TRANSACTIONS OF SOCIETY OF ACTUARIES
1984 VOL. 36

STATISTICAL ADJUSTMENT OF MORTALITY TABLES TO REFLECT KNOWN INFORMATION

PATRICK L. BROCKETT* AND SAMUEL H. COX, JR.

ABSTRACT

This paper presents a statistical methodology based upon information theory for adjusting mortality tables to obtain exactly some known individual characteristics, while obtaining a table that is as close as possible to a standard one. Applications to unisex pricing and incorporation of physician opinions are discussed.

I. INTRODUCTION

A common problem in actuarial science concerns the use of known information about an individual to adjust a standard mortality table to reflect the individual's underwriting characteristics. Using the adjusted table, the actuary can price life contingent financial instruments. In this paper we present a statistical approach to mortality table adjustment that simultaneously adjusts survival probabilities at all ages in a consistent, logical manner. We obtain a life table that is "as close as possible"[1] to the standard table and that exactly exhibits the given individual characteristics. A topical example of such a procedure involves the problem of using a unisex life table for pension calculations. One can start with the unisex table and then systematically adjust for the particular individual characteristics to reflect his or her expected life length, or a 50 percent confidence interval on the life length, and so on.

A prototype of the general situation we consider here has been discussed by Lumsden [5]. We summarize the problem as follows: In testifying as an expert witness about the economic loss due to the wrongful death of an individual, an actuary is asked to adjust a standard mortality table to obtain a table appropriate for a particular individual. A physician testifies, as an expert witness for the same side as the actuary, that the expected remaining life of the decedent at the date of an untimely death was m years. The actuary

* Mr. Brockett, not a member of the Society, is a member of the Department of Finance and Applied Research Laboratories at the University of Texas at Austin.

[1] The precise definition of the "closeness" of two tables will be given in the next section.

must value a temporary life annuity in order to testify about the present value of the future lost earnings. In order to do so in a manner consistent with the physician's testimony, the actuary must construct a mortality table which has $e_x = m$, where x is the age of the decedent. If the standard table satisfies this condition, then there is no problem. However, since this is usually not the case, we suppose that, for the standard table, $e_x \neq m$.

Here we show explicitly how to obtain an adjusted table that is as indistinguishable as possible from the standard table and that satisfies the physician's constraint $e_x = m$.

The method we use is based on the principle of minimum discrimination information explained in the next section. In section III the prototype problem of life table adjustment is carried to a numerical conclusion. Since our prototype example is discrete in character, we phrase all the formulas for the discrete case. Section IV gives extensions and further results.

II. MINIMUM DISCRIMINATION INFORMATION ESTIMATION FOR DISCRETE DISTRIBUTIONS

Consider the problem of distinguishing between two probability densities f and g after observing the value t of the random variable under study. In the application considered here, f and g will correspond to potential densities for the survival time of the individual. The technique presented here, however, is applicable to other problems of interest to the actuary (e.g., see Brockett [1]).

Now, for distinguishing between two densities f and g, the statistic $\ln [f(t)/g(t)]$ is a sufficient statistic and represents the log odds ratio in favor of the observation having come from f. It can be thought of as the amount of information contained in the particular observation t for discriminating in favor of f over g (cf. Kullback [4]). In a long sequence of observations from f, the long-run average log odds ratio is

$$E_t\left(\ln \frac{f(t)}{g(t)}\right) = \sum_l f(t_l) \ln \frac{f(t_l)}{g(t_l)}, \tag{2.1}$$

which represents the expected amount of information in an observation for discriminating between f and g. In the statistics and engineering literature this quantity is called the *divergence* between the densities f and g and is denoted by $I(f|g)$. It is not difficult to show that $I(f|g) \geq 0$, with $I(f|g) = 0$ if and only if $f = g$. Thus, the size of $I(f|g)$ is a measure of the closeness of the densities f and g. Such a global measure of closeness of densities will be very useful for adjusting mortality tables.

