A Uniform Asymptotic Estimate for Discounted Aggregate Claims with Subexponential Tails

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October 26, 2007

Abstract

In this paper we study the tail probability of discounted aggregate claims in a continuous-time renewal model. For the case that the common claim-size distribution is subexponential, we obtain an asymptotic formula, which holds uniformly for all time horizons within a finite interval. Then, with some additional mild assumptions on the distributions of the claim sizes and inter-arrival times, we further prove that this formula holds uniformly for all time horizons. In this way, we significantly extend a recent result of Tang (2007, J. Appl. Probab. 44, 285–294).

Keywords: Asymptotics; Poisson process; renewal process; subexponential distribution; uniformity.

1 Introduction

Consider the renewal risk model, in which claim sizes $X_k, k = 1, 2, \dots,$ constitute a sequence of independent, identically distributed (i.i.d.), and nonnegative random variables with common distribution F, while their arrival times $\tau_k, k = 1, 2, \dots,$ independent of $X_k, k = 1, 2, \dots,$ constitute a renewal counting process

$$N_t = \#\{k = 1, 2, \dots : \tau_k \le t\}, \qquad t \ge 0. \tag{1.1}$$

That is to say, the inter-arrival times $\theta_1 = \tau_1, \theta_k = \tau_k - \tau_{k-1}, k = 2, 3, \dots,$ constitute another sequence of i.i.d., nonnegative, and not-degenerate-at-zero random variables. If $\{N_t, t \ge 0\}$ is a homogeneous Poisson process, then this model reduces to the commonly used compound Poisson model. Aggregate claims form a random sum $X(t) = \sum_{k=1}^{N_t} X_k$, $t \ge 0$. Suppose that

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there is a constant interest force $r > 0$. The discounted aggregate claims are expressed as the stochastic process

$$D_r(t) = \int_{0-}^{t} \mathrm{e}^{-rs} \mathrm{d}X(t) = \sum_{k=1}^{\infty} X_k \mathrm{e}^{-r\tau_k} 1_{(\tau_k \leq t)}, \qquad t \geq 0, \tag{1.2}$$

where the symbol 1_E denotes the indicator function of an event E.

From (1.2) we see that $\{D_r(t), t \geq 0\}$ corresponds to a special case of the stochastic integral

$$Z_t = \int_{0-}^{t} \mathrm{e}^{-R_s} \mathrm{d}P_s, \qquad t \geq 0,$$

where $\{R_t, t \geq 0\}$ and $\{P_t, t \geq 0\}$ are two independent stochastic processes fulfilling certain requirements so that Z_∞ is well defined. When both of them are Lévy processes, Gjessing and Paulsen (1997) gave a wealth of examples showing the exact distribution or asymptotic tail probability of Z_∞. Related discussions on the distribution of Z_∞ can also be found in Dufresne (1990), Paulsen (1993, 1997), and Nilsen and Paulsen (1996), among others. However, we notice that all these references did not pay particular attention to the important case that $\{P_t, t \geq 0\}$ has heavy-tailed jumps.

In this paper, we are interested in the asymptotic tail behavior of $D_r(t)$ for all t for which the renewal function

$$\lambda_t = \mathrm{E}N_t = \sum_{k=1}^{\infty} \Pr(\tau_k \leq t)$$

is positive. Define $\Lambda = \{t : \lambda_t > 0\} = \{t : \Pr(\tau_1 \leq t) > 0\}$ for later use.

We shall assume that the claim-size distribution F on $[0, \infty)$ is subexponential, denoted by $F \in \mathcal{S}$. That is to say, $\overline{F}(x) = 1 - F(x) > 0$ holds for all $x \geq 0$ and the relation

$$\lim_{x \to \infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n$$

holds for all (or, equivalently, for some) $n = 2, 3, \ldots$, where F^{n*} denotes the n-fold convolution of F. It is well known that every subexponential distribution F is long tailed, denoted by $F \in \mathcal{L}$, in the sense that the relation

$$\lim_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1$$

holds for all (or, equivalently, for some) $y \neq 0$. Moreover, the class $\mathcal{S}$ covers the class ERV of distributions with extended-regularly-varying tails. By definition, a distribution F on $[0, \infty)$ is said to belong to the class ERV if $\overline{F}(x) > 0$ holds for all $x \geq 0$ and there are some constants α and β, $0 < \alpha \leq \beta < \infty$, such that the relations

