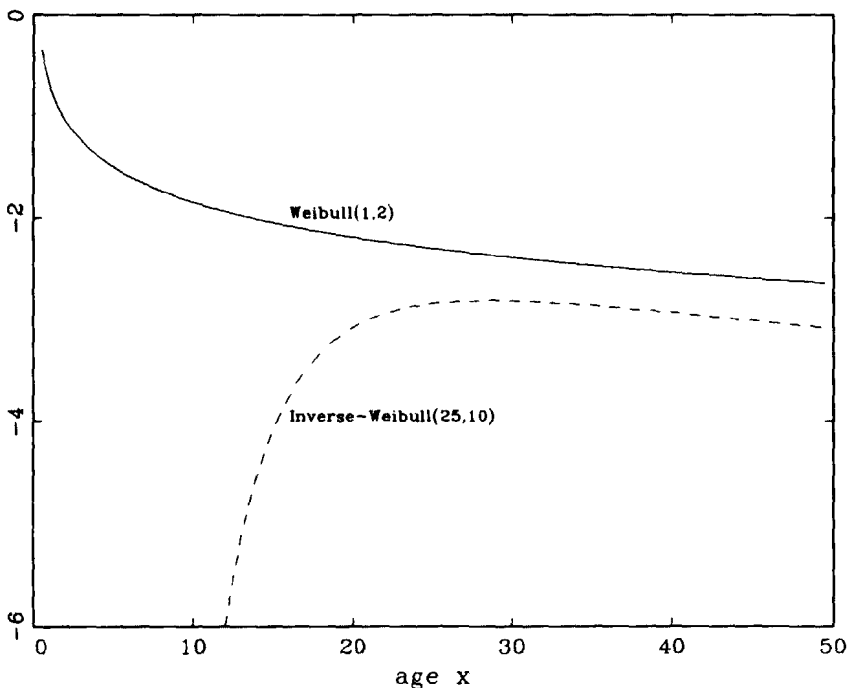
$$\mu_x = \frac{m^2}{\sigma^2} \frac{\phi \left(\frac{m}{\sigma} (1 - m/x) \right)}{\Phi \left(\frac{m}{\sigma} \right) - \Phi \left(\frac{m}{\sigma} (1 - m/x) \right)} \quad (2.15)$$

where

$$\Phi(t) = \int_{-\infty}^t \phi(z) dz \quad \text{and} \quad \phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2).$$

To understand the different shapes that $\log(\mu_x)$ may assume for the Weibull and Inverse-Weibull models, we present Figure 2. Note that $\log(\mu_x)$ is a decreasing function for the Weibull when $m=1$ and $\sigma=2$ and that the function is very concave for the Inverse-Weibull when $m=25$ and $\sigma=10$. Later we show that a mixture of a Weibull and an Inverse-Weibull fits the pattern of mortality at the childhood and teenage years. Including a Gompertz component yields a model that fits all ages.

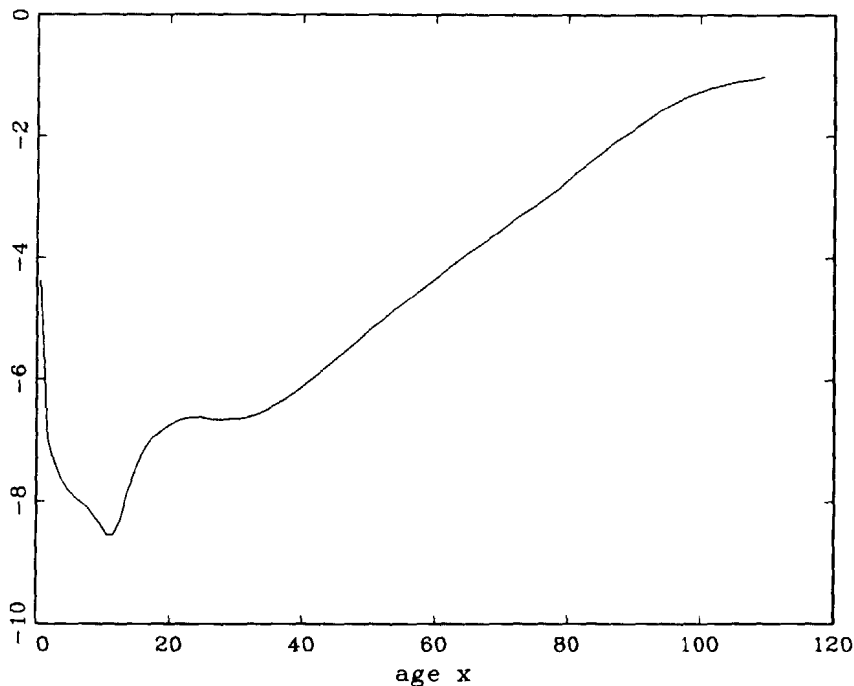
FIGURE 2
A PLOT OF $\log(\mu_x)$ FROM THE WEIBULL AND INVERSE-WEIBULL MODELS



3. APPLICATION TO A UNITED STATES LIFE TABLE

In this section, we fit a special case of (2.1) to a life table for the total population of the U.S., and we investigate several loss criteria for parameter estimation. The U.S. Decennial Life Tables for 1979–81 were prepared by the National Center for Health Statistics [8]. The pattern of mortality can be described by the function $\log(q_x)$. Figure 3 illustrates $\log(q_x)$ for the total U.S. population at the ages $x=0, \dots, 109$. This graph is very informative because we can immediately identify at least three components to the pattern of mortality. First, $\log(q_x)$ is decreasing at the early ages. Second, there appears to be a hump at around age 23. Third, there is a linear component at the adult ages.

