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RECURSIVE FORMULAS FOR COMPOUND DIFFERENCE DISTRIBUTIONS*

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ABSTRACT

Recursive formulas satisfied by the numbers of claims are lifted to recursive formulas satisfied by the amounts of aggregate claims. The derivation relies on only an elementary technique—power series solutions to differential equations. The formulas are useful in the application of risk theory and are computationally efficient.

I. INTRODUCTION

This paper reviews recursive formulas for aggregate claim distributions that have appeared recently in much actuarial literature ([2]; [5]; [17]; [20]; [21]; [22]; [24]). It presents a general method of lifting difference equations to derive this genre of formulas using one simple mathematical tool—the power series solutions to differential equations.

Section II is an exposition on recursive formulas. With occasional references to section I for definitions and notations, it can be worked through easily in detail. Sections III and IV give a thorough development of the method of lifting difference equations applied to other more general distributions. As the derivation gets heavier, following through the equations requires concentration and patience. For a picture of how the method works in general, a cursory browse would probably suffice.

Throughout this paper, all random variables will assume nonnegative integral values. For such a random variable Z, we denote its probability generating function by

$$G_Z(s) = E(s^Z) = \sum_{j=0}^{\infty} Pr\{Z = j\}s^j = Z(s).$$

Consider a sum of a random number of random variables

$$S_N = X_1 + X_2 + \ldots + X_N ,$$

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where $Pr\{N = k\} = n_k$, X_i's are identically distributed as X with $Pr\{X = k\} = x_k$; N and X_i's are independent. For example, when N is Poisson, S_N has the well-known, compound, Poisson distribution. In general, when $\{n_i\}$ satisfies a difference equation, we say S_N has a *compound difference distribution*. It is easy to show that

$$G_{S_N}(s) = G_N\,(G_X(s)). \tag{1}$$

See ([4], pp. 286–87).

We introduce these notations to be followed throughout:

$$\begin{aligned} G_{S_N}(s) &= \sum_{j=0}^{\infty} s_j\, s^j = S\,(s), \\ G_N(s) &= \sum_{j=0}^{\infty} n_j\, s^j = N\,(s), \\ G_X(s) &= \sum_{j=0}^{\infty} x_j\, s^j = X(s). \end{aligned}$$

For example, (1) will be simplified to

$$S(s) = N\,(X(s)),$$

a basic formula used many times later. What this formula says is that the power series $S(s)$ is a function (that is, N) of another power series $X(s)$. If N is a simple enough function, executing the function N on a power series would be easy. Remark 3 in section II describes that raising to a power and taking the exponential are easy executions on power series, and that these executions give recursive formulas for compound Poisson, compound binomial, and compound negative binomial distributions.

Executing certain N on a power series could be difficult. In fact, N functions, such as those considered in section III, are not even in closed form. However, we observe that difference equations (recursive formulas) for $\{n_i\}$ lead to differential equations satisfied by $N(s)$. These equations can then be lifted by using equation (1) and the chain rule to differential equations satisfied by $S(s)$. By standard techniques of power series solutions to differential equations, differential equations satisfied by $S(s)$ lead to difference equations (recursive formulas) for $\{s_i\}$. Thus, difference equations satisfied by the numbers of claims are lifted to difference equations satisfied by the amounts of aggregate claims. As an illustration of the general method, it is first applied in section II to cases where simpler methods would suffice. In section III, it is applied to cases where this general method would be necessary.

II. THE KATZ DISTRIBUTIONS

Katz ([13]; [14]; [12], pp. 37–43) studies a system of discrete distributions satisfying the following first order linear difference equation

$$(k + 1)\, n_{k+1} = (a + bk)\, n_k,\ k = 0, 1, 2, \ldots;\ a > 0;\ b < 1. \tag{2}$$

As mentioned in [13], equation (2) includes Poisson $[a + bk = \lambda]$, binomial $[a + bk = (p/q)\,(r - k)]$, and negative binomial $[a + bk = q(r + k)]$ distributions as special cases. The advantage of equation (2) is that a and b can be chosen to fit data without specifying any of the classical models. This can be done graphically by fitting a line through $\{(k, (k + 1)n_{k+1}/n_k),\ k = 0, 1, 2, \ldots\}$, as done in Ord [19]. Alternatively, a and b can be found by solving

$$E(N) = a/(1 - b),$$
$$\mathrm{Var}(N) = a/(1 - b)^2.$$

The above equations are derived from (i) in the following theorem and using ([4], p. 266, (1.9), (1.11)). The following theorem lifts equation (2) to a difference equation satisfied by $\{s_i\}$.