$$v^{-\beta} \leq \liminf_{x \to \infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq \limsup_{x \to \infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \leq v^{-\alpha} \tag{1.3}$$

hold for all $v \geq 1$. Note that relations (1.3) with $\alpha = \beta$ define the famous class $\mathcal{R}$ of regularly-varying-tailed distributions with regularity index $-\alpha$. Another useful distribution class is $\mathcal{A}$, which was introduced by Konstantinides *et al.* (2002). By definition, a distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{A}$ if $F \in \mathcal{S}$ and, for some $v > 1$,

$$\limsup_{x \to \infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < 1. \tag{1.4}$$

Since relation (1.4) is satisfied by almost all useful distributions with unbounded supports, we remark that the class $\mathcal{A}$ almost coincides the class $\mathcal{S}$. In conclusion,

$$\mathcal{R} \subset \mathrm{ERV} \subset \mathcal{A} \subset \mathcal{S} \subset \mathcal{L}.$$

For more details of heavy-tailed distributions in the context of insurance and finance, the reader is referred to Embrechts *et al.* (1997).

Hereafter, all limit relationships hold as x tends to ∞ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$. Furthermore, for two positive bivariate functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, we say that the asymptotic relation $a(x,t) \sim b(x,t)$ holds uniformly over all t in a nonempty set Δ if

$$\lim_{x \to \infty} \sup_{t \in \Delta} \left| \frac{a(x,t)}{b(x,t)} - 1 \right| = 0.$$

Tang (2007) investigated the tail probability of the stochastic process (1.2) and proved that, if $F \in \mathrm{ERV}$, then the relation

$$\Pr\left(D_r(t) > x\right) \sim \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s \tag{1.5}$$

holds uniformly for all $t \in \Lambda$. This formula transparently captures all stochastic information of the claim sizes and their arrival times. However, we point out that the assumption $F \in \mathrm{ERV}$ unfortunately excludes some important distributions such as lognormal and Weibull distributions. In the context of ruin theory, Tang (2005) and Wang (2007) obtained some similar asymptotic results as (1.5) for the finite-time ruin probability but for a fixed time horizon $t \in \Lambda$.

Our goal in this paper is to extend the work of Tang (2007) from the class ERV to the class $\mathcal{S}$ so that lognormal and heavy-tailed Weibull distributions are included. The class ERV enjoys some favorable properties like inequalities (3.1) and (3.2) in Tang (2007), which play a crucial role in establishing the main result of Tang (2007), but the class $\mathcal{S}$ does not possess such properties. Therefore, to achieve the desired extension we have to employ different approaches.

The rest of this paper consists of four sections: Section 2 presents our main results and Sections 3, 4, and 5 prove them, in turn, after preparing some necessary lemmas.

2 Main Results

The first main result of this paper is given below:

Theorem 2.1 *Consider the discounted aggregate claims described in relation (1.2). If* $F \in \mathcal{S}$*, then relation (1.5) holds uniformly for all* $t \in \Lambda_T = \Lambda \cap [0, T]$ *for arbitrarily fixed* $T \in \Lambda$*.*

In the next two main results below, we extend the set over which relation (1.5) holds uniformly to the maximal set Λ.

Theorem 2.2 *Consider the discounted aggregate claims described in relation (1.2). If* $F \in \mathcal{A}$ *and* $\Pr(\theta_1 > \delta) = 1$ *for some* $\delta > 0$*, then relation (1.5) holds uniformly for all* $t \in \Lambda$*.*

The technical assumption on the distribution of θ_1 in Theorem 2.2, though not nice-looking, causes no trouble for real applications since δ can be arbitrarily close to 0.

