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PERCENTILE PENSION COST METHODS: A NEW APPROACH TO PENSION VALUATIONS

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ABSTRACT

Traditional pension cost methods are based on the actuarial present value of future benefits (which is a mean value). As a consequence, these cost methods are deficient because they cannot provide plan sponsors with certain valuable information. For example, they cannot be used to determine the probability that the accumulation of a particular sequence of contributions will ultimately provide sufficient funds to pay benefits, or to determine the size of fund needed to ensure that retirees' lifetime benefits are paid with a specified probability.

This paper shows that, at age 65, there is a 45 percent chance that an amount equal to the (traditional) accrued liability will actually be sufficient to pay a lifetime benefit. This calls into question the notion that a fair value of a retiree's future benefits is the accrued liability and that a fully funded plan is one with no "unfunded liabilities."

Issues affecting the security of pension benefits are important to the Pension Benefit Guaranty Corporation (PBGC). Because of its role as an "insurer" of vested pension benefits, the PBGC is exposed to certain risks when a plan terminates. Traditional pension theory does not provide an adequate mechanism for determining the actual termination liability or the risk premiums for this type of termination insurance.

In response to these deficiencies, a new family of cost methods, called α-percentile cost methods, is developed. These cost methods are based on the probability of adequately covering all participants' benefits. Expressions for the normal cost, the accrued liability and the gain are provided.

1. INTRODUCTION

1.1 Problems with Traditional Cost Methods

The theory of traditional cost methods used in the valuation of *defined-benefit* pension plans, as applied to qualified plans under ERISA and the Internal Revenue Code, is adequately described in several standard texts, including McGill [7], Anderson [2] and Berin [4]. Examples of traditional cost methods are the projected unit credit method, the individual entry age normal method, the frozen initial liability method, attained age normal method, and the aggregate method. The cornerstone of traditional theory is the concept of the (actuarial) present value of future benefits.

The (actuarial) present value of future benefits is the discounted amount that, together with future interest, is expected to be sufficient to pay promised benefits. The discounting factors usually include: plan population decrements, interest and/or a salary scale. In other words, the actuarial present value is the mean (statistical expectation) of the present value, adjusted for projected salary changes.

The choice of pension cost method is important in the valuation of a defined-benefit plan because, among other things, it apprises the employer of the rate at which the plan's obligations are accruing as well as the level of contributions needed to meet these obligations. A "good" cost method thus enhances the security of the benefit rights of the plan's participants, which is, after all, the primary purpose of funding. The higher the ratio of plan assets to plan liabilities, the greater the assurance that the future benefits of active participants and pensioners will be satisfied; see McGill [7, chapter 17]. Ideally, the degree of assurance should be made known to all parties: employers, active participants and pensioners. In practice, the degree of assurance is never explicitly mentioned in terms of probabilities. Using traditional pension cost methods, actuaries cannot readily calculate such quantities as (1) the probability that the accumulation of contributions will ultimately provide sufficient funds to pay benefits or (2) the size of fund needed to ensure that benefits are paid with a specified probability. So even if actuaries and plan sponsors were so inclined, it would still be difficult to specify the degree of assurance (in terms of probabilities) because of the inadequacies of traditional cost methods.

For completeness, I must point out that the overall security of promised benefits is tied to other factors such as the employer's financial stability, the size and the timing of contributions, management of plan assets (including the investment performance of the fund), tax laws, and so on. These factors are crucial to the stability and viability of a plan. However, explicit consideration of these factors is not the major focus of this paper. In fact, it is assumed that the employer has taken the appropriate steps to ensure plan viability and that the plan is a trusteed noninsured plan.

Regardless of the level of funding, some plans are terminated for one reason or another. This highlights another purpose of funding: to protect the vested pension benefit insurance program, administered by the Pension Benefit Guaranty Corporation (PBGC) against abuse by terminating plans. For plans that are terminating (voluntarily or not), it is common practice to use the accrued liability (based on the actuarial present value) as the measure of the plan's liability to each participant; no contingency loading is included. This practice transfers all risks to either the participant or the PBGC. It is shown in Table 1 that for ages less than 75, the probability that an amount equal to the (traditional) accrued liability will actually be sufficient to pay a retiree's lifetime benefit is less than 50 percent. So at the younger ages, the accrued liability underestimates the risk associated with paying the retirees' lifetime annuities.

Since the PBGC is the ultimate insurer of qualified pension plans, it must be able to determine the premiums needed to cover its risks. Among the papers in this area that may be of interest to actuaries are those by Amoroso [1] and VanDerhei [12]. Amoroso discusses the arguments for and against the PBGC's pension termination program. VanDerhei provides estimates of risk-related premiums for the PBGC in the case of large single-employer defined-benefit plans. In addition, VanDerhei's paper contains numerous references.

Given a defined-benefit trusteed noninsured plan, three important and related questions arise concerning the valuation of such a plan:

(1) Should the degree of assurance be explicitly included in the actuarial cost methods used?
(2) Should the traditional accrued liability (measured in terms of the actuarial present value) be used as the sole measure of the plan's liability?
(3) How should the PBGC assess its potential liabilities with respect to the plan?

In my opinion, the answers to questions (1) and (2) are "yes" and "no," respectively. Certain aspects of question (3) are discussed by Amoroso and VanDerhei.

With respect to question (2), there are several arguments against using the traditional accrued liability, which is a mean, as the sole measure of a plan's liability. The mean is often used in statistical theory because it is easy to calculate, and for large samples, the underlying distribution of the sample mean is approximately normal. However, in the field of

TABLE 1

$\Pr[\ddot{Y}_x^{(12)} \leq \ddot{a}_x^{(12)}]$ FOR DIFFERENT INTEREST RATES*

x	5%	6%	7%	8%	9%	10%
60	0.4011	0.3886	0.3734	0.3585	0.3466	0.3324
61	0.4082	0.3924	0.3798	0.3646	0.3526	0.3406
62	0.4118	0.3990	0.3831	0.3703	0.3582	0.3460
63	0.4186	0.4019	0.3890	0.3761	0.3632	0.3540
64	0.4215	0.4079	0.3944	0.3846	0.3715	0.3585
65	0.4271	0.4133	0.4030	0.3893	0.3759	0.3660
66	0.4324	0.4180	0.4075	0.3971	0.3831	0.3728
67	0.4368	0.4259	0.4149	0.4005	0.3898	0.3792
68	0.4442	0.4329	0.4217	0.4105	0.3994	0.3884
69	0.4510	0.4393	0.4277	0.4162	0.4048	0.3933
70	0.4569	0.4450	0.4330	0.4211	0.4131	0.4014
71	0.4622	0.4498	0.4416	0.4293	0.4211	0.4088
72	0.4666	0.4581	0.4453	0.4368	0.4283	0.4199
73	0.4745	0.4658	0.4526	0.4438	0.4350	0.4262
74	0.4818	0.4682	0.4591	0.4500	0.4409	0.4363
75	0.4881	0.4788	0.4694	0.4600	0.4505	0.4410
76	0.4934	0.4837	0.4740	0.4642	0.4594	0.4496
77	0.4973	0.4874	0.4824	0.4722	0.4671	0.4569
78	0.5049	0.4944	0.4891	0.4787	0.4734	0.4629
79	0.5107	0.5000	0.4947	0.4840	0.4784	0.4729
80	0.5156	0.5099	0.4984	0.4927	0.4870	0.4755
81	0.5177	0.5118	0.5060	0.5002	0.4943	0.4827
82	0.5244	0.5185	0.5126	0.5067	0.5003	0.4940
83	0.5298	0.5233	0.5168	0.5103	0.5038	0.4973
84	0.5326	0.5260	0.5194	0.5127	0.5061	0.5061
85	0.5412	0.5345	0.5277	0.5210	0.5143	0.5069
86	0.5425	0.5349	0.5349	0.5273	0.5197	0.5121
87	0.5475	0.5397	0.5320	0.5320	0.5242	0.5164
88	0.5516	0.5437	0.5359	0.5359	0.5280	0.5280
89	0.5550	0.5471	0.5471	0.5392	0.5392	0.5313
90	0.5580	0.5580	0.5486	0.5392	0.5392	0.5298

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

actuarial risk theory, the mean by itself cannot be used as a measure of risk because the underlying distribution may be excessively skewed. For example, given two non-negative random variables with the same mean, one may be much more "dangerous," that is, riskier, than the other; see Beard, Pentikäinen and Pesonen [3, chapter 3.5.8]. A characteristic of dangerous distributions is their positive skewness and relatively thick right tails, for example, the Pareto distribution. A thick right tail suggests a relatively significant probability of very large claims (much larger than the mean) occurring and hence a significant probability of a catastrophe. On the other hand, a negative skewness implies that the mean is less than the median. This in turn implies a probability of at least 50 percent that the claim will exceed the mean, though not by very much. Fortunately, with a negatively skewed distribution, there is a relatively very insignificant probability of very large claims occurring. So, unless the distribution is normal or has negligible skewness with a light tail, there will be dangers associated with using the mean.

Pension actuaries have traditonally compensated for these inadequacies associated with the mean by using "conservative" assumptions, that is, assumptions with implicit safety margins (thus loading the mean), and by amortizing "gains." Since random fluctuations are inherent in a pension plan, the term "gain" serves as an actuarial sponge, absorbing the deviations from the "expected." The effect of amortizing gains is to dampen the amplitude of fluctuations in annual costs. The analysis of gains thus plays a very important role in the valuation of private pensions.

1.2 Objectives

The objective of this paper is to respond to questions (1) and (2) above and, in doing so, to address issues relating to question (3). To this end, a new family of pension cost methods, called α-percentile cost methods, is developed to replace the traditional cost methods. These cost methods are based on a modification of the traditional cost methods to *explicitly* include a degree of confidence (assurance) in the payment of future benefits. The confidence level is to be specified in terms of the probability of paying the participant's lifetime benefits. The essence of α-percentile cost methods is to shift the valuation process away from expected values to percentiles. In particular, they are designed to take into account the risk associated with the longevity of an annuitant.

Definition 1: *Given that a mortality table is determined to be "correct," the risk associated with the longevity of an annuitant is the risk that the actuarial present value of the retiree's future benefits is less than the actual amount needed to pay the annuitant's actual lifetime benefits.*

The longevity risk differs from the mortality risk, the latter being the risk associated with using the wrong set of mortality rates.

There is a growing concern over the "solvency" of pension funds and over the adequacy of the premiums charged by the PBGC. The α-percentile cost methods introduced in this paper can be useful in investigating these problems. They provide techniques that allow pension actuaries to routinely perform services that are rather difficult to perform by using current pension valuation techniques. For example, α-percentile cost methods will enable pension actuaries to calculate the size of the contributions (for a given amount of plan assets) needed to ensure that benefits are paid with a specified probability.

1.3 Overview

Throughout the rest of this paper, the development of the notation, theory and formulas is similar to Anderson's (in his chapter 2). The level of mathematics is just above that encountered in the professional pension EA 1B examination. The prefix "α" is appended when reference is being made to a specific quantity calculated (either explicitly or implicitly) according to a percentile funding method.

Warning 1: *Attention is focused only on developing the basic underlying theory behind α-percentile cost methods. Other topics such as contributions, vesting, early retirement, and other ancillary benefits are not dealt with in this paper.*

In Section 2, the basic random variable, $\ddot{Y}_x^{(m)}$, representing the present value of an m-thly whole life annuity due, is introduced. Expressions for its cumulative distribution function (cdf), mean, variance and skewness are given. In addition, a new function, ${}_{\alpha}\ddot{\xi}_x^{(m)}$, representing the 100α-percentile point of $\ddot{Y}_x^{(m)}$, is introduced. This function plays a key role in the development of the theory. Its role is analogous to that of $\ddot{a}_x^{(m)}$ in traditional cost methods.

In Section 3, two types of percentile cost methods are introduced: individual percentile and group percentile methods. The individual percentile methods are further broken down into individual and spread gain methods. The group method is inherently a spread gain method. As will be seen, each individual percentile cost method uses formulas very similar to those of its counterpart traditional method. In fact, the only change needed is to replace $\ddot{a}_y^{(12)}$ by the term ${}_{\alpha}\ddot{\xi}_y^{(12)}$. However, the group methods lead to more complex expressions and are ideally suited only for the valuation of medium or large plans.

In Section 4, the case of retirees is studied. Here two different approaches are used to calculate the accrued liability. In the first, called the individual approach, the liability associated with *each* retiree is calculated, ignoring the others. This liability is actually the amount needed to fund the retiree's lifetime benefits with probability α. The second approach, called the group approach, is based on funding the *entire* group of retirees' lifetime benefits with probability α.

Section 5 contains a modification for the valuation of entire plans (active and retired lives). Section 6 contains closing comments and is followed by the list of references. The Appendix contains an example of the valuation of an entire plan using both the traditional and percentile methods.

2. DISTRIBUTION THEORY

Let $\ddot{Y}_x^{(m)}$ be the present value of a life annuity due of 1 payable m times per annum issued to a life aged exactly x,

$$\ddot{Y}_x^{(m)} = \ddot{a}^{(m)}_{\overline{K_m+1/m}|} = \frac{1 - exp\left[-\delta\left(K_m + \frac{1}{m}\right)\right]}{d^{(m)}} \tag{1}$$

where i is the valuation rate of interest, $\delta = ln(1+i)$ is the force of interest,

$$d^{(m)} = m(1 - e^{-\delta/m}),$$

$$K_m = \frac{int[mT(x)]}{m},$$

and $T(x)$ measures the length of the future lifetime of a life currently aged x. For any real value t,

$$int[t] = \text{largest integer less than or equal to } t.$$

The cdf of $\ddot{Y}_x^{(m)}$ is defined as $H_x^{(m)}(u)$, where

$$H_x^{(m)}(u) = \begin{cases} 1 & \text{if } u \geq 1/d^{(m)} \\ {}_{k_m}q_x & \text{if } 0 \leq u < 1/d^{(m)} \\ 0 & \text{if } u < 0 \end{cases} \tag{2}$$

and

$$k_m = \frac{1}{m} int\left[-\frac{m}{\delta} ln(1 - ud^{(m)})\right].$$

From Bowers et al. [5, chapter 5], the mean and standard deviation of $\ddot{Y}_x^{(m)}$ are

$$E[\ddot{Y}_x^{(m)}] = \ddot{a}_x^{(m)}$$

$$\begin{aligned}\ddot{\sigma}_x^{(m)} &= \sqrt{Var\,[\ddot{Y}_x^{(m)}]}\\ &= \frac{1}{d^{(m)}}\sqrt{[{}^2A_x^{(m)} - (A_x^{(m)})^2]}\end{aligned} \tag{3}$$

where ${}^2A_x^{(m)}$ is calculated at twice the force of interest. Since the functions $\ddot{a}_x$ and A_x are usually known, the following approximations, based on the uniform distribution of deaths (UDD) in an age interval, are used to construct the tables:

$$\ddot{a}_x^{(m)} \approx \left(\frac{id}{i^{(m)}d^{(m)}}\right)\ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)}d^{(m)}} \tag{4}$$

and

$$A_x^{(m)} \approx \frac{i}{i^{(m)}} A_x \tag{5}$$

where $i^{(m)}=m(e^{\delta/m}-1)$.