Suppose that we are given a density function g, and we wish to find another density f that is as close as possible to g, and that satisfies certain moment constraints, such as

$$\begin{aligned} 1 = \theta_0 &= \sum f_l, \\ \theta_1 &= \sum a_1 (t_l) f_l, \\ &\vdots \\ \theta_k &= \sum a_k (t_l) f_l, \end{aligned} \tag{2.2}$$

For example, if $a_1(t) = t$, then the first constraint says that the mean for f is known to be θ_1. Similarly, by taking $a_1(t)$ to be unity on a certain interval and zero off the interval, we arrive at a constraint on the probability for that interval. This would be useful, for example, if one wanted to use a medical study that gives decennial survival probabilities; however, yearly (or more frequent) survival probabilities are required. One would then find a survival density that was as close as possible to a standard mortality table, and that reflected the decennial survival rates quoted by the medical study.

To phrase the problem mathematically, we desire to find a vector of probabilities $f = (f_1, f_2, \ldots)$ that solves the problem

$$\min I\,(f|g) \tag{2.3}$$

subject to the constraints (2.2). Here $\mathbf{g} = (g_1, g_2, \ldots)$ is the vector of probabilities corresponding to the standard probability distribution.

Brockett, Charnes, and Cooper [2] show that the problem (2.3) has a unique solution, which is of the form

$$f_l = g_l \exp\,[-(\beta_0 + 1) - \beta_1 a_1(t_l) - \ldots - \beta_k a_k(t_l)], \tag{2.4}$$

where the β_i's are constant parameters selected in such a way that the constraints (2.2) are all satisfied. They further show that the parameters β_i can be obtained easily as the dual variables in an unconstrained convex programming problem, viz.,

$$\min_\beta \sum \left\{ g_i \exp\,[-(\beta_0 + 1) - \beta_1 a_l (t_l) - \ldots - \beta_k a_k (t_l)] \atop - (\beta_0 + \theta_1\beta_1 + \ldots + \theta_k\beta_k). \right\} \tag{2.5}$$

The solution to (2.5) can be obtained easily by any of a number of efficient nonlinear programming codes. In the following section we use the Newton-Raphson technique.

III. INFORMATION-THEORETIC LIFE TABLE ADJUSTMENTS

The study of life contingencies is intrinsically a study of biostatistics. For example, the life expectation is the expected value of a random variable K that equals the integral number of years a person now aged x will live. We have $K = 0$ with probability q_x, $K = 1$ with probability $p_x\, q_{x+1}$, etc.

According to the standard mortality table, the distribution of the random variable K is given by the $(\omega - x + 1)$ dimensional probability vector $g = (g_0, g_1, \ldots, g_{\omega-x})$, where $g_k = {}_kp_x\, q_{x+k}$ for $k = 0, 1, \ldots, \omega - x - 1$.

Consider now the problem of finding the mortality table that is as close as possible to the standard table, and that satisfies certain given constraints. This translates into finding a probability distribution $f = (f_0, f_1, \ldots, f_{\omega-x})$ for the random variable K that satisifies the desired constraints. If, for example, the desired constraints involve the expectation of functions such as those given in section II, then the density (2.4) is the least distinguishable density from g among the class of all densities satisfying the constraints.

Returning to the problem considered by Lumsden [4], the physician has testified that the expectation of life for the decedent is m more years. Thus, the constraint set is

$$1 = \sum f_l, \qquad m = \sum k f_k \tag{3.1}$$

(all sums are over $\{0, 1, 2, \ldots, \omega - x\}$). Appealing to the principle of minimum discrimination information, we select the density f to satisfy

$$\min I(f|g) = \min \sum f_l \ln (f_l|g_l)$$

subject to the constraints (3.1).

We could now of course apply the result (2.4) directly; however, it is perhaps more instructive to show how to obtain the desired density directly by standard methods in this simple situation. Let $n = \omega - x$. The probability distributions that we are considering can be viewed as $n + 1$ vectors $f = (f_0, f_1, \ldots, f_n)$ that satisfy $f_k \geq 0$, $\Sigma f_k = 1$, and $\Sigma k f_k = m$. Letting β_0 and β_1 denote the Lagrangian multipliers for the equality constraints (3.1) allows us to replace the original problem: minimize the function

$$L(f, \beta) = \sum f_k \ln (f_k|g_k) - \beta_0 (1 - \sum f_k) - \beta_1 (m - \sum kf_k)$$