For a distribution F on $[0, \infty)$ with a finite positive expectation μ, denote by

$$F_e(x) = \frac{1}{\mu} \int_0^x \overline{F}(s)\mathrm{d}s, \qquad x \geq 0,$$

its equilibrium distribution function. Our third main result is given below:

Theorem 2.3 *Consider the discounted aggregate claims described in relation (1.2), in which* $\{N_t, t \geq 0\}$ *is a homogeneous Poisson process with intensity* $\lambda > 0$*. If* $F \in \mathcal{S}$ *and* $F_e \in \mathcal{A}$*, then the relation*

$$\Pr(D_r(t) > x) \sim \lambda \int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s \tag{2.1}$$

holds uniformly for all $t \in (0, \infty]$*.*

We remark that the assumptions $F \in \mathcal{S}$ and $F_e \in \mathcal{A}$ in Theorem 2.3 are satisfied by almost all useful heavy-tailed distributions such as Pareto (with finite expectation), lognormal, and Weibull distributions.

Let us illustrate the usefulness of the uniformity of (2.1). Denote by

$$\tau(x) = \inf\{t \geq 0 : D_r(t) > x\}, \qquad x > 0,$$

the time when $D_r(t)$ up-crosses the level x. Clearly, $\tau(x)$ is a defective random variable with total mass $\Pr(\tau(x) < \infty) = \Pr(D_r(\infty) \leq x) < 1$.

Let all conditions of Theorem 2.3 hold. We first consider the asymptotic behavior of the Laplace transform of $\tau(x)$. For every $u > 0$, use integration by parts and the identity $\Pr(\tau(x) \leq t) = \Pr(D_r(t) > x)$ for all $t \geq 0$ to get

$$\mathrm{E}\mathrm{e}^{-u\tau(x)} = u \int_0^\infty \Pr(D_r(t) > x)\,\mathrm{e}^{-ut}\mathrm{d}t.$$

Substituting the uniform asymptotic relation (2.1) into the above then changing the order of integrals, we have

$$\mathrm{E}\mathrm{e}^{-u\tau(x)} \sim \lambda \int_0^\infty \mathrm{e}^{-us}\overline{F}(x\mathrm{e}^{rs})\mathrm{d}s.$$

This gives an explicit asymptotic expression for the Laplace transform of $\tau(x)$.

We then consider the limiting distribution of $\tau(x)$ conditional on $(\tau(x) < \infty)$. For every fixed $t > 0$, by Theorem 2.3,

$$\Pr(\tau(x) \le t | \tau(x) < \infty) = \frac{\Pr(D_r(t) > x)}{\Pr(D_r(\infty) > x)} \sim \frac{\int_0^t \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s}{\int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}s}. \tag{2.2}$$

If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then using Theorem A3.2 of Embrechts *et al.* (1997) we see that the convergence

$$\frac{\overline{F}(x\mathrm{e}^{rs})}{\overline{F}(x)} \to \mathrm{e}^{-\alpha rs} \tag{2.3}$$

holds uniformly for all $s \in [0, \infty)$. Therefore, dividing both integrands on the right-hand side of (2.2) by $\overline{F}(x)$ then plugging (2.3), we obtain

$$\Pr(\tau(x) \le t | \tau(x) < \infty) \to 1 - \mathrm{e}^{-\alpha rt},$$

meaning that the limiting distribution under discussion is exponential.

3 Proof of Theorem 2.1

Lemma 3.1 *Let $X_k, 1 \le k \le n$ be n independent random variables, each X_k distributed by F_k. If there are n positive constants l_k, $1 \le k \le n$, and a distribution $F \in \mathcal{S}$ such that $\overline{F_k}(x) \sim l_k\overline{F}(x)$ holds for $k = 1, \ldots, n$, then for arbitrarily fixed numbers a and b, $0 < a \le b < \infty$, the relation*

$$\Pr\left(\sum_{k=1}^n c_k X_k > x\right) \sim \sum_{k=1}^n \overline{F_k}(x/c_k)$$

holds uniformly for all $(c_1, \ldots, c_n) \in [a, b] \times \cdots \times [a, b]$.

