TRANSACTIONS OF SOCIETY OF ACTUARIES
1993 VOL. 45

LOADING GROSS PREMIUMS FOR RISK WITHOUT USING UTILITY THEORY

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ABSTRACT

In this paper, I caution against using expected utility theory for measuring insurance risk, because of the numerous inherent difficulties associated with it. Among them is the fact that individuals do not seem to behave as described by the axioms of expected utility theory.

Instead, I develop an approach for loading gross premiums that is based on risk-measure functions. A risk-measure function, R, is a function that measures the level of "riskiness," r, inherent in an insurance risk X, where $r=R[X]$. Some basic properties of risk-measure functions are suggested. For an insurance risk X and a given risk-measure function $R[X]$, it is suggested that the insurance premium $\Pi[X]$ be calculated according to the premium calculation principle $\Pi[X]=\mu+\theta(r, \mu; R)$, where $\mu=E[X]$ and θ is the risk-loading function. The traditional properties of premium calculation principles are used to identify some possible functional forms of θ. The variance is the most commonly used risk-measure function, that is, $R[X]=Var[X]$; however, it does not adequately measure risk if the distribution of X is positively skewed. For positively skewed distributions, risk-measure functions depending on the third and/or fourth cumulants are provided. In particular, the normal power approximation is used to derive one such risk measure. Risk-measure functions based on the normal power approximation may be useful in pricing financial risks in portfolio analysis.

1. INTRODUCTION

1.1 The Premium Calculation Process

The calculation of insurance premiums is one of the most important functions of a practicing actuary. So it follows that the principles and techniques used to derive these premiums are very important. From a theoretical perspective, premiums can be split into two components: the "net (pure)" premium and the loading. It is generally agreed that the net premium is $E[X]$, where X is the amount at risk, and that loadings should take into account the "variability" or the "level of risk" in X. Unfortunately there is no general agreement on a method of loading premiums because no general principle for loading premiums has been shown to be the best one.

For this paper, the amount at risk X associated with a policy is the present value of all claims, expenses, dividends, and losses due to any special features (such as riders or options) in the policy. Note that since X includes expenses, the term "net premium" is used in a general context. Since it is generally agreed that the risk-loading should reflect the level of risk inherent in the amount at risk X, it seems reasonable to develop an explicit measure of the level of risk in X and to let the loading be a function of this quantity. As mentioned in Feldblum [19], there are different approaches to calculating the risk-loading. For example, one can use (1) the moments of X, (2) utility theory, (3) ruin theory, (4) the empirical costs of reducing risk via such methods as reinsurance, or (5) profitablility considerations in connection with modern portfolio analysis.

Once X is fully defined and the gross premium is determined, the required contingency reserve (surplus) must be found. The process of deriving the appropriate contingency reserve for a given gross premium is a very complex one; it is not the subject of this paper. However, ruin probabilities can be used to assist in the determination of the appropriate reserve level; see, for example, Brender [12].

The approach in this paper is to use risk-measure functions and risk-loading functions for calculating gross premium. Specifically, a new premium calculation principle is introduced. This principle requires the insurer to calculate risk-loaded gross single premium $\Pi[X]$ in five steps: given a risk X,

Step 1: Calculate $\mu=E[X]$.

Step 2: Choose a risk-measure function $R[\cdot]$.

Step 3: Calculate the level of riskiness, r, inherent in the insurance risk X, that is, $r=R[X]$.

Step 4: Choose a risk-loading function $\theta(\cdot, \cdot; R)$.

Step 5: Calculate the gross single premium $\Pi[X]$, as

$$\Pi[X] = \mu + \theta(r, \mu; R). \tag{1}$$

Steps 2 and 3 are very important because they require the insurer to *explicitly* choose R and to use R to calculate the risk in X. Since R will be defined (in Section 2.3) to be an objective measure of risk, this approach excludes premiums based on utility theory. In addition, see the comments (in Section 3.2) on the problems associated with expected utility theory.

The insurer also must choose a risk-loading function, θ, which is an *explicit* function of the level of risk as measured by R. Even though it is assumed for simplicity that θ depends only on μ and r, in general θ depends on other factors including the insurer's attitude towards risk-taking, the insurer's level of reserves, the overall size of the company, the size of its portfolio of risks, and the skewness and the tail behavior of the distribution of X. Of course, in a competitive environment, the actual premium charged ultimately depends on market conditions.

1.2 Objectives

This paper has three objectives:

1. To formally and explicitly include the element of risk analysis in the formulation of insurance premiums. To this end, the terms "insurance risk" and "risk-measure functions" are defined. Five desirable properties of risk-measure functions are given along with some of the well-known measures of risk in use today. A new measure of risk, based on the normal power approximation, is given.
2. To develop a risk-loaded premium using the third and/or fourth moments of X. Here the properties of premium calculation principles are discussed. Some of the well-known premium calculation principles are introduced, for example, the variance and standard deviation principles. A new premium calculation principle based on the normal power risk measure is developed. Since utility theory does not use risk-measure functions, it is not used in this paper.
3. To stimulate discussion and research on the nature and measurement of insurance risk; see the comments in Section 6.

2. RISK MEASURES

2.1 Definition of Risk

Before exploring the various measures of risk, I would like to define the term "risk." According to Williams et al. [58, pp. 6–10], risk is used in insurance to mean the following: (1) the possibility of loss, (2) the probability of loss, (3) a peril, (4) a hazard, (5) the property or person exposed to damage or loss, (6) potential losses, (7) variation in potential losses, and (8) uncertainty concerning losses.

There is a strong connection between the concepts of "risk" and "uncertainty." Freifelder [23, pp. 9–10] defines uncertainty as ". . . the lack of certainty; doubt as to the actual outcome of an event or trial of an experiment." He defines risk as ". . . the individual evaluation of the uncertainty surrounding the choice of a course of action or the outcome of an event." Thus he views risk as the subjective evaluation of uncertainty. Kaplan and Garrick [23] succinctly define risk as

$$risk = uncertainty + damage.$$

For a more philosophical discussion of risk, see Rescher [48].

The term "insurance risk," as used in this paper, is defined as follows:

Definition 1: *An insurance risk, X, is the non-negative real valued random variable that represents the present value of the insurer's total financial expenditures (attributable to the policy) over the life of the policy.*

The term "expenditure," as used in the definition above, includes the costs associated with the insurer's financial obligations to the insured and the expenses incurred over the life of the insurance contract. These costs are not confined only to claims; they also include the costs associated with all the policy's features such as riders and options. For example, features such as interest rate guarantees, policy loan provisions, guarantees against policy lapses, dividends options, waiver of premiums, and so on can be included in X.

To be precise, let $X^{(c)}$, $X^{(f)}$ and $X^{(e)}$ be the present value (at issue) of all future claims, all future costs due to special policy features, and all current and future expenses, respectively, generated by the risk over the life of the contract; then

$$X = X^{(c)} + X^{(f)} + X^{(e)}. \tag{2}$$

Not surprisingly, it may be very difficult to actually construct X for some interest-sensitive products (such as a universal life policy) with many of the options and riders mentioned above. Thus the actual implementation of Equation (1) may depend entirely on the technical skills of the pricing actuary.

All present values are assumed to be calculated under general conditions including the possibility of a stochastic interest rate. However, in practice, the interest rates used for *determining* gross premiums are usually nonstochastic.

Examples of the risks associated with X include claim size variation, investment risks, mortality/morbidity risks, lapse risks, and so on. See, for example, Lew [34] for an overview of the types of risks covered in insurance contracts. The Society of Actuaries Committee on Valuation and Related Matters classified the risks faced by insurance companies broadly into three parts, the C-1, C-2 and C-3 risks, as follows:

- The C-1 risk is the risk that an asset loses value because the borrower defaults, the asset is destroyed or the future earning power of the borrower has fallen; see Sega [51].
- The C-2 risk is the risk that the price (gross premium) is insufficient to pay losses; see Brender [12].
- The C-3 risk is the risk that there is a mismatch in timing between assets and liability cash-flow streams; see Mereu [38].

This paper is concerned primarily with the C-2 risk.

Definition 2: *A basic insurance contract consists of a single premium, Π, paid at the start of the contract and an insurance risk X. The contract may last for one period or several periods.*

2.2 Periodic Premiums

Definitions 1 and 2 may appear to be more suited to non-life insurance, in which contracts are short-term single-premium contracts, than to life insurance, in which premiums are usually paid more frequently and contracts can span several years. In addition, life insurance premiums can be level or variable (nonstochastic) or stochastic. Also, premiums can even be a part of the death benefit (as in the case of a premium refund feature). In any of these cases, the present value of the premium income is a random variable, Y. As a result, the insurer is faced with two sources of risk: X and Y.

When the premiums are scheduled to be paid more often than once, the gross premiums can be determined as follows: Let G_k be the periodic gross premium due at consecutive times t_k, $k=0, 1, 2, \ldots$ with $t_0=0$. Here Y is given by

$$Y = \sum_{k=0}^{K} G_k v_k \tag{3}$$

where $K+1$ is the actual number of premiums paid before the contract ends, and v_k is the discount factor, that is,

$$v_k = exp\left[- \int_0^{t_k} \delta(s)ds \right] \tag{4}$$

where $\delta(s)$ is the stochastic force of interest at time s. These premiums can be determined in three stages:

Stage 1: Calculate the single premium $\Pi[X]$ using Equation (1).