FIGURE 3

A PLOT OF $\log(q_x)$ FROM A LIFE TABLE OF THE TOTAL U.S. POPULATION

Carefully examining Figure 3, we notice that the linear component extends to age 94 where it becomes nonlinear for the ages 95, ..., 109. Examining the methodology used by the National Center for Health Statistics [1], we find that experience from the Social Security Medicare program was used to construct the table at these later ages. We believe that this experience is not representative of the total U.S. population, and so the nonlinear component from 95, ..., 109 is suspect. Furthermore, the experience from the Medicare program was blended with the population mortality rates at the ages 85, ..., 94. Therefore we exclude mortality rates above age 90 when modeling this U.S. life table.

The pattern of mortality in Figure 3 suggests that a mixture of a Weibull, an Inverse-Weibull and a Gompertz may be reasonable. In this case a survival function would have the form

$$s(x) = \psi_1 s_1(x) + \psi_2 s_2(x) + \psi_3 s_3(x) \quad (3.1a)$$

where

$$\psi_3 = 1 - \psi_1 - \psi_2,$$

$$s_1(x) = \exp\left\{-\left(\frac{x}{m_1}\right)^{m_1/\sigma_1}\right\}, \quad (3.1b)$$

$$s_2(x) = 1 - \exp\left\{-\left(\frac{x}{m_2}\right)^{-m_2/\sigma_2}\right\}, \quad (3.1c)$$

$$s_3(x) = \exp\{e^{-m_3/\sigma_3} - e^{(x-m_3)/\sigma_3}\}. \quad (3.1d)$$

Not that this is an eight-parameter model just like (1.2), but unlike (1.2) the parameters have a clear interpretation. For instance, ψ_1 is the probability that a new life will die from childhood causes; ψ_2 is the probability of dying from teenage causes; and ψ_3 is the probability of dying from adult causes. Moreover, the location and scale parameters provide some insightful statistical information.

To estimate the parameters ψ_k , m_k , σ_k for $k=1, 2, 3$, we investigated eight loss functions. All parameter estimates were calculated by a statistical computer program called SYSTAT. This system estimated the parameters by minimizing

$$\sum_{x=0}^{90} \left(1 - \hat{q}_x/q_x\right)^2, \quad (3.2a)$$

$$\sum_{x=0}^{90} \left\{ \log[\log(1 - \hat{q}_x)/\log(1 - q_x)] \right\}^2, \quad (3.2b)$$

$$\sum_{x=0}^{90} (q_x - \hat{q}_x)^2/q_x, \quad (3.2c)$$

$$\sum_{x=0}^{90} (q_x - \hat{q}_x) \log(q_x/\hat{q}_x), \quad (3.2d)$$

$$\sum_{x=0}^{90} \left(1 - \hat{d}_x/d_x \right)^2, \quad (3.2e)$$

$$\sum_{x=0}^{90} \left[\log(\hat{d}_x/d_x) \right]^2, \quad (3.2f)$$

$$\sum_{x=0}^{90} (d_x - \hat{d}_x)^2/d_x, \quad (3.2g)$$

$$\sum_{x=0}^{90} (d_x - \hat{d}_x) \log(d_x/\hat{d}_x). \quad (3.2h)$$

In these formulas q_x is a U.S. mortality rate and $d_x = s(x)q_x$ is the probability that a new life will die between the ages of x and $x + 1$. Estimates of q_x and d_x are

$$\hat{q}_x = 1 - \hat{s}(x + 1)/\hat{s}(x) \quad (3.3a)$$

and

$$\hat{d}_x = \hat{s}(x) - \hat{s}(x + 1) \quad (3.3b)$$

where $\hat{s}(x)$ is equal to (3.1a-d) evaluated at the estimated parameters.

Note that four of the loss functions are based on q_x , while the other four are based on d_x . The loss function in (3.2a) was used by Heligman and Pollard [5], while the function in (3.2b) was used by Tenenbein and Vanderhoof [11]. The loss functions (3.2d) and (3.2h) are based on Kullback's divergence measure, which is recommended by Brockett [4] because of its information theoretic interpretation. Parameter estimates for the eight loss functions are given in Table 1.

