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We will discuss mathematical modeling and pricing of "Stable Value" financial products offered to pension plan providers. These products target a risk-averse population of investors, who have a significant preference for a stability of returns. We offer a quantitative methodology that reflects a decision making process and applicable in the situation of incomplete market where traditional arbitrage free arguments are invalid

Pricing of Guaranteed Products for Defined Benefit Pension Funds.

We will discuss mathematical modeling and pricing of "Stable Value" financial products offered to pension plan providers. These products target a risk-averse population of investors, who have a significant preference for a stability of returns. We offer a quantitative methodology that reflects a decision making process and applicable in the situation of incomplete market where traditional arbitrage free arguments are invalid

As a first approximation these products can be described as debt instruments with some specific cashflow and contract arrangements. In a majority of cases, a contract does not terminate at once. Each contract is comprised of a number of participants who have a right to terminate (may be for unrelated personal reasons). For this reason, termination of the contract by the investor (put option) may randomly deviate from the optimal. Subsequent mathematical formalization will lead to a randomized stopping-time problem.

A predictability of returns is an essential component of portfolio manager performance. The cost of a product from the guarantee provider's point of view depends on volatility of return. At the same time the stability of return is also an investor's objective. Consequently, a portfolio management ability to generate reasonable return subduing volatility should be factored in the product pricing. This is a difficult problem since market is incomplete and inefficient, and arbitrage free arguments are not applicable

Fortunately, a corporate pricing routine gives us a clue of how to approach the problem. It turns out that required capital may serve as an indicator of a company's volatility tolerance and return on capital as a company's measure of profitability. As an important byproduct, a solution will necessary generate the investment strategy.

We will start from a small numerical example which we hope will clarify important points emphasizing necessity of proper quantitative portfolio management and exposing weaknesses of a traditional duration management approach.

Then we will show solution of some stochastic optimization models applicable to the above. Finally, we will discuss pricing models related to the Guarantee Investment Contracts.

Minimum Complexity Example

Consider a simplest asset-liability model with a stochastic interference. Assume that contract stipulates payments of two premiums - P_0 at the inception of the contract, and - P_1 a moment later . Liability payment L_T is paid at the time T_{mat} and contract therefore terminated.
Assume the following simplified financial environment.

- Yield curve is flat at r_0 = 5%
- Expect a random jump *N(m_0,sigma)* a second after investment decision is made. From that time on interest rate does not change.
- *m_0=7%,sigma =2%*
-
- Initial premium *P_0=$4 million* paid at time zero.
- Second and the last premium P_1 =$*6 million* expected at time *t_1=0.0001* right after the jump.
- Contract matured at *T_{mat} = 20* with a liability payment *L_T = $36 million*

One has to make an investment decision of how to allocate the existing assets (first premium) to maximize a present value of a future surplus. Second premium would be invested until maturity since the interest rate does not change after the first jump.

The following formulas are straight-forward results of the assumptions:
Present value of the ending surplus for a realized rate r

$$PV(t,r) = P_0 \cdot e^{r_0\cdot(t-T_{mat})+r\cdot(T_{mat}-t)} + P_1 \cdot e^{(r-r_0)\cdot(T_{mat})} - L\cdot e^{-r_0\cdot T_{mat}}$$

Expected value of the future surplus is

$$EV(t) = P_0 \cdot e^{m\cdot T+(r_0-m)\cdot t+\frac{\sigma^2}{2}(T-t)^2} + P_1 \cdot e^{m\cdot T+\frac{\sigma^2}{2}T^2} - L$$

Variance of the future surplus is a bit more cumbersome

$$Var(V(t)) = P_0{}^2 e^{2m\cdot T+2(r_0-m)t+2\sigma^2(T_2-t)^2} + P_1^2 \cdot e^{2m_1\cdot T+2\sigma^2T^2}$$
$$+2\cdot P_0\cdot P_1\cdot e^{2mT+(r_0-m)t+0.5\sigma^2\cdot(2T-t)^2} - (EV(t))^2$$

Old traditional Investment Decision 1.

Ignore premium. Invest in the bullet maturing at T_{mat}

Old traditional Investment Decision 2.