Stage 2: Calculate the risk-loaded single premium $\Pi[Y]$ using Equation (1) again. Here Y is being viewed as a risk as well. Note that R must be the same in stages 1 and 2.

Stage 3: By equating the two single premiums, that is, setting

$$\Pi[X] = \Pi[Y], \tag{5}$$

we can then solve for the G_k's. To solve Equation (5), the G_k's must all be known up to a single common unknown.

If $\theta \equiv 0$, then the well-known principle of equivalence results; see Bowers et al. [11, chapter 6].

Warning 1: *Given the three-stage approach leading up to Equation (5), the multiperiod case can be handled in a manner similar to the single-period case. So, without loss of generality, only a basic one-period contract is covered in this paper. It is assumed the insurer receives a fixed premium at the start of the period and in return pays all claims and expenses arising during the period.*

If premiums are stochastic, then, by definition, they cannot be determined a priori. Thus the techniques of this paper (or any other paper, as a matter of fact) cannot be used to completely specify stochastic premiums.

2.3 Properties of Risk-Measure Functions

Consider a portfolio of risks Ω, in which

$$\Omega = \{X : X \geq 0 \text{ is a random variable}\}.$$

Definition 3: *$X \in \Omega$ is said to be riskless if and only if X is a constant with probability 1; that is, there exists a constant μ such that* $\Pr[X=\mu]=1$.

For a given portfolio Ω, assume the insurer chooses a unique, real valued, risk-measure function $R[X]$, which measures the level of risk inherent in $X \in \Omega$. This function may be different for different portfolios.

Definition 4: *Given any insurance risk in Ω, a function R is called a risk-measure function if, and only if, it satisfies the following properties:*

(2.1) *Risklessness*: $X \in \Omega$ is riskless if and only if $R[X]=0$.

(2.2) *Non-negativity*: $R[X] \geq 0$ for all $X \in \Omega$.

(2.3) *Subadditivity*: If X_1, $X_2 \in \Omega$ are independent risks, then

$$R[X_1 + X_2] \leq R[X_1] + R[X_2].$$

(2.4) *Consistency*: For any $X \in \Omega$ and constant c,

$$R[X + c] = R[X].$$

(2.5) *Objectivity*: $R[X]$ depends on X only through the cumulative distribution function (cdf) $F_X(x)$ of X.

- According to the risklessness property, (2.1), risk exists if, and only if, there is a possibility of deviation from the expected.
- The subadditivity property, (2.3), reflects the fundamental principle of insurance: the overall level of risk must not increase if independent risks are pooled.
- The consistency property, (2.4), implies that the level of risk in a portfolio cannot be altered by adding "sure events" such as incurring riskless debt.
- The objectivity property, (2.5), is included to ensure that $F_X(x)$ contains *all* the information needed to measure the level of risk in X. One may argue that, by its very nature, risk is inherently subjective. However, given the cdf of X, any subjective element must lie in the insurer's choice of risk-measure function and/or the insurer's *attitude* towards risk. Once $R[\cdot]$ is chosen, the insurer's evaluation of the *level*

of the risk inherent in X must be based only on R and F. Another source of subjectivity in assessing risk lies in the choice of $F_X(x)$. Since, in practice, the cdf is usually not known completely, the insurer may have to arbitrarily assign a cdf to X. For a discussion of the objective/subjective nature of insurance risks, see Berliner [4], [5], [6]. Weirich [56] contains a discussion of an individual's attitude towards risk.

An axiomatic approach to the study of risk measures is given by Fishburn [22]. He defines X on the entire real line and uses the possibility of loss as his determinant of risk. Thus an absolutely certain loss of a constant amount μ is considered as having *positive* risk (in contrast to the risklessness property, (2.1), which would have assigned to μ a risk of zero). Fishburn's risk measure is objective in the sense of the objectivity property, (2.5).

2.4 Common Risk-Measure Functions

The well-known risk measures are as follows:

(1) *The variance*: $R_1[X]=\sigma^2[X]$. The variance is intuitively appealing as a risk measure. It has been used extensively in the economics and finance literature in the context of mean-variance analysis. The works by Markowitz [36] and Tobin [53] help to solidify the use of the variance as a risk measure. However, its use as a measure of risk is criticized by Borch [9] and Feldstein [20] for its lack of generality; that is, it is applicable only to cases in which the normal distribution is assumed or quadratic utility can be assumed. Markowitz criticizes the variance on the grounds that it considers extremely high and extremely low returns as equally undesirable. Brockett [13] proves that risk averse behavior does not always lead variance minimization when choosing between equal expected returns. According to Pentikäinen [43], in an insurance environment, the variance is not a good measure of risk because it does not give information on ruin probabilities unless a normal distribution is assumed. In addition, the variance does not give information on the degree of asymmetry (skewness) on the probability of large claims. The variance satisfies properties (2.1) to (2.5).

(2) *The standard deviation*: $R_2[X]=\sigma[X]$. Since the standard deviation contains the same information as the variance, the same comments apply as in the case of the variance.

(3) *The positive semivariance*: This measures the deviations in excess of the mean, μ; that is,

$$R[X] = \sigma_+^2[X] = \int_{\mu}^{\infty} (x - \mu)^2 \, dF_X(x).$$

This risk measure is not very popular because it is cumbersome. In addition, it does not, in general, satisfy the subadditivity property, (2.3); see [26, section 3.7]. In the area of finance, Markowitz advocates the use of the *negative semivariance* (for below-average returns),

$$\sigma_-^2[X] = \int_{-\infty}^{\mu} (\mu - x)^2 \, dF_X(x),$$

instead of the variance. Clarkson [15] develops measures of investment risk based on the investor's risk-weighting function, $W(r)$, and the negative semivariance. Since $W(r)$ is, in essence, the investor's "utility" for risk, Clarkson is in fact measuring the investor's attitude towards risk-taking, thus violating the objectivity property, (2.5). Fishburn [21] extends the semivariance concept by using a general targeted return instead of the mean.

(4) *Coefficient of variation*: $R[X]=\sigma[X]/E[X]$. This has often been called the measure of "relative" risk (as opposed to the measure of "absolute" risk in the case of the variance or standard deviation). However, it is often difficult to draw reliable conclusions about the riskiness of a heterogeneous portfolio of risks on the basis of this statistic alone. This risk measure has the attractive feature of being dimensionless, that is, scale invariant. However, it does not satisfy the consistency property, (2.4).

Stone [52] recommends the use of the coefficient of variation as a measure of the "capacity" of an insurance portfolio to accept more risk. A smaller coefficient of variation implies a larger capacity for risk acceptance.

(5) *The mean absolute deviation*: $R[X]=E|X-E[X]|$. This is similar to the variance and the standard deviation in that it assigns equal weight to deviations above and below the mean. Its advantage is that it does not stress larger deviations over smaller ones. Unfortunately, it is difficult to manipulate mathematically. The mean absolute deviation satisfies properties (2.1) to (2.5).

3. THE RISK-LOADING FUNCTION

In non-life insurance, the actual gross premium charged by an insurer for a risk X can usually be written in the form

$$\text{Gross Premium} = E[X] + \theta + \rho + \epsilon \tag{6}$$

where θ, ρ, and ϵ are the (non-negative) claim-risk, profit, and expense-risk loadings, respectively. Note that $E[X]$ already includes the expected costs of claims and expenses. Venezian [55] contains an overview of the use of risk loads in premium formulation in non-life insurance. Since the distribution of X is usually unknown, credibility theory is used to update the estimate of $E[X]$ from the past data generated by X or similar risks. However, credibility theory is *not* a risk-based technique in the sense of Equation (1); instead it focuses on approximating the mean and the variance of certain conditional random variables. See Venter [57, chapter 7] and Goovaerts et al. [27, chapter iv] for an introduction to credibility concepts.

In life insurance, deterministic techniques were devised for obtaining gross premiums because actuaries did not *explicitly* recognize the stochastic nature of life insurance; see, for example, the older life contingencies texts such as Jordan [31] and Neil [41]. Fortunately this is changing, as evidenced by the current standard life contingencies text *Actuarial Mathematics* by Bowers et al. [11]. Instead of using an equation-type of formula such as Equation (6), life actuaries have used asset share mathematics to calculate gross premiums; see Huffman [30] for more on asset share mathematics. These actuaries have compensated for random fluctuations by being "conservative," that is, assuming higher levels of mortality, expenses, and so on, and lower levels of interest rates. In general, they made assumptions about future trends that were inherently disadvantageous to the prospective insured; see [54, chapter viii]. Thus the claim-risk, profit and expense-risk loadings were involved in the gross premiums such that the gross premium could not be written in the form of Equation (6). In other words, insurers did not explicitly segregate these loading factors when deriving gross premiums. An exception to this is given in Pollard [45], in which a risk-loading factor L is explicitly given.

3.1 Properties of Premium Calculation Principles

Definition 5: *A premium calculation principle* Π *is a function that assigns a non-negative real number, called the premium, to a risk* $X \in \Omega$. *In other words,* $\Pi: F_X \to [0, \infty)$.