TABLE 1
PARAMETER ESTIMATES UNDER VARIOUS LOSS CRITERIA

Parameter	Equation							
	3.2a	3.2b	3.2c	3.2d	3.2e	3.2f	3.2g	3.2h
ψ_1	0.01632	0.01663	0.01641	0.01659	0.01665	0.01701	0.01624	0.01647
m_1	0.3107	0.3470	0.2481	0.2650	0.3293	0.3716	0.2325	0.2524
σ_1	1.127	1.289	0.9572	1.038	1.239	1.433	0.9000	1.006
ψ_2	0.01385	0.01356	0.01170	0.01172	0.01432	0.01408	0.01068	0.01070
m_2	22.12	22.06	21.49	21.51	22.26	22.21	21.17	21.19
σ_2	6.455	6.378	5.745	5.737	6.606	6.546	5.283	5.302
ψ_3	0.96983	0.96981	0.97189	0.97169	0.96903	0.96891	0.97308	0.97283
m_3	82.31	82.30	82.34	82.33	82.04	82.01	82.43	82.42
σ_3	11.40	11.44	11.68	11.68	11.26	11.28	11.81	11.81
Loss	0.495	0.464	0.00126	0.00125	0.473	0.441	0.00081	0.00080
Pollard	0.623	0.554	0.00185	0.00185	—	—	—	—

For an example of a fitted curve, see Figure 4 where $\log(q_x)$ and $\log(\hat{q}_x)$ are plotted for $x=0, \dots, 90$. This plot uses the parameter estimates given in column 3.2a of Table 1. Note that the estimated mortality rates fit the actual pattern fairly well, although the fit at the earlier ages could be better. Also, note that the parameter estimates are essentially the same for all the loss functions. Due to the similarity in the parameter estimates, we simply use (3.2a) for estimating parameters in the later models. Note that the parameters reveal some interesting information. The location parameters m_1 , m_2 , and m_3 clearly show that the Weibull component models the early ages; the Inverse-Weibull component models the ages from 10 to 35; and the Gompertz component models the late ages. The parameter ψ_3 reveals that the Gompertz component explains most of the deaths in the U.S. population.

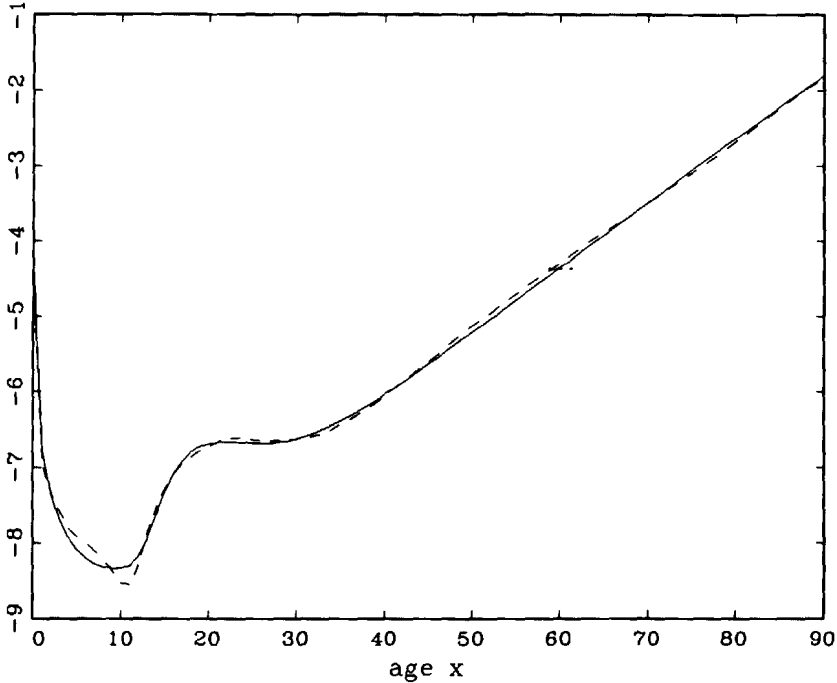
Let us compare the Heligman-Pollard model to our model. We estimated the parameters A, B, C, D, E, F, G , and H with the loss functions (3.2a-d). Finding estimates with (3.2e-h) is impractical because of the complexity that $s(x)$ would assume with model (1.2). Estimates based on (3.2a) are

$$A = 0.001095, B = 0.04413, C = 0.1412, D = 0.0008865, \\ E = 9.442, F = 21.24, G = 0.00006869, H = 1.092.$$

These values are similar to the estimates that Heligman and Pollard [5] calculated for the Australian national mortality. The last row in Table 1 shows the loss for (3.2a-d) when formula (1.2) is used to estimate q_x . Even though our model had a smaller loss for each of these four cases, there is practically no difference.