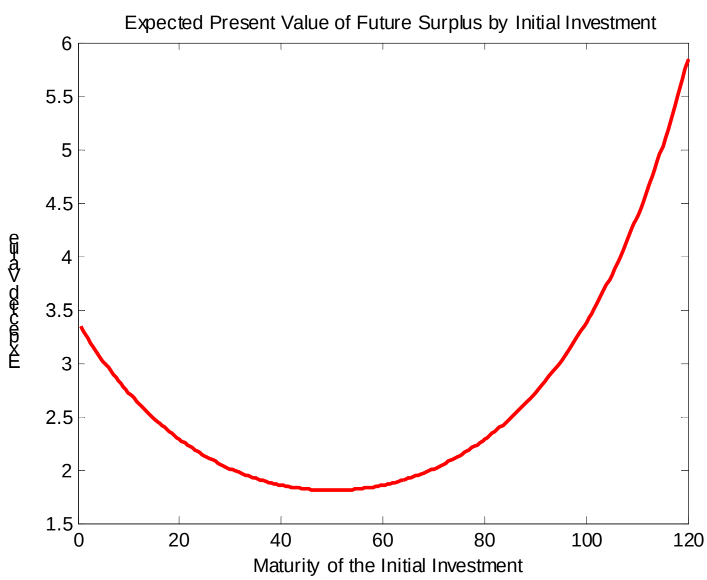
Calculate modified duration of the liability.

$$ModDur = \frac{(T_{mat} \cdot L_{T_{mat}} \cdot e^{-T_{mat} \cdot r_0} - t_1 \cdot P_1 \cdot e^{-t \cdot r_1})}{L_{T_{mat}} \cdot e^{-T_{mat} \cdot r_0} - P_1} = 65$$

Invest existing assets in the bullet maturing at 65 years.

Now we will investigate if indeed any of above solutions offers a reasonable strategy.

Figure 1



Above we plotted expected present values of a future surplus depending on a length of the initial investment. Clearly, the old traditional decisions bring about inadequate results. What follows from this observation that if long investment (>100) is not available- keep money in cash. However in this analysis we ignored randomness concentrating on the expected values. What about volatility of the return?

Figure 2

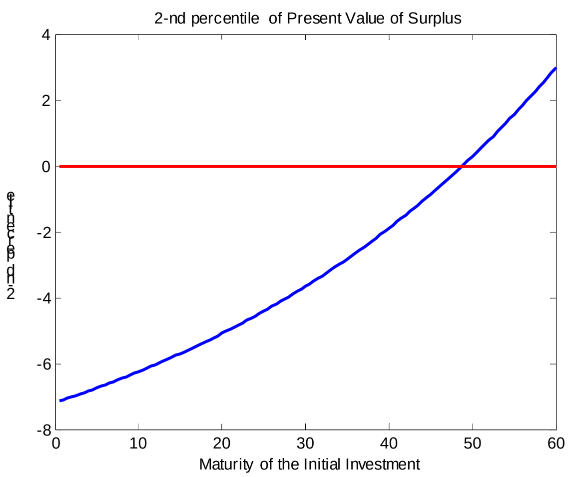
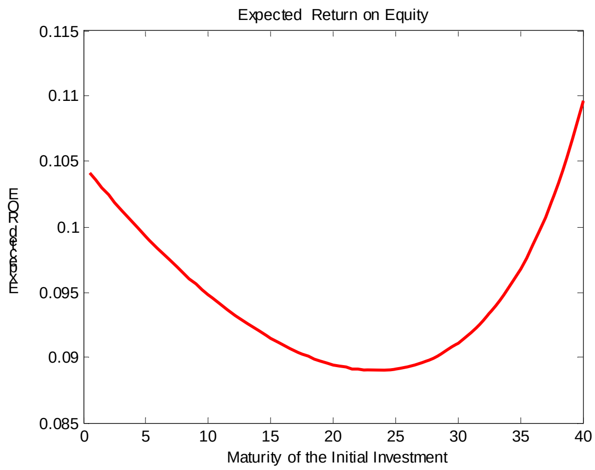


Figure 2 suggests that a short initial investment may generate a significant loss, or put it differently creates a significant value at risk. From this perspective, a risk averse investor would invest as long as possible however giving up an expected return.

At this time assume that company's internal requirement demands allocation of a risk capital covering losses with 98% confidence. Thus, a cash (very short) investment strategy requires $7 million of capital.

Consider now risk capital as an equity investment (*Figure 3*). More risky strategy, more capital should be allocated. Now, it turns out that short investment would bring better return on equity notwithstanding higher capital requirement, than investment into 20 years maturity. If exceptionally long maturities are not available, cash investment is superior again.