The well-known premium calculation principles are: (i) the expected value principle, (ii) the standard deviation principle, (iii) the variance principle, and (iv) the principle of exponential utility. See, for example, the texts by Bühlmann [14, chapter 4], Gerber [25, chapter 5], or Goovaerts et al. [26, chapter 2] for more on the theory of premium calculation principles.

Definition 6: *An insurance premium,* $\Pi[X]$, *is said to be loaded if, and only if,* $\Pi[X] \neq E[X]$. *The size of the loading is* $\Pi[X] - E[X]$.

Gerber [25, chapter 5] states the following five properties of premium calculation principles:

(3.1) *Non-negative loading*: $\Pi[X] \geq E[X]$.

(3.2) *No rip-off*: $\Pi[X] \leq \text{Max}(X)$.

(3.3a) *Consistency*: For any constant $c \geq 0$, $\Pi[c+X] = c + \Pi[X]$.

(3.4a) *Additivity*: If X_1 and X_2 are independent risks, then

$$\Pi[X_1 + X_2] = \Pi[X_1] + \Pi[X_2].$$

(3.5) *Iterativity*: If X and Y are arbitrary risks, then $\Pi[X] = \Pi[\Pi[X|Y]]$.

Clearly property (3.1) is not practical in some situations. For example, an insurer entering a new market may be forced by competition to price the product below the net premium and carry a deficiency reserve.

The consistency property, (3.3a), should be strengthened to give a new property:

(3.3b) *Linear Consistency*: For all non-negative constants a and c,

$$\Pi[aX + c] = a\Pi[X] + c.$$

This property is needed to ensure purchasing power parity and to prevent riskless arbitrage in international insurance markets. In such markets, in which premiums may be quoted in several different currencies, the premiums for identical risks must be equivalent. As an example, consider the U.S.-Canadian insurance market. If the premium for a risk X is \$$\pi$ *U.S.*, the exchange rate is \$1*U.S.*=\$*aCAN* and there is a perfect, frictionless market, the premium in Canadian dollars should be \$$a\pi$ *CAN*. As will be seen, then the variance principle violates this property. We are not suggesting that a similar property be developed for risk-measure functions; however, see the comments at the end of Section 6.

Reich [47] suggests the additivity property, (3.4a), be weakened to the one of subadditivity:

(3.4b) *Subadditivity*: If X_1 and X_2 are independent risks, then

$$\Pi[X_1 + X_2] \leq \Pi[X_1] + \Pi[X_2].$$

Subadditivity is a desirable property because it captures the spirit of insurance: the pooling of independent risks must not increase the total premium, while pooling may actually reduce it.

The iterativity property, (3.5), means that $\Pi[X]$ can be calculated in two stages: first, find the premiums for $[X|Y]$ as a function of Y; then find the premium for this function of Y. The premium calculation principle in Equation (1) is not iterative except in a few special cases; see [25, pp. 72–73]. As a result, property (3.5) is not used.

Finally, a new property is added to accommodate the risk-based approach of the premium calculation principle introduced in Equation (1).

(3.6) *Increased loading for increased risk*: This means that for fixed μ, as r increases, $\theta(r, \mu; R)$ must increase.

This property is needed to articulate a rational approach to risk-taking.

3.2 Problems of Traditional Expected Utility Theory

Deserving special mention is the class of premium calculation principles based on the traditional expected utility theory developed by von Neumann and Morgenstern (VNM) [42]. This expected utility theory is often used when there is a process of decision-making under uncertainty. The process of premium formulation is, in essence, one of decison-making under uncertainty. Not surprisingly, VNM's utility theory has been used by several authors in the actuarial literature, Borch being the leading proponent of the potential applications of this theory to various aspects of insurance; see Borch [8] and [10]. The Society of Actuaries now includes VNM's utility theory as a part of its Course 151 examination on risk theory.

Hammond [28] points out that, in practice, the top management of insurance companies does not usually articulate a clear risk-taking posture for the company. As a result, there may be several different "company positions," and hence, no consistent posture is demonstrated. The VNM utility theory can be used to resolve some of these difficulties because, once a utility function $u(x)$ is specified, consistent decisions can be made. Similarly, individual consumers of insurance products can use the VNM utility theory to make rational decisions about these products. Kahn [33] cautions that utility theory cannot be used to predict consumer behavior exactly, it can only provide clues about what is or is not "rational."

Another problem with VNM's utility theory is that utility functions for individuals or corporations are rarely known in practice. Methods for constructing utility functions have been provided by several authors, for example, Moore [39, chapter 5]. However, such empirically constructed utility functions must be suspect because it has been well-documented that individuals show inconsistencies when making even modestly complex decisions under uncertainty. In fact, McCord and de Neufville [37] demonstrate that very different utility functions are used by a decision-maker depending on the underlying probability distribution of the risk faced. In addition, there is strong evidence to the effect that decision-makers have preference orderings that can be represented only by utility functions that are convex in some intervals and concave in others. This point was made by Friedman and Savage [24] after observing that people who buy insurance are also willing to buy lottery tickets even though neither is sold at an actuarially fair price. So even a weak condition, such as a strictly concave utility function, may not suffice. A review of the empirical investigations of decision-making under uncertainty is contained in Schoemaker [50]. Recent experiments continue to show that individuals do not behave in the manner implied by VNM's utility theory and that individuals have more complex utility functions than those generally assumed; see Loomes [35] and Dahlbäch [16].

There are also theoretical problems with VNM's utility theory. It has long been argued that the axioms underlying expected utility theory, which are supposed to describe rational behavior, do not actually do so. Allais [2] shows how informed, intelligent persons choose actions that contradict the axioms of VNM's utility theory. In addition, there is confusion about the nature of the utility of X (a random variable) and of x (a certain event). Bernard [7] states this point as follows:

> "... $u(x)$ represents in the expression of the utility of a random variable the behavior in the simultaneous presence of the value x and of the *uncertainty* [author's emphasis added] about its obtaining, that is, risk aversion or risk love. So in the general case the cardinal utility of the value x [a certain event] is not the function $u(x)$ as specified in the utility $u(X)$ of the random variable, but a distinct function, ..., $v(x)$."

In view of these problems, premiums based on VNM's utility theory are suspect.

Warning 2: *Since VNM's utility theory does not explicitly calculate the level of risk in X, as required in Equation (1), it is not considered again in this paper.*

3.3 The Risk-Loading Function

The properties of risk-measure functions and premium calculation principles are now used to investigate the functional form of θ.

Theorem 1: *For any risk-measure function R satisfying properties (2.1)–(2.5) and for any premium calculation principle* Π *satisfying properties (3.1), (3.2) and (3.3b), the risk-loading function satisfies*

1. $\theta(0, \mu; R) = 0$, *and*

2. θ *is independent of* μ.

Proof: Let a and c be non-negative constants.

(1) From properties (3.1) and (3.2), $\Pi[c]=c$. Since there is no risk-loading on constants, the risklessness property, (2.1), implies

$$\theta(0, \mu; R) \equiv 0, \quad \mu \geq 0.$$

(2) From the consistency property, (2.4), $R[aX+c]=R[aX]$. Let

$$R[aX] = \rho(r, a), \tag{7}$$

where $r=R[X]=\rho(r,1)$ is the level of risk in X itself. The linear consistency property, (3.3b), says

$$\Pi[aX + c] = c + a\Pi[X]$$

$$(a\mu + c) + \theta[\rho(r, a), a\mu + c; R] = c + a[\mu + \theta(r, \mu; R)] \tag{8}$$

$$\theta[\rho(r, a), a\mu + c; R] = a\theta(r, \mu; R).$$

Since Equation (8) holds for any $c \geq 0$, it follows that $\theta(r, \mu; R)$ must be independent of μ. □

Since θ is independent of μ, the term μ will be dropped from $\theta(r, \mu; R)$ and the notation $\theta(r; R)$ will be used instead. Equation (8) can now be rewritten as the functional equation

$$\theta[\rho(r, a); R] = a\theta(r; R). \tag{9}$$

This functional equation must be satisfied in order for Equation (1) to be linearly consistent [property (3.3b)].

Theorem 2: *When the variance is used as the measure of risk and premiums are additive [property (3.4a)], then*

1. The variance principle results, that is,

$$\Pi[X] = E[X] + \beta_1 Var[X]; \quad \textit{and} \tag{10}$$

2. The linear consistency property, (3.3b), is violated.

Proof:

(1) For $i=1,2$, let $r_i=R_1[X_i]$, $\mu_i=E[X_i]$ and also let $\theta_1(r)=\theta(r; R_1)$. It follows that for independent $X_i \in \Omega$,

$$\Pi[X_1 + X_2] = \Pi[X_1] + \Pi[X_2]$$

$$\mu_1 + \mu_2 + \theta_1(r_1 + r_2) = \mu_1 + \theta_1(r_1) + \mu_2 + \theta_1(r_2)$$

$$\Rightarrow \theta_1(r_1 + r_2) = \theta_1(r_1) + \theta_1(r_2)$$

which is a functional equation in r_1 and r_2 with solution (see, for example, Aczél [1, pp. 11–12])

$$\theta_1(r) = \beta_1 r$$

where $\beta_1 > 0$ is a constant. The result follows.

(2) It is easily seen that $\rho(r, a) = a^2 r$ and that Equation (9) is violated.

