Adjustment Coefficient in the Sparre Anderson Model with Reinsurance

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Abstract:

Assumptions:

1. Reinsurance treaty: quota-share retention level a, retention limit M.

2. A diffusion process is added to the claim process.

Objective:

Choose the a and M to maximize the adjustment coefficient.

1. Reinsurance Forms

• Proportional (Pro Rata): premiums and losses are shared proportionately between the ceding company and the reinsurer.

Quota share: same percentage applies to all policies.

Surplus share: share varies for each risk.

• Non-proportional

Catastrophe: big losses involving more than one insured.

Stop loss: pays losses in excess of a predetermined amount.

Spread loss: ceding company's losses for a certain year are "spread" over a period of years.

2. Definitions

$X_{a,M} = min(aX, M)$: the insurer retains payment when a claim of size X occurs.

$\{T_i\}_{i=1}^{\infty}$: inter-claim time, i.i.d. non-negative r.v. exponential distribution with $E(T_i) = \frac{1}{\lambda}$

$S_n = T_1 + T_2 + \cdots + T_n$: the claim process with $S_0 = 0$.

$\{N(t)\}_{t\geq 0}$: the number of claims, an ordinary renewal process. $N(t) = \sup(n : S_n \leq t\}$.

$\{W(t)\}$: a diffusion process affects the surplus process, a Wiener process with infinitesimal drift 0 and infinitesimal variance $2D > 0$. Independent of $S(t)$ and $W(t) \sim N(0, 2Dt)$

$\{X_i\}_{i=1}^{\infty}$: claim amount, i.i.d. non-negative r.v.s, independent of $\{T_i\}_{i=1}^{\infty}$. With distribution function F and $E(X_i) = \mu$. Satisfies :

- $F(0) = 0$; $0 < F(x) < 1$ for $0 < x < +\infty$;

- $\frac{dF}{dx}$ exist and continuous;

- $M_X(r)$ (moment generating function) exists for $r \in (-\infty, \tau)$ for some $0 < \tau \leq +\infty$ and $\lim_{r \to \tau} M_X(r) = \lim_{r \to \tau} E[e^{rX}] = +\infty$.

P: the insurer's gross premium income per unit of time.

α: positive loading coefficient.

e: insurer's expenses rate.

c: commission payment rate.

u: non-negative initial surplus.

2. Assumptions

1. The reinsurer will pays the commission back to the insurer according to the business volume, which is $c(1-a)P$.

2. The insurer cannot reinsure the whole risk with a certain profit, i.e., $e > c$ and $(1-e)P - (1+\alpha)\gamma\mu < 0$. The reinsurance premium:

$$P_{a,M} = (1-c)(1-a)P + (1+\alpha)\gamma \int_{M/a}^{\infty} (ax - M)dF(x).$$

3. The surplus process can be expressed as

$$U_{a,M}(t) = u + [(1-e)P - P_{a,M}]t - \sum_{i=1}^{N(t)} \min(aX_i, M) + W(t).$$

Note that for $X \sim N(\mu, \sigma^2)$, $M(t) = E[e^{tX}] = e^{\mu t + \sigma^2 t^2/2}$

4. The insurer's net(of expenses and reinsurance) risk at time t:

$$Y_{a,M}(t) = [(1-e)P - P_{a,M}]t - \sum_{i=1}^{N(t)} \min(aX_i, M) + W(t).$$

$$Y^i_{a,M} = -[Y_{a,M}(S_i) - Y_{a,M}(S_{i-1})].$$

Note that for $X \sim \exp(\lambda)$, $E[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \frac{\lambda}{\lambda - t}$.

5. The insurer's expected net profit per period of time (after reinsurance and expenses):

$$E[W(a, M)] = (1-e)P - P_{a,M} - \lambda E[X_{a,M}].$$

3. Preliminaries

Centeno(1985) has proved the following lemma

LEMMA 1: Let $A = \{a : 0 < a \leq 1$ and there exists an M such that $E[W(a, M)] = 0\}$ and $a_0 = \frac{(e-c)P}{(1-c)P-\lambda E(X)}$. Under our assumption on the reinsurance premium $P_{a,M}$, we have

1. $A = (a_0, 1]$.