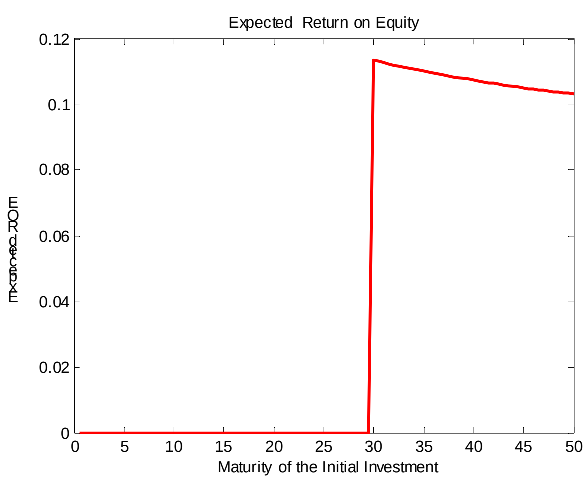
Figure 3



Finally consider a situation when risk capital decision is based on the availability of capital and capital is allocated regardless of the calculated risk. In this case we solve for an investment strategy that generates maximum return subject to loss restricted by the capital.

Assume that $2 million is allocated as risk capital. In this case *Figure 3* shows that an optimal investment is 30 years bullet.

Figure 4



Summary

- ***Risk Capital*** allocation is an indicator of a company risk tolerance. More risk averse company more risk capital required to open line of business.

- ***Return on Equity*** requirement is a company's desired profitability.

- ***Product is profitable*** if a maximum expected return is greater than the company's requirement.

- Ideally, an optimal control problem solves for the best strategy, maximizing expected return on equity and assuming that the initial capital is allocated according to the company requirement.

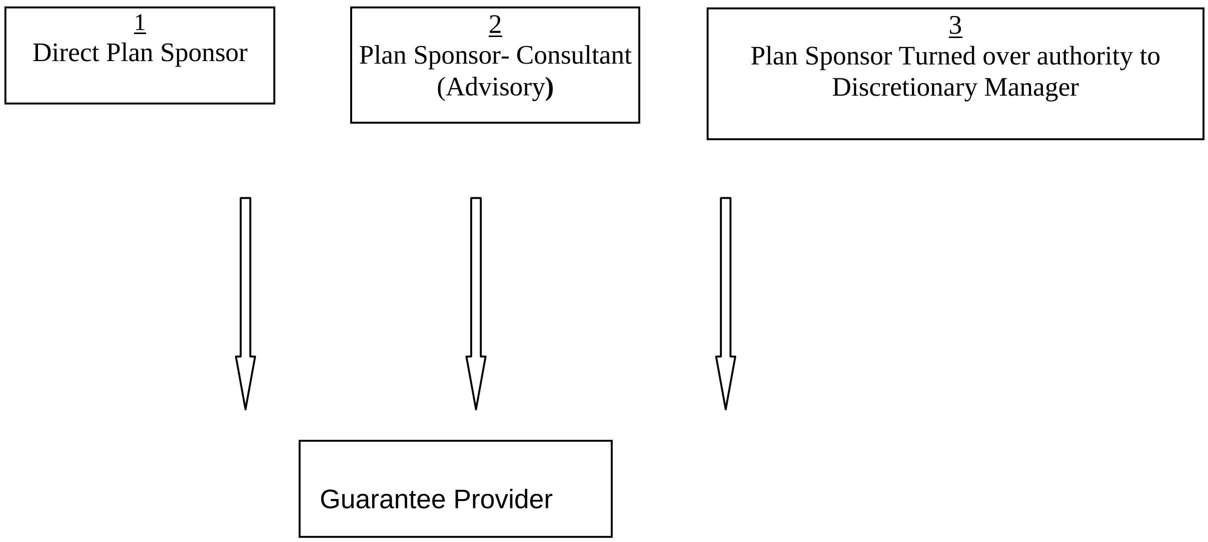
- In practice the optimization problem often too difficult to solve.

- A simplified problem solved with a hope that it's solution would produce a feasible result.

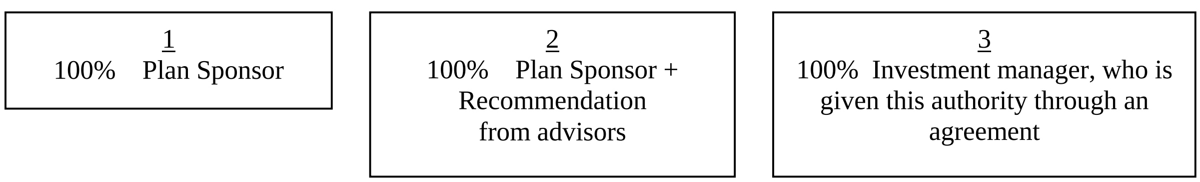
Mathematical Theory of a Stable Value Products

Logical Structure and Classifications of Guaranteed Investment Contracts

Structure of GIC Business



Decision making



The chart above classified GICs by the degree to which the investment manager is involved in the process of making economical decisions.