THEOREM. *If*

$$(k + 1)n_{k+1} = (a + bk)n_k,\ k = 0, 1, 2, \ldots, a > 0,\ b < 1, \tag{2}$$

then

(i) $\dfrac{dN}{ds} = \dfrac{aN}{1 - bs}$,

(ii) $\dfrac{S'}{X'} = \dfrac{aS}{1 - bX}$,

(iii) $$(1 - bx_0)\,(n + 1)s_{n+1} = \sum_{k=0}^{n} [a(k + 1) + b(n - k)]x_{k+1}s_{n-k}. \tag{3}$$

Proof. Multiply equation (2) by s^k and sum from $k = 0$ to $k = \infty$ to obtain (i). Use equation (1) and the chain rule on (i) to obtain (ii). Cross multiply (ii) and consider the coefficient of s^n to obtain (iii).

Many recursive formulas that have appeared recently in ([2]; [5]; [17]; [20]; [21], [22]; and [24]) can also be derived as special cases of equation (3), with appropriate values of a and b. More results, such as the first two remarks that follow, can be derived by imitating the easy proof of the theorem.

Remark 1: If

$$\frac{dN}{ds} = N(s)a(s) + b(s),$$

then

$$\frac{dS}{ds} = [S(s)a(X(s)) + b(X(s))]\frac{dX}{ds}.$$

For example, Crow and Bradwell [3] studied a system of discrete distributions satisfying

$$(k + c)n_{k+1} = an_k, \ k = 0, 1, 2, \ldots, a > 0, c > 0.$$

Imitating the proof of (i), we derive

$$s\frac{dN}{ds} = (as + 1 - c)N + (c - 1)n_0,$$

which can be lifted to

$$XS' = (aX + 1 - c)SX' + (c - 1)n_0X',$$

which generates a recursive formula for $\{s_i\}$. In general, an nth order, linear, differential equation on N can be lifted to an nth order, linear, differential equation on S.

Remark 2: If $n_0 = 0$,

$$(k + 1)n_{k+1} = (a + bk)n_k, \ k = 1, 2, \ldots, a > 0, b < 1,$$

then

(i) $\dfrac{dN}{ds} = \dfrac{aN + n_1}{1 - bs},$

(ii) $\dfrac{S'}{X'} = \dfrac{aS + n_1}{1 - bX},$

(iii) $$(1-bx_0)(n+1)s_{n+1} = (as_0+n_1)(n+1)x_{n+1} + \sum_{k=0}^{n-1}[a(k+1) + b(n-k)]x_{k+1}\, s_{n-k}.$$

For example, the logarithmic series distributions have $a+bk = qk$.

Remark 3: When $\{n_i\}$ satisfies equation (2), $N(s)$ can be solved explicitly. In fact,

$$N(s) = [(1-bs)/(1-b)]^{-a/b} \text{ if } b \neq 0 \text{ (see [12], p. 40)},$$

and

$$N(s) = e^{a(s-1)} \text{ if } b = 0 \text{ (see [12], p. 91)}.$$

By (1),

$$S(s) = [1-bX(s)/(1-b)]^{-a/b} \text{ if } b \neq 0, \tag{4}$$

and

$$S(s) = e^{a(X(s)-1)} \text{ if } b = 0. \tag{5}$$

As pointed out by Shiu [23], equation (4) leads to equation (3) by applying the J.C.P. Miller formula, a formula known to computer scientists (see [7], p. 12, (4)), and equation (5) leading to equation (3) is an exercise of medium difficulty ([16], p. 514, Exercise 4, solution on p. 656; [8], p. 43, Problem 6).

Remark 4: Many recursive formulas that have been rediscovered recently in actuarial literature have been known all along to statisticians. Compound Poisson is type A; compound binomial and compound negative binomial are type B; and compound logarithmic is type C in [15]. The recursive formula for the compound Poisson distribution made its debut even earlier ([18], p. 47, (48); [1], p. 357, (15)).