□

If property (3.4b) (subadditivity) is used instead, then

$$\Pi[X_1 + X_2] \leq \Pi[X_1] + \Pi[X_2],$$

which reduces to

$$\theta_1(r_1 + r_2) \leq \theta_1(r_1) + \theta_1(r_2).$$

Assuming $\theta_1(r)$ is differentiable for $r \geq 0$, this inequality implies

$$\frac{\theta_1(r_1 + r_2) - \theta_1(r_1)}{r_2} \leq \frac{\theta_1(r_2)}{r_2}.$$

Since $\theta_1(0)=0$ and property (3.6) implies that $\theta_1'(r) \geq 0$, then $r_2 \to 0$ implies

$$0 \leq \theta_1'(r) \leq \theta_1'(0), \quad r \geq 0. \tag{11}$$

One consequence of Equation (11) is the following:

Result 1: *If the variance is used as a measure of risk and premiums are subadditive [property (3.4b)], then any increasing, non-negative concave function of r passing through the origin can be used to represent the loading function.*

For example,

$$\theta_1(r) = \beta_2 r^{\alpha}\ 0 < \alpha \leq 1 \text{ and } \beta_2 > 0.$$

As a special case, when $\alpha = 1/2$, the *standard deviation principle* results, that is,

$$\Pi[X] = E[X] + \beta_2 \sqrt{Var[X]}. \tag{12}$$

Since $\rho(r, a) = a^2 r$ and $\theta_1(r) = \beta_2 r^{1/2}$, it is obvious from Equation (8) that the linear consistency property (3.3b) holds.

Another example of a concave function is

$$\theta_1(r) = \beta_3 \log(1 + r)$$

yielding a new premium calculation principle called the *log-variance principle*, that is,

$$\Pi[X] = E[X] + \beta_3 \log(1 + Var[X]). \tag{13}$$

The log-variance principle does not satisfy the linear consistency property, (3.3b).

When the standard deviation is used as the risk-measure, that is, $r=R_2[X]=\sqrt{Var[X]}$, it essentially yields the same risk-loading functions as the variance.

4. PREMIUMS USING HIGHER CUMULANTS

4.1 The Right-Tail Risk

The combination of the two consistency properties, (2.4) and (3.3b), has a very powerful effect on $\theta(r, \mu; R)$: it removes any dependence of the risk-loading on μ. Thus, if the variance or standard deviation is used as a measure of the level of risk in X, risks with the same mean and variance must be charged the same premium. This is somewhat disturbing because, even though they may have the same mean and variance, some risks are more "dangerous" than others. A distribution is said to be dangerous if there is a relatively significant probability of very large claims; see Beard et al. [3, sections 3.5.3 and 3.5.8]. From the criticisms of the variance given in Section 2.3, the mean and the variance jointly do not reflect the level of risk in X partly because the right tail contains information on the probabilities of obtaining excessively large claims. Such claims are, of course, a source of risk. Promislow [46] raises a similar point. Detailed discussions of the tail classification system of claim size distributions are contained in Embrechts and Goldie [17] and in Embrechts and Veraverbeke [18].

Definition 7: *The right-tail risk is the risk associated with the extreme right tail of a distribution; that is, it is the risk associated with the occurrence of claims that are much larger than the mean, that is, catastrophic claims.*

Since insurance claims are usually positively skewed and have relatively long right tails, then premiums based solely on mean/variance considerations will be inadequate: they will tend to underprice the long-tailed risks and overprice the short-tailed risks. In a competitive market, there is the danger that inadequate pricing may lead to insolvency. This danger is especially great for reinsurers because a large part of their business is related to accepting and pricing right-tail risks. As an example, suppose there are three non-negative risks, X_1, X_2 and X_3, each with the same mean $\mu=1$ and the same variance $\sigma^2=3$. In particular,

$$X_1 = \begin{cases} 4 & \text{with probability } 1/4 \\ 0 & \text{with probability } 3/4; \end{cases}$$

X_2 has a gamma distribution with pdf

$$f_2(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x \leq 0$$

where $\alpha=\beta=1/3$; and X_3 has a Pareto distribution with pdf

$$f_3(x) = (1 + b)(1 + bx)^{-(2+1/b)}, \quad x > 0$$

where $b=1/2$. Most insurers and reinsurers would assign the lowest premium to X_1 and the highest premium to X_3 because, of the three distributions, X_3 generates the longest right tail and X_1 the shortest; that is, the Pareto has the largest right-tail risk. However, if premiums are based solely on mean-variance considerations, each of these three risks must be charged the *same* premium! Since these risks essentially have very different characteristics and should be priced differently, relying only on mean-variance considerations forces the actuary to underprice X_3 and to overprice X_1.

A technique for including the right-tail risk in the premiums is now investigated.

4.2 A General Risk-Measure Function

Let $K_X(t)$ be the cumulant generating function of X, that is,

$$\begin{aligned} K_X(t) &= \ln E[e^{tX}] \\ &= \sum_{j=1}^{\infty} \kappa_j[X] \frac{t^j}{j!} \end{aligned}$$

where $\kappa_j[X]$ is the j-th cumulant of X. For independent risks the cumulants are additive, that is

$$\kappa_j[X_1 + X_2] = \kappa_j[X_1] + \kappa_j[X_2]. \tag{14}$$

When there is no chance of confusion, $\kappa_j[X]$ is written as κ_j. The first four cumulants of X are $\kappa_1=\mu$, $\kappa_2=\sigma^2$, $\kappa_3=\mu_3$, and $\kappa_4=\mu_4-3\sigma^4$, where the $\mu_n=E[(X-\mu)^n]$, the central moments of X.

Since the variance is an adequate measure of risk only in the case in which the distribution is normal (or nearly so), it does not adequately measure risk when the distribution is positively skewed. Clearly a risk-measure that adjusts the variance is needed. This adjustment can be written in additive form as

$$R[X] = \textit{variance} + \textit{right-tail risk}, \tag{15}$$

or in "relative risk" form as

$$R[X] = \textit{variance} \times (1 + \textit{relative right-tail risk}). \tag{16}$$

The factor

$$\zeta = 1 + \textit{relative right-tail risk}$$

is called the "variance adjustment factor."

The well-known and most accurate approaches to approximating the cdf of the aggregate claims (from an insurance portfolio) use at least the first three or four cumulants; see [3, chapter 3], [40], or [44]. Since these approximations to the cdf are accurate, it follows from the objectivity property, (2.5), that these cumulants should be used in developing risk-measure functions. To this end, a general risk-measure function is defined as

$$R[X] = r(\kappa_2, \kappa_3, \kappa_4).$$

An example of such a risk-measure function is

$$R[X] = \kappa_2 + \omega_3\kappa_3^{a_3} + \omega_4\kappa_4^{a_4} \tag{17}$$

where the $\omega_j \geq 0$ and $0 \leq a_j \leq 1$, $j=3,4$.

The use of fractional powers may be problematic because κ_3 can be negative. To avoid such problems, only risks that have non-negative skewness are considered, that is, risks in the set

$$\Omega^+ = \{X : X \in \Omega \text{ and } \kappa_3[X] \geq 0\}.$$

Fortunately in practice most insurance risks are non-negatively skewed, making this assumption a reasonable one.

Suggestion 1: *If X is negatively skewed, the variance/standard deviation can be used as a conservative measure of its level of riskiness.*

We have now established that Equation (17) defines a risk-measure function.

Theorem 3: *If $X \in \Omega^+$, the risk-measure function defined in Equation (17) with $\omega_j \geq 0$ and $0 \leq a_j \leq 1$, $j=3,4$ satisfies properties (2.1) to (2.5).*

Proof: It is obvious that these properties, except perhaps the subadditivity property, (2.3), must hold. Recall Equation (14). To prove that the subadditivity property, (2.3), holds, we only need to prove that for any pair of constants $x \geq 0$ and $y \geq 0$, the inequality is $(x+y)^a \leq x^\alpha + y^\alpha$, for $0 \leq a \leq 1$. However, from an application of the Minkowski inequality (see, for example, Hardy et al. [29, p. 31, equation (2.11.4)]), this inequality holds. Hence Equation (17) satisfies property (2.3). □

Another problem with $R[X]$ in Equation (17) is that it may suffer from hyperinflation; that is, if $c>1$ is a constant and $j=2, 3, \ldots$, then $\kappa_j[cX]=c^j\kappa_j[X]$. This gives an undue amount of weight to the higher-order cumulants. To combat this, we can use

$$\begin{aligned} R_3[X] &= \kappa_2 + \omega_{31}\kappa_3^{2/3} + \omega_{32}\kappa_4^{1/2} \\ &= \sigma^2(1 + \omega_{31}\gamma_1^{2/3} + \omega_{32}\gamma_2^{1/2}) \end{aligned} \tag{18}$$

where

$$\gamma_1 = k_3/k_2^{3/2} \quad \text{and} \quad \gamma_2 = k_4/k_2^2 \tag{19}$$

are the coefficients of skewness and kurtosis, respectively. The condition $R_3[cX]=c^2R_3[X]$ now holds. Equation (18) suggests a general way of avoiding this type of inflation: simply define $R[X]$ as

$$R[X] = \sigma^2[1 + h(\gamma_1, \gamma_2)] \tag{20}$$

where $h(\cdot,\cdot) \geq 0$ is the relative right-tail risk. The proper choice of $h(\cdot,\cdot)$ is left to the insurer, the only condition being that the resulting risk-measure must satisfy properties (2.1) to (2.5).