Consider how it works in its simplest form. We look at the case 3 as the most general one.

Case 3 (Fully Discretional Pension Plan Manager)

A *plan sponsor* - PS corporation, on behalf of its employees – *plan participants*, has to invest $\$M$ in the stable value sector of the 401K pension plan. We assume $M = \sum_{i \in I} M_i$ where I is a set of all pension *plan participants*. PS hires an investment manager –

company IM to manage the entire process. IM enters into a contract with an insurance company GP (Guarantee Provider). In the resulting agreement GP gets money in return for an obligation to pay $M \cdot (1+r)^T$ upon maturity *T* of the contract assuming no withdrawals have been made. For the *illustrative* purposes assume that the interest rate[1] r is a constant and is stipulated by the participating sides at the inception of the contract. The plan sponsor has the right to withdraw money at any time. For each dollar invested at the inception, the amount available for withdrawal at time *t* is equal to $(1+r)^t$. However there are usually important strings attached. If the withdrawal is initiated by the IM, the plan sponsor has to pay an early withdrawal fee F_t. No fee is paid if the withdrawal is a *plan participant* initiated event. The money could be withdrawn (before maturity) from the guaranteed account at time $t < T$ for three different reasons.

1. At time *t*, IM realizes that an obligation of the Guarantee Provider GP is worth less than $M \cdot (1+r)^t - F_t$. In this case a sound economic decision is to initiate a termination of the contract and withdraw money.
2. An employee *x* of the PS decides that the guaranteed rate is too low for the current market situation and requests the IM to withdraw his portion of the account. Since this is an action initiated by an employee, no early withdrawal penalty is imposed. Accordingly, the employee gets back his investment at a guaranteed value of $M_x \cdot (1+r)^t$ and the account value is reduced by the same amount. $M_x \cdot (1+r)^t$. Here M_x denotes the initial investment by *x* at time *t.*
3. An employee *x* of PS decides to withdraw money for reasons not related to the market situation (this may include retirement or change of employment.)

As we shall see later, the mathematical tractability of the withdrawal is very different for each case.

Open window option

The next common feature, which may be added to the contract, is an "open window" option. This feature is familiar to many homebuyers as the "lock-in-rate" option for house financing. Under the open window clause the plan manager (or plan sponsor) may deposit money during a certain period of time (open window) with the crediting rate established at the beginning of the period.

Synthetic GIC

This product separates guarantee on the book value from the ownership of the assets. In the case of a regular GIC, the guarantor has an obligation to pay the entire sum requested for the withdrawal by the plan sponsor. In the case of a Synthetic GIC the guarantor has to *subsidize* a withdrawal if the assets portfolio (not owned by the guarantor) has

[1] The constant interest rate assumption is local and is made here only for an illustrative purpose. In fact the methodology of assigning contractual interest rates is essential partin certain types of contract classifications.

insufficient funds to cover the withdrawal request and the amount of withdrawal does not exceed the guaranteed value.

Participating (Par) and Non-Participating (Non-Par) GICs

The Guaranteed Investment Contract is said to be "participating" if the plan sponsor participates in the profit or loss of the assets portfolio through the adjustment of the guaranteed contract rate. If the guaranteed rate does not depend on portfolio performance we say that the contract is non-participating.

Mathematical Models

For all the considerations below we assume that interest rate term-structure is described by a stochastic differential equation

$$dr_t = b_t(r_t)\cdot dt + \sigma_t(r_t)dw_t \tag{1}$$

with respect to a standard Wiener process w_t, which is defined on a complete probability space $\{\Omega, P\}$ with filtration. $\{F_t\}_{t>0}$, r_t is an instantaneous interest rate, i.e. a risk free interest rate paid for short term borrowing.

We also assume that the probability measure P is risk neutral. Accordingly we will calculate the price of a security as a discounted cashflow.

There are two different mathematical problems, which from the client's (plan sponsor) point of view are associated with the type of the contract. For the non-participating contracts the client is interested only in the size of the fee he is charged by the guarantee provider. If the contract is participating, the client would request that the guarantee provider's management of the portfolio would maximize plan sponsor's investment. Therefore we will discuss below a *Pricing Problem* – the calculation of the fair market value of the fee charged by the guarantee provider and a *Portfolio Management Problem* - the optimal strategy the portfolio manager has to follow to benefit his client the most.