subject to $f_k \geq 0$, $k = 0, \ldots, n$. The $n + 3$ first-order conditions found by differentiating with respect to $f_0, f_1, \ldots, f_n$, β_0, and β_1 are as follows:

$$\ln (f_k|g_k) + 1 + \beta_0 + k\beta_1 = 0, \qquad k=0, \ldots, n;$$

$$-1 + \sum f_k = 0;$$

$$-m + \sum kf_k = 0.$$

The first $n + 1$ equations give $f_k = g_k \exp(-1 - \beta_0 - k\beta_1)$ for $k = 0, \ldots, n$. The last two equalities allow us to find the parameters $1 + \beta_0$ and β_1. Consider the function $\phi(\beta_1) = \Sigma g_k e^{-k\hat{\beta}}$. Since $\Sigma f_k = 1$, we have $1 = \Sigma g_k e^{-l-\beta_0-k\beta_1} = e^{-1-\beta_0} \phi(\beta_1)$. Therefore, $1 + \beta_0 = \ln \phi(\beta_1)$. Because $\phi'(\beta) = - \Sigma g_k k e^{-k\beta}$, we obtain

$$\phi'(\beta_1) = - \Sigma k g_k e^{-k\beta_1} = - e^{(1 + \beta_0)} \Sigma k g_k e^{-1 - \beta_0 - k\beta_1} =$$

$$- e^{(1 + \beta_0)} \Sigma k f_k = - e^{(1 + \beta_0)} m = - \phi(\beta_1) m.$$

Thus, in order to find the precise numerical value for β_1, we solve

$$\phi'(\beta_1) = - \phi(\beta_1) m,$$

or equivalently,

$$\frac{d}{d\beta} \ln [\phi(\beta)] = - m$$

for $\beta = \beta_1$. We then may obtain the other parameter β_0 through the equation $1 + \beta_0 = \ln \phi(\beta_1)$. After we have the two parameters β_0 and β_1, we easily calculate the desired density $f_k = g_k e^{-(1+\beta_0+\beta_1 k)}$.

We used Newton's method to solve $d[\ln \phi(\beta)]/d\beta = -m$ for β_1. Recall that to solve an equation of the form $F(\beta) = 0$ by Newton's method, one uses the recursion relation

$$\beta^{j+1} = \beta^j - F(\beta^j) / F'(\beta^j).$$

In our case $F(\beta) = d[\ln \phi(\beta)]/d\beta + m$, and this reduces to

$$\beta^{j+1} = \beta^j - \frac{\phi'(\beta)/\phi(\beta) + m}{\{\phi''(\beta)\phi(\beta) - [\phi'(\beta)]^2\} / \phi(\beta)^2}\Big|_{\beta=\beta^j}$$

$$= \beta^j - \frac{\phi'(\beta^j)\,\phi(\beta^j) + m\phi(\beta^j)^2}{\{\phi''(\beta^j)\phi(\beta^j) - [\phi'(\beta^j)]^2\}},$$

where $\phi(\beta) = \Sigma g_k e^{-k\beta}$, $\phi'(\beta) = -\Sigma k g_k e^{-k\beta}$, and $\phi''(\beta) = \Sigma k^2 g_k e^{-k\beta}$.

For illustrative purposes, we shall do a numerical example that is a special case of the above. The standard table used is the U.S. Population Table for 1978. The life lost is a male, aged 45, having a life expectancy of $m = 8$ years. The standard and adjusted tables are shown below (Table 1). In this example we found $1 + \beta_0 = -3.0806$ and $\beta_1 = 0.18417$, so that the adjusted table satisfies $f_k = (21.7715)g_k\,(0.838)^k$.

TABLE 1

ADJUSTED AND STANDARD MORTALITY TABLE FOR MALE AGE 45.