To accurately approximate the cdf of a certain aggregate distribution, the coefficient of skewness of $\ddot{Y}_x^{(m)}$ is needed. Let $\ddot{\gamma}_x^{(m)}$ be the coefficient of skewness of $\ddot{Y}_x^{(m)}$, that is,

$$\ddot{\gamma}_x^{(m)} = \gamma[\ddot{Y}_x^{(m)}] = \frac{E[(\ddot{Y}_x^{(m)} - \ddot{a}_x^{(m)})^3]}{(Var[\ddot{Y}_x^{(m)}])^{3/2}}. \tag{6}$$

The following properties of the coefficient of skewness can easily be proved:

1. If a random variable Y is a linear function of a random variable X, that is, $Y=a+bX$ where a and b are known constants,

$$\gamma[Y] = \begin{cases} \gamma[X] & \text{if } b > 0 \\ 0 & \text{if } b = 0 \\ -\gamma[X] & \text{if } b < 0. \end{cases} \tag{7}$$

2. If Y_j, $j=1, \ldots, n$ are mutually independent random variables with standard deviation σ_j and skewness γ_j, then the coefficient of skewness of the sum is

$$\gamma\left[\sum_{j=1}^{n} Y_j\right] = \sum_{j=1}^{n}\left(\frac{\sigma_j}{\sigma}\right)^3 \gamma_j \tag{8}$$

where

$$\sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

From Equations (1), (6) and (7), it follows that

$$\ddot{\gamma}_x^{(m)} = \frac{-[{}^3A_x^{(m)} - 3({}^2A_x^{(m)})(A_x^{(m)}) + 2(A_x^{(m)})^3]}{[{}^2A_x^{(m)} - (A_x^{(m)})^2]^{3/2}} \tag{9}$$

where ${}^3A_x^{(m)}$ is calculated using 3δ. Tables 2 and 3 display values of $\ddot{\sigma}_x^{(12)}$ and $\ddot{\gamma}_x^{(12)}$, respectively, for different ages and interest rates. Notice that for ages less than 75, the skewness is negative. For non-negative unimodal random variables, a negative skewness suggests that the mean is less than the mode and that there is a short right tail. Thus there is a significant probability of obtaining values in excess of the mean, though not much greater than the mean. See, for example, Table 1 for a confirmation of this.

Finally, a new function, ${}_\alpha\ddot{\xi}_x^{(m)}$, called the *α-confidence function*, or simply the *confidence function*, is introduced. As will be seen, this function replaces $\ddot{a}_x^{(m)}$ in the traditional pension cost methods formulas.

Definition 2: *For a given confidence level α, age x and frequency of payments m, ${}_\alpha\ddot{\xi}_x^{(m)}$ is defined as the amount needed to ensure that a whole life annuity due of* 1 *per annum (payable m times per annum) to x is paid in its entirety with probability of α, that is,*

$$Pr[\ddot{Y}_x^{(m)} \leq {}_\alpha\ddot{\xi}_x^{(m)}] = \alpha.$$

This definition and Equation (2) result in the following equation:

$${}_\alpha\ddot{\xi}_x^{(m)} = \begin{cases} \ddot{a}_{\overline{k}|}^{(m)} & \text{if } \alpha = {}_kq_x \text{ and } k = 1/m, 2/m, \ldots \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{10}$$

TABLE 2

$\ddot{a}_x^{(12)}$ EVALUATED AT DIFFERENT INTEREST RATES*

x	5%	6%	7%	8%	9%	10%
60	3.861105	3.339495	2.913578	2.562681	2.271112	2.026857
61	3.895804	3.382897	2.961994	2.613610	2.322873	2.078337
62	3.927929	3.424488	3.009298	2.664046	2.374675	2.130306
63	3.956513	3.463315	3.054569	2.713108	2.425680	2.181975
64	3.980512	3.498336	3.096783	2.759799	2.474928	2.232429
65	3.998787	3.528399	3.134786	2.802983	2.521311	2.280589
66	4.010160	3.552288	3.167350	2.841429	2.563605	2.325252
67	4.013554	3.568872	3.193304	2.873942	2.600606	2.365213
68	4.008424	3.577524	3.211957	2.899784	2.631542	2.399676
69	3.994800	3.578179	3.223165	2.918747	2.656151	2.428338
70	3.973210	3.571263	3.227267	2.931096	2.674635	2.451347
71	3.944658	3.557683	3.225086	2.937577	2.687677	2.469331
72	3.910427	3.538643	3.217751	2.939258	2.696289	2.483251
73	3.871400	3.514956	3.206020	2.936846	2.701136	2.493740
74	3.828007	3.487001	3.190223	2.930633	2.702478	2.501034
75	3.780318	3.454791	3.170328	2.920551	2.700221	2.505016
76	3.728097	3.418033	3.145996	2.906221	2.693954	2.505253
77	3.670945	3.376265	3.116706	2.887075	2.683070	2.501107
78	3.608693	3.329247	3.082157	2.862758	2.667167	2.492135
79	3.541435	3.277007	3.042312	2.833175	2.646099	2.478148
80	3.469422	3.219730	2.997299	2.798395	2.619883	2.459116
81	3.393009	3.157717	2.947358	2.758607	2.588658	2.435131
82	3.312593	3.091314	2.892788	2.714061	2.552627	2.406353
83	3.228630	3.020939	2.833965	2.665088	2.512079	2.373030
84	3.141643	2.947085	2.771346	2.612110	2.467398	2.335510
85	3.052241	2.870342	2.705496	2.555662	2.419090	2.294269
86	2.961198	2.791476	2.637167	2.496481	2.367872	2.250001
87	2.868675	2.710634	2.566490	2.434674	2.313826	2.202765
88	2.775191	2.628337	2.493978	2.370744	2.257443	2.153037
89	2.681066	2.544907	2.419948	2.304999	2.199021	2.101100
90	2.586475	2.460520	2.344575	2.237609	2.138717	2.047103

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

The fact that ${}_{\alpha}\ddot{\xi}_x^{(m)}$ is undefined in some areas is not surprising because $\ddot{Y}_x^{(m)}$ is a discrete random variable, so its cdf [Equation (2)] is not invertible. For those values of $0 \le \alpha \le 1$ where the function ${}_{\alpha}\ddot{\xi}_x^{(m)}$ does not exist, then ${}_{\alpha}\ddot{\xi}_x^{(m)}$ can be defined in *any way* that is convenient. Since $\bar{a}_{\overline{t}|}$ is defined for all $t \ge 0$, then ${}_{\alpha}\ddot{\xi}_x^{(m)}$ can be defined as:

Definition 3: *For given α, let t be such that ${}_tq_x = \alpha$, then*

$$ {}_{\alpha}\ddot{\xi}_x^{(m)} = \frac{\delta}{d^{(m)}} \bar{a}_{\overline{t}|} \quad if\, 0 \le \alpha \le 1. \tag{11}$$

TABLE 3

$\ddot{Y}_x^{(12)}$ EVALUATED AT DIFFERENT INTEREST RATES*

x	5%	6%	7%	8%	9%	10%
60	−0.988183	−1.158547	−1.324760	−1.486953	−1.645198	−1.799537
61	−0.920462	−1.082963	−1.241065	−1.394974	−1.544841	−1.690778
62	−0.854372	−1.009517	−1.160029	−1.306177	−1.448175	−1.586194
63	−0.789832	−0.938144	−1.081614	−1.220563	−1.355255	−1.485908
64	−0.726720	−0.868725	−1.005715	−1.138052	−1.266037	−1.389925
65	−0.664843	−0.801060	−0.932130	−1.058443	−1.180332	−1.298080
66	−0.603927	−0.734847	−0.860529	−0.981389	−1.097778	−1.210003
67	−0.543616	−0.669682	−0.790461	−0.906386	−1.017827	−1.125105
68	−0.483579	−0.605172	−0.721467	−0.832909	−0.939882	−1.042718
69	−0.423530	−0.540963	−0.653115	−0.760443	−0.863341	−0.962149
70	−0.363248	−0.476757	−0.585028	−0.688523	−0.787643	−0.882736
71	−0.302652	−0.412414	−0.516990	−0.616852	−0.712405	−0.804001
72	−0.241874	−0.348024	−0.449049	−0.545425	−0.637561	−0.725813
73	−0.181136	−0.283796	−0.381395	−0.474411	−0.563254	−0.648284
74	−0.120718	−0.220016	−0.314316	−0.404097	−0.489772	−0.571701
75	−0.060930	−0.157007	−0.248152	−0.334841	−0.417489	−0.496453
76	−0.002056	−0.095072	−0.183223	−0.266985	−0.346768	−0.422929
77	0.055693	−0.034430	−0.119763	−0.200774	−0.277870	−0.351405
78	0.112138	0.024737	−0.057953	−0.136392	−0.210983	−0.282076
79	0.167115	0.082270	0.002052	−0.073992	−0.146256	−0.215086
80	0.220494	0.138048	0.060139	−0.013677	−0.083786	−0.150525
81	0.272193	0.192001	0.116248	0.044503	−0.023610	−0.088421
82	0.322206	0.244137	0.170403	0.100587	0.034323	−0.028709
83	0.370611	0.294555	0.222724	0.154714	0.090173	0.028789
84	0.417564	0.343437	0.273417	0.207117	0.144197	0.084357
85	0.463236	0.390982	0.322711	0.258053	0.196682	0.138310
86	0.507644	0.437225	0.370660	0.307596	0.247722	0.190761
87	0.550836	0.482233	0.417346	0.355843	0.297427	0.241837
88	0.592827	0.526036	0.462819	0.402863	0.345887	0.291644
89	0.633511	0.568542	0.506998	0.448584	0.393040	0.340129
90	0.672764	0.609636	0.549774	0.492906	0.438788	0.387201

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

Notice that Equations (10) and (11) yield the same values when $\alpha = {}_kq_x$ for $k=1/m, 2/m, \ldots$. However, Equation (11) effectively defines $\ddot{Y}_x^{(m)}$ in terms of the present value of a continuous whole life annuity to x, $\bar{Y}_x$, that is,

$$\ddot{Y}_x^{(m)} = \frac{\delta}{d^{(m)}} \bar{Y}_x. \tag{12}$$

This leads to the equation

$$_\alpha\ddot{\xi}_x^{(m)} = \frac{\delta}{d^{(m)}} \, _\alpha\bar{\xi}_x \tag{13}$$

where ${}_{\alpha}\bar{\xi}_x$ is the 100α percentile point of $\bar{Y}_x$. In Tables 4 and 5, ${}_{\alpha}\ddot{\xi}_x^{(12)}$ and $\ddot{a}_x^{(12)}$ are compared. Notice that for percentiles of at least 50 percent ($\alpha \geq 50\%$), ${}_{\alpha}\ddot{\xi}_x^{(m)} > \ddot{a}_x^{(m)}$ for most ages.

A recursive equation for ${}_{\alpha}\ddot{\xi}_x^{(m)}$ can be obtained as follows: let

$$I_x = \begin{cases} 1 & \text{if } (x) \text{ survives to age } x+1; \\ 0 & \text{otherwise;} \end{cases}$$

TABLE 4

COMPARING $\ddot{a}_x^{(12)}$ AND ${}_{\alpha}\ddot{\xi}_x^{(12)}$ WITH $i = 5\%$*

x	$\ddot{a}_x^{(12)}$	${}_{\alpha}\ddot{\xi}_x^{(12)}/\ddot{a}_x^{(12)}$				
		$\alpha = 50\%$	$\alpha = 60\%$	$\alpha = 70\%$	$\alpha = 80\%$	$\alpha = 90\%$
60	12.242980	1.076182	1.143713	1.205331	1.266518	1.335358
61	11.941884	1.076025	1.147694	1.213329	1.278590	1.352255
62	11.633875	1.075593	1.151646	1.221597	1.291209	1.370091
63	11.319905	1.074822	1.155496	1.230057	1.304347	1.388841
64	11.001114	1.073640	1.159158	1.238623	1.318051	1.408453
65	10.678852	1.071964	1.162532	1.247188	1.332109	1.428842
66	10.354637	1.069709	1.165589	1.255613	1.346388	1.449880
67	10.030005	1.066765	1.168121	1.263746	1.360732	1.471414
68	9.706082	1.063052	1.170025	1.271458	1.375012	1.493669
69	9.383460	1.058574	1.171231	1.278658	1.389144	1.516380
70	9.062226	1.053328	1.171703	1.285756	1.403093	1.539502
71	8.741965	1.047369	1.171726	1.292448	1.416887	1.563121
72	8.421983	1.040775	1.171281	1.298746	1.431135	1.587385
73	8.102136	1.033477	1.170204	1.304637	1.445773	1.612371
74	7.782952	1.025560	1.168411	1.310313	1.460456	1.638046
75	7.465579	1.017285	1.166775	1.316336	1.474996	1.664555
76	7.151727	1.008156	1.164300	1.321635	1.489065	1.692622
77	6.843457	0.998897	1.160745	1.325812	1.504181	1.720732
78	6.542550	0.988997	1.157698	1.330522	1.518426	1.748341
79	6.250362	0.978887	1.153028	1.333964	1.531079	1.774846
80	5.967909	0.968429	1.148614	1.335813	1.543728	1.802123
81	5.695871	0.958160	1.142964	1.338531	1.555799	1.828658
82	5.434631	0.947008	1.137580	1.338059	1.564712	1.852527
83	5.184203	0.937490	1.130563	1.339553	1.575711	1.875577
84	4.944212	0.926594	1.125027	1.337026	1.583160	1.899257
85	4.713908	0.916323	1.116549	1.337662	1.591124	1.918381
86	4.492156	0.907335	1.110825	1.333872	1.596677	1.940115
87	4.279150	0.896901	1.104039	1.333771	1.602488	1.958885
88	4.074074	0.885473	1.094113	1.331032	1.604933	1.976374
89	3.876624	0.878593	1.090231	1.325925	1.610525	1.994959
90	3.686742	0.871095	1.084219	1.324559	1.608776	2.009363

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

TABLE 5

COMPARING $\ddot{a}_x^{(12)}$ AND $_{\alpha}\ddot{\xi}_x^{(12)}$ WITH i = 8%*

x	$\ddot{a}_x^{(12)}$	$_{\alpha}\ddot{\xi}_x^{(12)}/\ddot{a}_x^{(12)}$ α = 50%	α = 60%	α = 70%	α = 80%	α = 90%
60	9.619892	1.086419	1.132435	1.171594	1.207661	1.244669
61	9.437403	1.088063	1.137865	1.180424	1.219693	1.260136
62	9.247381	1.089508	1.143383	1.189651	1.232400	1.276620
63	9.050352	1.090677	1.148918	1.199211	1.245761	1.294115
64	8.847020	1.091481	1.154378	1.209022	1.259788	1.312589
65	8.638290	1.091818	1.159650	1.218978	1.274326	1.331976
66	8.425244	1.091574	1.164665	1.228937	1.289249	1.352169
67	8.209032	1.090614	1.169220	1.238739	1.304409	1.373037
68	7.990508	1.088833	1.173188	1.248247	1.319677	1.394680
69	7.770124	1.086205	1.176485	1.257362	1.334972	1.416935
70	7.547924	1.082713	1.179061	1.266370	1.350263	1.439788
71	7.323526	1.078402	1.181142	1.275061	1.365584	1.463335
72	7.096291	1.073347	1.182729	1.283461	1.381402	1.487737
73	6.866018	1.067496	1.183693	1.291567	1.397707	1.513095
74	6.633064	1.060912	1.183950	1.299518	1.414256	1.539405
75	6.398337	1.053819	1.184213	1.307742	1.430881	1.566756
76	6.163263	1.045753	1.183612	1.315343	1.447274	1.595604
77	5.929630	1.037344	1.181888	1.321926	1.464523	1.624832
78	5.699054	1.028118	1.180388	1.328796	1.481094	1.653953
79	5.472834	1.018464	1.177223	1.334422	1.496333	1.682426
80	5.252015	1.008258	1.174036	1.338469	1.511431	1.711532
81	5.037382	0.998016	1.169486	1.343015	1.525891	1.740098
82	4.829479	0.986733	1.164929	1.344532	1.537541	1.766587
83	4.628533	0.976827	1.158631	1.347500	1.550675	1.792314
84	4.434424	0.965440	1.153478	1.346649	1.560594	1.818389
85	4.246685	0.954515	1.145382	1.348375	1.570705	1.840734
86	4.064467	0.944731	1.139695	1.345913	1.578547	1.865035
87	3.888111	0.933441	1.132855	1.346584	1.586376	1.886801
88	3.717031	0.921091	1.122888	1.344673	1.591051	1.907376
89	3.551090	0.913142	1.118560	1.340398	1.598286	1.928710
90	3.390371	0.904542	1.112099	1.339416	1.598783	1.946388