1. *Participating GIC with no withdrawal options (Portfolio Management Problem)*

Assume that the contract starts at time $t = 0$ when client makes a deposit L_0. The guarantee provider purchases portfolio A of assets. Denote by M_t a market price of such a portfolio at moment *t*. We assume that $L_0 = M_0$. Assume that investment income from portfolio A_t is equal to $r_t + s$ where r_t -is a spot interest rate and *s* is a quality spread. Dynamic of the market value of the portfolio A is described by a following equation

$$dM_t = M_t\cdot((r+s)dt - D_t\cdot dr_t)$$

This equation simply states that change in portfolio price consists of two components. The first component is an investment income, which is defined by the amount of interest earned through a coupon payment over the time interval dt. The second component is a change of price due to the interest rate shift. This change is proportional to a portfolio effective duration D_t (By definition) at the time t.

The guaranteed value of liability L_t is defined as $L_t = L_0 \cdot e^{\int_0^t l_s ds}$. Here r_t^L is a guaranteed rate calculated traditionally as

$$r_t^L = MAX(\hat{r}^L, \frac{1}{T-t}\ln(\frac{M_t}{L_t}) + (r_t + s))$$

Where $\hat{r}^L$ is a minimum guaranteed rate. The equation is derived from

$$L_t \cdot e^{(T-t)(r_t^L)} = M_t \cdot e^{(T-t)(r_t+s)} \text{ if } L_t e^{(T-t)(\hat{r}^L)} < M_t \cdot e^{(T-t)(r_t+s)} \qquad (2)$$

The last equation makes a continuous readjustment to the guaranteed rate in such a way that final value of assets and liability portfolios would be equal each other if the market conditions remain unchanged until maturity of the contract.

The equation (2) implies that the final liability value should converge to a final market value of the assets portfolio in a case of an adequate portfolio performance.

Denote $\quad l_t = \ln(L_t);\ m_t = \ln(M_t)$

By the Ito formula we have

$$dm_t = (-D_t \cdot dr_t + (r+s)dt) + 0.5 \cdot D_t^2 \cdot (dr_t)^2$$

Thus we are getting a following equation for a market value dynamic

$$dm_t = (-D_t \cdot b_t(r_t) \cdot dt + (r+s)dt) + 0.5 \cdot D_t^2 \cdot (\sigma_t(r_t))^2 \cdot dt - D_t \cdot \sigma_t(r_t) \cdot dw_t$$

$$dm_t = (-D_t \cdot b_t(r_t) + (r+s) + 0.5 \cdot D_t^2 \cdot (\sigma_t(r_t))^2) \cdot dt - D_t \cdot \sigma_t(r_t) \cdot dw_t \qquad (3)$$

Equation (3) describes a fixed income portfolio dynamic. Even though it looks quite simplistic, there are no significant aberrations from the reality. Most restrictive (implicit) assumption that has to be made to justify (3) is a continuous rebalancing of the portfolio.

Assets Management Optimization Problem

The next step is a choice of criterions that evaluate a portfolio manager and a guarantee provider performance. The most straightforward one is to maximize liability value at the end of the horizon.

***Optimization Problem 1. 1* (Maximize Ending Liability Value)**

$$\max_{u \in \mathcal{U}} E(l_T) \tag{4}$$

$$r_t^L = MAX(\hat{r}^L, \frac{1}{T-t}(m_t - l_t) + (r_t + s))$$

$$dl_t = r_t^l \cdot dt$$

$$dm_t = (-u_t \cdot b_t(r_t) + (r+s) + 0.5 \cdot u_t^2 \cdot (\sigma_t(r_t))^2) \cdot dt - u \cdot \sigma_t(r_t) \cdot dw_t$$

$$\mathcal{U} = \{u_t : u_t \in [\breve{D}_t, \widehat{D}_t]$$

Here $\breve{D}_t, \widehat{D}_t$ are low and upper boundary allowed for the duration D_t of the assets portfolio. Those functions are deterministic and usually are the contract-stipulated values

Since we consider a participating contract, the final total earning would be in large degree defined by a portfolio performance itself. Therefore as an approximation and as a reasonable compromise for the investor would be a following problem, which is easier to solve analytically.