YEAR k	AGE x	STANDARD TABLE RATE q_x	STANDARD TABLE PROBABILITY g_k	STANDARD TABLE SURVIVAL FUNCTION l_k	ADJUSTED TABLE RATE q'_x	ADJUSTED TABLE PROBABILITY f_k	ADJUSTED TABLE SURVIVAL FUNCTION l'_k
0	45	0.00444	0.00444	1.00000	0.09667	0.09667	1.00000
1	46	0.00493	0.00491	0.99556	0.09840	0.08888	0.90333
2	47	0.00548	0.00543	0.99065	0.10041	0.08178	0.81445
3	48	0.00608	0.00599	0.98522	0.10244	0.07506	0.73267
4	49	0.00673	0.00659	0.97923	0.10444	0.06868	0.65762
5	50	0.00746	0.00726	0.97264	0.10681	0.06290	0.58893
6	51	0.00825	0.00796	0.96539	0.10918	0.05743	0.52603
7	52	0.00905	0.00866	0.95742	0.11091	0.05197	0.46860
8	53	0.00985	0.00935	0.94876	0.11191	0.04662	0.41663
9	54	0.01068	0.01003	0.93941	0.11253	0.04164	0.37000
10	55	0.01153	0.01072	0.92938	0.11265	0.03699	0.32837
11	56	0.01249	0.01147	0.91866	0.11307	0.03295	0.29138
12	57	0.01370	0.01243	0.90719	0.11486	0.02968	0.25843
13	58	0.01525	0.01365	0.89476	0.11850	0.02711	0.22875
14	59	0.01706	0.01503	0.88112	0.12319	0.02484	0.20164
15	60	0.01907	0.01652	0.86608	0.12840	0.02270	0.17680
16	61	0.02113	0.01795	0.84957	0.13318	0.02052	0.15410
17	62	0.02315	0.01925	0.83162	0.13706	0.01831	0.13358
18	63	0.02504	0.02034	0.81236	0.13959	0.01609	0.11527
19	64	0.02688	0.02129	0.79202	0.14124	0.01401	0.09918

TABLE 1—*Continued*

Year k	Age x	Standard Table Rate q_x	Standard Table Probability g_k	Standard Table Survival Function l_k	Adjusted Table Rate q'_x	Adjusted Table Probability f_k	Adjusted Table Survival Function l'_k
20	65	0.02875	0.02216	0.77073	0.14239	0.01213	0.08517
21	66	0.03085	0.02309	0.74857	0.14394	0.01051	0.07304
22	67	0.03323	0.02411	0.72548	0.14599	0.00913	0.06253
23	68	0.03603	0.02527	0.70137	0.14905	0.00796	0.05340
24	69	0.03922	0.02652	0.67610	0.15289	0.00695	0.04544
25	70	0.04264	0.02770	0.64959	0.15682	0.00604	0.03849
26	71	0.04628	0.02878	0.62189	0.16072	0.00522	0.03246
27	72	0.05030	0.02983	0.59311	0.16513	0.00450	0.02724
28	73	0.05483	0.03088	0.56327	0.17033	0.00387	0.02274
29	74	0.05982	0.03185	0.53239	0.17606	0.00332	0.01887
30	75	0.06528	0.03268	0.50054	0.18234	0.00284	0.01555
31	76	0.07113	0.03328	0.46787	0.18899	0.00240	0.01271
32	77	0.07740	0.03364	0.43459	0.19587	0.00202	0.01031
33	78	0.08400	0.03368	0.40095	0.20288	0.00168	0.00829
34	79	0.09100	0.03342	0.36727	0.21003	0.00139	0.00661
35	80	0.09829	0.03281	0.33385	0.21720	0.00113	0.00522
36	81	0.10591	0.03188	0.30103	0.22403	0.00092	0.00409
37	82	0.11366	0.03059	0.26915	0.23070	0.00073	0.00317
38	83	0.13421	0.03202	0.23856	0.26092	0.00064	0.00244
39	84	0.13845	0.02860	0.20654	0.26276	0.00047	0.00180
40	85	0.13979	0.02488	0.17795	0.25731	0.00034	0.00133
41	86	0.15229	0.02331	0.15307	0.26945	0.00027	0.00099
42	87	0.16524	0.02144	0.12976	0.28346	0.00020	0.00072
43	88	0.17778	0.01926	0.10832	0.29318	0.00015	0.00052
44	89	0.18964	0.01689	0.08906	0.30897	0.00011	0.00036
45	90	0.20152	0.01454	0.07217	0.31864	0.00008	0.00025
46	91	0.21481	0.01238	0.05763	0.33171	0.00006	0.00017
47	92	0.28758	0.01301	0.04525	0.41086	0.00005	0.00012
48	93	0.27789	0.00896	0.03224	0.40325	0.00003	0.00007
49	94	0.18381	0.00428	0.02328	0.28013	0.00001	0.00004
50	95	0.26173	0.00497	0.01900	0.36107	0.00001	0.00003
51	96	0.27789	0.00390	0.01403	0.35313	0.00001	0.00002
52	97	0.29289	0.00297	0.01013	0.44711	0.00000	0.00001
53	98	0.30562	0.00219	0.00716	0.27441	0.00000	0.00001
54	99	0.32054	0.00159	0.00497	1.00000	0.00000	0.00000
55	100	0.33192	0.00112	0.00338	1.00000	0.00000	0.00000
56	101	0.34447	0.00078	0.00226	1.00000	0.00000	0.00000
57	102	0.35560	0.00053	0.00148	1.00000	0.00000	0.00000
58	103	0.36789	0.00035	0.00095	1.00000	0.00000	0.00000
59	104	0.38009	0.00023	0.00060	1.00000	0.00000	0.00000
60	105	0.38838	0.00015	0.00037	1.00000	0.00000	0.00000
61	106	0.40000	0.00009	0.00023	1.00000	0.00000	0.00000
62	107	0.38750	0.00005	0.00014	1.00000	0.00000	0.00000
63	108	0.45578	0.00004	0.00008	1.00000	0.00000	0.00000
64	109	0.33750	0.00002	0.00005	1.00000	0.00000	0.00000
65	110	0.50943	0.00002	0.00003	1.00000	0.00000	0.00000
66	111	1.00000	0.00001	0.00001	1.00000	0.00000	0.00000
Totals			1.00000	28.47310 $= e_x$		1.00000	7.99995 $= e'_x$