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

then

$$\alpha = Pr[\ddot{Y}_x^{(m)} \leq {_{\alpha}\ddot{\xi}_x}]$$
$$= q_x Pr[\ddot{Y}_x^{(m)} \leq {_{\alpha}\ddot{\xi}_x^{(m)}} | I_x = 0] + p_x Pr[\ddot{Y}_x^{(m)} \leq {_{\alpha}\ddot{\xi}_x^{(m)}} | I_x = 1]. \quad (14)$$

However, for all but the very oldest ages, ${_{\alpha}\ddot{\xi}_x^{(m)}} > \ddot{a}_{\overline{1}|}^{(m)}$. So given that (x) dies within a year, it must be the case that ${_{\alpha}\ddot{\xi}_x^{(m)}} > \ddot{a}_{\overline{1}|}^{(m)} \geq \ddot{Y}_x^{(m)}$ with probability 1, so

$$Pr[\ddot{Y}_x^{(m)} \leq {_{\alpha}\ddot{\xi}_x^{(m)}} | I_x = 0] = 1.$$

If, on the other hand, (x) survives to age $x+1$,

$$[\ddot{Y}_x^{(m)} | I_x = 1] = \ddot{a}_{\overline{1}|}^{(m)} + v\ddot{Y}_{x+1}^{(m)}.$$

It follows that Equation (14) can be rewritten as

$$\alpha = q_x + p_x\, Pr[\ddot{a}_{\overline{1}|}^{(m)} + v\ddot{Y}_{x+1}^{(m)} \leq {}_\alpha\ddot{\xi}_x^{(m)}],$$

which implies

$$(\alpha - q_x)/p_x = Pr[\ddot{Y}_{x+1}^{(m)} \leq (1 + i)({}_\alpha\ddot{\xi}_x^{(m)} - \ddot{a}_{\overline{1}|}^{(m)})].$$

This yields the following recursion:

$${}_\alpha\ddot{\xi}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} + v\, {}_\beta\ddot{\xi}_{x+1}^{(m)} \tag{15}$$

where $\beta=(\alpha-q_x)/p_x$. This result, though exact, is not a recursion involving ${}_\alpha\ddot{\xi}_x^{(m)}$ and ${}_\alpha\ddot{\xi}_{x+1}^{(m)}$.

The equation $\ddot{a}_x=1+vp_x\ddot{a}_{x+1}$ forms the basis of the traditional analysis of gains for pensioners; see Anderson's equation (2.10.4). So to derive equations for retiree gains that are similar to the traditional equations, Equation (15) must be put in the following form:

$${}_\alpha\ddot{\xi}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} + vp_x\, {}_\alpha\ddot{\xi}_{x+1}^{(m)} + {}_\alpha\ddot{\theta}_x^{(m)} \tag{16}$$

where ${}_\alpha\ddot{\theta}_x^{(m)}$ is the balancing item required to ensure that the right-hand sides of Equations (15) and (16) are equal. It follows from Equations (13) and (16) that

$${}_\alpha\ddot{\theta}_x^{(m)} = \frac{\delta}{d^{(m)}}\, {}_\alpha\bar{\theta}_x \tag{17}$$

where ${}_\alpha\bar{\theta}_x$ is the balancing item in the continuous case. Tables 6 and 7 give values of ${}_\alpha\ddot{\theta}_x^{(12)}$ for two interest rates (i=5 percent and 8 percent, respectively), and for various values of x and α.

3. PERCENTILE COST METHODS

3.1 Introduction

As pointed out in Section 1.1, the primary objective of funding is to enhance the security of benefits, thus improving the likelihood that the promised benefits will be paid throughout each participant's retirement years. Also, the ultimate security of benefits is tied to other factors such as the employer's financial stability. The objective of percentile cost methods is to explicitly include the level of security in the funding equation. This is done by specifying the level of longevity risk the plan is willing to *explicitly fund.*

TABLE 6

BALANCING ITEM, ${}_{\alpha}\ddot{\theta}_x^{(12)}$, WITH $i = 5\%$*

	α Values				
x	50%	60%	70%	80%	90%
60	0.071905	0.091030	0.105778	0.119447	0.132122
61	0.074273	0.095971	0.112514	0.128332	0.142824
62	0.076824	0.101564	0.120492	0.138351	0.155278
63	0.079495	0.107820	0.129576	0.148665	0.169693
64	0.082208	0.114669	0.139844	0.161936	0.186300
65	0.084841	0.121226	0.151314	0.176952	0.205265
66	0.087449	0.129075	0.163853	0.193630	0.226563
67	0.089559	0.136891	0.176944	0.211425	0.246447
68	0.090292	0.144189	0.190008	0.229695	0.269599
69	0.089760	0.150573	0.198686	0.247858	0.293683
70	0.087517	0.153402	0.209554	0.265372	0.317561
71	0.083492	0.155602	0.219597	0.277785	0.340931
72	0.078875	0.158182	0.228974	0.291240	0.364546
73	0.072435	0.160191	0.236141	0.307557	0.389286
74	0.063052	0.154895	0.238874	0.324904	0.413735
75	0.056155	0.156092	0.248809	0.343764	0.434295
76	0.043219	0.157159	0.259669	0.351968	0.466074
77	0.032924	0.146915	0.258645	0.372581	0.500388
78	0.019112	0.149125	0.268131	0.395328	0.536525
79	0.005605	0.138676	0.277021	0.406659	0.560224
80	−0.011166	0.135923	0.270808	0.420322	0.591878
81	−0.022589	0.124035	0.284862	0.444866	0.630734
82	−0.046608	0.120956	0.270799	0.440728	0.656601
83	−0.055452	0.102227	0.284774	0.463102	0.674400
84	−0.074226	0.103295	0.264058	0.463768	0.713222
85	−0.094552	0.078692	0.276331	0.475896	0.717977
86	−0.103438	0.071323	0.253295	0.474716	0.745387
87	−0.115768	0.069444	0.256297	0.485692	0.762487
88	−0.146941	0.034272	0.254681	0.471212	0.769638
89	−0.156935	0.031072	0.231576	0.491462	0.793052
90	−0.165437	0.029496	0.236987	0.455484	0.791123

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

Definition 4: *An α-percentile cost method (also called a percentile cost method or an α-cost method) is an actuarial cost method that funds promised benefits so that the ideal fund balance for* retirees *is the* lump-sum amount *such that there is a 100α% chance of paying the promised lifetime benefits if no further contributions are paid into the fund.*

An important concept resulting from this definition is that, at the time of retirement, the total accrued liability is related to the amount of assets needed to pay the present value of the retirees' future benefits *with probability* α. As a consequence, these cost methods shift the focus of funding away from the mean and to the α percentile point of the benefit

TABLE 7

BALANCING ITEM, ${}_{\alpha}\theta_x^{(12)}$, WITH $i = 8\%$*

x	α Values				
	50%	60%	70%	80%	90%
60	0.064868	0.076341	0.084556	0.091511	0.097358
61	0.068006	0.081345	0.090812	0.099062	0.105924
62	0.071447	0.087025	0.098128	0.107700	0.115920
63	0.075152	0.093413	0.106490	0.117084	0.127535
64	0.079069	0.100492	0.115989	0.128555	0.140975
65	0.083097	0.107747	0.126665	0.141606	0.156405
66	0.087215	0.116054	0.138429	0.156201	0.173846
67	0.090989	0.124509	0.150862	0.171928	0.191373
68	0.093697	0.132658	0.163470	0.188277	0.211008
69	0.095272	0.140110	0.173492	0.204771	0.231487
70	0.095290	0.145011	0.184681	0.220952	0.252077
71	0.093621	0.149147	0.195183	0.234132	0.272533
72	0.091149	0.153292	0.205196	0.247954	0.293450
73	0.087036	0.156902	0.213813	0.263735	0.315545
74	0.080443	0.155255	0.219554	0.280553	0.338216
75	0.075436	0.158098	0.230250	0.298822	0.359464
76	0.065522	0.160781	0.241783	0.310659	0.388164
77	0.057289	0.154865	0.244987	0.330878	0.419242
78	0.045882	0.157987	0.255514	0.352774	0.452117
79	0.034290	0.151092	0.265459	0.366924	0.477611
80	0.019618	0.149746	0.263919	0.382521	0.508343
81	0.008936	0.140786	0.277220	0.405676	0.543945
82	−0.012611	0.138420	0.268678	0.407845	0.570927
83	−0.021754	0.122955	0.281343	0.428966	0.592136
84	−0.039649	0.123270	0.266324	0.433532	0.627519
85	−0.059162	0.101919	0.277012	0.446221	0.638594
86	−0.068962	0.094697	0.258959	0.448407	0.665563
87	−0.081989	0.091808	0.261680	0.459576	0.684715
88	−0.111734	0.060238	0.260292	0.450306	0.696175
89	−0.122761	0.056026	0.240701	0.468360	0.719493
90	−0.132558	0.053028	0.244816	0.440698	0.723311

*Based on GAM 1983 male mortality rates with deaths uniformly distributed across each age.

distribution. This definition does not (implicitly or explicitly) imply that there is a probability of $1-\alpha$ of a retiree not receiving his/her promised retirement benefit; the only probability affecting the ultimate payment of benefits is the probability of plan terminaton. The probability α is used only to determine funding levels.

Warning 2: *Under a percentile (individual or group) cost method, a retiree who "lives too long" generates an actuarial loss that is added to the gains/losses for the entire plan. Similarly, a retiree who "dies too soon" generates an actuarial gain that is added to the gains/losses for* *the entire plan. The total gain/loss is then amortized and used to adjust future plan contributions. At no time is a retiree's benefits jeopardized because, unless the plan is terminated,* all *promised benefits are paid!*

A fundamental notion in pension valuation theory is the *actuarial present value of future benefits at time t* ($PVFB_t$). For percentile cost methods, there are two ways of looking at the $PVFB_t$: (1) the individual percentile approach, and (2) the group percentile approach. As will be seen, the difference between these approaches reflects the difference in the funding objectives. The individual percentile approach requires the $PVFB_t$ to be such that each participant's $PVFB_t$ is individually and separately calculated. Specifically, each $PVFB_t$ is calculated so that his/her accrued liability upon retirement is the lump-sum amount needed to fund the participant's retirement life annuity with some specified probability α. On the other hand, the group percentile approach considers the entire plan's PVFB; it does not produce PVFBs for individual participants. Specifically, the $PVFB_t$ is the amount needed on hand to fund the projected retirement annuities for all participants with a specified overall probability α.

In addition to the way it is used above, the term "individual" also is used to denote those cost methods that define the normal cost and accrued liability for each participant, for example, unit credit, entry age normal and individual level premium methods. In contrast, there are "spread gain" methods that define the normal cost and accrued liability for the entire plan, for example, frozen initial liability, attained age normal and aggregate methods.

The table below shows the possible combination of percentile methods (individual versus group) and cost allocation methods (individual versus spread gain). The subsections of Section 3 in which they are studied are also shown. Note that it is not possible to have the combination "group percentile individual cost method."

Cost Allocation	Individual Percentile	Group Percentile
Individual Gain	Section 3.2	Impossible
Spread Gain	Section 3.3	Section 3.4

The family of individual percentile cost methods is expected to produce more conservative funding levels than the family of group percentile cost methods. Since the former cost method requires the employer to focus on funding *each* participant's benefits at the α-level, it thus increases the plan's accrued liabilities, normal costs and contributions. However, because of its inherent conservative nature, individual percentile cost methods are ideally suited for setting regulatory standards. For example, individual percentile costs methods can be used:

- To establish new minimum funding standards under Internal Revenue Code Section 412
- To determine quantities such as the accrued benefit obligation (ABO), pension benefit obligation (PBO), pension costs, and so on under *FAS 87*
- To calculate settlement, curtailment or termination liabilities under *FAS 88*
- To fairly determine the value of the pension to the individual in the case of a plan termination where employees are offered lump-sum payments in lieu of future pension benefits, and very importantly
- To value small plans. Shapiro [9] points out that plan termination rates are much higher for smaller plans. As such, these plans may pose a significant threat to the security of their participants' benefits.

Because the group percentile approach is less conservative than the individual percentile approach, it may be viewed more favorably by plan sponsors. However, to fully enjoy the benefits of the group approach, the plan must be large. The group percentile method is *not* suited for use by small plans. Unfortunately, the group approach leads to a more complicated analysis of gains and requires a more sophisticated calculation of liabilities and costs.

The idea behind using the percentiles of a distribution (rather than its mean) to fund benefits is not a new one. For example, Gerber [6, chapter 5] proposes a premium calculation principle based on percentiles. In addition, Bowers et al. [5, chapter 2.5] provides examples using percentiles in insurance calculations.

Throughout the rest of this section, the following sets of employees are used: Symbolically,

$$A_{t+1} = A_t - D_t - T_t - R_t + N_t$$

where $t = 0, 1, 2, \ldots$

A_t = set of active employees in plan at time t

D_t = members of A_t who died in $(t,t+1)$

R_t = members of A_t who retired in $(t,t+1)$

T_t = members of A_t who withdrew in $(t,t+1)$

N_t = set of new entrants into plan in $(t,t+1)$.

New entrants are assumed to occur at the end of the year, that is, at $(t+1)-$, with no past service credit. If new entrants were given past service credit at the valuation date, this past service credit must be included in the valuation.

All lives are assumed to be mutually independent. Employee j (also referred to as "(j)") is assumed to be age w_j at hire and is currently age x_j. As is the standard practice, no symbol j is appended to these ages; however, j is implied. The normal retirement age is y for all employees.

The general case in which the projected pension benefit is based on salary is considered. The case in which benefits are independent of salary clearly is a special case with salary assumed to be constant throughout the active life of the employee. Let S_t^j be employee j's annual salary at time t for year $(t,t+1)$ and $\{s_x\}$ be the sequence of salary scale indexes. If employee j is age x at time t, the projected salary at age z is $S_t^j\, s_z/s_x$.