Proof. The proof can be given by going along the same lines of the proof of Proposition 5.1 of Tang and Tsitsiashvili (2003) with some obvious modifications. ■

Lemma 3.2 *Consider the renewal counting process $\{N_t, t \ge 0\}$ defined in (1.1). There exits some $h > 0$ such that $\mathrm{E}\mathrm{e}^{hN_t} < \infty$ holds for all $t \ge 0$.*

Proof. It is shown in Stein (1946) that, for arbitrarily fixed $t_0 > 0$, there exists some $h > 0$ such that $\mathrm{E}\mathrm{e}^{hN_{t_0}} < \infty$. For every $t \geq 0$, we can find a positive integer k such that $(k-1)\,t_0 \leq t < kt_0$. Inductively applying Lemma 2.2 of Cai and Kalashnikov (2000), we can obtain i.i.d. random variables $\widehat{N}_{t_0}(1), \ldots, \widehat{N}_{t_0}(k)$ with common distribution as that of N_{t_0} such that

$$N_t \leq N_{kt_0} \leq_p \sum_{i=1}^{k} \widehat{N}_{t_0}(i) + k - 1,$$

where for two random variables X and Y, the relation $X \leq_p Y$ means that $\Pr(X > x) \leq \Pr(Y > x)$ for all x. Therefore, $\mathrm{E}\mathrm{e}^{hN_t} < \infty$, as claimed. ■

Proof of Theorem 2.1:

Arbitrarily choose some positive integer N. Clearly, for $t \in \Lambda_T$,

$$\begin{aligned}\Pr(D_r(t) > x) &= \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} > x, N_t = n\right) \\ &= I_1(x,t,N) + I_2(x,t,N).\end{aligned}$$

First consider $I_2(x,t,N)$. We have

$$\begin{aligned}I_2(x,t,N) &\leq \sum_{n=N+1}^{\infty} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_1} > x, \tau_n \leq t < \tau_{n+1}\right) \\ &= \sum_{n=N+1}^{\infty} \int_{0-}^{t} \Pr\left(\sum_{k=1}^{n} X_k \mathrm{e}^{-rs} > x, \tau_n - \tau_1 \leq t - s < \tau_{n+1} - \tau_1\right) \Pr(\tau_1 \in \mathrm{d}s) \\ &= \sum_{n=N+1}^{\infty} \int_{0-}^{t} \Pr\left(\sum_{k=1}^{n} X_k > x\mathrm{e}^{rs}\right) \Pr(N_{t-s} = n-1) \Pr(\tau_1 \in \mathrm{d}s) \\ &\leq \sum_{n=N}^{\infty} \int_{0-}^{t} \Pr\left(\sum_{k=1}^{n+1} X_k > x\mathrm{e}^{rs}\right) \Pr(N_{t-s} = n) \,\mathrm{d}\lambda_s.\end{aligned}$$

Applying Lemma 1.3.5(c) of Embrechts *et al.* (1997) to the above, for every $\varepsilon > 0$ and some $C_\varepsilon > 0$,

$$\begin{aligned}I_2(x,t,N) &\leq C_\varepsilon (1+\varepsilon) \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs}) \mathrm{E}(1+\varepsilon)^{N_{t-s}} 1_{(N_{t-s} \geq N)} \mathrm{d}\lambda_s \\ &\leq C_\varepsilon (1+\varepsilon) \mathrm{E}(1+\varepsilon)^{N_T} 1_{(N_T \geq N)} \int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs}) \mathrm{d}\lambda_s.\end{aligned}$$

By Lemma 3.2, we can choose some ε sufficiently small such that $\mathrm{E}(1+\varepsilon)^{N_T} < \infty$. It follows that $\mathrm{E}(1+\varepsilon)^{N_T} 1_{(N_T \geq N)} \to 0$ as $N \to \infty$. Therefore, for all $x > 0$,

$$\lim_{N\to\infty} \sup_{t \in \Lambda_T} \frac{I_2(x,t,N)}{\int_{0-}^{t} \overline{F}(x\mathrm{e}^{rs})\,\mathrm{d}\lambda_s} = 0. \tag{3.1}$$

Next consider $I_1(x,t,N)$. Using Lemma 3.1, it holds uniformly for all $t \in \Lambda_T$ that

$$\begin{aligned} I_1(x,t,N) &\sim \left(\sum_{n=1}^{\infty} \sum_{k=1}^{n} - \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \right) \Pr\left(X_k \mathrm{e}^{-r\tau_k} > x, N_t = n \right) \\ &= I_{11}(x,t) - I_{12}(x,t,N). \end{aligned}$$