III. THE GENERALIZED WARING DISTRIBUTIONS

As illustrated by Remark 3 of the last section, going from difference equation on $\{n_i\}$ to differential equation on $N(s)$ to differential equation on $S(s)$ to difference equation on $\{S_i\}$ is unnecessary when $N(s)$ is expressible in closed form. Indeed, the theorem of the last section is presented in a format to illustrate a general systematic method, valid even when $N(s)$ is not in closed form. The case considered in this section is as such.

Consider the second order linear difference equation

$$n_{k+1} = \frac{(\alpha + k)(\beta + k)}{(\gamma + k)(k + 1)} n_k \tag{6}$$

satisfied by the hypergeometric, the negative hypergeometric, and the beta-Pascal distributions. Distributions satisfying equation (6) were considered by Guldberg ([6], p.45) and were named *generalized Waring* by Irwin [10]. Ord [19] described a graphical method to differentiate a discrete distribution for satisfying equations (2) or (6). Irwin ([9]; [10]; [11]) studied the generalized Waring distribution and its application to accident theory. When N has a generalized Waring distribution, S_N is said to have a *compound* generalized Waring distribution.

With the boundary condition $\sum_{k=0}^{\infty} n_k = 1$, the solution to the difference equation (6) is

$$n_k = [F(\alpha, \beta; \gamma; 1)]^{-1} \frac{\alpha^{[k]} \beta^{[k]}}{\gamma^{[k]} k!}$$

Recall that $\alpha^{[k]}$, the ascending factorial, is defined by

$$\alpha^{[k]} = \alpha(\alpha+1) \ldots (\alpha + k-1).$$

Thus,

$$N(\alpha, \beta; \gamma; s) = \sum_{k=0}^{\infty} n_k s^k = F(\alpha, \beta; \gamma; s)/F(\alpha, \beta; \gamma; 1). \tag{7}$$

Here,

$$F(\alpha, \beta; \gamma; s) = \sum_{k=0}^{\infty} \frac{\alpha^{[k]} \beta^{[k]}}{\gamma^{[k]} k!} s^k \tag{8}$$

is the hypergeometric function.

In this section, we shall show that the difference equation (6) leads to a first order differential equation on N. (Compare with (i) in the theorem.) This equation can be lifted by using equation (1) to a differential equation on S. (Compare with (ii) in the theorem). It then leads to difference equations (recursive formulas) for $\{s_i\}$. (Compare with (iii) in the theorem.) Thus, again, difference equations on $\{n_i\}$ are lifted to difference equations on $\{s_i\}$.

The differential equation on N promised earlier is

$$N'(\alpha, \beta; \gamma; s) = [\alpha\beta/\gamma - \alpha - \beta - 1]N(\alpha + 1, \beta + 1; \gamma + 1; s). \tag{9}$$

Verification requires equations (7), (8), and the following standard fact

$$F(\alpha,\ \beta;\ \gamma;\ 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)},$$

but is straightforward. By using equation (1) for parameters $\{\alpha,\beta,\gamma\}$, $\{\alpha+1,\beta+1,\gamma+1\}$, . . ., i.e.,

$$S(\alpha+j,\ \beta+j;\ \gamma+j;\ s) = N(\alpha+j,\ \beta+j;\ \gamma+j;\ X(s))\ , \qquad j = 0,1,2,\ \ldots, \tag{10}$$

and the chain rule, we lift the differential equation (9) for parameters $\{\alpha,\beta,\gamma\}$, $\{\alpha+1,\beta+1,\gamma+1\}$, . . . , i.e.,

$$\begin{aligned} N'\ (\alpha+j,\beta+j;\gamma+j;s) &= [(\alpha+j)\ (\beta+j)/\gamma-\alpha-\beta-j-1] \\ &\quad N(\alpha+j+1,\beta+j+1;\gamma+j+1;s)\ , \\ j &= 0,1,2,\ \ldots, \end{aligned}$$

to

$$\begin{aligned} S'(\alpha+j,\beta+j;\gamma+j;\ \mathrm{s}) &= [(\alpha+j)(\beta+j)/\gamma-\alpha-\beta-j-1]X'\ (s) \\ &\quad S(\alpha+j+1,\beta+j+1;\gamma+j+1;s)\ , \\ j &= 0,1,2,\ \ldots\ . \end{aligned} \tag{11}$$