FIGURE 4

A PLOT OF $\log(q_x)$ FROM A U.S. LIFE TABLE AND OF $\log(\hat{q}_x)$ USING FORMULAS (3.1a-d)
 (THE U.S. VALUES ARE GIVEN AS A DASHED LINE)

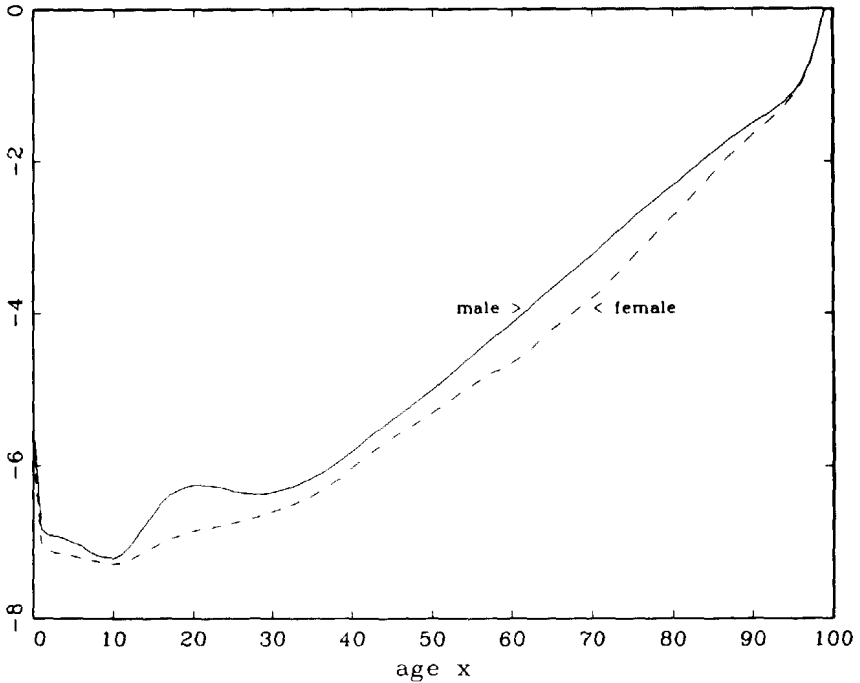


4. APPLICATION TO THE 1980 CSO TABLES

In this section, we fit special cases of (2.1) to the male and female 1980 CSO tables found in [10]. The pattern of mortality for the male and female tables can be described by the function $\log(q_x)$, which is plotted in Figure 5 at the ages of $x=0, \dots, 99$. Note that the pattern of mortality for the male 1980 CSO table is similar to the pattern for the total U.S. population. Also, note that the female pattern is very different from the male pattern. To estimate the parameters in this section, we minimize

$$\sum_{x=0}^{90} (1 - \hat{q}_x/q_x)^2 \quad (4.1)$$

FIGURE 5

A PLOT OF $\text{LOG}(q_x)$ FROM THE MALE AND FEMALE 1980 CSO TABLES

where q_x is a 1980 CSO mortality rate and \hat{q}_x is an estimated rate. The CSO valuation table gives rates for $x=0, \dots, 99$, but our loss function excludes $x=91, \dots, 99$. We believe that CSO rates at these ages are unreliable because at ages 85 and over they were graded smoothly into a rate of 1.0 at age 99. This property is evident in Figure 5.

We used a mixture of a Weibull, an Inverse-Gompertz and a Gompertz to model the male 1980 CSO table. This eight-parameter model has the form

$$s(x) = \sum_{k=1}^3 \psi_k s_k(x) \quad (4.2)$$

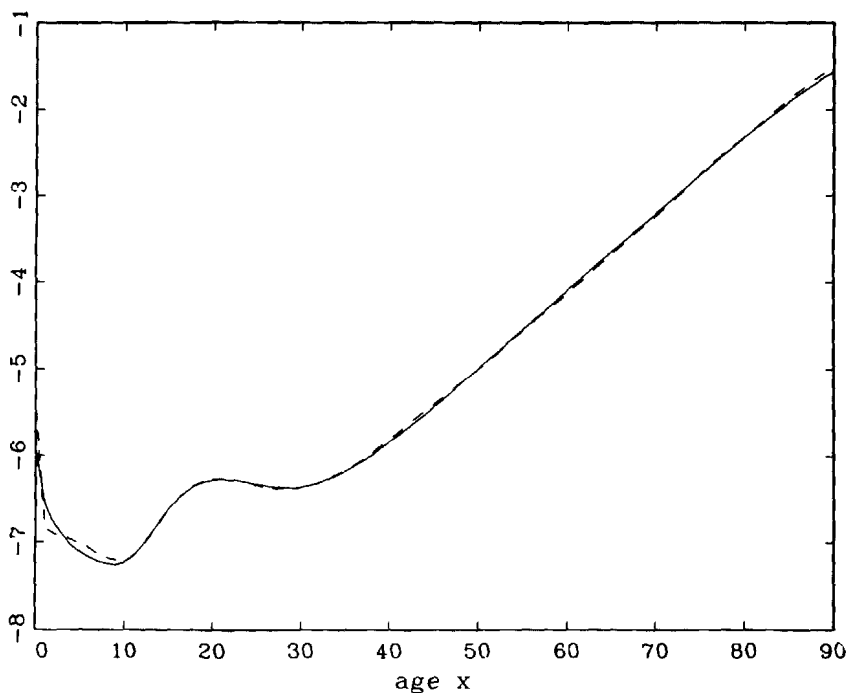
where $s_1(x)$ is equal to (2.9), $s_2(x)$ is equal to (2.6), and $s_3(x)$ is equal to (2.3). To estimate the parameters ψ_k, m_k, σ_k for $k=1, 2, 3$, we minimized the loss function in (4.1). Estimates of the parameters are