Clearly, for all $t \in \Lambda_T$,

$$I_{11}(x,t) = \sum_{k=1}^{\infty} \Pr\left(X_k \mathrm{e}^{-r\tau_k} > x, N_t \geq k \right) = \int_{0-}^{t} \overline{F}\left(x\mathrm{e}^{rs} \right) \mathrm{d}\lambda_s. \tag{3.2}$$

For $I_{12}(x,t,N)$, similarly to the derivation for $I_2(x,t,N)$, we have

$$\begin{aligned} I_{12}(x,t,N) &\leq \sum_{n=N+1}^{\infty} \sum_{k=1}^{n} \Pr\left(X_k \mathrm{e}^{-r\tau_1} > x, N_t = n \right) \\ &\leq \sum_{n=N}^{\infty} \sum_{k=1}^{n+1} \int_{0-}^{t} \overline{F}\left(x\mathrm{e}^{rs} \right) \Pr\left(N_{t-s} = n \right) \mathrm{d}\lambda_s \\ &\leq \int_{0-}^{t} \overline{F}\left(x\mathrm{e}^{rs} \right) \mathrm{d}\lambda_s \sum_{n=N}^{\infty} (n+1) \Pr\left(N_T \geq n \right). \end{aligned}$$

It follows that, for all $x > 0$,

$$\lim_{N\to\infty} \sup_{t\in\Lambda_T} \frac{I_{12}(x,t,N)}{\int_{0-}^{t} \overline{F}\left(x\mathrm{e}^{rs} \right) \mathrm{d}\lambda_s} = 0. \tag{3.3}$$

From (3.1), (3.2), and (3.3) we conclude that the asymptotic relation (1.5) holds uniformly for all $t \in \Lambda_T$.

4 Proof of Theorem 2.2

Lemma 4.1 *If a distribution F on $[0,\infty)$ satisfies (1.4) for some $v > 1$, then there are positive constants p, C, and x_0 such that the inequality*

$$\frac{\overline{F}(xy)}{\overline{F}(x)} \leq Cy^{-p} \tag{4.1}$$

holds uniformly for $xy \geq x \geq x_0$.

Proof. This is a restatement of Proposition 2.2.1 of Bingham *et al.* (1989). See also (2.3) in Chen *et al.* (2005). ■

Lemma 4.2 *If a distribution F on $[0,\infty)$ satisfies (1.4) for some $v>1$, then*

$$\lim_{t\to\infty}\limsup_{x\to\infty}\frac{\int_t^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s}{\int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s}=0, \tag{4.2}$$

where the positive constant r and the renewal function λ_s, $s\geq 0$, are the same as introduced in Section 1.

Proof. For every $t\in\Lambda$, apply inequality (4.1) to obtain that, for $x\geq x_0$,

$$\frac{\int_t^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s}{\int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}\lambda_s}=\frac{\int_t^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)/\overline{F}\left(x\mathrm{e}^{rt}\right)\mathrm{d}\lambda_s}{\int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right)/\overline{F}\left(x\mathrm{e}^{rt}\right)\mathrm{d}\lambda_s}\leq C^2\frac{\int_t^\infty \mathrm{e}^{-pr(s-t)}\mathrm{d}\lambda_s}{\int_0^t \mathrm{e}^{pr(t-s)}\mathrm{d}\lambda_s}=C^2\frac{\int_t^\infty \mathrm{e}^{-prs}\mathrm{d}\lambda_s}{\int_0^t \mathrm{e}^{-prs}\mathrm{d}\lambda_s}.$$

This implies (4.2). ■

Lemma 4.3 *Under the conditions of Theorem 2.2, we have*

$$\Pr\left(D_r(\infty)>x\right)\lesssim\int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s. \tag{4.3}$$

Proof. Arbitrarily choose some positive integer N such that $N\delta\in\Lambda$. Since $\Pr\left(\theta_1>\delta\right)=1$, we have

$$\Pr\left(D_r(\infty)>x\right)\leq\Pr\left(\sum_{k=1}^N X_k\mathrm{e}^{-r\tau_k}+\left(\sum_{k=N+1}^\infty X_k\mathrm{e}^{-r(k-N)\delta}\right)\mathrm{e}^{-r\tau_N}>x\right). \tag{4.4}$$