*Standard table is U.S. Population Table for 1978. Adjusted table is the result of requiring the expectation of life to be eight years, while selecting the table as close as possible to the standard.

IV. CONCLUSIONS AND EXTENSIONS

The technique presented in the previous section also could be used in underwriting individual life insurance or annuities. If an underwriter had an estimate of remaining life expectancy as the result of a medical examination, this method could be used to construct a mortality table that then would be used to calculate the gross premium. If the expectation was close to the tabular value, then the resulting gross premium would be close to the standard. If the expectation was less than this value, this method would produce a gross premium that is larger. However, the resulting deviation from the standard value would reflect only the known information that the life expectancy of the candidate for insurance is not average. If the standard table is a unisex mortality table, and the information gathered from the medical examination (e.g., family history, smoking habits, drinking habits, medical abnormalities, and the like) is translated into relative risk measures by consultation with pertinent medical studies, then this information easily may be incorporated into the constraint set (2.2). The resulting adjusted life table reflects only individual characteristics and hence is not sex-biased. In a sense we are giving the candidate the benefit of the doubt by making the individual's mortality table as close as possible to the standard. The numerical computations involved in implementing the proposed procedure, even when there are many constraints (much information gathered about the potential insured), are carried out easily using the unconstrained dual mathematical programming problem (2.5). One can envision an underwriter adjusting a standard table in his or her own office interactively and obtaining premium quotes immediately that are simultaneously statistically valid for risk pooling purposes, and also sufficiently individually tailored to satisfy recent court rulings on sexual discrimination.

The technique can be extended in other directions as well. It may be true that different mortality tables have different shapes that reflect the various different causes of death and their rising and falling incidence at different ages of life. For example, if a person has cirrhosis of the liver, the shape of this person's mortality curve may be different from the standard. If the actuary actually has concrete information about this shape, then the shape of the desired curve can be input easily into the constraint set (2.2). Increasing mortality rates, for example, are input in the form $f_i - f_{i+1} \leq 0$. Smoothness also can be input in the form of second-difference constraints. In fact, the information-theoretic method exposed here also can be used as a mortality graduation technique with monotonicity and smoothness constraints forced by (2.2). We shall pursue this topic elsewhere. If the actuary does not have any concrete information on how the curve shape should be changed, then it would be incorrect to input any forced shape other than the standard. In effect our technique allows the actuary to input any prior information actually possessed, just as Bayesian analysis would do, without the subjectivity of Bayesian prior distributions. At any rate, the mortality table adjusting techniques allows for adjustment in a statistically valid manner.