$$\begin{aligned} \psi_1 &= 0.03170, & m_1 &= 49.05, & \sigma_1 &= 77.55, \\ \psi_2 &= 0.01721, & m_2 &= 20.39, & \sigma_2 &= 5.656, \\ \psi_3 &= 0.95109, & m_3 &= 78.97, & \sigma_3 &= 10.89. \end{aligned}$$

With $\psi_3=0.95109$ we find that most deaths are due to a Gompertz component. Figure 6 plots $\log(q_x)$ and $\log(\hat{q}_x)$ for $x=0, \dots, 90$. The estimated mortality rates fit the actual pattern very well.

FIGURE 6

A PLOT OF $\log(q_x)$ FROM THE MALE CSO TABLE AND OF $\log(\hat{q}_x)$ USING FORMULA (4.2)
(THE CSO VALUES ARE GIVEN AS A DASHED LINE)



For the female 1980 CSO table we used a mixture of two Weibull and two Gompertz components. This eleven-parameter model has the form

$$s(x) = \sum_{k=1}^4 \psi_k s_k(x) \quad (4.3)$$

where $s_1(x)$ and $s_2(x)$ are equal to (2.9) and where $s_3(x)$ and $s_4(x)$ are equal to (2.3). To estimate the parameters ψ_k, m_k, σ_k for $k=1, 2, 3, 4$, we minimized the loss function in (4.1). Estimates of the parameters are

$$\begin{array}{lll} \psi_1 = 0.007797, & m_1 = 5.922, & \sigma_1 = 12.36, \\ \psi_2 = 0.04466, & m_2 = 47.00, & \sigma_2 = 28.92, \\ \psi_3 = 0.05913, & m_3 = 55.97, & \sigma_3 = 9.029, \\ \psi_4 = 0.888413, & m_4 = 84.87, & \sigma_4 = 8.777. \end{array}$$

With $\psi_4=0.888413$, we find that most deaths are due to a Gompertz component. Figure 7 plots $\log(q_x)$ and $\log(\hat{q}_x)$ for $x=0, \dots, 90$. The estimated mortality rates fit the actual pattern very well. Note that $\psi_1 \leq \psi_i$ for $i=1, 2, 3, 4$, and so we investigated a model without the first Weibull component. The resulting eight-parameter model had a loss of 0.49, which was substantially greater than the loss 0.17 for the eleven-parameter model. In our opinion the eleven-parameter model fit the pattern of mortality much better than the reduced model.

5. APPLICATION TO THE MALE AND FEMALE 1983 TABLE *a*

In this section, we present some other special cases of (2.1) and fit them to the male and female 1983 Table *a* found in [9]. Figure 8 depicts the pattern of mortality for this valuation table. This graph plots the function $\log(q_x)$ for $x=5, \dots, 115$.

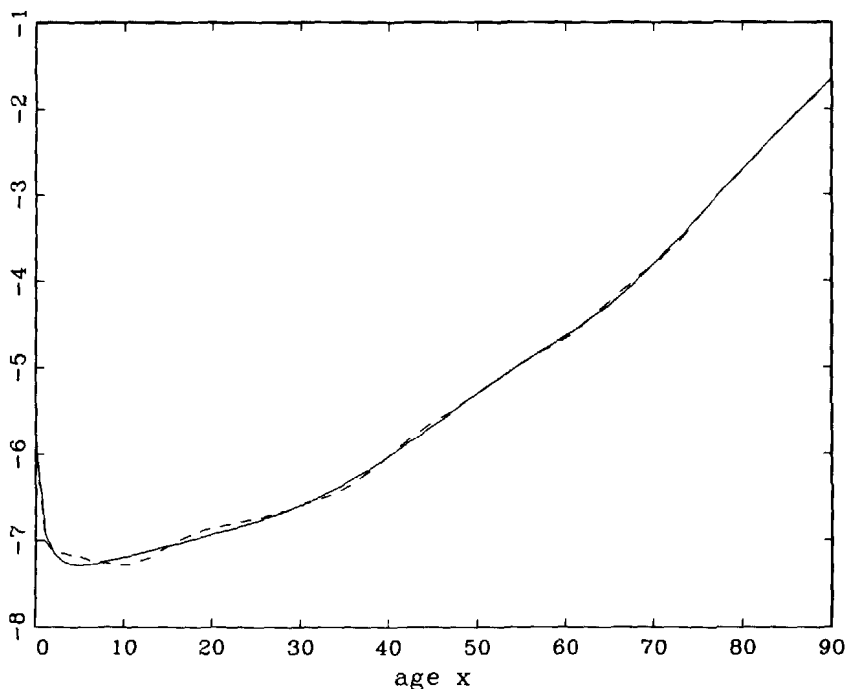
To estimate the parameters in this section, we minimize