4.3 The Normal Power Risk-Measure Function

The normal power (NP) approximation can be used to provide a relative risk-measure function, h. For a moderately positively skewed random variable, X, the NP approximation essentially states that when standardized, X can be approximated in the short version as

$$\tilde{X} \approx Y + \frac{1}{6}\gamma_1(Y^2 - 1)$$

or in the long version as

$$\tilde{X} \approx Y + \frac{1}{6}\gamma_1(Y^2 - 1) + \frac{1}{24}\gamma_2(Y^3 - 3Y) - \frac{1}{36}\gamma_1^2(2Y^3 - 5Y)$$

where $Y \sim N(0,1)$ and $\tilde{X} = (X-\mu)/\sigma$. In view of the definition of $\tilde{X}$, one would expect its variance to be 1; however, $Var[\tilde{X}] = \zeta_{NP}$ where

$$\zeta_{NP} = \begin{cases} \left(1 + \dfrac{1}{18}\gamma_1^2\right) & \text{short} \\[2ex] \left[1 + \left(\dfrac{5}{36}\gamma_1^2 - \dfrac{1}{10}\gamma_2\right)^2 + \dfrac{1}{2400}\gamma_2^2\right] & \text{long.} \end{cases} \tag{21}$$

The fact that $Var[\tilde{X}] \geq 1$ suggests the NP approximation is adjusting the variance of X to accommodate the skewness and kurtosis of X. This gives the adjusted variance of X as

$$Adjusted\ Var[X] = \sigma^2\zeta_{NP},$$

which suggests a new risk-measure function, called the NP risk-measure function, given by

$$R_{NP}[X] = \sigma^2\zeta_{NP}. \tag{22}$$

I know of no other risk-measure function in the actuarial literature or the finance literature that explicitly uses the first three or four cumulants. This risk measure also may be useful in determining the risk in financial securities.

Theorem 4: *For any* $X \in \Omega^+$,

$$R_{NP}[X] = \sigma^2 \left(1 + \frac{1}{18}\gamma_1^2\right) \tag{23}$$

is a risk-measure function satisfying properties (2.1) to (2.5).

Proof: It is obvious that these properties, except perhaps the subadditivity property, (2.3), must hold. To prove that the subadditivity property, (2.3), holds, proceed as follows: Recall Equation (14). Consider two independent risks X_1, $X_2 \in \Omega^+$. Assume X_i has j-th cumulant κ_{ij}; then subadditivity implies

$$(\kappa_{12} + \kappa_{22})\left[1 + \frac{(\kappa_{13} + \kappa_{23})^2}{18(\kappa_{12} + \kappa_{22})^3}\right] \leq \kappa_{12}\left(1 + \frac{\kappa_{13}^2}{18\kappa_{12}^3}\right) + \kappa_{22}\left(1 + \frac{\kappa_{23}^2}{18\kappa_{22}^3}\right)$$

$$\Rightarrow \left(\frac{\kappa_{13} + \kappa_{23}}{\kappa_{12} + \kappa_{22}}\right)^2 \leq \left(\frac{\kappa_{13}}{\kappa_{12}}\right)^2 + \left(\frac{\kappa_{23}}{\kappa_{22}}\right)^2.$$

Let $\kappa_{i3} = x_i \kappa_{i2}$ and $r_i = \kappa_{i2}/(\kappa_{12} + \kappa_{22})$; then subadditivity implies

$$(r_1 x_1 + r_2 x_2)^2 \leq x_1^2 + x_2^2$$

where r_1, $r_2 \geq 0$ and $r_1 + r_2 = 1$. Since $y = x^2$ is a convex function of x, then from Hardy et al. [29, chapter 3.5], it follows that

$$\begin{aligned}(r_1 x_1 + r_2 x_2)^2 &\leq r_1 x_1^2 + r_2 x_2^2 \\ &\leq x_1^2 + x_2^2,\end{aligned}$$

so subadditivity holds. □

The long version in Equation (21) does not generally produce a risk-measure that satisfies the subadditivity property, (2.3). However, if the risks belong to a homogeneous portfolio (that is, one consisting of i.i.d. random variables) or a portfolio that consists of risks that are linear transformations of i.i.d. random variables, the long version becomes subadditive. Even though the long version often improves the approximation, Beard et al. [3, section 3.11(g), p. 116] caution against its use because, in the right tail of $F_X(x)$, the long version may produce worse results.

Suggestion 2: *In general situations X may be negatively skewed. In such cases, assume both γ_1 and γ_2 are zero in R_{NP}. This can be used as a* conservative *measure of its level of risk.*

The obvious generalization of the variance principle is the corresponding NP-premium calculation principle given by

$$\Pi_{NP}[X] = \mu + \beta_4\sigma^2\left(1 + \frac{1}{18}\gamma_1^2\right) \tag{24}$$

where $\beta_4 > 0$ is a constant to be determined by the insurer. This premium is subadditive but is not linearly consistent. The "standard deviation" version of the NP-premium satisfies the linear consistent property, (3.3b), and is given by

$$\Pi_{NP}[X] = \mu + \beta_5\sigma\sqrt{1 + \frac{1}{18}\gamma_1^2} \tag{25}$$

where $\beta_5 > 0$ is a constant to be determined by the insurer.

5. AN EXAMPLE

To demonstrate how the risk-based approach can be used, the following example is provided.

Consider a \$100,000 whole life insurance policy issued to a life age 30 with level gross premiums paid for life and death benefit paid at end of year of death. For simplicity, assume the following:

- Mortality follows Makeham's law with

  $$1000\mu_x = 0.7 + 0.05(10^{0.04})^x$$

  for $x=0, 1, \ldots$. This is the same mortality assumption used by Bowers et al. in their text *Actuarial Mathematics*.
- Interest is $i=0.06$ and $d=i/(1+i)$.

- Expenses (based on table 14.8 in *Actuarial Mathematics*) are as follows

Expenses	First Year	Renewal
Percentage of premium	87%	10%
Per \$1000 of benefits	\$5	\$1
Per policy	\$46	\$6

- Settlement expenses: \$18 plus \$0.10 per \$1000.
- Data:

$$\ddot{a}_{30} = 15.85612$$

$$1000A_{30} = 102.4835$$

$$1000 \times {}^2A_{30} = 25.3113$$

$$1000 \times {}^3A_{30} = 11.9269$$

$$\gamma_1 = \frac{{}^3A_{30} - 3({}^2A_{30})A_{30} + 2(A_{30})^3}{[{}^2A_{30} - (A_{30})^2]^{3/2}}$$

$$= 3.4948$$

- Find: (1) the net level premium P and
  (2) the gross premium G using:
    (a) the equivalence principle
    (b) the standard deviation principle of Equation (12)
    (c) the normal power-standard deviation principle of Equation (25).
  (3) Repeat (2) using a group of n identical and independent lives age 30.

Solution:

1. Clearly

$$P = 100{,}000A_{30}/\ddot{a}_{30} = 646.34.$$

2. Let X and Y be the present values of claims plus expenses and gross premiums, respectively.

$$X = 440 + 100{,}028v^{K+1} + 0.77G + (0.1G + 106)\ddot{a}_{\overline{K+1}|} \tag{26}$$

$$Y = G\ddot{a}_{\overline{K+1}|} \tag{27}$$

where K is the curtate future lifetime of (30). The key is Equation (5).

(a) Solve the equation $E[X]=E[Y]$, that is, $\theta\equiv 0$ in Equation (1). This leads to

$$G = \frac{440 + 100{,}028A_{30} + 106\ddot{a}_{30}}{0.9\ddot{a}_{30} - 0.77} = 916.41.$$

(b) Solve the equation

$$E[X] + \beta_2\sigma[X] = E[Y] + \beta_2\sigma[Y]. \tag{28}$$

But

$$E[X] = 440 + 100{,}028A_{30} + 0.77G + (0.1G + 106)\ddot{a}_{30}$$

$$= 12{,}371.97 + 2.35561G$$

$$\sigma[X] = \left[100{,}028 - \frac{(0.1G + 106)}{d}\right]\sqrt{{}^2A_{30} - A_{30}^2}$$

$$= 11{,}944.51 - 0.21499G$$

$$E[Y] = G\ddot{a}_{30}$$

$$= 15.85612G$$

$$\sigma[Y] = \frac{G}{d}\sqrt{{}^2A_{30} - A_{30}^2}$$

$$= 2.14985G$$

Solving Equation (28) yields

$$G = 916.41\left(\frac{1 + 0.96545\beta_2}{1 + 0.17517\beta_2}\right).$$

Recall that 916.41 was the gross premium using the equivalence principle. The insurer must determine the appropriate level of β_2.