Write $\Sigma_\delta=\sum_{k=N+1}^\infty X_k\mathrm{e}^{-r(k-N)\delta}$, whose distribution does not depend on N. Applying Corollary 3.1 of Chen *et al.* (2005),

$$\Pr\left(\Sigma_\delta>x\right)=\Pr\left(\sum_{k=1}^\infty X_k\mathrm{e}^{-rk\delta}>x\right)\sim\overline{F}\left(x\right)\sum_{k=1}^\infty\frac{\overline{F}\left(x\mathrm{e}^{rk\delta}\right)}{\overline{F}\left(x\right)}.$$

Hence, by inequality (4.1), there is some constant $C_*>0$ such that $\Pr\left(\Sigma_\delta>x\right)\leq C_*\overline{F}\left(x\right)$ for all $x\in[0,\infty)$. Next we come back to (4.4). Introduce a new random variable $\widetilde{\Sigma_\delta}$ independent of $\{X_k,k=1,2,\ldots\}$ and $\{\tau_k,k=1,2,\ldots\}$ with a tail satisfying

$$\Pr\left(\widetilde{\Sigma_\delta}>x\right)=\min\left\{C_*\overline{F}\left(x\right),1\right\},\qquad x\geq 0.$$

Therefore, $\Sigma_\delta\leq_p\widetilde{\Sigma_\delta}$, and

$$\Pr\left(D_r(\infty)>x\right)\lesssim\Pr\left(\sum_{k=1}^N X_k\mathrm{e}^{-r\tau_k}+\widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N}>x\right). \tag{4.5}$$

To apply Lemma 3.1, we choose some $M_1 > 0$ and derive

$$\begin{aligned}
\Pr\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N} > x\right) &= \Pr\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N} > x, \bigcup_{i=1}^{N}(\theta_i \geq M_1)\right) \\
&\quad + \Pr\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N} > x, \bigcap_{i=1}^{N}(\theta_i < M_1)\right) \\
&= J_1(x, N, M_1) + J_2(x, N, M_1). \qquad (4.6)
\end{aligned}$$

It is easy to prove by induction on N that

$$J_1(x, N, M_1) \leq \Pr\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N} > x\right) \Pr\left(\bigcup_{i=1}^{N}(\theta_i \geq M_1)\right). \qquad (4.7)$$

Substituting (4.7) into (4.6) and rearranging the resulting inequality, we have

$$\Pr\left(\sum_{k=1}^{N} X_k \mathrm{e}^{-r\tau_k} + \widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N} > x\right) \leq \frac{J_2(x, N, M_1)}{1 - \Pr\left(\bigcup_{i=1}^{N}(\theta_i \geq M_1)\right)}.$$

Further substituting this into (4.5), applying Lemma 3.1 to $J_2(x, N, M_1)$, and letting $M_1 \to \infty$, we obtain that

$$\begin{aligned}
\Pr(D_r(\infty) > x) &\lesssim \sum_{k=1}^{N} \Pr\left(X_k \mathrm{e}^{-r\tau_k} > x\right) + \Pr\left(\widetilde{\Sigma_\delta}\mathrm{e}^{-r\tau_N} > x\right) \\
&\leq \sum_{k=1}^{\infty} \Pr\left(X_k \mathrm{e}^{-r\tau_k} > x\right) + \int_{N\delta}^{\infty} \Pr\left(\widetilde{\Sigma_\delta} > x\mathrm{e}^{rs}\right) \Pr(\tau_N \in \mathrm{d}s) \\
&\leq \int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s + C_* \int_{N\delta}^{\infty} \overline{F}(x\mathrm{e}^{rs}) \Pr(\tau_N \in \mathrm{d}s). \qquad (4.8)
\end{aligned}$$

Apply inequality (4.1) again to obtain that, for some $M_2 \in \Lambda \cap (0, N\delta]$ and all large x,

$$\frac{\int_{N\delta}^{\infty} \overline{F}(x\mathrm{e}^{rs}) \Pr(\tau_N \in \mathrm{d}s)}{\int_0^\infty \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s} \leq \frac{C\overline{F}\left(x\mathrm{e}^{rM_2}\right) \mathrm{E}\mathrm{e}^{-pr(\tau_N - M_2)}}{\int_0^{M_2} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s} \leq \frac{C}{\lambda_{M_2}} \mathrm{E}\mathrm{e}^{-pr(\tau_N - M_2)} \to 0, \qquad (4.9)$$

as $N \to \infty$. From (4.8) and (4.9), the asymptotic relation (4.3) follows immediately. ■