2. For each $a \in A$, there is a unique M such that $E[W(a, M)] = 0$, i.e. there is a function Φ mapping A into $(0, \infty)$, such that $M = \Phi(a)$ is equivalent to $E[W(a, M)] = 0$.

3. $\Phi(a)$ is convex.

4. $\lim_{a \to a_0} \Phi(a) = +\infty$.

4. Two Models

The adjustment coefficient, $R_{a,M}$ is the positive solution of:

1. Modified Centeno's model

$$\begin{aligned} g_{a,M}(r) &= M_{Y^i_{a,M}}(r) = E[e^{rY^i_{a,M}}] \\ &= E[e^{rX_{a,M}}]E[e^{-[(1-e)P-P_{a,M}]rT}]E[e^{-rW(1)}] = 1 \end{aligned}$$

2.Modified Dufresne's model

$$\lambda \int_0^\infty e^{rx_{a,M}} dF(x) + Dr^2 = \lambda + [(1-e)P - P_{a,M}]r.$$

We want to figure out M and a to maximize $R_{a,M}$. We will discuss both models.

5. Modified Centeno's model

LEMMA 2.1: $R_{a,M}$ is the one and the only one positive solution of

$$E[e^{rX_{a,M}}]E[e^{-[(1-e)P-P_{a,M}]rT}]E[e^{-rW(1)}] = 1.$$

Proof:

1. $E[e^{rX_{a,M}}] = \int_0^\infty e^{rx_{a,M}} dF(x)$ is a non-decreasing convex function;

2. For $T \sim \exp(\lambda)$, $E[e^{-[(1-e)P-P_{a,M}]rT}] = \frac{\lambda}{\lambda+[(1-e)P-P_{a,M}]r}$;

3. For $W(1) \sim N(0, 2D)$, $E[e^{-rW(1)}] = e^{Dr^2}$ is a non-decreasing convex function.

So the above equation is equivalent to

$$\lambda e^{Dr^2} \int_0^\infty e^{rx_{a,M}} dF(x) = \lambda + [(1-e)P - P_{a,M}]r \qquad (1)$$

LEMMA 2.2: The adjustment coefficient is positive if and only if $(a, M) \in L$. L is the set of points for which the insurer's net expected profit is positive, i.e.

$$L = \{(a, M) : 0 \leq a \leq 1, M \geq 0 \text{ and } E[W(a, M)] > 0\}.$$

And for any $(a, M) \in L$, $H'_{a,M}(r)$ is positive at $r = R_{a,M}$.

Proof:

$$H_{a,M}(r) = \lambda e^{Dr^2 \int_0^\infty e^{rx_{a,M}} dF(x)} - [(1-e)P - P_{a,M}]r - \lambda = 0,$$

$$H'_{a,M}(0) = \frac{\partial H_{a,M}(r)}{\partial r}|_{r=0} = \lambda E[x_{a,M}] - [(1-e)P - P_{a,M}].$$

$$\xi = \begin{cases} +\infty & \text{if } M < +\infty \\ \tau & \text{for } M = +\infty \end{cases}, \longrightarrow \begin{cases} H_{a,M}(0) = 0 \\ \lim H_{a,M}(r)_{r \to \xi} = +\infty \end{cases}.$$

Adjustment coefficient is positive $\Longleftrightarrow H'_{a,M}(0) < 0$.

RESULT 1.1: For a fixed value of $a \in (a_0, 1]$, $R_{a,M}$ is a unimodal function of M, attaining its maximum value at the only point satisfying $M = \frac{1}{R_{a,M}} \ln[(1+\alpha)] - DR_{a,M}$, where $R_{a,M}$ is the only positive solution of (1). Let $\widehat{R}_a = \max(R_{a,M})$.