(c) From Equation (25), solve the equation

$$E[X] + \beta_5\sigma[X]\sqrt{1 + \frac{1}{18}\gamma_1^2[X]}$$

$$= E[Y] + \beta_5\sigma[Y]\sqrt{1 + \frac{1}{18}\gamma_1^2[Y]}. \quad (29)$$

Note that if W and Z are any pair of random variables, a and b are known constants and $Z=a+bW$, then their coefficients of skewness are connected as follows:

$$\gamma_1[Z] = \begin{cases} \gamma_1[W] & \text{if } b > 0 \\ 0 & \text{if } b = 0 \\ -\gamma_1[W] & \text{if } b < 0 \end{cases}$$

In other words, the skewness is invariant under positive linear transformations and changes signs under negative linear transformations. Assuming that

$$\left[100{,}028 - \frac{(0.1G + 106)}{d}\right] > 0,$$

which will hold true in practice, it follows that

$$\gamma_1[X] = 3.394794 = -\gamma_1[Y].$$

Since Y is negatively skewed, suggestion 2 implies that $\gamma_1[Y]$ should be set to zero. This leads to the equation

$$E[X] + \beta_5\sigma[X]\sqrt{1 + \frac{1}{18}\gamma_1^2[X]} = E[Y] + \beta_5\sigma[Y], \qquad (30)$$

which has the solution

$$G = 916.41\left(\frac{1 + 1.25082\beta_5}{1 + 0.17987\beta_5}\right).$$

The insurer must determine the appropriate level of β_5.

3. Let X_i and Y_i be the present values of claims plus expenses and gross premiums, respectively, for the i-th life, $i=1, 2, \ldots, n$.

$$X_i = 440 + 100{,}028v^{K_i+1} + 0.77G + (0.1G + 106)\ddot{a}_{\overline{K_i+1}|}$$

$$Y_i = G\ddot{a}_{\overline{K_i+1}|}$$

and

$$X = \sum_{i=1}^{n} X_i \qquad (31)$$

$$Y = \sum_{i=1}^{n} Y_i. \qquad (32)$$

It follows that

$$\gamma_1[X] = \frac{\gamma_1[X_1]}{\sqrt{n}} \qquad (33)$$

because the X_i's are i.i.d. It can easily be proved that

$$G = 916.41\left(\frac{1 + \dfrac{0.96545}{\sqrt{n}}\beta_2}{1 + \dfrac{0.17517}{\sqrt{n}}\beta_2}\right).$$

under the standard deviation case, and

$$G = 916.41 \left(\frac{1 + 0.96545\beta_5 \sqrt{1 + \frac{0.67853}{n}}}{1 + 0.15924\beta_5 \sqrt{1 + 0.1 \times \frac{0.67853}{n}}} \right).$$

under the standard deviation-normal power case.

6. COMMENTS

The following are interesting areas for actuaries to explore.

1. The development of an axiomatic approach (similar to Fishburn's) to study risk-measure functions would be helpful. It is hoped that properties (2.1) to (2.5) can be used as a part of this axiomatic approach. A key axiom will be that of mixing: if X_1 and X_2 are in Ω and

$$X = \begin{cases} X_1 & \text{with probability } p_1; \\ X_2 & \text{with probability } p_2; \end{cases}$$

with $p_1+p_2=1$, how should $R[X]$ be defined?
If X and Y are in Ω, can we define $R[X]$ as

$$R[X] = E[R[X|Y]] + R[E[X|Y]]? \tag{34}$$

I think that only the variance satisfies this equation. I would welcome other suggestions for dealing with mixtures. Note, using

$$R[X] = R[R[X|Y]], \tag{35}$$

which is analogous to $E[X]=E[E[X|Y]]$, will not be useful because if X is discrete, then $R[X]=0$.

2. In developing the axiomatic approach, the $R[\cdot]$ induces an order of risks. It would be interesting to determine whether the normal power risk measure results in the same definition of increasing risk as given by Rothschild and Stiglitz [49]. Rothschild and Stiglitz attempt to answer the question: When is a random variable Y "more variable" than a random variable X? They give four plausible answers: (1) Y is equal to X plus noise; (2) every risk averter prefers X to Y; (3) Y

has more weight in the tails than X; and (4) Y has a greater variance than X. They prove that the first three are equivalent but that they are not equivalent to the fourth.

3. I am not sure whether $\rho(r, a)$ [from Equation (7)] should be increasing in a or should be independent of a for $r>0$. Should the properties of $\rho(r, a)$ be listed as a part of the properties of R? Since a can be viewed as an inflation factor, should it affect the inherent risks involved? Do the comments on purchasing power parity, immediately after property (3.3b), apply? For example, suppose the prices of all items (goods, services and the factors of production) in an economy were inflated by the same factor, say, a. Will this affect the risk in X as it changes to aX? Suppose instead, that only one risk, say, X_1, was inflated; will the risk change as X_1 becomes aX_1? These questions may lead to an analog of the CAPM analysis used in the theory of finance. For example, one can think of the entire market of insurable risks as Ω, and $r_m=R[\Omega]$ as the market risk. This may lead to the notions of "the market risk" (associated with the economy as a whole) and the "relative risk," which is the proportion of X's risk $R[X]$ in r_m. Answers to these questions are welcomed.
4. Further research into the NP-risk measure is needed, especially from the point of view of portfolio analysis.

I hope this paper will stimulate discussions on the foundations of the nature of insurance risk.

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DISCUSSION OF PRECEDING PAPER

JACQUES F. CARRIERE:

I enjoyed reading Dr. Ramsay's article. The purpose of this discussion is to show that *utility functions* can be used to create *risk measures*. The article suggests that a general risk measure should be a function of the cumulants κ_2, κ_3 and κ_4, and as an example the article presents a risk-measure of the form $R(X)=\kappa_2+\omega_3\kappa_3^a+\omega_4\kappa_4^b$. I claim that prices that are calculated according to the utility principle are always functions of κ_2, κ_3 and κ_4, if they exist. Let me demonstrate this claim with the exponential utility function, $u(x)=-\exp(-\gamma x)$.

Let ω denote the insurer's initial wealth and let P denote the smallest premium that the insurer will accept for a risk with a loss random variable, X. The value P is found by solving the identity $u(\omega)=E\{u(\omega+P-X)\}$. For the exponential utility, this means that

$$P = \gamma^{-1} \log_e\{E(e^{\gamma X})\} = \gamma^{-1} \sum_{i=1}^{\infty} \kappa_i\gamma^i/i!$$

where κ_i is the i-th cumulant of X. Therefore,

$$P \approx \kappa_1 + \kappa_2\gamma/2 + \kappa_3\gamma^2/6 + \kappa_4\gamma^3/24 = \mu + \beta R_u(X),$$

where $\mu=\kappa_1$, $\beta=\gamma/2$ and $R_u(X)=\kappa_2+\kappa_3\gamma/3+\kappa_4\gamma^2/12$. Note that $R_u(X)$ is simply a special case of the risk measure $\kappa_2+\omega_3\kappa_3^a+\omega_4\kappa_4^b$.

In general, the utility principle implies that the premium $P(\boldsymbol{\kappa}|u)$ is always a function of the cumulants $\boldsymbol{\kappa}=\{\kappa_1, \kappa_2, \kappa_3, \ldots\}$ and that the functional relationship between the cumulants is determined by the utility function, u. A weakness of the utility approach is finding an explicit expression for the premium function $P(\boldsymbol{\kappa}|u)$. An advantage of using a risk-measure approach is the elimination of the thorny problems of (1) choosing an appropriate utility function and (2) calculating $P(u, \boldsymbol{\kappa})$. But the axiomatic development of the risk-measure approach is much weaker than that of utility theory. In conclusion, I think that the risk-measure approach would benefit tremendously by drawing on the strengths of the utility approach.

CHARLES S. FUHRER:

Dr. Ramsay has made an important contribution to the way actuaries think about risk and premium loads. At the Colorado Springs 1991 meeting [2], I used a different method of loading premiums for group health insurance that, like the author's method, was based on a risk measure. The risk measure I used was based on ruin theory and was suggested as a measure of required surplus by Brender [1] and others. I present this measure, my premium loads based on it, and discuss whether they meet the author's principles.

My measure of risk is defined as $R(X)=\inf\{r\geq 0|F(\mu+r)>\alpha\}$ for a fixed constant $0<\alpha<1$. Usually $\alpha=0.99$ or 0.98; that is, $R(X)$ is the minimum amount of surplus that avoids ruin with a probability of α. Unlike the author's measures of risk, it can be expressed in monetary units. Now let us see if this R is a risk measure function as defined by the author in Definition 4. $R=0$ for a riskless random variable. However, R can equal 0 for a risk that is not riskless. Thus Property (2.1), risklessness, is only half satisfied. This R considers low-risk contracts as riskless. Property (2.2), non-negativity, is satisfied by definition. I was somewhat surprised to see that R does not always satisfy Property (2.3), subadditivity. I have not determined the minimum sufficient conditions for subadditivity. I have shown that R is subadditive if the two random variables are continuous and have the same distribution except for location and scale and if the distribution is reproductive with respect to a positive power of the scale parameter. These conditions are satisfied, for example, by the normal and the gamma distributions. The function R also satisfies Property (2.4), consistency, and Property (2.5), objectivity.