A final extension of our technique also should be mentioned: the inequality constrained case. If, for example, the medical doctor projected a life expectancy of between five and twelve years as opposed to exactly eight years in our numerical example, then the second constraint in (3.1) would become an inequality rather than an equality constraint. This may be handled by introducing slack variables into the convex programming problem (2.3). Charnes, Cooper, and Seiford [3] show that the numerical computation of the Lagrange multipliers is accomplished easily in this case by reference to the dual convex programming problem that has only nonnegativity constraints. Thus, the computations in the inequality constrained situation are carried out easily on any computer.

REFERENCES

1. BROCKETT, PATRICK L. "Using a Standard Distribution and Client Data to Obtain a Client Loss Distribution," *PCAPP*, XXXIV (1984–85), 503–11.
2. BROCKETT, PATRICK L., CHARNES, A., and COOPER, W. W. "MDI Estimation via Unconstrained Convex Programming," *Communications in Statistics*, IX, Ser. B, No. 3 (1980), 223–34.
3. CHARNES, A., COOPER, W. W., and SEIFORD, L. "Extremal Principles and Optimization Dualities for Khinchine-Kullback-Leibler Estimation," *Mathematische Operationsforschung und Statistik, Series Optimization*, IX, no. 1 (1978), 21–29.
4. KULLBACK, S. *Information Theory and Statistics*. New York: John Wiley & Sons, 1959.
5. LUMSDEN, WILLIAM F. "Why to Avoid Testifying to an Annuity-certain for Life Expectancy," *PCAPP*, XXXI (1981–82), 416–33.

DISCUSSION OF PRECEDING PAPER

ALLAN C. WEAVER:

This interesting paper raises some issues upon which I'd like to comment.

In section I, does the actuary and the physician have the same understanding of terms used in testimony? What does the physician mean exactly by a life expectancy of eight years? He might mean that half the number of lives in a group similarly afflicted (as the decedent was afflicted) would die before eight years. Note that the authors' adjusted table shows that half the group dies after six and a half years. In that case, it seems the "known" information is different for the actuary and the physician.

In section III, one of the problems with Newton's method in testing for convergence is when it is programmed on a computer, it has the tendency for the algorithm to oscillate between two or more final values rather than to converge to one final result. This tendency can be alleviated by establishing tolerance limits and by testing convergence by changes on the largest parameter, which in this case is Ø″ (β). The nature of the problem also suggests an initial trial value of β_1 as zero. As much precision as possible is desirable. While the procedure described is ideally suited for APL, in other languages looping and indexing time can be saved by separately defining the summation element $g_k \cdot e^{-k \cdot \beta i}$ or $g_k / e^{k \cdot \beta i}$ within the loop as, say, term t since the program will need Σt, Σkt and $\Sigma k \cdot k \cdot t$ for the iteration. The adjustment (negative) then becomes:

$$\frac{(\Sigma t)\ (\Sigma k \cdot t)\ -\ m(\Sigma k \cdot t)^2}{(\Sigma k)\ (\Sigma k \cdot k \cdot t)\ -\ (\Sigma k \cdot t)^2},$$

and convergence can be tested on the $\Sigma\ k \cdot k \cdot t$ value. After the result β_1 is obtained, $1 + \beta_0 = \ln \Sigma t$.

I suggest some techniques to complete the authors' table at high ages. After the f_k values are calculated, normalize them by f_k (adjusted) $= f_k$ (formula)$/\Sigma f_k$ since one constraint was that $\Sigma f_k = 1$. Rounding should be avoided until the final values are printed. The adjusted table survival probability can then be determined by cumulative subtraction of the f_k value from 1, ensuring an ultimate zero rather than a residual. The adjusted q values can be determined by dividing the f_k's by the adjusted table survival probabilities. The differences become significant toward the end of the table. For example, I found q values of .41393 and .42784 at ages 98 and 99, respectively.