Warning 3: *For ease of presentation and to follow Anderson's approach, it is assumed that the pension plan keeps separate funds for retirees and active lives.*

3.2 Individual Percentile Individual Cost Method

The distinguishing feature of individual cost methods is that their normal costs and accrued liabilities are first computed for the typical employee j and then summed to get the totals. The total α-accrued liability and total α-normal cost are defined as

$$_{\alpha}AL_t = \sum_{j\in A_t} {_{\alpha}AL_t^j} \tag{18}$$

$$_{\alpha}NC_t = \sum_{j\in A_t} {_{\alpha}NC_t^j}. \tag{19}$$

These definitions are valid only for individual percentile individual cost methods. For each α-percentile method, the approach used by Anderson is followed and results similar to Anderson's are established. The only change needed is to replace $\ddot{a}_y^{(12)}$ with ${}_\alpha\ddot{\xi}_y^{(12)}$.

3.2.1 α-Percentile Projected Unit Credit Method

For employee j, let $B_t^j(x)$ be the annual retirement benefit accrued to time t. In keeping with the traditional projected unit credit method,

$$B_t^j(x) = \frac{(x - w)}{(y - w)} B_t^j(y), \; w \le x \le y, \tag{20}$$

where $B_t^j(y)$ is the projected retirement benefit at age y based on the information available at time t. The α-accrued liability for employee j is defined as

$${}_\alpha AL_t^j = B_t^j(x) \, {}_\alpha\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_x^{(\tau)}}. \tag{21}$$

Employee j's normal cost (due at time t) is defined as

$${}_\alpha NC_t^j = \Delta B_t^j(x) {}_\alpha\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} \tag{22}$$

where

$$\Delta B_t^j(x) = B_t^j(x + 1) - B_t^j(x)$$

is the expected change in accrued benefit over $(t, t+1)$. Total α-accrued liabilities and normal costs are calculated according to Equations (18) and (19). If no further benefits were to accrue, the amount needed to cover the lifetime retirement benefit with probability α when (j) retires at age y would be $B_t^j(x){}_\alpha\ddot{\xi}_y^{(12)}$. So the α-accrued liability [Equation (21)] is defined as the *actuarial present value* of this amount.

Warning 4: *Note that the probability of survival to age y as an active participant,* ${}_{y-x}p_x$, *is* not *included in the probability* α. *This is because, if (j) does not survive to age y, there is no risk associated with paying pension benefits. Another reason for not including* ${}_{y-x}p_x$ *in* α *is that, if there are high termination rates in the early years, the normal cost and accrued liability will be very small or even zero in the early years and then escalate very rapidly as the employee approaches retirement.*

3.2.2 α-Percentile Entry Age Normal Method

Based on the information at time t, (j)'s projected (to age y) retirement benefit is defined as $B_t^j(y)$. For convenience, this projected benefit is written without the y as in B_t^j. The ideal fund balance at retirement will be $B_{t\,\alpha}^j\ddot{\xi}_y^{(12)}$. Assuming that employee j was hired at age w and that the α-normal cost is a constant fraction of salary, ${}_\alpha U_t^j$, at each year of age, then

$$ {}_\alpha U_t^j S_t^j \frac{S_w}{S_x}\left(\frac{{}^sN_w^{(\tau)} - {}^sN_y^{(\tau)}}{sD_w^{(\tau)}}\right) = B_{t\,\alpha}^j\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_w^{(\tau)}} $$

where

$$ {}^sD_x^{(\tau)} = s_x D_x^{(\tau)} \qquad \text{and} \qquad {}^sN_x^{(\tau)} = \sum_{z=x}^{\infty} {}^sD_z^{(\tau)}. $$

This leads to

$$ \begin{aligned} {}_\alpha NC_t^j &= {}_\alpha U_t^j S_t^j \\ &= B_{t\alpha}^j\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_w^{(\tau)}}\left(\frac{{}^sD_w^{(\tau)}}{{}^sN_w^{(\tau)} - {}^sN_y^{(\tau)}}\right)\frac{s_x}{s_w}. \end{aligned} \tag{23} $$

The α-accrued liability is given as

$$ {}_\alpha AL_t^j = B_{t\alpha}^j\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} - {}_\alpha NC_t^j\left(\frac{{}^sN_x^{(\tau)} - {}^sN_y^{(\tau)}}{{}^sD_x^{(\tau)}}\right) $$

which can be simplified to give

$$ {}_\alpha AL_t^j = B_{t\alpha}^j\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_x^{(\tau)}}\left(\frac{{}^sN_w^{(\tau)} - {}^sN_x^{(\tau)}}{{}^sN_w^{(\tau)} - {}^sN_y^{(\tau)}}\right). \tag{24} $$

Total α-accrued liabilities and α-normal costs are calculated according to Equations (18) and (19).

3.2.3 α-Individual Level-Premium Method

Like its traditional counterpart, the α-normal cost is defined recursively. Let x_0 be (j)'s age at the inception of the plan, that is, at $t=0$, then

$$ {}_{\alpha}NC_0^j = B_{0\alpha}^j \ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_{x_0}^{(\tau)}} \left(\frac{{}^sD_{x_0}^{(\tau)}}{{}^sN_{x_0}^{(\tau)} - {}^sN_y^{(\tau)}} \right) \tag{25} $$

and for $t=1, 2, \ldots$

$$ {}_{\alpha}NC_t^j = {}_{\alpha}NC_0^j \left(\frac{S_{x_0+t}}{S_{x_0}} \right) + \sum_{k=1}^{t} (\Delta_{\alpha}NC_k^j) \left(\frac{S_{x_0+t}}{S_{x_0+k}} \right) \tag{26} $$

where

$$ \Delta_{\alpha}NC_k^j = \Delta B_{k-1}^j \, {}_{\alpha}\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_{x_0+k}^{(\tau)}} \left(\frac{{}^sD_{x_0}^{(\tau)}}{{}^sN_{x_0+k}^{(\tau)} - {}^sN_y^{(\tau)}} \right), \tag{27} $$

and

$$ \Delta B_{k-1}^j = B_k^j - B_{k-1}^j \; k = 1, 2, \ldots $$

If (j) was not in the plan at its inception, then $B_{x_0+k}^j=0$ for $k=0, 1, \ldots$ until (j) entered the plan. The α-accrued liability for (j) is defined as

$$ {}_{\alpha}AL_t^j = B_{t\alpha}^j \ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} - {}_{\alpha}NC_t^j \left(\frac{{}^sN_x^{(\tau)} - {}^sN_y^{(\tau)}}{{}^sD_x^{(\tau)}} \right). \tag{28} $$

Total α-accrued liabilities and α-normal costs are calculated according to Equations (18) and (19).

3.2.4 Gains for Individual Percentile Individual Methods

Let F_t be the fund balance at time t for *active lives* and C_t be the contribution made during year t. The α-unfunded accrued liability at t, ${}_{\alpha}U\,AL_t$, is defined to be

$$ {}_{\alpha}U\,AL_t = {}_{\alpha}AL_t - F_t. \tag{29} $$

In keeping with tradition, the gain over year t, ${}_{\alpha}G_t$, is defined as

$$ {}_{\alpha}G_t = ({}_{\alpha}U\,AL_t + {}_{\alpha}NC_t)(1 + i) - (C_t + I_t^c) - {}_{\alpha}U\,AL_{t+1} \tag{30} $$

where i is the plan's valuation interest rate and I_t^c is the expected interest earned on C_t during year $(t, t+1)$.

Following the overall approach used in Anderson's chapter 2, the gain in Equation (30) can be split into its components as follows:

$$ {}_{\alpha}G_t = {}_{\alpha}G_t^{(i)} + {}_{\alpha}G_t^{(d)} + {}_{\alpha}G_t^{(w)} + {}_{\alpha}G_t^{(r)} + {}_{\alpha}G_t^{(n)} + {}_{\alpha}G_t^{(s)} \tag{31} $$

where

$$_{\alpha}G_t^{(i)} = I_t - i\,F_t - I_t^{(c)} + {}_{\alpha}I_t^{(p)} \tag{32}$$

$$_{\alpha}G_t^{(d)} = \sum_{j\in D_t} {}_{\alpha}\widetilde{AL}_{t+1}^{j} - \sum_{j\in A_t} q_x^{(d)}\,{}_{\alpha}\widetilde{AL}_{t+1}^{j} \tag{33}$$

$$_{\alpha}G_t^{(w)} = \sum_{j\in T_t} {}_{\alpha}\widetilde{AL}_{t+1}^{j} - \sum_{j\in A_t} q_x^{(w)}\,{}_{\alpha}\widetilde{AL}_{t+1}^{j} \tag{34}$$

$$_{\alpha}G_t^{(r)} = \sum_{j\in R_t} {}_{\alpha}\widetilde{AL}_{t+1}^{j} + B_{New} + I_{New} - ({}_{\alpha}PP_t + {}_{\alpha}I_t^{(p)}) \tag{35}$$

$$_{\alpha}G_t^{(n)} = -\sum_{j\in N_t} {}_{\alpha}\hat{AL}_{t+1}^{j} \tag{36}$$

$$_{\alpha}G_t^{(s)} = -\sum_{j\in A_{t+1}\cap A_t} [{}_{\alpha}\hat{AL}_{t+1}^{j} - {}_{\alpha}\widetilde{AL}_{t+1}^{j}], \tag{37}$$

which are the gains due to interest, mortality, withdrawal, retirement, new entrants, and unexpected salary (or benefit) changes, respectively.

In the above equations, ${}_{\alpha}\widetilde{AL}_{t+1}^{j}$ is the accrued liability calculated using the expected benefit at $t+1$. On the other hand, ${}_{\alpha}\hat{AL}_{t+1}^{j}$, which is more applicable to spread gain methods, is an "accrued liability" calculated using the actual benefit at $t+1$. In addition, ${}_{\alpha}PP_t$ is the amount of assets withdrawn to "purchase" pensions and ${}_{\alpha}I_t^{(p)}$ is the expected interest earned on ${}_{\alpha}PP_t$; $I_t = F_{t+1} - (F_t + C_t - {}_{\alpha}PP_t)$ is the actual interest earned during year $(t, t+1)$; and $B_{New} + I_{New}$ is the amount of benefits accumulated with interest at the assumed rate i, paid to newly retired lives during year $(t, t+1)$.

This decomposition of the gain for active lives is valid for every cost method, be it an individual or a spread gain method. The only items that change are the definition of ${}_{\alpha}\widetilde{AL}_{t+1}^{j}$ and ${}_{\alpha}\hat{AL}_{t+1}^{j}$. However, for the α-aggregate cost method, an extra term must be added; see Equation (60).

Excellent discussions of the calculation of gains and losses under traditional methods are given by Small [11], Anderson [2] and Berin [4]. Small and Berin use a less theoretical approach than Anderson.

In developing his expression for the gain under the unit credit method, Anderson makes an error in dropping the subscripts "t" from B_t^j. This results in the term for the salary scale gain, ${}_{\alpha}G_t^{(s)}$, being missed. The correct expression for the gain can be derived by noting that the accrued liability at time $t+1$ is actually:

$$ {}_{\alpha}AL_{t+1} = \sum_{j \in A_{t+1}} B^j_{t+1}(x+1)\, {}_{\alpha}\ddot{\xi}^{(12)}_y \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}}. \tag{38} $$

Note Anderson (his first equation on page 8) uses $B^j_t(x+1)$ instead. For notational convenience, let

$$ B^j_{t+1}(x+1) = B^j_t(x) + \Delta_t B^j_t(x+1) + \Delta B^j_t(x) $$

where

$$ \Delta_t B^j_t(x+1) = B^j_{t+1}(x+1) - B^j_t(x+1). $$

This leads to the following results for the percentile projected unit credit method:

$$ {}_{\alpha}\widetilde{AL}^j_{t+1} = B^j_t(x+1)\, {}_{\alpha}\ddot{\xi}^{(12)}_y \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}} \tag{39} $$

$$ {}_{\alpha}\widehat{AL}^j_{t+1} = B^j_{t+1}(x+1)\, {}_{\alpha}\ddot{\xi}^{(12)}_y \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}} \tag{40} $$

and

$$ {}_{\alpha}G^{(s)}_t = -\sum_{j \in A_{t+1} \cap A_t} \Delta_t B^j_t(x+1)\, {}_{\alpha}\ddot{\xi}^{(12)}_y \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}}. \tag{41} $$

For the α-entry cost method,

$$ {}_{\alpha}\widetilde{AL}^j_{t+1} = B^j_t\, {}_{\alpha}\ddot{\xi}^{(12)}_y \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}} \left(\frac{{}^sN^{(\tau)}_w - {}^sN^{(\tau)}_{x+1}}{{}^sN^{(\tau)}_w - {}^sN^{(\tau)}_y} \right) $$

$$ {}_{\alpha}G^{(n)}_t = 0, \tag{42} $$

and

$$ {}_{\alpha}G^{(s)}_t = \sum_{j \in A_{t+1} \cap A_t} \Delta B^j_t\, \ddot{\xi}^{(12)}_y \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}} \left(\frac{{}^sN^{(\tau)}_w - {}^sN^{(\tau)}_{x+1}}{{}^sN^{(\tau)}_w - {}^sN^{(\tau)}_y} \right) \tag{43} $$

where $\Delta B^j_t = B^j_{t+1} - B^j_t$ is the change in the projected annual retirement benefit.

For the α-individual level premium method, the expressions for the gains are very similar to those given for the α-entry age normal method. The only difference is that ${}_{\alpha}\widetilde{AL}^{j}_{t+1}$ is calculated by using Equation (28) with $x+1$ replacing x and with B^{j}_{t} unchanged. The new entrant gain also is zero.

3.3 Individual Percentile Spread Gain Methods

For traditional spread gain cost methods, the normal cost and the accrued liability are defined for the entire group of active lives. As explained in Section 3.1, this feature is retained under the individual percentile spread gain approach.

3.3.1 Individual α-Frozen Initial Liability Method

Here the α-normal cost is defined recursively, for $t=0, 1, 2, \ldots$, as

$$ {}_{\alpha}NC_{t} = {}_{\alpha}U_{t} \sum_{j \in A_{t}} S^{j}_{t} \tag{44} $$

where ${}_{\alpha}U_{t}$ is the normal cost percentage. In particular,

$$ {}_{\alpha}U_{t} = \frac{{}_{\alpha}PVFB_{t} - {}_{\alpha}UAL_{t} - F_{t}}{PVFS_{t}} \tag{45} $$

where

$$ PVFS_{t} = \sum_{j \in A_{t}} PVFS^{j}_{t} \tag{46} $$

with

$$ PVFS^{j}_{t} = S^{j}_{t} \left(\frac{{}^{s}N^{(\tau)}_{x} - {}^{s}N^{(\tau)}_{y}}{{}^{s}D^{(\tau)}_{x}} \right) \tag{47} $$

and

$$ {}_{\alpha}PVFB_{t} = \sum_{j \in A_{t}} {}_{\alpha}PVFB^{j}_{t} \tag{48} $$

with

$$ {}_{\alpha}PVFB^{j}_{t} = B^{j}_{t} \, {}_{\alpha}\ddot{\xi}^{(12)}_{y} \frac{D^{(\tau)}_{y}}{D^{(\tau)}_{x}}. \tag{49} $$

Assuming the gain is always zero, Equation (30) yields

$$_{\alpha}UAL_{t+1} = (_{\alpha}UAL_t + {}_{\alpha}NC_t)(1 + i) - (C_t + I_t^{(c)}). \tag{50}$$

Once $_{\alpha}UAL_0$ is known, $_{\alpha}NC_0$ can be determined and Equations (44) and (50) can be used recursively for $t=1, 2, \ldots$. In particular, the α-*entry age normal method* is used to find the initial unfunded liability, that is,

$$_{\alpha}UAL_0 = \sum_{j\in A_0} B_0^j \, {}_{\alpha}\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_{x_0}^{(\tau)}} \left(\frac{{}^sN_w^{(\tau)} - {}^sN_{x_0}^{(\tau)}}{{}^sN_w^{(\tau)} - {}^sN_y^{(\tau)}} \right) - F_0 \tag{51}$$

where x_0 is defined in Section 3.2.3.