The risk loading function is $\theta=c\mu R_T/\mu_T$, where R_T and μ_T are determined from a portfolio of similar risks and c is a constant. I suggest that $c=(t+r-i)/(1+t+r-i)$, where t is the trend of premiums, r is the return-to-stockholders rate, and i is the interest rate. This is derived from assuming that the loading plus interest on surplus needs to be large enough to provide the stockholder's return on surplus and to allow for surplus growth at the trend rate of premiums. Another assumption is that the loading plus required surplus needs to be equal to an insurance contract's share of the risk R. Let us see whether this loading function satisfies the properties of premium calculation stated by the author in Section 3.1. Property (3.1) is satisfied since $R\geq 0$. Property (3.2) is satisfied as long as the risk is as risky as the portfolio used. Property (3.3a) is satisfied if the constant is added to all the risks in the portfolio. Property (3.3b) is satisfied. Property (3.4a) is satisfied if the portfolio is not assumed to change. If the portfolio is changed, then (3.4b) is satisfied whenever (2.3) is satisfied. Property (3.5) is not satisfied.

I believe that most insurance company actuaries are using R above as a measure of risk for required surplus levels. Most are also using a risk-loading function that is proportional to this required surplus.

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JOSÉ GARRIDO:

It is well-known that the variance is a poor measure of risk for heavy-tailed distributions. The originality in Dr. Ramsay's work lies in his axiomatic approach to risk measurement and its possibilities for estimating the underpricing that results from the use of variance-type premium principles with positively skewed risks. The paper raises many issues on the nature of risk and a number of interesting questions that the author deliberately leaves unanswered. My comment focuses on the scale invariance property of the risk measure $R(X)$ and the related linear consistency of the premium principle $\Pi(X)$. I hope these thoughts can help answer one such question.

Dr. Ramsay illustrates well that "to prevent riskless arbitrage in an international insurance market," linear consistency of premiums $\Pi(aX+c)=a\Pi(X)+c$ must be imposed. It avoids situations as in his example of a risk in American dollars ($U.S.) being priced differently than if the same risk had been underwritten in Canadian dollars ($CAN) by the same insurer. However, he does not clearly discuss how this differs from a scale-invariance requirement on the risk measure $R(X)$. The problem is subtle and best understood through simple examples.

Consider the above currency problem. Is it reasonable that the measure of risk $R(X)$ be dependent of the currency in which the risk is measured? If $R(X)$ is an index of the riskiness inherent to X, should it be affected by the units of X, be it $U.S., $CAN or Swiss francs? The answer seems to be, "no," if based solely on intuition. This implies that $R(aX)=R(X)$, for any constant a and positive risk X. Together with property (2.4) of the paper, it means that R must be scale and location invariant; that is, $R(aX+c)=R(X)$. None of the usual, nor the new risk measures discussed in the paper, is invariant under both scale and location transforms! Perhaps this indicates that the requirement is not appropriate for risk measures; in some way it must be related to the linear consistency requirement $\Pi(aX+c)=a\Pi(X)+c$ imposed on premiums. It is this relation that needs to be clarified before the theory can be reconciled with our intuition.

Additional hindsight on the desirable properties of $R(X)$ can be gained by looking at the partial ordering it induces on risks. The value of $R(X)$ is especially informative when it is compared with that of another risk. It then defines an order relation; we will say that X is more risky than Y if $R(X)>R(Y)$. Pushing this reasoning further, we could say that an "absolute" risk measure like $R(X)$ is informative only when given in reference to some standard like (1) a reference scale (for example, the minimum and/or maximum $R(X)$ value), or (2) the "absolute" risk measure $R(Y)$ of another risk Y. The simple currency problem above with its dependence of $R(X)$ on the currency units provides another example of a risk measure relative to a scale (here the \$U.S.).

From this discussion, it seems clear that two different concepts of a risk measure need to be distinguished: (1) an "absolute" risk measure, $R(X)$, necessarily location-invariant, as in Ramsay's axioms, and (2) a "relative" risk measure, say $RR(X)$, defined through $R(X)$, that gives the riskiness of X relative to a reference scale (or to that of another risk). Intuitively it is the latter that should be scale-invariant, not $R(X)$. This probably adds other necessary requirements on $R(X)$. To identify them, the idea of a "relative" risk measure first needs to be formalized.

Let us define, as an example, a relative measure of risk as the ratio of $R(X)$ to the premium $\Pi(X)$,

$$RR(X) = \frac{R(X)}{\Pi(X)}.$$

This is similar to a well-known relative risk index, the loss ratio. Consider a risk X, with absolute risk measure $R(X)$ and charged premium $\Pi(X)$. Then the ratio

$$\frac{X}{\Pi(X)}$$

is also a risk and its expected value, the expected loss ratio,

$$\frac{E(X)}{\Pi(X)},$$

is a relative index of risk. Note that it is not a risk measure in Ramsay's sense because it does not satisfy Properties 2.1–2.5 of the paper. It is easy to show that proposed relative risk measure satisfies Properties 2.1–2.3 and 2.5 but not 2.4, and hence is not a true risk measure either. It is, however, a simple generalization of the expected loss ratio, where the numerator can be any risk measure $R(X)$. Scale invariance of $RR(X)$ means that

$$RR(aX) = \frac{R(aX)}{a\Pi(X)} = RR(X),$$

which, in turn, implies that $R(aX)=aR(X)$. This is precisely the relation we were after; intuition suggested scale-invariance for $R(X)$, but it lacked consistency with the other axioms of Ramsay. This analysis shows that linearity for $R(X)$ [and the resulting scale-invariance of the relative risk measure $RR(X)$] is the only possible requirement that can agree with the linear consistency of premiums.

This conclusion is based on a particular definition of a relative risk measure. We conjecture that the above linearity property of the absolute risk measure $R(aX)=aR(X)$ is the appropriate scale requirement with any relative risk measure.

S. DAVID PROMISLOW:

Dr. Ramsay is to be congratulated for a thought-provoking paper. Particularly interesting is the axiomatic approach to measuring risk. In my discussion I comment on this aspect of the paper and attempt to answer or partially answer some of the questions that Dr. Ramsay has posed.

The Consistency Property

The axioms that one chooses for a risk-measure function must necessarily reflect one's attitude towards the meaning of risk. The paper brings out the fact that there is no universal viewpoint. There appear to be two main ingredients to risk (which admittedly are somewhat linked). One is the possibility of something "bad happening," to quote the work of Fishburn referenced in the paper. The other is the presence of uncertainty. Dr. Ramsay's so-called *consistency* property (I am not sure if the name is accurate), (2.4), gives much greater emphasis to the latter. Essentially it says that the amount of risk in a random variable does not depend on its mean. One can look upon it in the following way. Instead of considering the set of non-negative risks, denoted by Ω in the paper, let us consider

$$\Omega_0 = \{X: X \text{ is a random variable with mean } 0\}.$$

Property (2.4) says that we need only define our risk function R on Ω_0. We can then extend it to all random variables (not just non-negative ones) by simply taking $R(X)=R[X-E(X)]$.

This property seems appropriate for insurance purposes. After all, the insurer charges a net premium of $E(X)$ for the coverage and so is really faced with the mean zero random variable, $X-E(X)$, to form the basis for the risk charge. It would not be an appropriate axiom for the case in which the predominant viewpoint is on possibility of loss. This occurs in much of financial decision-making. Consider a situation in which investors have a 50-50 chance of either doubling their present capital or losing it all. Most people would consider this to be much more risky than a situation in which there is a 50-50 chance of either doubling present capital or quadrupling it. To put this another way, the issue in financial decision-making is usually not just to measure risk alone, but to decide on the trade-off between risk and expected return.

An example of a function used to measure risk, which is highly mean dependent and therefore violates the consistency property, is the adjustment coefficient of ruin theory; see Bowers et al. [1, Chapter 12]. This is as it should be, since one is not concerned here with uncertainly, but rather with the possibility of "something bad," namely, ruin. (Note that this moves in the other direction to Dr. Ramsay's functions as it assigns higher value to the *safer* possibilities.)

Partial versus Complete Orderings

The paper comments on comparing risk as defined by a risk measure with the Rothschild-Stiglitz definition (which is perhaps better known as *second-order stochastic dominance*). Of course, these can never result in exactly the same ordering. Rothschild-Stiglitz results in a *partial order*, as do most of the commonly used methods for ordering risks. In general, a given pair of random variables will not be comparable. This reflects the viewpoint that assessment of risk depends on individual preferences. If two individuals are faced with choices X and Y, it may well be that one finds alternative X more risky, while the other thinks so of alternative Y. This is a well-accepted principle, notwithstanding the fact, indicated in the paper, that various writers have questioned the axioms of the classical VNM utility theory. They are not questioning the existence of individual preferences, but rather the particular methodology used for measuring such preference. (There is in fact much active research involved in modifying utility theory, so as to more accurately represent the actual decision-making principles that people seem to use; see the work by Schmeidler [3] as an example.) I therefore find the words in the title of the paper, "without using utility theory," as somewhat inaccurate. The use of a real valued function to measure risk necessarily imposes a *completeness* on the ordering. Given two random variables, X and Y, we will be able to unequivocally state that one is more risky than the other or else that they are both of equal risk. An insurer adopting a risk-measure function is therefore automatically implying certain preferences and is in effect using a utility theory of some sort.