Proof of Theorem 2.2:

According to Lemma 4.2, for every $\varepsilon > 0$ there exists some $T_0 > 0$ such that the inequality

$$\int_{T_0}^{\infty} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s \leq \varepsilon \int_0^{T_0} \overline{F}(x\mathrm{e}^{rs})\mathrm{d}\lambda_s \qquad (4.10)$$

holds for all large x. By Theorem 2.1 and inequality (4.10), it holds uniformly for all $t \in (T_0, \infty]$ that

$$\begin{aligned}\Pr\left(D_r(t) > x\right) &\geq \Pr\left(D_r(T_0) > x\right) \\ &\sim \int_0^{T_0} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\geq \left(\int_0^t - \int_{T_0}^{\infty}\right) \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\geq (1-\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s.\end{aligned}$$

Likewise, by Lemma 4.3 and inequality (4.10), it holds uniformly for all $t \in (T_0, \infty]$ that

$$\begin{aligned}\Pr\left(D_r(t) > x\right) &\leq \Pr\left(D_r(\infty) > x\right) \\ &\lesssim \int_0^{\infty} \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\leq \left(\int_0^t + \int_{T_0}^{\infty}\right) \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \\ &\leq (1+\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s.\end{aligned}$$

Hence, for all $t \in (T_0, \infty]$ and all large x,

$$(1-2\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s \leq \Pr\left(D_r(t) > x\right) \leq (1+2\varepsilon) \int_0^t \overline{F}\left(x\mathrm{e}^{rs}\right) \mathrm{d}\lambda_s. \tag{4.11}$$

By Theorem 2.1 again, the inequalities in (4.11) also hold for all $t \in \Lambda_{T_0}$ (hence for all $t \in \Lambda$) and all large x. As $\varepsilon > 0$ is arbitrary, we complete the proof.

5 Proof of Theorem 2.3

Konstantinides *et al.* (2002) investigated the asymptotic behavior of the ruin probability of the compound Poisson model. In their model, the surplus process is expressed as

$$S_r(t) = x\mathrm{e}^{rt} + c\int_0^t \mathrm{e}^{r(t-s)}\mathrm{d}s - \sum_{k=1}^{\infty} X_k \mathrm{e}^{r(t-\tau_k)} 1_{(\tau_k \leq t)}, \qquad t \geq 0,$$

where $x \geq 0$ is the initial surplus, $c > 0$ is the constant rate at which the premiums are continuously collected, and $\{X_k, k = 1, 2, \ldots\}$, $\{\tau_k, k = 1, 2, \ldots\}$, and r are the same as appearing in relation (1.2). The counting process $\{N_t, t \geq 0\}$ generated by $\{\tau_k, k = 1, 2, \ldots\}$ is a homogeneous Poisson process with intensity $\lambda > 0$. The ruin probability is defined as

$$\psi_r(x) = \Pr\left(\inf_{0<t<\infty} S_r(t) < 0\right).$$

Theorem 2.1 of Konstantinides *et al.* (2002) shows that, if $F_e \in \mathcal{A}$, then

$$\psi_r(x) \sim \frac{\lambda}{r} \int_x^\infty \frac{\overline{F}(y)}{y} \mathrm{d}y. \tag{5.1}$$

Based on relation (5.1) we produce the following result:

Lemma 5.1 *Consider the discounted aggregate claims described in relation (1.2), in which $\{N_t, t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$. If $F_e \in \mathcal{A}$, then*

$$\Pr(D_r(\infty) > x) \sim \frac{\lambda}{r} \int_x^\infty \frac{\overline{F}(y)}{y} \mathrm{d}y. \tag{5.2}$$

Proof. In terms of the model of Konstantinides *et al.* (2002),

$$\psi_r(x) = \Pr\left(\sup_{0<t<\infty} \left(D_r(t) - c \int_0^t \mathrm{e}^{-rs} \mathrm{d}s \right) > x \right).$$