Whenever there is a change in either the plan or assumptions, the unfunded liability $_{\alpha}UAL_t$ is adjusted by adding to it the change (increase or decrease) in the α-entry age normal accrued liability due to the change.

3.3.2 Individual α-Attained Age Normal

This method uses the (projected) unit credit method to find $_{\alpha}UAL_0$

$$_{\alpha}UAL_0 = \sum_{j\in A_0} B_0^j(x_0) \, {}_{\alpha}\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_{x_0}^{(\tau)}} - F_0. \tag{52}$$

Once the initial unfunded is known, proceed with Equations (44) and (50) as in the α-frozen initial liability method.

Whenever there is a change in either the plan or assumptions, the unfunded liability, $_{\alpha}UAL_t$, is adjusted by adding to it the change (increase or decrease) in the α-(projected) unit credit accrued liability due to the change.

3.3.3 Individual α-Aggregate Method

Here, as in the traditional method, the α-accrued liability is defined to be the actual fund balance, that is,

$$_{\alpha}AL_t = F_t. \tag{53}$$

Similarly, the α-normal cost is as in Equation (44), that is,

$$_{\alpha}NC_t = {}_{\alpha}U_t \sum_{j\in A_t} S_t^j$$

where

$$_{\alpha}U_t = \frac{_{\alpha}PVFB_t - F_t}{PVFS_t} \tag{54}$$

and $PVFS_t$ and $_{\alpha}PVFB_t$ are defined in Equations (46) and (48), respectively.

3.3.4 Gains for Individual Percentile Spread Gain Methods

For the α-frozen initial liability, α-attained age normal and the α-aggregate cost methods, a term "gain" can be defined in a manner similar to that used for traditional spread gain methods:

$$_{\alpha}G_t = (_{\alpha}U_t - {_{\alpha}U_{t+1}})\, PVFS_{t+1} \tag{55}$$

where $_{\alpha}U_t$ is the normal cost percentage at time t for the appropriate method.

The various components of the gain can be determined in a manner similar to that given by Anderson's equation (2.8.3): let

$$_{\alpha}\widetilde{AL}^j_{t+1} = {_{\alpha}P\widetilde{VFB}^j_{t+1}} - {_{\alpha}U_t}\, P\widetilde{VFS}^j_{t+1} \tag{56}$$

$$_{\alpha}\widehat{AL}^j_{t+1} = {_{\alpha}PVFB^j_{t+1}} - {_{\alpha}U_t}\, PVFS^j_{t+1} \tag{57}$$

where

$$P\widetilde{VFS}^j_{t+1} = S^j_t \left(\frac{S_{x+1}}{S_x}\right)\left(\frac{{^sN^{(\tau)}_{x+1}} - {^sN^{(\tau)}_y}}{{^sD^{(\tau)}_{x+1}}}\right) \tag{58}$$

$$_{\alpha}P\widetilde{VFB}^j_{t+1} = B^j_t\, {_{\alpha}\ddot{\xi}^{(12)}_y}\, \frac{D^{(\tau)}_y}{D^{(\tau)}_{x+1}}. \tag{59}$$

The terms $_{\alpha}\widetilde{AL}^j_{t+1}$ and $_{\alpha}\widehat{AL}^j_{t+1}$ can now be substituted into Equations (32) to (37) to determine the various components of the gain. For the aggregate cost method in Section 3.3.3, an additional component of the gain must be added. This term represents the gain due to excess contributions, $_{\alpha}G^{(c)}_t$, and is defined as

$$_{\alpha}G^{(c)}_t = C_t + I^{(c)}_t - {_{\alpha}NC_t}(1 + i). \tag{60}$$

3.4 Group Percentile Spread Gain Methods

3.4.1 Problems in Defining ${}_{\alpha}PVFB_t$

So far ${}_{\alpha}PVFB_t$ has been defined as in Equation (48), that is, as the sum of the individual ${}_{\alpha}PVFB_t^j$ given in Equation (49). This is consistent with the objective of individual percentile funding. However, for group percentile methods, ${}_{\alpha}PVFB_t$ must be defined differently. Unfortunately, there is no unique way to define ${}_{\alpha}PVFB_t$. To see this, let $L_t^{(act)}$ be the random variable representing the present value of the future benefits of the active participants. How should $L_t^{(act)}$ be defined? First, it is clear that the expectation of $L_t^{(act)}$ must be the traditional measure of $PVFB_t$, that is,

$$\mu_t = E[L_t^{(act)}] = PVFB_t = \sum_{j \in A_t} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j \ddot{a}_y^{(12)}. \tag{61}$$

Second, ${}_{\alpha}PVFB_t$ must be defined so that

$$Pr[L_t^{(act)} \leq {}_{\alpha}PVFB_t] = \alpha. \tag{62}$$

Even though ${}_{\alpha}PVFB_t$ is now "defined," $L_t^{(act)}$ is still undefined.

There may be many ways to define $L_t^{(act)}$ while still ensuring that Equation (61) holds; however, only the two most obvious ways are considered: one way is to set

$$L_t^{(act)} = \sum_{j \in A_t} v^{y-x} I_t^j(x)\, B_t^j\, \ddot{Y}_y^{(12)} \tag{63}$$

where $I_t^j(x)$ is the indicator random variable

$$I_t^j(x) = \begin{cases} 1 & \text{if } (j) \text{ remains an active participant until retirement at age } y; \\ 0 & \text{otherwise.} \end{cases}$$

Another way is to replace the $I_t^j(x)$ term in Equation (63) by its mean, ${}_{y-x}p_x^{(\tau)}$, to give a new random variable

$$L_t^{(act)} = \sum_{j \in A_t} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j\, \ddot{Y}_y^{(12)}. \tag{64}$$

The $L_t^{(act)}$'s in Equations (63) and (64) have the same mean because $E[v^{y-x} I_t^j(x)] = D_y^{(\tau)}/D_x^{(\tau)}$, but they treat the probability of retiring as an active participant quite differently. To see this, first recall warning 4 in

Section 3.2.1; then consider the extreme case in which there is only a single (one) active participant (j) in the plan. Then Equation (64) implies

$$Pr\left[\frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j \ddot{Y}_y^{(12)} \leq {}_{\alpha}PVFB_t\right] = \alpha.$$

This further implies that

$${}_{\alpha}PVFB_t = \frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j \, {}_{\alpha}\ddot{\xi}_y^{(12)}, \tag{65}$$

which, interestingly, is the same as the individual percentile definition of ${}_{\alpha}PVFB_t^j$ given in Equation (49).

On the other hand, Equation (63) yields

$$Pr[v^{y-x} I_t^j(x) \, B_t^j \, \ddot{Y}_y^{(12)} \leq {}_{\alpha}PVFB_t] = \alpha.$$

Following the technique used to derive Equation (15),

$$\begin{aligned} \alpha = {}_{y-x}q_x^{(\tau)} \, Pr[v^{y-x} I_t^j(x) B_t^j \, \ddot{Y}_y^{(12)} \leq {}_{\alpha}PVFB_t | I_t^j(x) = 0] \\ + {}_{y-x}p_x^{(\tau)} \, Pr[v^{y-x} I_t^j(x) B_t^j \, \ddot{Y}_y^{(12)} \leq {}_{\alpha}PVFB_t | I_t(x) = 1]. \end{aligned} \tag{66}$$

But since ${}_{\alpha}PVFB_t \geq 0$, then Equation (66) reduces to

$$\alpha = {}_{y-x}q_x^{(\tau)} + {}_{y-x}p_x^{(\tau)} \, Pr[v^{y-x} \, B_t^j \, \ddot{Y}_y^{(12)} \leq {}_{\alpha}PVFB_t | I_t^j(x) = 1],$$

which can be further reduced to

$$Pr[v^{y-x} \, B_t^j \, \ddot{Y}_y^{(12)} \leq {}_{\alpha}PVFB_t | I_t^j(x) = 1] = \beta \tag{67}$$

where

$$\beta = \frac{\alpha - {}_{y-x}q_x^{(\tau)}}{{}_{y-x}p_x^{(\tau)}} \leq \alpha.$$

But as $\ddot{Y}_y^{(12)}$ and $I_t^j(x)$ are (by definition) independent, Equation (67) implies

$$\begin{aligned} {}_{\alpha}PVFB_t &= v^{y-x} \, B_t^j \, {}_{\beta}\ddot{\xi}_y^{(12)} \\ &= \frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j \, {}_{\alpha}\ddot{\xi}_y^{(12)} \left(\frac{{}_{\beta}\ddot{\xi}_y^{(12)}}{{}_{y-x}p_x^{(\tau)} \, {}_{\alpha}\ddot{\xi}_y^{(12)}} \right). \end{aligned} \tag{68}$$

Since $\beta \leq \alpha$ and ${}_{y-x}p_x^{(\tau)} \leq 1$, it is not clear which of Equations (65) and (68) yields the larger $PVFB$.

This extreme case (with a single participant) illustrates the key difference between the constructions in Equation (63) and (64). Equation (63) includes the probability of not surviving to retirement age y as an active participant, while Equation (64) excludes this probability and considers only the risk of longevity *after* retirement age y. For example, using the service table (in Table 8), one can see that if (j) was a newly hired employee aged 25 and $\alpha=0.5$, then under Equation (63) the ${}_{\alpha}PVFB_t$ is *zero* because ${}_{40}q_{25}^{(\tau)} < 0.5$. In fact, it remains zero for more than 10 years until $x > 35$ when ${}_{65-x}q_x^{(\tau)} < 0.5$! On the other hand, using Equation (64) results in an ${}_{\alpha}PVFB_t$ that is positive and increasing. However, notice that as (j) approaches retirement age y, the ${}_{\alpha}PVFB_t$ values in Equations (65) and (68) converge to the same quantity, $B_t^j\, {}_{\alpha}\ddot{\xi}_y^{(12)}$. It follows that, compared to Equation (64), Equation (63) results in a more rapid increase in costs and liabilities as the participant approaches retirement. This is not an attractive feature of Equation (63). The anomalies due to Equation (63)'s approach become insignificant when the number of active participants is very large.

To evaluate the probability in Equation (62) [using either Equation (63) or (64)], the distribution of $L_t^{(act)}$ must be approximated because $L_t^{(act)}$ is a sum of independent (though not necessarily identically distributed) random variables. For simplicity it is assumed that the number of active participants is so large that the skewness of $L_t^{(act)}$ is small, that is, less than 0.30 in *absolute* value. When the skewness is that small, an accurate approximation is the Haldane Type A approximation. From Pentikäinen [8, equations (3.10)–(3.12)]:

Approximation 1 (Haldane's Type A): *If X is a random variable with mean μ_X, standard deviation σ_X and skewness γ_X, then*

$$Pr[X \le x_0] \approx \Phi\,\{[(1 + s\tilde{x}_0)^h - \mu(h, s)]/\sigma(h, s)\} \tag{69}$$

where

$$\tilde{x}_0 = \frac{(x_0 - \mu_X)}{\sigma_X}$$

$$s = \frac{\sigma_X}{\mu_X}$$

TABLE 8

SERVICE TABLE FUNCTIONS WITH $i = 8\%$

x	$D_{65}^{(\tau)}/D_x^{(\tau)}$	$\ddot{a}_{x:\overline{65-x}\rvert}$	${}^sD_{65}^{(\tau)}/{}^sD_x^{(\tau)}$	${}^s\ddot{a}_{x:\overline{65-x}\rvert}$	s_x
25	0.009862	7.409626	0.146534	17.570684	0.0673
26	0.011627	7.556903	0.153796	17.391816	0.0756
27	0.013656	7.701320	0.161041	17.164034	0.0848
28	0.015988	7.845558	0.169186	16.981548	0.0945
29	0.018658	7.988696	0.178033	16.817306	0.1048
30	0.021704	8.129647	0.187264	16.637416	0.1159
31	0.025167	8.267207	0.197542	16.495664	0.1274
32	0.029090	8.400001	0.208380	16.345794	0.1396
33	0.033500	8.521956	0.219962	16.198783	0.1523
34	0.038479	8.639803	0.232220	16.045721	0.1657
35	0.044082	8.752366	0.245174	15.885071	0.1798
36	0.050319	8.849172	0.258710	15.706859	0.1945
37	0.057260	8.931796	0.272794	15.507511	0.2099
38	0.064956	8.997935	0.287543	15.291863	0.2259
39	0.073500	9.049893	0.303091	15.064645	0.2425
40	0.082957	9.085734	0.319067	14.805978	0.2600
41	0.093399	9.103449	0.335725	14.526798	0.2782
42	0.104896	9.100928	0.353065	14.225424	0.2971
43	0.117521	9.075982	0.371197	13.904656	0.3166
44	0.131419	9.031058	0.389968	13.557223	0.3370
45	0.146615	8.959642	0.409652	13.191060	0.3579
46	0.163356	8.868527	0.430678	12.816760	0.3793
47	0.181782	8.756059	0.452531	12.416371	0.4017
48	0.202040	8.620415	0.475276	11.990185	0.4251
49	0.224290	8.459625	0.499310	11.545928	0.4492
50	0.248830	8.275812	0.524959	11.087658	0.4740
51	0.275743	8.062743	0.550935	10.586831	0.5005
52	0.305540	7.825947	0.579113	10.077153	0.5276
53	0.338359	7.559150	0.609108	9.547296	0.5555
54	0.374683	7.263285	0.641141	8.996801	0.5844
55	0.414676	6.931820	0.675258	8.422336	0.6141
56	0.458689	6.561419	0.711257	7.818026	0.6449
57	0.498685	6.046347	0.736501	7.060016	0.6771
58	0.542452	5.489242	0.763050	6.278463	0.7109
59	0.590406	4.886097	0.790793	5.470375	0.7466
60	0.643029	4.232470	0.820400	4.637746	0.7838
61	0.700891	3.523335	0.851629	3.776218	0.8230
62	0.764657	2.752907	0.884918	2.884736	0.8641
63	0.835127	1.914453	0.920352	1.960206	0.9074
64	0.913254	1.000000	0.958495	1.000000	0.9528
65	1.000000	0.000000	1.000000	0.000000	1.0000

$$h = 1 - \frac{\gamma_X}{3s}$$

$$\mu(h, s) = 1 - \frac{1}{2}h(1 - h)\left[1 - \frac{1}{4}(2 - h)(1 - 3h)s^2\right]s^2$$

$$\sigma(h, s) = hs\sqrt{1 - \frac{1}{2}(1 - h)(1 - 3h)s^2}$$

and Φ is standard cdf, that is,

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2}}{\sqrt{2\pi}}\, dy.$$

The Haldane approximation is chosen because Pentikäinen's results suggest that it may be the most accurate of the well-known approximations to the cdf of compound distributions. If the skewness is zero, then Haldane's approximation is the normal approximation. It therefore seems reasonable to use the simpler normal approximation if the skewness is negligible (very close to zero):