The NP Ordering

We can pose a one-sided version of the ordering question introduced above. Given a particular partial order $\leq$ on a collection of random variables and a risk-measure function R, is it true that $X \leq Y$ implies that $R(X) \leq R(Y)$? Given two random variables of equal means, $X \leq Y$ in the Rothschild-Stiglitz sense (that is, X is less risky than Y), if and only if $E[g(X)] \leq E[g(Y)]$ for all convex g. Therefore, taking this partial order, the answer to our question is in the affirmative for $R(X) = \sigma^2(X)$. However, x^3 is not convex, and it turns out that the short form of the NP risk function does *not* respect the Rothschild-Stiglitz ordering. To see this, consider the following two sequences of random variables

$$X_n = \begin{cases} 2n \text{ with probability } \dfrac{1}{2n} \\ n \text{ with probability } \dfrac{n-1}{n} \\ 0 \text{ with probability } \dfrac{1}{2n} \end{cases} \quad Y_n = \begin{cases} 2n \text{ with probability } \dfrac{1}{2n} \\ n \text{ with probability } \dfrac{n-1}{2n}. \\ n-1 \text{ with probability } \dfrac{1}{2} \end{cases}$$

The relevant facts are

$$E(X_n) = E(Y_n) = n$$

$$\sigma^2(X_n) = n, \quad \sigma^2(Y_n) = \frac{n-1}{2}, \quad k_3(X_n) = 0, \quad k_3(Y_n) = \frac{n^2-1}{2}.$$

It is not hard to verify that Y_n is less risky than X_n in the Rothschild-Stiglitz sense; see, for example, [2, Theorem 1.1]. On the other hand, the condition that $R_{NP}(X_n) \leq R_{NP}(Y_n)$ fails dramatically. Noting that $\gamma_1(Y_n)$ is $O(\sqrt{n})$, we see that

$$\frac{R_{NP}(Y_n)}{R_{NP}(X_n)} \rightarrow \infty \text{ as } n \rightarrow \infty,$$

using the "short form" of R_{NP}.

I am not sure what happens with the long form of the formula. In fact, I am puzzled by the fact that the fourth cumulant is used at all. It was pointed out in the paper that the third cumulant can be negative, but this is also true of the fourth. Dr. Ramsay makes a case for excluding risks with negative skewness, but I cannot see any possible reason for excluding risks with negative kurtosis.

Mixtures

Dr. Ramsay wonders about a proper axiom to cover mixing. One has to be careful here, as it was precisely the fact that individuals often make what appear to inconsistent choices regarding mixtures that that led to the re-examination of classical utility theory. Nonetheless, one natural axiom that comes to mind is simply a weaker version of Requirement (34). To state this in the two component case, if

$$X = \begin{cases} X_1 \text{ with probability } p \\ X_2 \text{ with probability } 1-p \end{cases}$$

then

$$R(X) \geq pR(X_1) + (1-p)\,R(X_2).$$

We could also require the obvious extension to countable or more general mixing distributions. This axiom provides an interesting counterpart to the subadditivity property (2.3). The motivating idea is simply that the measure of risk in the mixture is *at least as much* as the corresponding mixture of the respective risk measures and possibly more due to the uncertainly arising from the fact that we don't know which of the random variables, X_1 or X_2, will occur.

A stronger axiom arises if we wish to postulate that this additional uncertainty depends *only* on the means of X_1 and X_2. This seems reasonable because the internal uncertainty has already been accounted for. If so (switching to the more general formulation used by Dr. Ramsay), we could require that for all X and Y, $R(X)-E[R(X|Y)]$ is some non-negative function of $R[E(X|Y)]$. In the case of (34), the function is simply the identity, but other functions could be possibly be pertinent.

Other Axioms

Here are a few suggestions for other features that we may want of a risk-measure function. We could introduce these directly as axioms or provide other axioms from which they could be deduced. Some of these follow from familiar criteria that have been used for ordering risks. A major reference for this topic is the book by Stoyan [4]; see particularly definition 1.1.1, which motivated the following ideas.

(1) One natural requirement is that comparison of risk should be independent of the units involved. We would want then that

$$R(X) \leq R(Y) \text{ implies that } R(cX) \leq R(cY)$$

for all $c>0$.

(2) We may believe that adding an independent summand will not affect relative risk. To state this precisely, suppose that X and Z are independent and that Y and Z are independent. Then we will require that

$$R(X) \leq R(Y) \text{ implies that } R(X + Z) \leq R(Y + Z).$$

(3) In most axiomatic schemes, some sort of continuity property is required to obtain suitable representation theorems. A natural one here is simply to postulate that for a sequence, F_n, of distribution functions that converge weakly to a distribution function, F,

$$R(F_n) \text{ converges to } R(F).$$

[The objectivity property, (2.5), justifies the notation used here.]

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ELIAS S.W. SHIU:

Dr. Ramsay is to be thanked for cautioning us in applying the expected utility theory and for proposing alternative risk measures. Recently, another actuarial paper, by Clarkson [5], also voices objection to the expected utility maxim. However, this field of study is still being actively researched, as evidenced by the number of articles under the heading "Choice under Risk" in the book of classified bibliography [9]. An excellent source on the development of utility theory is [6], a collection of articles on utility and probability from the four volumes of *The New Palgrave: A Dictionary of Economics*.

Perhaps the most eloquent introduction to the expected utility theory is Raiffa's book [10]. Raiffa also discusses the Allais paradox mentioned by Dr. Ramsay. The following is from pages 127 and 128 of Raiffa's book [10].

> We have considered the problem faced by a person who on most occasions makes decisions intuitively and more or less inconsistently, but who on some one particular occasion wishes to make one particular decision in a reasoned, deliberate manner. We have assumed he starts by structuring the anatomy of his problem in a decision-flow diagram that depicts the chronological interaction between his decision alternatives at any stage and the information he obtains in the dynamic evolution of his problem. We have shown that if the decision maker adopts two principles of consistent behavior, transitivity of preferences and substitutability of indifferent consequences in a lottery, then he is pretty well fenced in. . . .
>
> Our conclusions represent the foundations of the so-called "Bayesian" position. Nowhere in our analysis did we refer to the behavior of an "idealized, rational, economic man" who always acts in a perfectly consistent manner as if somehow there were embedded in his

> very soul coherent utility and probability evaluations for all eventualities. Rather, our approach has been *constructive*: We have prescribed the way in which an individual who is faced with a problem of choice under uncertainty should go about choosing an act that is consistent with his basic judgments and preferences. He must consciously police the consistency of his subjective inputs and *calculate* their implications for action. These lectures do not present a descriptive theory of actual behavior, nor a positive theory of behavior for a fictitious superintelligent being, but rather present an approach designed to help us erring folk to reason and act a bit more systematically—when we choose to do so!

My next comment is motivated by Section 4.2, in which Dr. Ramsay points out that, "[f]or independent risks the cumulants are additive." Borch [3] wrote that the cumulants were introduced by Thiele at the end of the last century "just because they are additive for independent variables." That is,

$$\Pi[X] := \sum_{j=1}^{\infty} \omega_j \kappa_j [X] \tag{1}$$

satisfies the additive Formula (3.4a). By citing a theorem of Luckas [8], Borch ([1], [3]) also pointed out the remarkable fact that any premium formula, additive for independent risks, is given by Equation (1).

I would like to conclude this discussion by quoting the last paragraph in the first section of [7]:

> The philosophy is not to claim that economical agents have a utility function. Instead, we deliberately assume that they have a utility function, and discuss the resulting conclusions, which often could not be obtained otherwise. Thus a utility function plays the role of a deus from my china (ex machina)!

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(AUTHOR'S REVIEW OF DISCUSSIONS)

COLIN M. RAMSAY:

I thank Mr. Fuhrer and Drs. Carriere, Garrido, Promislow, and Shiu for their insightful comments. I respond to each discussant's remarks separately and in alphabetical order.

My own aversion to expected utility theory not withstanding, I agree with Dr. Carriere's assertion that the axiomatic development of the risk-measure approach (as it now stands) is weaker than that of utility theory and that it would benefit from drawing on the strengths of utility theory. However, he should bear in mind that this theory is still in its incipient stages, so there still is a lot of work to be done. For example, the discussions by Drs. Garrido and Promislow represent a step in that direction.

Mr. Fuhrer's use of $R(X)$ as a measure of risk is more in line with Fishburn's [22] approach; that is, it deals with the possibility of something "bad" happening, such as ruin. So it is not surprising that $R[X]$ may be zero when a nondegenerate random variable is "safe."

I was very pleased to receive Dr. Garrido's discussion. About five or six years ago, Dr. Garrido had invited me to give a talk at Concordia University. Though I had only just started thinking about the problems associated with loading premiums for risk and measuring risk, I presented a (very early) version of this paper. The discussions following that initial presentation have influenced my thinking about the nature of risk. Dr. Garrido's current discussion deals with one of the central dilemmas that I faced when developing this axiomatic approach: How must we define $R[aX]$? I agree with him that, intuitively, we should define $R[aX]=aR[X]$. His supporting arguments are persuasive.

Dr. Promislow's discussion deals directly with some of the issues raised at the end of the paper. In addition, he has explored the mathematical underpinnings of some of the axioms. I was delighted to see his proof that the short form of the *NP* measure was not consistent with the Rothschild and Stiglitz ordering. Dr. Promislow's suggestions on mixtures and his other axioms point us in the direction that the axiomatic development of the risk-measure function should take.

I thank Dr. Shiu for providing me with additional references on utility theory and for bringing to my attention the result in his Equation (1). (This may be the result that Dr. Carriere was referring to.)

In closing, I hope that these discussants will continue to be interested in the development of the axiomatic approach to risk-measure functions and to make contributions to the literature.