$$\sum_{x=5}^{100} (1 - \hat{q}_x/q_x)^2 \quad (5.1)$$

where q_x is a 1983 Table *a* mortality rate and \hat{q}_x is an estimated rate. Note that this valuation table gives rates for $x=5, \dots, 115$, but our loss function in (5.1) excludes $x=101, \dots, 115$. We believe that the rates from 101 to 115 are unreliable because the rates at ages 97 and over were adjusted to grade smoothly into a rate of 1.0 at age 115.

FIGURE 7

A PLOT OF $\log(q_x)$ FROM THE FEMALE CSO TABLE AND OF $\log(\hat{q}_x)$ USING FORMULA (4.3)
(THE CSO VALUES ARE GIVEN AS A DASHED LINE)

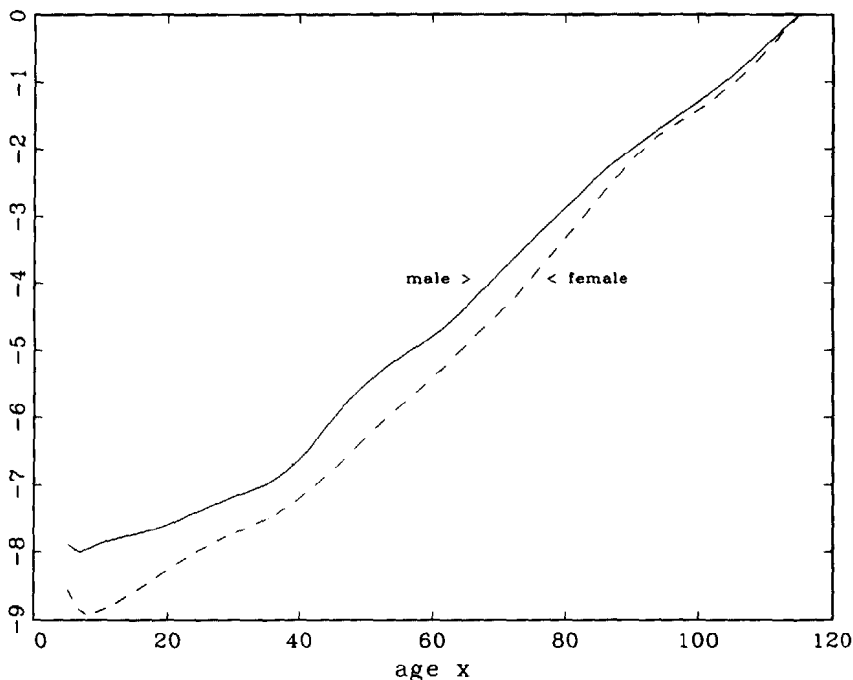


The eight-parameter model that we used for the male Table a is different from that used for the U.S. life table and the male CSO table. It is equal to

$$s(x) = \sum_{k=1}^3 \psi_k s_k(x) \quad (5.2)$$

where $s_1(x)$ and $s_3(x)$ are equal to (2.3) and $s_2(x)$ is equal to (2.9). Note that this is a mixture of a Weibull and two Gompertz components. To estimate the parameters ψ_k, m_k, σ_k for $k=1, 2, 3$, we minimized the loss function in (5.1). Estimates of the parameters are

FIGURE 8
A PLOT OF $\log(q_x)$ FROM THE MALE AND FEMALE 1983 TABLE *a*



$$\begin{array}{lll} \psi_1 = 0.01077, & m_1 = 16.82, & \sigma_1 = 18.07, \\ \psi_2 = 0.008842, & m_2 = 53.28, & \sigma_2 = 3.469, \\ \psi_3 = 0.980388, & m_3 = 86.20, & \sigma_3 = 10.65. \end{array}$$