It follows that

$$\psi_r(x) \leq \Pr(D_r(\infty) > x) \leq \psi_r(x - c/r). \tag{5.3}$$

By (5.1) and integration by parts,

$$\psi_r(x) \sim \frac{\mu\lambda}{r} \left(\frac{\overline{F_e}(x)}{x} - \int_x^\infty \frac{\overline{F_e}(y)}{y^2} \mathrm{d}y \right) = \frac{\mu\lambda}{r} \left(K_{11}(x) - K_{12}(x) \right).$$

Changing x into $x - c/r$ in the above yields that

$$\psi_r(x - c/r) \sim \frac{\mu\lambda}{r} \left(\frac{\overline{F_e}(x - c/r)}{x - c/r} - \int_{x-c/r}^\infty \frac{\overline{F_e}(y)}{y^2} \mathrm{d}y \right) = \frac{\mu\lambda}{r} \left(K_{21}(x) - K_{22}(x) \right).$$

Since $F_e \in \mathcal{A} \subset \mathcal{L}$,

$$K_{11}(x) \sim K_{21}(x), \qquad K_{12}(x) \sim K_{22}(x).$$

In order to infer $\psi_r(x) \sim \psi_r(x - c/r)$, it suffices to show that

$$\limsup_{x \to \infty} \frac{K_{12}(x)}{K_{11}(x)} < 1. \tag{5.4}$$

Since $F_e \in \mathcal{A}$, there exits some ε, $0 < \varepsilon < 1$, such that $\overline{F_e}(vx) / \overline{F_e}(x) \leq 1 - \varepsilon$ holds for all large x. Hence, for all large x,

$$\begin{aligned} \frac{K_{12}(x)}{K_{11}(x)} &= \sum_{n=1}^\infty \int_{xv^{n-1}}^{xv^n} \frac{\overline{F_e}(y)}{\overline{F_e}(x)} \frac{x}{y^2} \mathrm{d}y \leq \sum_{n=1}^\infty \int_{xv^{n-1}}^{xv^n} \frac{\overline{F_e}(xv^{n-1})}{\overline{F_e}(x)} \frac{x}{y^2} \mathrm{d}y \\ &\leq \sum_{n=1}^\infty (1-\varepsilon)^{n-1} \int_{xv^{n-1}}^{xv^n} \frac{x}{y^2} \mathrm{d}y = \frac{v-1}{v-1+\varepsilon}. \end{aligned}$$

This proves (5.4). Therefore by (5.1) and (5.3), relation (5.2) follows immediately. ■

Lemma 5.2 *For a distribution F on $[0,\infty)$ with a finite positive expectation, if relation (1.4) with F replaced by F_e holds for some $v>1$, then*

$$\lim_{t\to\infty}\limsup_{x\to\infty}\frac{\int_t^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}s}{\int_0^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}s}=0. \tag{5.5}$$

Proof. Clearly,

$$\frac{\int_t^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}s}{\int_0^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}s}=\frac{-\int_{x\mathrm{e}^{rt}}^\infty \frac{1}{y}\mathrm{d}\overline{F_e}\left(y\right)}{-\int_x^\infty \frac{1}{y}\mathrm{d}\overline{F_e}\left(y\right)}=\frac{\frac{\overline{F_e}\left(x\mathrm{e}^{rt}\right)}{x\mathrm{e}^{rt}}-\int_{x\mathrm{e}^{rt}}^\infty \frac{\overline{F_e}(y)}{y^2}\mathrm{d}y}{\frac{\overline{F_e}(x)}{x}-\int_x^\infty \frac{\overline{F_e}(y)}{y^2}\mathrm{d}y}.$$

By (5.4), there is some constant $C^*>0$ such that, uniformly for all $t>0$,

$$\frac{\int_t^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}s}{\int_0^\infty \overline{F}\left(x\mathrm{e}^{rs}\right)\mathrm{d}s}\le C^*\frac{\frac{\overline{F_e}\left(x\mathrm{e}^{rt}\right)}{x\mathrm{e}^{rt}}}{\frac{\overline{F_e}(x)}{x}}\le C^*\mathrm{e}^{-rt}.$$

Therefore, (5.5) holds. ■

Proof of Theorem 2.3:

The proof can be given by copying the proof of Theorem 2.2 with the only modification that we use Lemmas 5.1 and 5.2 instead of Lemmas 4.2 and 4.3.

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