***Optimization Problem 1.2* (Expected Ending Portfolio Value)**

$$\max_{u \in \mathcal{U}} E(m_T) \tag{4'}$$

$$dm_t = (-u_t \cdot b_t(r_t) + (r+s) + 0.5 \cdot u_t^2 \cdot (\sigma_t(r_t))^2) \cdot dt - u \cdot \sigma_t(r_t) \cdot dw_t$$

$$\mathcal{U} = \{u_t : u_t \in [\breve{D}_t, \widehat{D}_t]$$

The problem has to be reformulated for an infinite time horizon if an evergreen[2] contract is considered

[2] With no expiration date

Optimization Problem 1.3 **(Infinite time horizon)**

$$\max_{u\in \mathscr{U}} \lim_{T\to\infty} \frac{\left(E(m_T)\right)}{T} \text{ or } \max_{u\in \mathscr{U}} \lim_{T\to\infty} \frac{\left(E(l_T)\right)}{T}$$

$$dm_t = (-u_t \cdot b_t(r_t) + (r+s) + 0.5\cdot u_t^2 \cdot (\sigma_t(r_t))^2)\cdot dt - u\cdot \sigma_t(r_t)\cdot dw_t$$

$$\mathscr{U} = \{u_t : u_t \in [\breve{D}_t, \widehat{D}_t]$$

The criteria 1.1 and 1.2 may not adequately reflect the basic premises of the product. The guaranteed investment contracts continue to occupy a significant portion of the pension fund market because of the higher return-to-volatility ratio, not to the return on the investment per se as would suggested by 1.1 or 1.2. Considering this, we will introduce a CAPM[3]- type criteria. We assume that the market is risk averse and demands an additional return from more volatile securities. This is to say that the plan sponsor would have estimated a performance of the portfolio manager by looking not only at the portfolio return $\frac{(m_T - m_0)}{T}$ but also at the historically estimated volatility. The last conception couldn't be directly introduced within the continuous time framework. Assume that the return is measured at times $t_0 = 0, t_1 = \Delta t, \ldots t_n = T$. A historical estimate of the volatility is

$$\frac{1}{n}\cdot\sum_{i=0}^{n-1}\left(\frac{m^{i+1}-m^i}{\Delta t}\right)^2 - \left(\frac{m^n - m^0}{\Delta t\cdot n}\right)^2 \quad (5).$$

Here in order to shorten the notations we denote $m^i = m_{t_i}$

It is naturally therefore to state that portfolio management purposes to minimize a mathematical expectation of (5) together with (4) or (4') . A different optimal strategy is generated by a new problem where the criterions (4) and (5) are mixed together.

Optimization Problem 1.4

$$\max_{u\in \mathscr{U}}\{\alpha\cdot E(l_T) + \beta\cdot(\frac{1}{n}\cdot\sum_{i=0}^{n-1}(\frac{m^{i+1}-m^i}{\Delta t})^2 - E\left(\frac{m^T - m^0}{T}\right)^2)\}$$

$$dm_t = (-u_t \cdot dr_t + (r+s)dt) + 0.5\cdot u_t^2\cdot (dr_t)^2$$

$$\mathscr{U} = \{u_t : u_t \in [\breve{D}_t, \widehat{D}_t]$$

[3] Capital Asset Pricing Model

There are at least two problem with the problem 1.3. The first is a robustness and stability of the solution.. If Δt is small the volatility component $\frac{1}{n}\cdot\sum_{i=0}^{n-1}(\frac{m^{i+1}-m^{i}}{\Delta t})^{2}-E\left(\frac{m^{T}-m^{0}}{T}\right)^{2}$ could significantly supersede the market value component $E(m_t)$. Another problem is that minimization of historical volatility does not necessary coincides with the client goals. It is more in sync with a GIC salesman arguments than with a client interest.. Assuming that the client main objective is to maximize profit and minimize volatility the optimal problem could be reinstated as follows:

Optimization Problem 1.6

$$\max_{u\in \mathcal{U}}\{\alpha\cdot E(l_T)+\beta\cdot\left((E(l_T^2)-(E(l_T))^2\right)\}$$

$$dm_t=(-u_t\cdot dr_t+(r+s)dt)+0.5\cdot u_t^2\cdot(dr_t)^2$$

$$\mathcal{U}=\{u_t: u_t\in[\breve{D}_t,\widehat{D}_t]$$

Another version is may be considered is

Optimization Problem 1.6

$$\max_{u\in \mathcal{U}}\left\{\frac{E(m_T)}{\sqrt{(E(m_T^2)-(E(m_T))^2}}\right\}$$

It looks like a more difficult problem, though more in line with CAPM arguments.