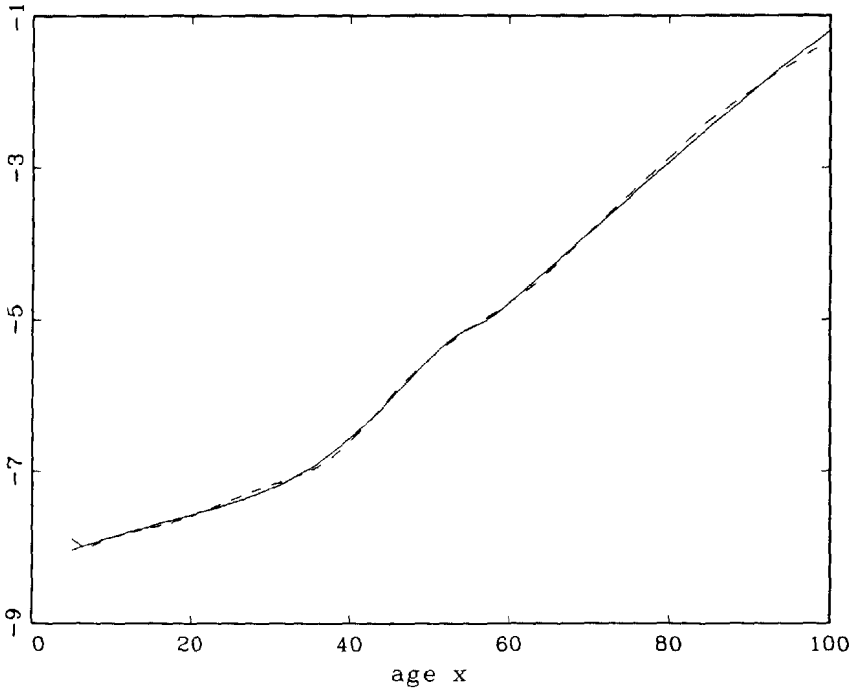
With $\psi_3 = 0.980388$, we find that most deaths are due to a Gompertz component. Figure 9 plots $\log(q_x)$ and $\log(\hat{q}_x)$ for $x = 5, \dots, 100$. The estimated mortality rates fit the actual pattern very well.

For the female 1983 Table *a*, we used a mixture of two Weibull and two Gompertz components. This eleven-parameter model was also used on the female 1980 CSO, and it has the form

$$s(x) = \sum_{k=1}^4 \psi_k s_k(x) \quad (5.3)$$

FIGURE 9

A PLOT OF $\text{LOG}(q_x)$ FROM THE MALE TABLE a AND OF $\text{LOG}(\hat{q}_x)$ USING FORMULA (5.2)
(THE TABLE a VALUES ARE GIVEN AS A DASHED LINE)



where $s_1(x)$ and $s_2(x)$ are equal to (2.9) and where $s_3(x)$ and $s_4(x)$ are equal to (2.3). To estimate the parameters ψ_k, m_k, σ_k for $k=1, 2, 3, 4$, we minimized the loss function in (5.1). Estimates of the parameters are

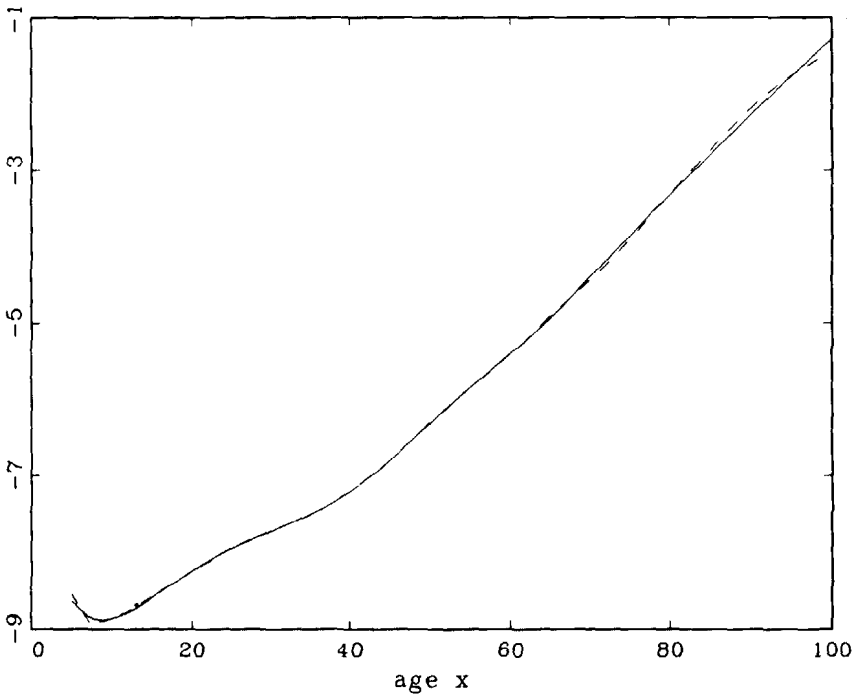
$$\begin{array}{lll} \psi_1 = 0.01473, & m_1 = 0.3388, & \sigma_1 = 1.904, \\ \psi_2 = 0.006268, & m_2 = 33.30, & \sigma_2 = 10.52, \\ \psi_3 = 0.008959, & m_3 = 55.76, & \sigma_3 = 6.670. \\ \psi_4 = 0.970043, & m_4 = 90.46, & \sigma_4 = 9.128. \end{array}$$

With $\psi_3=0.970043$, we find that most deaths are due to a Gompertz component. Figure 10 plots $\text{log}(q_x)$ and $\text{log}(\hat{q}_x)$ for $x=5, \dots, 100$. The estimated rates fit the actual pattern very well. Note that $\psi_2 \leq \psi_i$ for $i=1, 2, 3, 4$, and so we investigated a model without the second Weibull component. The

resulting eight-parameter model had a loss of 0.35, which was somewhat greater than the loss of 0.15 for the eleven-parameter model. In our opinion the eleven-parameter model fit the pattern of mortality much better than the reduced model.