It then leads to difference equations for $\{s_i\}$ as follows. Recall that

$$S(\alpha,\ \beta;\ \gamma;\ s) = \sum_{k=0}^{\infty} s_k\, s^k.$$

Let $s^+{}_k$ denote the coefficient of s^k in $S(\alpha\ +\ 1,\ \beta\ +\ 1;\ \gamma\ +\ 1;\ s)$,

$$S(\alpha\ +\ 1,\ \beta\ +\ 1;\ \gamma\ +\ 1;\ s) = \sum_{k=0}^{\infty} s^+{}_k\, s^k,$$

and let $s^{+2}{}_k$ denote the coefficient of s^k in $S(\alpha + 2, \beta + 2; \gamma + 2; s)$,

$$S(\alpha + 2, \beta + 2; \gamma + 2; s) = \sum_{k=0}^{\infty} s^{+2}{}_k\, s^k,$$

and so on. Consider the coefficient of s^n in equation (11) with $j=0$ to obtain

$$(n+1)s_{n+1} = [\alpha\beta/\gamma - \alpha - \beta - 1] \sum_{j=0}^{n} (j+1)x_{j+1}\, s^{+}{}_{n-j}. \tag{12}$$

With $j=1,2,\ldots,$

$$(n+1)s^{+}{}_{n+1} = [(\alpha+1)(\beta+1)/\gamma - \alpha - \beta - 2] \sum_{j=0}^{n} (j+1)x_{j+1}\, s^{+2}{}_{n-j}\,, \tag{12$^+$}$$

$$(n+1)s^{+2}{}_{n+1} = [(\alpha+2)(\beta+2)/\gamma - \alpha - \beta - 3] \sum_{j=0}^{n} (j+1)x_{j+1}\, s^{+3}{}_{n-j}\,, \tag{12^{+2}}$$

and so on. The system of equations (12), (12^+), (12^{+2}), . . . are used to compute $\{s_i\}$ recursively:

$$s_0 = \frac{F(\alpha,\beta;\gamma;x_0)}{F(\alpha,\beta;\gamma;1)} \qquad s^{+}{}_0 = \frac{F(\alpha+1,\beta+1;\gamma+1;x_0)}{F(\alpha+1,\beta+1;\gamma+1;1)} \qquad s^{+2}{}_0 = \frac{F(\alpha+2,\beta+2;\gamma+2;x_0)}{F(\alpha+2,\beta+2;\gamma+2;1)}$$

$$s_1 = \frac{\alpha\,\beta}{\gamma-\alpha-\beta-1}\, x_1\, s^{+}{}_0 \qquad s^{+}{}_1 = \frac{(\alpha+1)(\beta+1)}{\gamma-\alpha-\beta-2}\, x_1 s^{+2}{}_0 \qquad \vdots$$

$$2s_2 = \frac{\alpha\,\beta}{\gamma-\alpha-\beta-1}\,(x_1 s^{+}{}_1 + x_2 s^{+}{}_0) \qquad \vdots$$

$$\vdots$$

The first row of this system of equations comes from equation (10) with $j=0,1,2,\ldots$ and equation (7). The first column of equations, starting from the second equation, comes from equation (12) with $n=0,1,2,\ldots$; the second column, equation (12^+) with $n=0,1,2,\ldots$; the third column, equation (12^{+2}) with $n=0,1,2,\ldots$, and so on. For example, to compute s_2, we need the six equations displayed. To compute s_i, we need $(i+1)(i+2)/2$ equations arranged in a triangular shape following the scheme that is displayed by these equations.

IV. GENERAL CASES

In sections II and III, we illustrate the method of lifting difference equations on $\{n_i\}$ to difference equations on $\{s_i\}$. The method will work on lifting equations other than (2) and (6).

Exercise: Note that the Waring distribution ([12], p.250) satisfies

$$(k + 1 + \lambda)n_{k+1} = (a + k)n_k.$$

Lift it to a difference equation on $\{s_i\}$.

However, difference equations on $\{n_i\}$ involving more parameters than in (2) or (6) should be introduced only when simpler models cannot fit the data. Here, we borrow the principle of parsimony from the time series theory that the smallest number of parameters should be used for adequate representation. We refer again to [14] for fitting models described by equations (2) and (6) to data.