Approximation 2 (Normal): *If X is a random variable with mean μ_X and standard deviation σ_X and the skewness γ_X is very small, then*

$$Pr[X \leq x_0] \approx \Phi\,[\bar{x}_0]. \tag{70}$$

Since the plan is assumed to be large, one may use either the normal approximation or the Haldane Type A approximation. Each approximation requires the evaluation of the first two moments of $L_t^{(act)}$, while the Haldane Type A approximation requires the third moment. Let

$$\sigma_t^2 = Var[L_t^{(act)}]$$

and

$$\gamma_t = \gamma[L_t^{(act)}]$$

be the variance and skewness terms of $L_t^{(act)}$. Using the definition of $L_t^{(act)}$ given in Equation (63), it is not surprising that the moments are complicated expressions:

$$\sigma_t^2 = \sum_{j \in A_t} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} (B_t^j)^2 \left[v^{y-x} \sigma_y^2 + (\ddot{a}_y^{(12)})^2 \left(v^{y-x} - \frac{D_y^{(\tau)}}{D_x^{(\tau)}} \right) \right] \tag{71}$$

$$\gamma_t = \sum_{j \in A_t} \frac{(B_t^j)^3}{\sigma_t^3} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} \left((\ddot{\sigma}_y^{(12)})^3 \ddot{\gamma}_y^{(12)} v^{2(y-x)} + \left(v^{y-x} - \frac{D_y^{(\tau)}}{D_x^{(\tau)}} \right) \right.$$
$$\left. \times \left\{ 3 \left[v^{y-x} \ddot{a}_y^{(12)} (\ddot{\sigma}_y^{(12)})^2 + \left(v^{y-x} - 2\frac{D_y^{(\tau)}}{D_x^{(\tau)}} \right) (\ddot{a}_y^{(12)})^3 \right] \right\} \right). \tag{72}$$

On the other hand, using the definition in Equation (64) yields

$$\sigma_t^2 = \sum_{j \in A_t} \left(\frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j \, \ddot{\sigma}_y^{(12)} \right)^2 \tag{73}$$

$$\gamma_t = \ddot{\gamma}_y^{(12)} \left(\frac{\ddot{\sigma}_y^{(12)}}{\sigma_t} \right)^3 \sum_{j \in A_t} \left(\frac{D_y^{(\tau)}}{D_x^{(\tau)}} B_t^j \right)^3. \tag{74}$$

Notice that the variance terms given in Equations (71) and (73) are different and that the former variance is the larger of the two. This is to be expected because Equation (63) includes more "uncertainty" (due to the extra $I_t^j(x)$ random variable) in $L_t^{(act)}$ than does Equation (64), even though they have the same mean. The γ_t term in Equation (74) is always negative and is expected to be small even for medium-size plans. However, for Equation (72), the size and sign of γ_t will not be easy to predict because they will depend on the plan's demographics. Younger actives will tend to add a positive skewness, while older actives tend to add a negative skewness to γ_t. So, for example, if the plan's active participants are predominantly young employees, then γ_t in Equation (72) will be positive.

Recall the problems associated with using Equation (63):

1. The improper handling of the probability α
2. A larger variance
3. A rapid escalation in costs and liabilities as participants approach retirement
4. An unpredictable and substantial skewness.

In view of these problems, Equation (64) is used to define $L_t^{(act)}$. In particular,

Definition 5:

$$
{}_{\alpha}PVFB_t = \begin{cases} PVFB_t + z_{\alpha}\sigma_t & \text{if } |\gamma_t| \leq 0.01 \\ PVFB_t + \sigma_t\{[\mu(h_t, s_t) + z_{\alpha}\sigma(h_t, s_t)]^{1/h_t} - 1\}/s_t & \\ & \text{if } 0.01 < |\gamma_t| \leq 0.30 \end{cases} \tag{75}
$$

where μ_t, σ_t and γ_t are defined in Equations (61), (73) and (74), respectively.

3.4.2 Group α-Frozen Initial Liability Method

Like the α-frozen initial liability method of Section 3.3, here the α-normal cost is defined recursively, for $t=0, 1, 2, \ldots$, as

$$
{}_{\alpha}NC_t = {}_{\alpha}U_t \sum_{j \in A_t} S_t^j
$$

where ${}_{\alpha}U_t$ is the normal cost percentage

$$
{}_{\alpha}U_t = \frac{{}_{\alpha}PVFB_t - {}_{\alpha}UAL_t - F_t}{PVFS_t}
$$

where ${}_{\alpha}PVFB_t$ is given by Equation (75).

Assuming the gain is always zero, then Equation (30) yields

$$
{}_{\alpha}UAL_{t+1} = ({}_{\alpha}UAL_t + {}_{\alpha}NC_t)(1 + i) - ({}_{\alpha}C_t + {}_{\alpha}I_t^{(c)})
$$

which is similar to Equation (50). Once ${}_{\alpha}UAL_0$ is known, ${}_{\alpha}NC_t$ and ${}_{\alpha}UAL_{t+1}$ can be determined recursively for $t=0, 1, 2, \ldots$. In keeping with the analogy to the individual percentile case, the *α-group aggregate entry age normal method* is used to find the initial accrued and initial unfunded liability. The α-group aggregate entry age normal accrued liability is defined in a manner analogous to Anderson's definition in his equations (2.8.8) and (2.8.9),

$$
{}_{\alpha}AL_0 = {}_{\alpha}PVFB_0 - \frac{{}_{\alpha}PVFBW_0}{PVFSW_0} PVFS_0 \tag{76}
$$

where ${}_{\alpha}PVFB_0$ is calculated using Equation (75) with x replaced by x_0; ${}_{\alpha}PVFBW_0$ also uses Equation (75) but with x replaced by the age at hire w; in $PVFSW_0$ the salaries are discounted to the age w, that is,

$$PVFSW_t = \sum_{j \in A_t} S_t^j \frac{s_w}{s_x} \left(\frac{{}^sN_w^{(\tau)} - {}^sN_y^{(\tau)}}{{}^sD_w^{(\tau)}} \right) \tag{77}$$

while $PVFS_0$ is defined in Equation (46).

Whenever there is a change in either the plan or assumptions, the unfunded liability ${}_\alpha UAL_t$ is adjusted by adding to it the change (increase or decrease) in the α-group aggregate entry age normal accrued liability [Equation (76)] due to the change.

3.4.3 Group α-Attained Age Normal

This method uses the *α-group projected unit credit method* to find ${}_\alpha UAL_0$. This method has not been defined in the traditional literature. However, after a moment's thought, it is clear that the α-group projected unit credit method's accrued liability is given by

$${}_\alpha AL_0 = {}_\alpha PV\, AB_0 \tag{78}$$

where ${}_\alpha PV\, AB_0$ is the present value of the benefits *actually accrued* at time 0 and is calculated by using Equation (75) with $t=0$, $x=x_0$ and B_t^j replaced by $B_0^j(x_0)$, which is defined in Equation (20). Once the initial unfunded liability is known, proceed as in the group α-frozen initial liability method.

Whenever there is a change in either the plan or assumptions, the unfunded liability ${}_\alpha UAL_t$ is adjusted by adding to it the change (increase or decrease) in the α-group aggregate projected unit credit accrued liability [Equation (78)] due to the change.

3.4.4 Group α-Aggregate Method

Here, as in the individual percentile method, the α-accrued liability is defined to be the actual fund balance, that is,

$${}_\alpha AL_t = F_t.$$

Similarly, the α-normal cost is

$${}_\alpha NC_t = {}_\alpha U_t \sum_{j \in A_t} S_t^j$$

where

$$_{\alpha}U_t = \frac{_{\alpha}PVFB_t - F_t}{PVFS_t}$$

and $_{\alpha}PVFB_t$ are defined in Equation (75).

3.4.5 Gains for Group Percentile Spread Gain Methods

For the group α-frozen initial liability, group α-attained age normal and the group α-aggregate cost methods, the gains can be defined in a manner similar to the definition used for individual percentile spread gain methods in Equation (55),

$$_{\alpha}G_t = (_{\alpha}U_t - {_{\alpha}U_{t+1}})\, PVFS_{t+1}$$

where $_{\alpha}U_t$ is the normal cost percentage at time t for the appropriate method.

Now it is much more difficult to determine the individual components of the gain because $_{\alpha}PVFB^j_t$ is not defined for individuals under group percentile cost methods. Since it is not defined, I define the term $_{\alpha}PVFB^j_t$ so that (j)'s proportion of the traditional $PVFB_t$ is the same as his/her proportion of the group $_{\alpha}PVFB_t$, that is,

$$\begin{aligned} _{\alpha}PVFB^j_t &= \frac{PVFB^j_t}{PVFB_t}\, {_{\alpha}PVFB_t} \\ &= \frac{_{\alpha}PVFB_t}{PVFB_t}\, PVFB^j_t \\ &= {_{\alpha}\psi_t}\, PVFB^j_t \end{aligned} \tag{79}$$

where

$$_{\alpha}\psi_t = \frac{_{\alpha}PVFB_t}{PVFB_t}. \tag{80}$$

In a similar manner,

$$_{\alpha}\widetilde{PVFB}^j_{t+1} = {_{\alpha}\psi_t}\, \widetilde{PVFB}^j_{t+1}.$$

Throughout this paper $_{\alpha}\psi_t$ is called the "proportional adjustment factor."

This definition has two very attractive features:

(1) If the skewness is so small that the normal approximation is used, then for α=0.5, the traditional method results; that is,

${}_{\alpha}PVFB_t = PVFB_t$. In such situations the individual $PVFB$s also must match; that is, ${}_{\alpha}PVFB_t^j = PVFB_t^j$ as well.

(2) Anderson's approach [see his equations (2.6.6) to (2.6.10) and (2.9.6) to (2.9.9)] can easily be used to develop expressions for the components of the gain.

In fact, the components of the gain can be written down quite easily if one uses the following notation: let

$${}_{\alpha}\psi_{t+1} = {}_{\alpha}\psi_t + \Delta_{\alpha}\psi_t \qquad \text{and} \qquad B_{t+1}^j = B_t^j + \Delta B_t^j.$$

In addition, let

$${}_{\alpha}\widetilde{AL}_{t+1}^j = {}_{\alpha}\psi_t\, \widetilde{PVFB}_{t+1}^j - {}_{\alpha}U_t\, \widetilde{PVFS}_{t+1}^j \tag{81}$$

$${}_{\alpha}AL_{t+1}^j = {}_{\alpha}\psi_{t+1}\, PVFB_{t+1}^j - {}_{\alpha}U_t\, PVFS_{t+1}^j. \tag{82}$$

These can be substituted into Equations (32) to (37) and the components of the gain determined. An extra term must be added to Equation (31) for the gain due to the change in the proportional adjustment factor ${}_{\alpha}\psi_t$:

$${}_{\alpha}G_t^{(\psi)} = - \sum_{j \in A_{t+1} \cap A_t} (\Delta_{\alpha}\psi_t) PVFB_{t+1}^j. \tag{83}$$

For the group aggregate cost method (Section 3.3.3), the gain due to excess contributions [Equation (60)] must be included.

4. PENSIONERS

4.1 Introduction

So far it has been assumed that, upon retirement, an amount ${}_{\alpha}P_t^j$ is transferred to a separate fund for pensioner j from which his/her lifetime benefits are paid. This need not be the case. Upon retirement, the retiree's benefit may be paid by a third party. For example, for some split funded plans and terminal funded plans, the retirement benefit obligations are transferred to an insurance company via the purchase of a single-premium life annuity with monthly benefits equal to that accrued under the plan, thus absolving the plan of any risks. However, if, as is often the case, the plan pays the benefits directly, then all risks associated with the retirees are borne by the plan. The plan then must determine the ideal amount needed to cover these benefits.

At this point I must reiterate my comments in warning 2 of Section 3.1: the ultimate payment of all benefits is guaranteed unless the plan is terminated.

For the remainder of this section, it is assumed that the plan keeps separate funds for retirees and active lives. Anderson uses a similar approach. This assumption is lifted in Section 5. Following Anderson, let P_t be the set of retirees at time t, D_t be the set of those who die during time $(t, t+1)$, and R_t be the set of lives who retire during $(t, t+1)$. In a symbolic sense,

$$P_{t+1} = P_t + R_t - D_t.$$

All lives are assumed to be mutually independent.

4.2 Percentile Accrued Liabilities

In discussing the impact of retirees on the pension fund, Anderson (chapter 2.10, p. 38) asserts that there can be no question that the desired fund balance (accrued liability) for (j) at time t *must* be $B_t^j \ddot{a}_x^{(12)}$, giving the traditional method of determining the accrued liability for retirees as

$$AL_t = \sum_{j \in P_t} B_t^j \ddot{a}_x^{(12)}. \tag{84}$$

Is Anderson's assertion true? In general, the answer is no! This is especially true for small plans for which there is a significant degree of skewness in their benefit distributions. From Table 1, one sees that, for younger pensioners, $Pr[\ddot{Y}_x^{(12)} \leq \ddot{a}_x^{(12)}] < 50\%$. This means that in the majority of cases the accrued liability, computed by using traditional methods, will not be sufficient to pay the accrued lifetime benefits. In view of this observation, actuaries need to reformulate the notion of a "fully funded plan," especially for small plans with members who are younger than age 80. For large plans, the skewness is usually negligible so the mean and the median are very close, but the mean is still less than the median. It is for these plans that Anderson's assertion will be credible.

The desired fund balance can be computed in other ways. In particular, the desired fund balance can be viewed from two additional perspectives:

(i) Individual Percentile Approach: Let ${}_{\alpha}AL_t^j$ be the α-accrued liability for the retired life aged x at time t, then ${}_{\alpha}AL_t^j$ satisfies

$$Pr[B_t^j \ddot{Y}_x^{(12)} \leq {}_{\alpha}AL_t^j] = \alpha. \tag{85}$$

The aggregate of these accrued liabilities is

$$ {}_{\alpha}AL_t = \sum_{j \in P_t} {}_{\alpha}AL_t^j. \tag{86} $$

(ii) Group Percentile Approach: Here the α-accrued liability for *all* retired lives is ${}_{\alpha}AL_t$ where

$$ Pr\left[\sum_{j \in P_t} B_t^j \ddot{Y}_x^{(12)} \leq {}_{\alpha}AL_t\right] = \alpha. \tag{87} $$

In comparing these approaches, it is clear that the individual approach is the more conservative; that is, it yields a higher accrued liability than the group approach (the results from the example in the Appendix bear this out). This is due to the fact that the individual approach does not, in its definition of accrued liability, take into account the benefits inherent in pooling the longevity risk of each retiree. As a result, it produces larger experience gains than the group approach. However, since the gains are derived for the entire group of retirees, the individual approach reaps the benefits of pooling through sharing these gains among all retirees. On the other hand, the group approach takes into account the benefits of pooling in its definition of accrued liability and as such will lead to smaller experience gains than the individual approach.

4.3 Analysis of Gains

In what follows, the individual and the group approaches are investigated separately. The individual approach yields results that are analogous to those given by the traditional methods. The group approach, though quite intuitive, leads to complex formulas and an ambiguous notion of "gains and losses."