Proof:

From the implicit function theorem, we have,

$$\frac{\partial R_{a,M}}{\partial M} = -\frac{(\partial/\partial M)H_{a,M}(r)}{(\partial/\partial r)H_{a,M}(r)}|_{r=R_{a,M}}$$

$$\frac{\partial^2 R_{a,M}}{\partial M^2}|_{\frac{\partial R_{a,M}}{\partial M}=0} = -\frac{(\partial^2/\partial M^2)H_{a,M}(r)}{(\partial/\partial r)H_{a,M}(r)}|_{r=R_{a,M},\frac{\partial R_{a,M}}{\partial M}=0}$$

From lemma 2.2, $\frac{\partial H_{a,M}(r)}{\partial r}|_{r=R_{a,M}} > 0$.

Target: find out M, s.t. $\frac{\partial R_{a,M}}{\partial M} = 0$ where $\frac{\partial^2 R_{a,M}}{\partial M^2}|_{\frac{\partial R_{a,M}}{\partial M}=0} < 0$.

$\Longleftrightarrow$ M satisfies

$$\frac{\partial H_{a,M}(r)}{\partial M}|_{r=R_{a,M}} = 0$$

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}|_{r=R_{a,M},\frac{\partial R_{a,M}}{\partial M}=0} > 0.$$

Since

$$\frac{\partial H_{a,M}(r)}{\partial M} = \lambda r[1 - F(\frac{M}{a})][e^{Dr^2+rM} - (1+\alpha)],$$

implies $\frac{\partial R_{a,M}}{\partial M} = 0$ at least has one solution and one solution is

$$M = \frac{1}{R_{a,M}} \ln[(1+\alpha)] - DR_{a,M} \quad (2)$$

Since

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2} = \lambda r^2 e^{rM+Dr^2}[1-F(\frac{M}{a})] + \frac{\lambda r f(\frac{M}{a})}{a}(1+\alpha-e^{Dr^2+rM}),$$

When M satisfies (2), we have

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}\Big|_{r=R_{a,M},\frac{\partial R_{a,M}}{\partial M}=0} = \lambda r^2(1+\alpha)[1-F(\frac{M}{a})] > 0$$

Hence the second derivative with respect to M of $R_{a,M}$ is negative whenever the first derivative is zero, which implies that for fixed $a \in (a_0, 1]$, $R_{a,M}$ has at most one turning point, and that when such a point exists it is a maximum. The maximum will exist and be finite at the only point satisfying $M = \frac{1}{R_{a,M}} \ln[(1+\alpha)] - DR_{a,M}$.

Compared to the original model, the excess of loss retention limit M decreased due to the diffusion process, which means that for increasing uncertainty, the insurance company should cede more business to the reinsurer.

RESULT 1.2: $\widehat{R}_a$ is a unimodal function of a, for $a \in (a_0, 1]$, attaining it's maximum at $a = 1$, if and only if $\lim_{a \to 1^-} \frac{d}{da}\widehat{R}_a \geq 0$

Proof:

At equation(2), defines M as a function of a:$\Upsilon(a)$. Let $\widehat{R}_a = R_{a,\Upsilon(a)}$.

First we have $\frac{d\widehat{R}_a}{da} = -\frac{(\partial/\partial a)H_{a,M}(r)}{(\partial/\partial r)H_{a,M}(r)}|_{M=\Upsilon(a),r=\widehat{R}_a}$

$$\frac{d^2\widehat{R}_a}{da^2}|_{\frac{d\widehat{R}_a}{da}=0} = -\frac{(\partial^2/\partial a^2)H_{a,M}(r)\times(\partial^2/\partial M^2)H_{a,M}(r)-[(\partial^2/\partial a\partial M)H_{a,M}(r)]^2}{(\partial/\partial r)H_{a,M}(r)\times(\partial^2/\partial M^2)H_{a,M}(r)}|_{M=\Upsilon(a),r=\widehat{R}_a,\frac{d\widehat{R}_a}{da}=0}$$

Note here we already proved that

$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}|_{M=\Upsilon(a),r=\widehat{R}_a} > 0$ and $\frac{\partial H_{a,M}(r)}{\partial r}|_{M=\Upsilon(a),r=\widehat{R}_a} > 0$.