2. *Non-Participating GIC with a withdrawal option (Pricing Problems)*

For non-participating GICs the liability is a guaranteed contact value calculated on the base of the guaranteed contact rate. This rate is a parameter of the contract and is assumed to be a constant until the maturity of the contract. Plan sponsor of such a GIC has an option (put) to withdraw money at any time at his discretion. However, two cases of withdrawal are identified and separated. If a *plan participant* initiates withdrawal, no penalty for an early withdrawal is imposed. In this case the Guaranteed Contract Value is reduced by the amount withdrawn. If the plan sponsor initiated a withdrawal on behalf of the plan participants the Guaranteed Contract value is reduced by the amount withdrawn plus early withdrawal penalty.

Notations, Assumptions and Preliminary Information

Consider a Guaranteed Investment Contract paying coupon f_t with a continuously compounded interest[4].

Assume that

- Level of a risk free interest rate is equal to r_t at the time $t > 0$ as described by (1) .

[4] This means that coupon payment over the infinitesimal interval Δt is equal to $\Delta t\cdot f_t\cdot M_t$ Δt

- The contract-holder is entitled to a guaranteed value g_t at time t if decided to withdraw money.
- Contract has a par value of \$1 and g_T is a guaranteed value that will be mandatory withdrawn at maturity T. We assume that if termination of the contract is happened due to the plan manager decision then there is a penalty which reduces the guaranteed value by g_t^f .
- Non-arbitrage transaction frees trading in a risk-neutral world.

Denote by τ a fixed random moment (not necessarily optimal) when the contract holder decided to withdraw money. Denote $V_\tau(x,t)$ the price for such a contract, calculated as a mathematical expectation of a discounted future cashflow.
We therefore have

$$V_\tau(x,t) = E\left\{ \int_t^{\tau\wedge T} \exp\left(-\int_t^s x_u \cdot du\right) \cdot f_s + \exp\left(-\int_t^{\tau\wedge T} x_u \cdot du\right) \cdot \left(I_{\tau\geq T} \cdot g(\tau) + I_{\tau<T} \cdot g^f(\tau)\right) \right\}$$

It is naturally to assume that the manager would chose moment τ to maximize $V_\tau(x,t)$. Therefore if he has a full discretion over the process he will behave accordingly. In this case the price for the contract should be calculated as a solution of the optimization problem

Optimization Problem 2.1 (Withdrawal is on the plan manager own discretion)

$$dx_s = b ds + \sigma dw_s \qquad x_0 = x$$

$$V(x,t) = \max_\tau E\left\{ \int_t^{\tau\wedge T} \exp\left(-\int_t^s x_u \cdot du\right) \cdot f_s + \exp\left(-\int_t^{\tau\wedge T} x_u \cdot du\right) \cdot \left(I_{\tau\geq T} \cdot g(\tau) + I_{\tau<T} \cdot g^f(\tau)\right) \right\}$$

Optimality Conditions

It is shown in [1] that *v(x,t)* satisfies the Non-Linear Partial Differential Equation.

Equation 1

$$\overline{g} - V + [\tfrac{1}{2} \cdot \sigma^2 \frac{\partial^2 V}{\partial x^2} + b \cdot \frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} - x \cdot V + f_t + v - \overline{g}(t)]_+ = 0$$

$$[a]_+ = \max(a,0)$$

$$\overline{g}(t) = I_{t\geq T} \cdot g(t) + I_{t<T} \cdot g^f(t)$$

This in turn is equivalent to the following three conditions:

Equation 2

$$V - \bar{g} \geq 0$$
$$v - \bar{g} > 0 \Rightarrow Lv = 0$$
$$\bar{g} = v \Rightarrow Lv \leq 0$$

where $Lv = \frac{1}{2}\sigma\frac{\partial^2 v}{\partial x^2} + b\cdot\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - x\cdot v$

This problem is very difficult to solve numerically. We consider a modification of the problem which drastically reduces the complexity.

Contractholder is entitles to request a withdrawal.

Assume that *plan participant's* withdrawal decision is not based on a market situation. He may withdraw money because of a contingency related to a retirement, job security, and other social events. For the simplification of the analysis assume that if *plan participant* decided to withdraw money, he will withdraw a total guaranteed value and the Guaranteed Contract is terminated. Assume that the intensity of withdrawal is $r(x,t)$. This means that at time t_0 probability that the *plan participant* will terminate the contract on the interval (*t*,*t*+*dt*) is

$$r(x_t, t)\cdot \exp(-\int_{t_0}^{t} r(x_u, u)\cdot du)\cdot dt$$

For the sake of simplicity assume that $t_0 = 0$. We reformulate the optimization problem by introducing an additional variable y_t, and assuming perpetuity in the payments period T=∞. It is not difficult to see that the last assumption does not cause a loss of generality. To achieve the actual maturity at T one has to choose unrestrictedly high intensity function in the small vicinity of T. The optimization problem 2.1 may be rewritten as