In recent papers, recursive formulas are derived either by power series methods ([2]; [17]) or by Laplace transforms ([21]; [22]; [24]). In this paper, we choose to use power series because it is more elementary to do so. As a final remark, we wish to note the origin of similarity between the two methods. Let

$$L_Z(t) = E(e^{-tZ}) = G_Z(e^{-t})$$

denote the Laplace transform of the distribution of a nonnegative, integral-valued, random variable Z. We note that equation (1), the starting point for the power series method, is paired with

$$L_{S_N}(t) = G_N(L_X(t)),$$

the starting point for the Laplace transform method via the transformation $s = e^{-t}$.

REFERENCES

1. BEALL, G., AND RESCIA, R. R. "A Generalization of Neyman's Contagious Distributions," *Biometrics*, Vol. 9, 1953, 354–86.
2. CHAN, B. "Recursive Formulas for Discrete Distributions," *Insurance: Mathematics and Economics*, Vol. 1, 1982, 241–43.
3. CROW, E. L., AND BRADWELL, G. E. "Estimation of the Parameters of the Hyper-Poisson Distributions," *Classical and Contagious Discrete Distributions*, G. P. Patil (ed.), London: Pergamon Press, 1965, 127–40.
4. FELLER, W. *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd ed., New York: Wiley, 1968.
5. GERBER, H. U. "On the Numerical Evaluation of the Distribution of Aggregate

Claims and Its Stop Loss Premiums," *Insurance: Mathematics and Economics*, Vol. 1, 1982, 13–18.

6. GULDBERG, A. "On Poisson's Frequency Function," *Skandinavisk Aktuarietidskrift, 14*, 1931, 43–48.
7. HENRICI, P. "Automatic Computations with Power Series," *Journal of the Association for Computing Machinery*, Vol. 3, 1956, 10–15.
8. ———. "*Applied and Computational Complex Analysis, Vol. I: Power Series—Integration—Conformal Mapping—Location of Zeros*, New York: Wiley Interscience, 1974.
9. IRWIN, J. O. "The Place of Mathematics in Medical and Biological Statistics," *Journal of the Royal Statistical Society*, A, 126, 1963, 1–44.
10. ———. "The Generalized Waring Distribution Applied to Accident Theory," *Journal of the Royal Statistical Society*, A, 131, 1968, 205–225.
11. ———. "The Generalized Waring Distribution," *Journal of the Royal Statistical Society*, A, 138, 1975, 18–31, 204–227, 374–84.
12. JOHNSON, N. L., AND KOTZ, S. *Distribution in Statistics: Discrete Distributions*, New York: Wiley, 1969.
13. KATZ, L. "On the Class of Functions Defined by the Difference Equation $(x+1)f(x+1) = (a+bx)f(x)$," Abstract, *Annals of Mathematical Statistics*, Vol. 17, 501 (1946).
14. ———. "Unified Treatment of a Broad Class of Discrete Distributions," *Classical and Contagious Discrete Distributions*, G. P. Patil (ed.), London: Pergamon Press, 1965, 175–82.
15. KHATRI, C. G., AND PATEL, I. R. "Three Classes of Univariate Discrete Distributions," *Biometrics*, Vol. 17, 1961, 567–75.
16. KNUTH, D. E. *The Art of Computer Programming Vol. II: Seminumerical Algorithms*, 2d ed. Reading, MA: Addison-Wesley, 1981.
17. KORNYA, P. S. "Distribution of Aggregate Claims in the Individual Risk Theory Model," *TSA*, XXXV (1984), pp. 000–00.
18. NEYMAN, J. "On a New Class of 'Contagious' Distributions, Applicable in Entomology and Bacteriology," *Annuals of Mathematical Statistics*, Vol. 10, 1939, 35–57.
19. ORD, J. K. "Graphical Methods for a Class of Discrete Distributions," *Journal of the Royal Statistical Society*, Series A, Vol. 130, 1967, 232–38.
20. PANJER, H. H., The Aggregate Claims Distribution and Stop-Loss Reinsurance," *TSA*, XXXII (1980), 523–35.
21. ———. "Recursive Evaluation of a Family of Compound Distributions," *Astin Bulletin*, Vol. 12, 1981, 22–26.
22. PANJER, H. H., AND WILLMOT, G. E. "Recursions for Compound Distributions," *Astin Bulletin*, Vol. 13, 1982, 1–11.
23. SHIU, E. S. Discussion on Kornya's paper, "Distribution of Aggregate Claims in the Individual Risk Theory Model," *TSA*, XXXV (1984), pp. 000–00.
24. SUNDT, B., AND JEWELL, W. S. "Further Results on Recursive Evaluation of a Family of Compound Distributions," *Astin Bulletin*, Vol. 12, 1981, 27–39.