After some calculations, we have

$$(\partial^2/\partial a^2)H_{a,M}(r)\times(\partial^2/\partial M^2)H_{a,M}(r)-[(\partial^2/\partial a\partial M)H_{a,M}(r)]^2|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial a}H_{a,M}(r)=0}$$
$$= \lambda^2 r^4 e^{rM+2Dr^2}[1-F(\frac{M}{a})]\int_0^{M/a} x^2 e^{rax}dF(x)>0,$$

which means $\frac{d^2\widehat{R}_a}{da^2}|_{\frac{d\widehat{R}_a}{da}=0}>0$.

On the other hand, when $a\to a_0$, $\widehat{R}_a$ goes to zero and we can say that the maximum of $\widehat{R}_a$ is 1, if and only of if $\lim_{a\to 1^-}\frac{d}{da}\widehat{R}_a\geq 0$. Hence the result is proved. It is the same as the original result.

Example 1. Exponential Distribution

Claim amount distribution: $F(x) = 1 - e^{-\beta x}$ and $f(x) = \beta e^{-\beta x}$.

Let $\alpha = 0.9, c = 0.2, e = 0.3, P = 1.7, \beta = 0.5, \lambda = 1, D = 0.004$.
It satisfies $e > c$ and $(1-e)P - (1+\alpha)\lambda\mu = -2.61 < 0$

In this case, we have:

$$\int_0^\infty e^{rx_{a,M}} dF(x) = \frac{ra}{ra-\beta} e^{rM - \beta\frac{M}{a}} - \frac{\beta}{ra-\beta},$$

$$(1-e)P - P_{a,M} = (a + c - ca - e)P - (1+\alpha)\lambda\frac{a}{\beta}e^{-\beta\frac{M}{a}}$$

Hence (1) becomes

$$\lambda e^{Dr^2}\left(\frac{ra}{ra-\beta} e^{rM-\beta\frac{M}{a}} - \frac{\beta}{ra-\beta}\right) = \lambda + [(a+c-ca-e)P - (1+\alpha)\lambda\frac{a}{\beta}e^{-\beta\frac{M}{a}}]r$$

After some simplification, it becomes:

$$\frac{\lambda r^2 a^2}{\beta}(1+\alpha)^{1-\frac{\beta}{ar}} e^{D\beta r/a} = \lambda\beta e^{Dr^2} + (ra-\beta)(a+c-ca-e)rP + \lambda(ra-\beta).$$

The solution is $r = \beta/a$.

When $a = 1$, the adjustment coefficient $R_{a,M}$ attains its maximum value which satisfies:

$$\{\frac{\lambda r^2}{\beta(r-\beta)}(1+\alpha)^{1-\beta/r} e^{D\beta r} = \frac{\lambda\beta}{r-\beta} e^{Dr^2} + (1-e)rP + \lambda\} \mid_{r=R_{a,M}} .$$

$R_{a,M} = 0.5$ and $M = 1.2817$.

6. Modified Dufresne's model

LEMMA 3.1: $R_{a,M}$ is the one and the only one positive solution of

$$\lambda \int_0^\infty e^{rx_{a,M}} dF(x) + Dr^2 = \lambda + [(1-e)P - P_{a,M}]r.$$

The function can be rewritten as

$$\lambda \int_0^\infty e^{rx_{a,M}} dF(x) = -Dr^2 + [(1-e)P - P_{a,M}]r + \lambda. \quad (3)$$

LEMMA 3.2: The adjustment coefficient is positive if and only if $(a, M) \in L$. L is the set of points for which the insurer's net expected profit is positive, i.e.

$L = \{(a, M) : 0 \leq a \leq 1, M \geq 0$ and $E[W(a, M)] > 0\}$.

And for any $(a, M) \in L$, $H'_{a,M}(r)$ is positive at $r = R_{a,M}$.