4.3.1 Individual Percentile Approach

From Equations (85) and (86), the α-accrued liability at time t is

$$ {}_{\alpha}AL_t = \sum_{j \in P_t} B_t^j \, {}_{\alpha}\ddot{\xi}_x^{(12)} \tag{88} $$

while at time $t+1$ it is

$$ {}_{\alpha}AL_{t+1} = \sum_{j \in P_{t+1}} B_{t+1}^j \, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)}. $$

Let $B^j_{t+1}=B^j_t+\Delta B^j_t$, and for convenience, let $\tilde{B}^j_{t+1}=B^j_{t+1}=B^j_t$ for deaths. From Equation (16)

$$
\begin{aligned}
{}_\alpha AL_{t+1} &= \sum_{j\in P_t} B^j_t \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} + \sum_{j\in P_t} \Delta B^j_t \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} \\
&\quad - \sum_{j\in D_t} B^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} + \sum_{j\in R_t} B^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} \\
&= \sum_{j\in P_t} B^j_{t+1} \left[(1+i)_\alpha\ddot{\xi}^{(12)} - \ddot{s}^{(12)}_{\overline{1|}} + q_x \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} - {}_\alpha\theta^{(12)}_x\right] \\
&\quad + \sum_{j\in P_t} \Delta B^j_t \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} - \sum_{j\in D_t} B^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} + \sum_{j\in R_t} B^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} \\
&= {}_\alpha AL_t(1+i) + \sum_{j\in P_t} \Delta B^j_t \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} \\
&\quad - \left[\sum_{j\in D_t} \tilde{B}^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} - \sum_{j\in P_t} q_x \tilde{B}^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1}\right] \\
&\quad + \sum_{j\in R_t} B^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} - \sum_{j\in P_t} B^j_t \left[\ddot{s}^{(12)}_{\overline{1|}} + (1+i)_\alpha\theta^{(12)}_x\right]. \qquad (89)
\end{aligned}
$$

If F_t is the fund balance at time t for *retirees*, the α-unfunded accrued liability satisfies

$$
\begin{aligned}
{}_\alpha UAL_{t+1} &= {}_\alpha AL_{t+1} - F_{t+1} \\
&= {}_\alpha UAL_t(1+i) - [I_t - i\,F_t - {}_\alpha I^{(p)}_t + I^{(b)}_t] \\
&\quad + \left[\sum_{j\in P_t\cap P_{t+1}} \Delta B^j_t \, {}_\alpha\ddot{\xi}^{(12)}_{x+1}\right] \\
&\quad - \left[\sum_{j\in D_t} \tilde{B}^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} - \sum_{j\in P_t} q_x \, \tilde{B}^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1}\right] \\
&\quad - \left[{}_\alpha PP_t + {}_\alpha I^{(p)}_t - \left(\sum_{j\in R_t} B^j_{t+1} \, {}_\alpha\ddot{\xi}^{(12)}_{x+1} + B_{New} + I_{New}\right)\right] \\
&\quad - \left[\sum_{j\in P_t} B^j_{t+1} \left(\ddot{s}^{(12)}_{\overline{1|}} + (1+i)_\alpha\theta^{(12)}_x\right) - (B_{Old} + I_{Old})\right] \qquad (90)
\end{aligned}
$$

where $B_{New}+I_{New}$ is the amount of benefits accumulated with interest at the assumed rate i, paid to newly retired lives during year $(t, t+1)$. Similarly $B_{Old}+I_{Old}$ is the amount of benefits plus interest paid to those already retired in P_t. Note $B_{Old}+B_{New}=B_t$, which is the actual amount of pension payments made from the fund during year t, and $I_{Old}+I_{New}=I_t^{(b)}$.

If assumptions work out exactly, then $\Delta B_t^j=0$ and all terms in square brackets in Equation (90) also are zero. Thus it is appropriate to define the gain as

$$_{\alpha}G_t = {}_{\alpha}UAL_t(1+i) - {}_{\alpha}UAL_{t+1}. \tag{91}$$

Interestingly, the terms within the brackets on the right-hand side of Equation (90) can be viewed as the interest gain, the loss due to benefit changes, the mortality gain, the new retirement gain, and the gain from pension payments, respectively. Specifically, for the individual percentile method:

$$_{\alpha}G_t^{(d)} = \sum_{j\in D_t} \tilde{B}_{t+1}^j \, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} - \sum_{j\in P_t} q_x \tilde{B}_{t+1}^j \, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} \tag{92}$$

$$_{\alpha}G_t^{(r)} = {}_{\alpha}PP_t + {}_{\alpha}I_t^{(p)} - \left(\sum_{j\in R_t} B_{t+1}^j \, {}_{\alpha}\ddot{\xi}_{x+1}^{(12)} + B_{New} + I_{New} \right) \tag{93}$$

$$_{\alpha}G_t^{(p)} = \sum_{j\in P_t} B_{t+1}^j \, [\ddot{s}_{\overline{1}|}^{(12)} + (1+i)_{\alpha}\theta_x^{(12)}] - (B_{Old} + I_{Old}) \tag{94}$$

are the mortality, retirement and pension payment gains, respectively. From Anderson's equation (2.10.7), the traditional method yields:

$$G_t^{(d)} = \sum_{j\in D_t} \tilde{B}_{t+1}^j \, \ddot{a}_{x+1}^{(12)} - \sum_{j\in P_t} q_x \tilde{B}_{t+1}^j \ddot{a}_{x+1}^{(12)} \tag{95}$$

$$G_t^{(r)} = PP_t + I_t^{(p)} - \left(\sum_{j\in R_t} B_{t+1}^j \ddot{a}_{x+1}^{(12)} + B_{New} + I_{New} \right) \tag{96}$$

$$G_t^{(p)} = \sum_{j\in P_t} B_t^j \left(1 + \frac{13}{24} i - \frac{11}{24} q_x \right) - (B_{Old} + I_{Old}). \tag{97}$$

Notice that Anderson, unlike many authors, does not include the $11q_x/24$ term in the mortality gain [Equation (95)]; it appears instead in the expression for the pension payment gain [Equation (97)].

Note that Equation (97) is based on the approximation

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}.$$

However, the tables in the Appendix are all based on the UDD approximation of Equation (4). It can be proved easily that UDD approximation yields the following expression for the traditional pension payment gain:

$$G_t^{(p)} = \sum_{j \in P_t} B_t^j \left[\frac{i}{d^{(12)}} - \frac{(i - i^{(12)})}{i^{(12)} d^{(12)}} q_x \right] - (B_{Old} + I_{Old}). \tag{98}$$

Equation (98) is used in the example in the Appendix.

Finally, from Equation (90), there is a term for the gain due to benefit changes, $G_t^{(b)}$, which is given by

$$G_t^{(b)} = - \sum_{j \in P_t \cap P_{t+1}} \Delta B_t^j \,{}_{\alpha}\ddot{\xi}_{x+1}^{(12)}. \tag{99}$$

Since retiree benefits are not changed very often, this gain is usually zero.

4.3.2 Group Percentile Approach

Let the random variable $L_t^{(ret)}$ be the sum of the present values of future annuity benefits for all retirees (which is a plan liability), that is,

$$L_t^{(ret)} = \sum_{j \in P_t} B_t^j \ddot{Y}_x^{(12)}. \tag{100}$$

Obviously $\ddot{Y}_x^{(12)}$ is implicitly a function of j. In a sense, $L_t^{(ret)}$ is the random variable that measures the plan's current aggregate retirement liabilities for the entire group of retirees. Unlike the individual percentile approach, which has as its objective the security of each retirees' benefits, the group approach focuses on the security of the retirees as a whole. It directly exploits the benefits of spreading the risks among the retirees. This results in a lower total accrued liability and higher gains than the individual percentile approach. The example in the Appendix shows this to be the case.

The traditional measure of the accrued liability for retirees [Equation (84)] is actually the mean of $L_t^{(ret)}$. Like $L_t^{(act)}$, the distribution of $L_t^{(ret)}$ is very complicated, so its cdf must be approximated. For simplicity, it is assumed that the number of retirees is large enough so that the skewness of $L_t^{(ret)}$ is small, that is, less than 0.30 in *absolute* value, and that the cdf of $L_t^{(ret)}$ can be approximated using the Haldane Type A approximation or normal approximation.

Let μ_t, σ_t^2 and γ_t be the mean, variance and skewness, respectively, of $L_t^{(ret)}$, in particular

$$\mu_t = AL_t = \sum_{j \in P_t} B_t^j \ddot{a}_x^{(12)} \tag{101}$$

$$\sigma_t^2 = \sum_{j \in P_t} (B_t^j \ddot{\sigma}_x^{(12)})^2 \tag{102}$$

and

$$\gamma_t = \sum_{j \in P_t} \left(\frac{\sigma_{tj}}{\sigma_t}\right)^3 \ddot{\gamma}_x^{(12)}. \tag{103}$$

This leads to an equation similar to Equation (75)

$${}_\alpha AL_t = \begin{cases} AL_t + z_\alpha \sigma_t & \text{if } |\gamma_t| \leq 0.01 \\ AL_t + \sigma_t\{[\mu(h_t, s_t) + z_\alpha \sigma(h_t, s_t)]^{1/h_t} - 1\}/s_t & \\ & \text{if } 0.01 < |\gamma_t| \leq 0.30. \end{cases} \tag{104}$$

For the group approach, analysis of the α-gain by source is a much more difficult task than it was for the individual case. This is because the relationship among the various sources of gains, that is, interest, death, retirement, and so on, is made much more complicated by the presence of the variance and skewness terms in the α-accrued liability of Equation (104). To make matters worse, it is not clear just how the various components of the gain, such as mortality and retirement, should be defined. This is in fact a serious problem because, for the group percentile approach, individual α-accrued liabilities are not even defined!

One way out of this dilemma is to *define* the "individual" accrued liability by using the proportional adjustment factor as in Section 3.4.5.

This means that retiree (j)'s α-accrued liability is assumed to be in the same proportion as his/her traditional accrued liability, that is,

$$
\begin{aligned}
{}_{\alpha}AL_t^j &= \frac{AL_t^j}{AL_t}\,{}_{\alpha}AL_t \\
&= {}_{\alpha}\psi_t\,AL_t^j \qquad (105)
\end{aligned}
$$

where

$$
{}_{\alpha}\psi_t = \frac{{}_{\alpha}AL_t}{AL_t} \qquad (106)
$$

and ${}_{\alpha}AL_t$ is calculated using Equation (104). Similarly,

$$
{}_{\alpha}\widetilde{AL}_{t+1}^j = {}_{\alpha}\psi_t\,\widetilde{AL}_{t+1}. \qquad (107)
$$

As will be seen in Section 5 below, the ${}_{\alpha}\psi_t$'s defined in Equations (80) and (106) are the same.

Now the gain terms can be defined. However, two extra terms are needed, ${}_{\alpha}G_t^{(b)}$ and ${}_{\alpha}G_t^{(\psi)}$, which are the gains due to retiree benefit changes and gains due to changes in the proportional adjustment factor ${}_{\alpha}\psi_t$, respectively:

$$
{}_{\alpha}G_t^{(d)} = \sum_{j\in D_t} {}_{\alpha}\widetilde{AL}_{t+1}^j - \sum_{j\in P_t} q_x\,{}_{\alpha}\widetilde{AL}_{t+1}^j \qquad (108)
$$

$$
{}_{\alpha}G_t^{(r)} = {}_{\alpha}PP_t + {}_{\alpha}I_t^{(p)} - \left(\sum_{j\in R_t} {}_{\alpha}AL_{t+1}^j + B_{New} + I_{New}\right) \qquad (109)
$$

$$
{}_{\alpha}G_t^{(p)} = \sum_{j\in P_t} {}_{\alpha}\psi_t\,B_t^j\left(1 + \frac{13}{24}i - \frac{11}{24}q_x\right) - (B_{Old} + I_{Old}) \qquad (110)
$$

$$
{}_{\alpha}G_t^{(b)} = -\sum_{j\in P_t\cap P_{t+1}} {}_{\alpha}\psi_t\,\Delta B_t^j\,\ddot{a}_{x+1}^{(12)} \qquad (111)
$$

$$
{}_{\alpha}G_t^{(\psi)} = -\sum_{j\in P_t\cap P_{t+1}} (\Delta_{\alpha}\psi_t)\,B_t^j\,\ddot{a}_{x+1}^{(12)}. \qquad (112)
$$

These components can be derived by using the method of Anderson's equation (2.10.5).

5. THE ENTIRE PLAN

So far Anderson's approach has been followed; that is, it was assumed that the plan consists of two completely autonomous groups: active lives and retired lives (pensioners), each with its own separate fund. This splitting of the plan's funds was done for mathematical convenience: it facilitates the separate analysis of active lives and retired lives.

In practice, actives' and retirees' funds are not separated. To perform a valuation of the entire plan, these groups must be treated as one. This can be accomplished very easily by following Berin's approach (see his chapters 3 to 6); no new theory is needed. In addition, the Appendix shows how this can be done by way of a detailed example.

When the entire plan's gains are calculated, items such as the accrued liability, the fund and the unfunded liability must be found for the entire plan. This is not a problem for individual percentile valuation methods. However, for spread gain methods certain items must be defined separately. Let

$$ {}_{\alpha}PVFB_t = {}_{\alpha}PVFB_t^{(ACT)} + {}_{\alpha}AL_t^{(RET)} \tag{113} $$

$$ {}_{\alpha}AL_t = {}_{\alpha}AL_t^{(ACT)} + {}_{\alpha}AL_t^{(RET)} \tag{114} $$

$$ F_t = F_t^{(ACT)} + F_t^{(RET)} \tag{115} $$

$$ {}_{\alpha}UAL_t = {}_{\alpha}AL_t - F_t, \tag{116} $$

where the superscripts (ACT) and (RET) refer to the quantities calculated for active and retired lives, respectively. Expressions for these quantities are given in Sections 3 and 4, respectively.

When the plan as a whole is being considered, the retirement gains for active lives and for retired lives offset each other; that is, they sum to zero. Also, the sum of the actual individual components of the gain is called the "explained gain." The difference between the explained gain and the total gain is called the "unexplained gain." From a theoretical perspective, the unexplained gain should be zero. However, due to rounding errors this term is usually non-zero. In addition, for the group percentile spread gain methods, the unexplained gain is non-zero because of the proportional assumption used to generate the "individual" expected liabilities. Fortunately, from the results of the example in the Appendix, the unexplained gain is expected to be a relatively small percentage (<0.5%) of the total gain.

Finally, when the spread gain methods of Sections 3.3 and 3.4 are used, the expression ${}_{\alpha}PVFB_t$ must now include the α-accrued liability for pensioners. This must be done differently for the individual percentile and the group percentile methods. In particular, the individual percentile spread gain approach yields:

$$ {}_{\alpha}PVFB_t = \sum_{j\in A_t} B_t^j \, {}_{\alpha}\ddot{\xi}_y^{(12)} \frac{D_y^{(\tau)}}{D_x^{(\tau)}} + {}_{\alpha}AL_t^{(RET)}. \tag{117} $$

where ${}_{\alpha}AL_t^{(RET)}$ is calculated according to Equations (104).