FIGURE 10

A PLOT OF $\text{LOG}(q_x)$ FROM THE FEMALE TABLE a AND OF $\text{LOG}(\hat{q}_x)$ USING FORMULA (5.3)
(THE TABLE a VALUES ARE GIVEN AS A DASHED LINE)



6. CONCLUSION

In this paper we presented a general law of mortality that is a mixture of Gompertz, Weibull, Inverse-Gompertz, and Inverse-Weibull survival functions. We also presented parsimonious special cases of this general law that

fit a U.S. population table and various valuation mortality tables. We presented a parametrization of our law that emphasizes demographic and statistical information. Using this parametrization, we demonstrated that the Gompertz law explains most of the patterns of mortality in all the analyzed tables. Moreover, in the Appendix we demonstrate that our model is equivalent to a parametric multiple decrement model.

In the derivation of the 1983 Table *a* [9], an attempt was made to define the table in terms of a reasonable mathematical formula, but the effort was reluctantly abandoned. We developed some reasonable formulas for the male and female 1980 CSO tables and the male and female 1983 Table *a* that perform well up to very high ages.

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APPENDIX

EQUIVALENCE OF A MIXTURE AND A MULTIPLE DECREMENT MODEL

In this appendix we demonstrate that a mixture of survival functions is equivalent to a multiple decrement model. The discussion about multiple decrement models draws extensively from Jordan [6] and Bowers et al. [2]. Under the Balducci hypothesis $\mu_x = q_x/p_x$, and so the Heligman-Pollard model in (1.2) can be interpreted as a force of mortality. Actually we can interpret (1.2) as a total force of decrement that is equal to the sum of three forces of decrement from different causes. That is, (1.2) is equal to the sum of

$$\begin{aligned}\mu_x^{(1)} &= A^{(x+B)^c}, \\ \mu_x^{(2)} &= D \exp \{-E(\log x - \log F)^2\}, \\ \mu_x^{(3)} &= GH^x.\end{aligned}$$

and so (1.2) may be interpreted as a parametric multiple decrement model. To demonstrate that a multiple decrement model is actually equivalent to a mixture of survival functions, we first assume that we know the forces of decrement $\mu_x^{(k)} \geq 0$ for all the causes of decrement $k=1, \dots, m$. Define

$$\mu_x = \sum_{k=1}^m \mu_x^{(k)}. \quad (\text{A.1})$$

Then the survival function is

$$s(x) = \exp \left\{ - \int_0^x \mu_t dt \right\}, \quad (\text{A.2})$$

and it will be well-defined as long as $\int_0^x \mu_t dt = \infty$. By knowing $\mu_x^{(k)}$ and $s(x)$, we can calculate the number of lives aged x that will eventually die from cause k given a radix of l_0 . This is equal to

$$l_x^{(k)} = l_0 \int_x^\infty s(t) \mu_t^{(k)} dt. \quad (\text{A.3})$$

With this we can calculate the probability that a life aged 0 will eventually die from cause k . This is denoted as ψ_k , and it is equal to

$$\psi_k = l_0^{(k)} / l_0. \quad (\text{A.4})$$

We can also calculate the probability that a life will survive to age x given that the life will eventually die from cause k . This will be denoted as $s_k(x)$,

and it is equal to

$$s_k(x) = l_x^{(k)} / l_0^{(k)}. \quad (\text{A.5})$$

Using the relations $l_x = l_x^{(1)} + \dots + l_x^{(m)}$ and $l_x = l_0 s(x)$ along with the definitions in (A.4) and (A.5), we find that (A.2) can be written as

$$s(x) = \sum_{k=1}^m \psi_k s_k(x). \quad (\text{A.6})$$

This demonstrates that $s(x)$ can be written as a mixture of survival functions when μ_x is equal to (A.1). Now suppose we know ψ_k and $s_k(x)$; then we can find $\mu_x^{(k)}$ as follows. Assuming that $s_k(x)$ is differentiable, we can calculate the probability density function

$$f_k(x) = -\frac{d}{dx} s_k(x). \quad (\text{A.7})$$

Using the relations in (A.4) and (A.5) and the fact that $s(x) = l_x / l_0$, we find that

$$\frac{\psi_k f_k(x)}{s(x)} = -\frac{1}{l_x} \frac{d}{dx} l_x^{(k)}. \quad (\text{A.8})$$

Using the definition for $\mu_x^{(k)}$ we finally find that

$$\mu_x^{(k)} = \frac{\psi_k f_k(x)}{s(x)}. \quad (\text{A.9})$$

We have just shown that knowing $\mu_x^{(k)}$ for $k=1, \dots, m$ is equivalent to knowing ψ_k and $s_k(x)$. This paper modeled life tables with mixtures of parametric survival functions. This is equivalent to modeling life tables with parametric multiple decrement tables.