RESULT 2.1: For a fixed value of $a \in (a_0, 1]$, $R_{a,M}$ is a unimodal function of M, attaining its maximum value at the only point satisfying $M = \frac{1}{R_{a,M}} \ln[(1+\alpha)]$, where $R_{a,M}$ is the only positive solution of (2.2). Let $\hat{R}_a = \max(R_{a,M})$.

Proof:

Similar to result 1.1,

$$\frac{\partial H_{a,M}(r)}{\partial M} = \lambda r[1 - F(\frac{M}{a})][e^{rM} - (1+\alpha)]$$

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2} = \lambda r[-\frac{1}{a} f(\frac{M}{a})][e^{rM} - (1+\alpha)] + \lambda r^2 [1 - F(\frac{M}{a})] e^{rM}.$$

which implies,

$$\frac{\partial R_{a,M}}{\partial M} = -\frac{(\partial/\partial M) H_{a,M}(r)}{(\partial/\partial r) H_{a,M}(r)} |_{r=R_{a,M}} = 0$$

at lease has one solution and one of solution is

$$M = \frac{1}{R_{a,M}} \ln[(1+\alpha)]. \tag{4}$$

When the above relationship satisfied,

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}|_{r=R_{a,M}, \frac{\partial R_{a,M}}{\partial M}=0} = \lambda r^2 [1 - F(\frac{M}{a})] e^{rM} > 0$$

which means.

$$\frac{\partial^2 R_{a,M}}{\partial M^2}|_{\frac{\partial R_{a,M}}{\partial M}=0} = -\frac{(\partial^2/\partial M^2) H_{a,M}(r)}{(\partial/\partial r) H_{a,M}(r)}|_{r=R_{a,M}, \frac{\partial R_{a,M}}{\partial M}=0} < 0$$

Hence the second derivative with respect to M of $R_{a,M}$ is negative whenever the first derivative is zero, which implies that for fixed $a \in (a_0, 1]$, $R_{a,M}$ has at most one turning point, and that when such a point exists it is a maximum. The maximum will exist and be finite at the only point satisfying $M = \frac{1}{R_{a,M}} \ln[(1+\alpha)]$.

RESULT 2.2:

$\widehat{R}_a$ is a unimodal function of a, for $a \in (a_0, 1]$, attaining it's maximum at $a = 1$, if and only of $\lim_{a \to 1^-} \frac{d}{da}\widehat{R}_a \geq 0$

Proof:

Here $\widehat{R}_a = R_{a,\Upsilon(a)}$, where $\Upsilon(a) = M = \frac{1}{R_{a,M}} \ln[(1+\alpha)]$.

Similar to result 1.2, we first have $\frac{d^2\widehat{R}_a}{da^2}|_{\frac{d\widehat{R}_a}{da}=0} > 0$.

On the other hand, when $a \to a_0$, $\widehat{R}_a$ goes to zero and we can say that the maximum of $\widehat{R}_a$ is 1, if and only of if $\lim_{a \to 1^-} \frac{d}{da}\widehat{R}_a \geq 0$. Hence the result is proved. It is the same as the original result.

Example 2. Exponential Distribution

Use the same assumptions as example 1, (3) becomes

$$\lambda(\frac{ra}{ra-\beta}e^{rM-\beta\frac{M}{a}}-\frac{\beta}{ra-\beta})+Dr^2=\lambda+(a+c-ca-e)rP-(1+\alpha)\lambda r\frac{a}{\beta}e^{-\beta\frac{M}{a}}$$

Consider M satisfying (4), after some simplification, we have :

$$\lambda a(1+\alpha)^{1-\frac{\beta}{ra}}\frac{ra}{\beta}+(ra-\beta)Dr=\lambda a+(a+c-ca-e)(ra-\beta)P.$$

The solution is $r=\beta/a$.

When $a=1$, the adjustment coefficient $R_{a,M}$ attains its maximum value which satisfies:

$$(1+\alpha)^{1-\frac{\beta}{r}}\frac{\lambda r}{\beta(r-\beta)}+Dr=\frac{\lambda}{r-\beta}+(1-e)P.$$

$R_{a,M}=0.5$ and $M=1.2837$.

References:

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Any Questions?

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