The results in developing values on the adjusted table seem counterintuitive. It is clear that $f_k = g_k$ when $e^{(-1-\beta_0-K\beta_1)} = 1 = e^0$ or $k = \dfrac{-1-\beta_0}{\beta_1}$ where the result should be greater than $\omega - x$, but this does not always seem assured. Since the initial g values were higher on the adjusted table, this relationship implies that the life gets more hardy as it ages relative to the standard table. It seems difficult to build in the constraint $q^1x \geq q_x$, but the constraint $f_k \geq g_k$ could be handled as described in the text.

The choice of the standard table is lacking in that the table is not smooth, particularly between ages 90-95. The expectancy in the adjusted table is so much less than the standard table that possibly a disabled life table should be considered instead. Such tables exhibit altogether different characteristics from standard life tables.[1] Unless the standard table to be used (as recommended by the authors) is stipulated by law or by agreement between the parties, another table should have been chosen. Using PBGC Table V as the standard produces intuitively more acceptable results. Another point that should be remembered is that the value from a U.S. population table for age $x + 10$ will represent the rate for a generation born ten years earlier than the rate for age x. A single life will follow the mortality of the cohort. The bias involved may be considerable.

(AUTHORS' REVIEW OF DISCUSSION)

PATRICK L. BROCKETT AND SAMUEL H. COX, JR.:

We appreciate Mr. Weaver's considered discussion of our paper. We have only a few comments to add.

In section I, the problem of communication between the doctor (or other expert) and the actuary must be resolved in any framework, whether it be the information theoretic method we propose, or a Bayesian method, or some other method of analysis. The actuary must determine what the doctor means before "known information" truly has been obtained. If it turns out that the doctor truly means that the median is eight years rather than the mean, then a median constraint must be used. This is easily done. The constraint $f_8 + f_9 + \ldots + f_{\omega-x} = 0.5$ would replace $\Sigma k f_k = 8$, but the rest of the analysis would be identical. The function $a_1(t)$ used in (2.2) changes as noted in the paragraph following (2.2). Thus, the solution (2.4) is changed accordingly. Communication, as Mr. Weaver notes, is an important aspect of this analysis.

In the computation, we used the Newton-Raphson method as the numerical technique because our readers are very likely to be familiar with it since it has been on the Society of Actuaries examination syllabus for a long time.

[1]Robert T. McCrory, "Mortality Risk in Life Annuities," *TSA* XXXVI (1984) 309–49.

It is simple and easy to program. Mr. Weaver's suggestions are important for those who wish to write their own program using this method. However, Newton-Raphson may not be the best computational method available. The numerical analysis publications on nonlinear mathematical programming form a rapidly growing and widely available literature. We have taken advantage of this in other work where we have had many constraints of both the equality and inequality type. In such problems, we have approached the computational portion from the dual convex programming side given in (2.5) and used highly developed nonlinear programs such as GRGII and SUMT alluded to at the end of section 2. Using these canned computer packages alleviates the computational problems.

It is interesting that Mr. Weaver finds the results in developing values on the adjusted table counterintuitive. As Mr. Weaver notes, the constraint

$$f_k \geq g_k$$

can easily be included in our method. The constraints

$$q'_x \geq q_x$$

are satisfied in our particular example, however they may not be in some other application of our technique. If it is necessary that the annual probabilities (or rates) do not increase, then we suggest applying the technique we outlined on the annual rates directly. The technique is not restricted to probability distributions. The objective function would be

$$I(q'_x|q_x) = \Sigma q'_x \ln(q'_x/eq_x),$$

and we would minimize it subject to the constraints expressed in terms of the annual probabilities. The paper by Brockett and Zhang [1] considers graduation methods using the annual probabilities with linear equality, inequality and quadratic inequality constraints on an information theoretic functional. It applies to the situation described by Mr. Weaver. This is a much deeper problem which our paper does not address.

Mr. Weaver's comment on our standard table is very pertinent. Anyone desiring to adjust a standard table must first select an appropriate standard table to adjust. Certainly the tables Mr. Weaver mentions should be considered as candidates for the standard table. In our paper, we present a new statistical technique and illustrate it with a numerical example. Our choice of an illustrative table was not meant as a recommendation for its use in a particular problem.

REFERENCE

1. BROCKETT, P.L., AND ZHANG, J. "Information Theoretic Quadratically Constrained Graduation Techniques," Working paper, Department of Finance, The University of Texas at Austin.