Optimization Problem 2.2

$$v(x_0, y_0) = \max_{\tau} E[\int_t^{\tau} \exp(-y_s) \cdot f_s\, ds + \exp(-y_{\tau}) \cdot \bar{g}(\tau)]$$

$$dx_t = bds + \sigma dw_s \qquad x_0 = x_0$$

$$dy_t = x_t dt \qquad y_0 = y_0$$

Now consider an individual trajectory ω where the bond is scheduled to be called at time τ(ω). The conditional contribution of this trajectory to the criteria of the Optimization Problem 2.2 is

$$\int_0^{\tau} \exp(-y_s) \cdot f_s\, ds + \exp(-y_{\tau}) \cdot \bar{g}(\tau)]$$

Assume now that together with management call τ, the bond may be called due to the irrational cause defined by intensity function $r(x,t)$ Therefore the expected contribution from the individual trajectory is

$$\exp(-\int_0^{\tau} r(x_t,t)dt) \cdot \{\int_0^{\tau} \exp(-y_s) \cdot f_s\, ds + \exp(-y_{\tau}) \cdot g^{f}(\tau)\} +$$

$$\int_0^{\tau} [r(x_t,t) \cdot \exp(-\int_0^{t} r(x_s,s)ds) \cdot \{\int_0^{t} \exp(-y_s) \cdot f_s\, ds + \exp(-y_t) \cdot g(t)\}]dt =$$

$$= I + II$$

For the given trajectory the contract will be terminated either due to the manager's call or the *plan participants* decision to withdraw money. If the plan manager decides that τ is an optimal time to withdraw, two different events may happen. First takes place when the contractholder does not withdraw before τ and the first part (I) of the expression above evaluates the expected contribution from this event. In this case $\exp(-\int_0^{\tau} r(x_t,t)dt)$ is the probability that the withdrawal of the bond will be a plan manager initiated event. The second event takes place when the contractholder decided to withdraw before τ. Accordingly the second part (II) is a contribution from such an event. After some transformations we have

$$I + II = \exp(-\int_0^\tau r(x_t,t)dt)\cdot \exp(-y_\tau) g^f(\tau) + \{\int_0^\tau \exp\left(-y_t - \int_0^t r(x_s,s)ds\right)\cdot (f_t + r(x_t,t)\cdot g(t))\, dt\}$$

Now we may get rid of *y* and return to previous notations.

$$I + II = \exp(-\int_0^\tau (r(x_t,t)+x_t)dt)\cdot g^f(\tau) + \{\int_0^\tau \exp\left(-\int_0^t (r(x_s,s)+x_s)ds\right)\cdot (f_t + r(x_t,t)\cdot g(t))\, dt\}$$

The price of the entire contract with given random withdrawal time τ is a mathematical expectation of the contributions of the individual trajectories

$$v_\tau(x,0) = E(\mathrm{I} + \mathrm{II})$$

Effective market will price the contract by choosing the call time to a maximum disadvantage of a bondholder. Therefore we obtain price of the bond as a result of the

Optimization Problem 2.3

$$v(x,t) = \max_\tau v_\tau(x,t)$$

or

$$v(x,t) = \max_\tau E[\exp(-\int_0^\tau (r(x_t,t)+x_t)dt)\cdot g^f(\tau) +$$
$$\{\int_0^\tau \exp\left(-\int_0^t (r(x_s,s)+x_s)ds\right)\cdot (f_t + r(x_t,t)\cdot g(t))\, dt\}]$$

Applying Equation 3 to this problem we obtain a Differential Equation for $v(x,t)$

Equation 3

$$g^f - v(t,x) + [\tfrac{1}{2}\cdot\sigma^2 \frac{\partial^2 v}{\partial x^2} + b\cdot\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - (x+r)\cdot v + f_t + rg + v - g^f]_+$$
$$[a]_+ = \min(a,0)$$

Pricing With Stationary Processes

Assume that x_t is a stationary process. This means that coefficients σ and b in the equation 1 do not depend on time. Assume also that the intensity function $r(x,t)$ is a function only of the state x, i.e. $r(x,t) = r(x)$ We have

$$\frac{1}{2}\sigma^2 v'' + bv' + (x + r)v + f + rg = 0$$
$$where \quad x < x_c$$
$$and \quad v(x_c) = g; \quad v'(x_c) = 0;$$

It is a second order *ordinary* differential equation with free boundary conditions.
It can be shown that this equation has a unique bounded solution.

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