DISCUSSION OF PRECEDING PAPER

ELIAS S. W. SHIU:

Dr. Chan is to be complimented for a mathematically elegant paper. As he points out at the end of section II, various recursive formulas have been rediscovered many times. In fact, the J. C. P. Miller formula was known to Euler as early as 1748 ([1], p. 3); other occurrences of the formula are also reported in [1]. On the other hand, the most recent rediscovery of the recursive formulas is [3]. I would also like to mention that the recursive formula for the compound Poisson distribution can be found in [2], since the late H. L. Seal had referred to this fact twice ([4], [5]).

REFERENCES

1. GOULD, H. W. "Coefficient Identities for Powers of Taylor and Dirichlet Series," *American Mathematical Monthly*, LXXXI (1974), 3–14.
2. PLACKETT, R. L. "Stochastic Models of Capital Investment," *Journal of the Royal Statistical Society*, B, XXXI (1969), 1–20; Discussion 20–28.
3. POURAHMADI, M. "Taylor Expansion of exp $(\Sigma_{k=0}^{\infty} a_k z^k)$ and Some Applications," *American Mathematical Monthly*, XCI (1984), 303–307.
4. SEAL, H. L. Comment on "Actuarial Note: The Distribution Functions of Collective Risk Theory as Linear Compounds," *ARCH*, 1979.1, 89.
5. ———. "Distributions of Claim Amounts—Continuous or Discrete?" *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, LXXXIII (1983), 128–29.

(AUTHOR'S REVIEW OF DISCUSSION)

BEDA CHAN:

I wish to thank Professor Shiu for adding valuable references and contributing his encyclopedic knowledge of the literature. One recent addition to the power series derivation of recursive formulas is [27] where formulas in [2] of the paper for random variables are generalized to random vectors. The paper lacks sufficient discussion on computational efficiency of recursive formulas. Fortunately Bühlmann's recent paper [25] comparing computational efficiency of recursive formulas and fast Fourier transforms fills this gap.

As mentioned in section IV, the method of lifting difference equations on $\{n_i\}$ to difference equations on $\{s_i\}$ will work in general. This can be described by the following diagram:

$$\begin{tikzcd} \Delta E \text{ on } \{s_i\} \arrow[r, leftrightarrow] & DE \text{ on } S(s) \\ \Delta E \text{ on } \{n_i\} \arrow[r, leftrightarrow] & DE \text{ on } N(s) \arrow[u, "\text{chain rule}"'] \end{tikzcd}$$

where the horizontal one-to-one correspondences are consequences of the theory of characteristic functions, or the general theory of Fourier transforms. (In the diagram, ΔE stands for difference equation and DE for differential equation.)

We now discuss briefly two ideas related to the paper. From (1) we obtain

$$K_{S_N}(s) = K_N(K_X(s))$$

where K_X is the cumulant generating function for X. Recursive formulas involving cumulants are thus obtained. These formulas are useful for asymptomatic analysis.

Next, consider

$$G_N(s) = G_M(G_C(s))$$

where N comes from M i.i.d. clusters C's. If each cluster C has i.i.d. claims X's, then

$$G_S(s) = G_M(G_C(G_X(s))) .$$

The recursive formulas in the paper can be extended to two stage formulas for compound cluster random variables. Clustering processes have been used to study computer failures [26] and earthquakes [28].

REFERENCES
(Continued from paper)

25. BÜHLMANN, H. "Numerical Evaluation of the Compound Poisson Distribution: Recursion of Fast Fourier Transform?" *Scandinavian Actuarial Journal*, 1984, 116–26.
26. LEWIS, P. "A Branching Poisson Process Model for the Analysis of Computer Failure Patterns," *Journal of the Royal Statistical Society*, *B*, 26, 1964, 389–456.
27. PAPAGEORGIOU, H. "On Recursive Formulas for Aggregate Claims," *Scandinavian Actuarial Journal*, 1984, 102–104.
28. VERE-JONES, D. "Stochastic Models for Earthquake Occurrences," *Journal of the Royal Statistical Society*, *B*, 32, 1970, 1–62.