On the other hand, for the group percentile spread gain approach, ${}_{\alpha}PVFB_t$ must now be defined using the equation

$$ Pr[L_t^{(act)} + L_t^{(ret)} \leq {}_{\alpha}PVFB_t] = \alpha. \tag{118} $$

From Equations (75) and (104),

$$ {}_{\alpha}PVFB_t = \begin{cases} \mu_t + z_\alpha\sigma_t & \text{if } |\gamma_t| \leq 0.01 \\ \mu_t + \sigma_t\{[\mu(h_t, s_t) + z_\alpha\sigma(h_t, s_t)]^{1/h_t} - 1\}/s_t & \\ & \text{if } 0.01 < |\gamma_t| \leq 0.30 \end{cases} \tag{119} $$

where μ_t, σ_t and γ_t are the mean, standard deviation and skewness, respectively, of $L_t^{(act)}+L_t^{(ret)}$.

Once ${}_{\alpha}PVFB_t$ has been determined from Equation (119), the analysis of gains requires that ${}_{\alpha}PVFB_t^{(ACT)}$ and ${}_{\alpha}AL_t^{(RET)}$ be determined. This can be done by once again using the proportional split, that is,

$$ \begin{aligned} {}_{\alpha}PVFB_t^{(ACT)} &= \frac{PVFB_t^{(ACT)}}{PVFB_t} \, {}_{\alpha}PVFB_t \\ &= \frac{{}_{\alpha}PVFB_t}{PVFB_t} PVFB_t^{(ACT)} \end{aligned} \tag{120} $$

and

$$ \begin{aligned} {}_{\alpha}AL_t^{(RET)} &= \frac{AL_t^{(RET)}}{PVFB_t} \, {}_{\alpha}PVFB_t \\ &= \frac{{}_{\alpha}PVFB_t}{PVFB_t} AL_t^{(RET)}. \end{aligned} \tag{121} $$

Given the above definitions, the proportional adjustment term for active lives [Equation (79)] and for retired lives [Equation (106)] can be shown to be equal as follows: let ${}_{\alpha}PVFB_t^j$ be the $PVFB$ assigned to an active life (j) in a group percentile valuation method. It follows that

$$\begin{aligned}
{}_{\alpha}PVFB_t^j &= \frac{{}_{\alpha}PVFB_t^{(ACT)}}{PVFB_t^{(ACT)}} PVFB_t^j \\
&= \frac{{}_{\alpha}PVFB_t}{PVFB_t} PVFB_t^{(ACT)} \times \frac{PVFB_t^j}{PVFB_t^{(ACT)}} \\
&= \frac{{}_{\alpha}PVFB_t}{PVFB_t} PVFB_t^j \\
&= {}_{\alpha}\psi_t \, PVFB_t^j. \qquad (122)
\end{aligned}$$

Similarly, let ${}_{\alpha}AL_t^j$ be the accrued liability assigned to an active life (j) in a group percentile valuation method. It follows that

$$\begin{aligned}
{}_{\alpha}AL_t^j &= \frac{{}_{\alpha}AL_t^{(RET)}}{AL_t^{(RET)}} AL_t^j \\
&= \frac{{}_{\alpha}PVFB_t}{PVFB_t} AL_t^{(RET)} \times \frac{AL_t^j}{AL_t^{(RET)}} \\
&= \frac{{}_{\alpha}PVFB_t}{PVFB_t} AL_t^j \\
&= {}_{\alpha}\psi_t \, AL_t^j. \qquad (123)
\end{aligned}$$

Equations (122) and (123) give the expression

$$ {}_{\alpha}\psi_t = \frac{{}_{\alpha}PVFB_t}{PVFB_t}, \qquad (124)$$

which proves the assertion.

Finally, the gain for the entire plan is the sum of the gains for the active lives and the retired lives. Note that the sum of the gains due to retirement for active lives [Equation (35)] and pensioners [Equation (109)] must be zero. See the Appendix for a detailed example.

6. COMMENTS

As mentioned in Section 3.4.5, the group percentile approach does not lend itself to a straightforward analysis of gains. However, the definition of the components of the gain using the proportional method to calculate for the "individual" accrued liabilities will explain almost all the gain for group percentile spread gain methods. The unexplained gain is expected to be a small part of the total gain. The example in the Appendix confirms this; the unexplained gains are less than 0.5 percent of the total gain.

I hope that the ideas of this paper will stimulate discussion on the continued reliance on expected values in pension calculations and encourage a shift to a more probabilistic approach. The theory presented here is intended to serve as an alternative to the expected value approach. From the detailed example in the Appendix, this theory will be easy to implement because its formulas are very similar to the traditional formulas. We hope that pension actuaries and important agencies, such as the IRS and the PBGC, will find merit in this family of cost methods.

Concerning the practical implementation of the percentile cost methods, there are likely to be a few problems. Shapiro [10] points out several potential problem areas inherent in using a risk-based approach to pension valuations. Among them are:

(i) IRS stipulation that expected values must be used in pension valuations even though the underlying process is stochastic

(ii) Employers' reluctance to sponsor a plan unless they can deduct the cost or receive a credit for doing so

(iii) IRS refusal to allow for a contingency reserve (as in life insurance)

(iv) The arbitrary funding limit on the deducibility of contributions.

However, given the recent S&L disaster and its impact on the federal deposit insurance system, the time may be ripe for a more risk-sensitive approach to pension funding and plan termination insurance.

Finally, I must express my appreciation for the work done by the early researchers in pension mathematics. I am particularly indebted to Anderson and Berin for their texts on this subject. It should be clear from my exposition that I have borrowed very heavily from them.

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REFERENCES

1. AMOROSO, V. "Termination Insurance for Single-Employer Pension Plans: Cost and Benefits," *TSA XXXV* (1983): 93–105.
2. ANDERSON, A.W. *Pension Mathematics for Actuaries*. Windsor Press Inc., 1985; now published by Actex Publications, Winsted, Conn.
3. BEARD, R.E., PENTIKÄINEN, T., AND PESONEN, E. *Risk Theory*. London: Chapman and Hall, 1984.
4. BERIN, B.N. *The Fundamentals of Pension Mathematics*. Schaumburg, Ill.: Society of Actuaries, 1989.
5. BOWERS, N.L., GERBER, H.U., HICKMAN, J.C., JONES, D.A., AND NESBITT, C.J. *Actuarial Mathematics*. Itasca, Ill.: Society of Actuaries, 1986.
6. GERBER, H.U. "An Introduction to Mathematical Risk Theory." *Huebner Foundation Monograph 8* (1979). Distributed by Irwin, Homewood, Ill.
7. MCGILL, D.M. *Fundamentals of Private Pensions*. Homewood, Ill.: Irwin, 1984.
8. PENTIKÄINEN, T. "Approximate Evaluation of the Distribution of Aggregate Claims," *ASTIN Bulletin* (1987): 15–39.
9. SHAPIRO, A.F. "Modified Cost Methods for Small Pension Plans," *TSA XXXV* (1983): 11–29.
10. SHAPIRO, A.F. "Pension Risk," *ARCH* 1986.1: 25–43.
11. SMALL, H.J. "A Unified Approach to Pension Plan Gain and Loss Analysis," *TSA XXXIV* (1982): 511–36.
12. VAN DERHEI, J.L. "An Empirical Analysis of Risk-Related Insurance Premiums for the PBGC," *Journal of Risk and Insurance* 57, no. 2 (1990): 240–59.

APPENDIX A

AN EXAMPLE

This example is based on the data provided by Berin in his Appendix 1. Some aspects of his data have been modified for this example. Modifications are pointed out as they occur. For convenience, the plan is assumed to have begun on January 1, 1991.

1. Plan Information

Throughout this example, the following are used:

- **Plan effective date:** January 1, 1991 ($t=0$).
- **Normal retirement benefit:** 1.5 percent of salary at plan inception times past service plus 1.5 percent of total future salary, that is,

$$B_t^j = 0.015(x_0 - w)S_0^j + 0.015\left[\sum_{k=0}^{t-1} S_k^j + \sum_{z=x}^{y-1} S_t^j \frac{S_z}{S_x}\right]. \qquad (125)$$

- **Preretirement death or termination benefit:** None.
- **Actuarial assumptions:**
  - — Interest rate: i=8 percent.
  - — Confidence level: α=50 percent, the median.
  - — Salary scale: See s_x in Table 8.
  - — Preretirement deaths or terminations: See Tables 8 and A-1.

TABLE A-1

SERVICE TABLE RATES

x	q_x	w_x	x	q_x	w_x
25	0.000464	0.0835	45	0.002183	0.0285
26	0.000488	0.0800	46	0.002471	0.0270
27	0.000513	0.0770	47	0.002790	0.0255
28	0.000542	0.0740	48	0.003138	0.0240
29	0.000572	0.0710	49	0.003513	0.0230
30	0.000607	0.0680	50	0.003909	0.0215
31	0.000645	0.0650	51	0.004324	0.0210
32	0.000687	0.0615	52	0.004755	0.0200
33	0.000734	0.0590	53	0.005200	0.0195
34	0.000785	0.0565	54	0.005660	0.0185
35	0.000860	0.0530	55	0.006131	0.0175
36	0.000907	0.0500	56	0.006618	0.0000
37	0.000966	0.0470	57	0.007139	0.0000
38	0.001039	0.0445	58	0.007719	0.0000
39	0.001128	0.0420	59	0.008384	0.0000
40	0.001238	0.0395	60	0.009158	0.0000
41	0.001370	0.0370	61	0.010064	0.0000
42	0.001527	0.0345	62	0.011133	0.0000
43	0.001715	0.0325	63	0.012391	0.0000
44	0.001932	0.0300	64	0.013686	0.0000
			65		

  - — Retirement age: y=65.
  - — Postretirement mortality: GAM 1983 males, see Table A-2, with deaths uniformly distributed across each age.

    F_0= \$2,950,000 and F_1=\$3,350,000.

    Contributions are made on July 1 each year.

    Interest is exact, that is, use $(1+i)^t$ for $0<t<1$.

    \$134,000 in total pension payments in 1991 (Berin used \$140,000).

$\ddot{a}_{65}^{(12)} = 8.63829$ and ${}_{0.5}\ddot{\xi}_{65}^{(12)} = 9.431436$ (Berin used $\ddot{a}_{65}^{(12)} = 8.64681$).

TABLE A-2

VALUATION FACTORS FOR RETIRED LIVES, WITH $i = 8\%$

x	q_x	$\ddot{a}_x^{(12)}$	$\ddot{\sigma}_x$	$\ddot{\gamma}_x^{(12)}$	${}_\alpha\ddot{\xi}_x^{(12)}$
60	0.009158	9.619892	2.562681	−1.486953	10.451231
61	0.010064	9.437403	2.613610	−1.394974	10.268488
62	0.011133	9.247381	2.664046	−1.306177	10.075092
63	0.012391	9.050352	2.713108	−1.220563	9.871007
64	0.013868	8.847020	2.759799	−1.138052	9.656351
65	0.015592	8.638290	2.802983	−1.058443	9.431436
66	0.017579	8.425244	2.841429	−0.981389	9.196778
67	0.019804	8.209032	2.873942	−0.906386	8.952887
68	0.022229	7.990508	2.899784	−0.832909	8.700327
69	0.024817	7.770124	2.918747	−0.760443	8.439948
70	0.027530	7.547924	2.931096	−0.688523	8.172237
71	0.030354	7.323526	2.937577	−0.616852	7.897702
72	0.033370	7.096291	2.939258	−0.545425	7.616784
73	0.036680	6.866018	2.936846	−0.474411	7.329446
74	0.040388	6.633064	2.930633	−0.404097	7.037099
75	0.044597	6.398337	2.920551	−0.334841	6.742688
76	0.049388	6.163263	2.906221	−0.266985	6.445248
77	0.054758	5.929630	2.887075	−0.200774	6.151069
78	0.060678	5.699054	2.862758	−0.136392	5.859302
79	0.067125	5.472834	2.833175	−0.073992	5.573882
80	0.074070	5.252015	2.798395	−0.013677	5.295389
81	0.081484	5.037382	2.758607	0.044503	5.027387
82	0.089320	4.829479	2.714061	0.100587	4.765407
83	0.097525	4.628533	2.665088	0.154714	4.521276
84	0.106047	4.434424	2.612110	0.207117	4.281169
85	0.114836	4.246685	2.555662	0.258053	4.053524
86	0.124170	4.064467	2.496481	0.307596	3.839827
87	0.133870	3.888111	2.434674	0.355843	3.629321
88	0.144073	3.717031	2.370744	0.402863	3.423725
89	0.154859	3.551090	2.304999	0.448584	3.242651
90	0.166307	3.390371	2.237609	0.492906	3.066732

- **Valuation objectives:** The following tasks will be accomplished:
  - Compute, as of January 1, 1991, the normal cost and the accrued liability for active lives using each of the following cost methods: projected unit credit method, entry age normal, frozen initial liability, attained age normal, and aggregate methods. Each method is to be used in its α-percentile form and its traditional form. The accrued liability for retired lives is calculated by using the traditional, individual percentile, and group percentile methods of Section 4.
  - Compute, as of January 1, 1992, the items above by using the actual salaries as of January 2, 1992.
  - Compute the gains for the entire plan in 1991, and perform an analysis of gains by source.

- **Employee data:** Here # ee's denotes the number of employees with the same age at hire, current age and annual salary S_t^j. All ages are exact on the valuation dates. All deaths, terminations and retirements actually occurred on December 31, 1991.

EMPLOYEE DATA

Active Lives at Time $t = 0$				Active Lives at Time $t = 1$			
w	x	# ee's	S_0^j	w	x	# ee's	S_1^j
25	27	90	$20,000.00	25	25	20	$20,000.00
25	39	40	30,000.00	25	28	89	24,000.00
25	51	50	35,000.00	25	40	40	33,000.00
25	64	10	40,000.00	25	52	49	36,000.00
35	39	60	25,000.00	35	40	59	30,000.00
35	51	80	30,000.00	35	52	80	34,000.00
45	51	30	25,000.00	45	52	30	28,000.00

EMPLOYEE DATA
SUMMARY OF FIRST-YEAR ACTIVITY

w	x	Event
25	25	20 new hires (N_0)
25	28	1 termination (T_0)
25	52	1 death (D_0)
25	65	1 death (D_0)
25	65	9 retirements (R_0)
35	40	1 termination (T_0)

- **Retiree data:** An analogous definition of # ee's applies to retired lives with annual pension benefit B_t^j.
During 1991, one retiree from the age 67 group died on December 31, 1991.

Retired lives at time $t=0$			Retired lives at time $t=1$		
x	# ee's	B_0^j	x	# ee's	B_1^j
			65	9	$24,000.00
67	7	$12,000.00	68	6	12,000.00
70	5	10,000.00	71	5	10,000.00

2. *Valuation Results*

The following notation is used:

PUC = Projected unit credit cost method
EAN = Entry age normal cost method
FIL = Frozen initial liability cost method
AAN = Attained age normal cost method
AGG = Aggregate cost method
INDV = Individual percentile cost method
GROUP = Group percentile cost method

My calculations using the traditional methods do not agree with Berin's; I have not been able to duplicate Berin's results.

2.1 Valuation on January 1, 1991 (t=0)

TABLE A-3

TRADITIONAL VALUATION RESULTS, t=0

				PUC		EAN	
w	x	# ee's	$B_0^j(65)$	NC_0^j	AL_0^j	NC_0^j	AL_0^j
25	27	90	56,872.64	167.73	335.45	347.44	745.64
25	39	40	32,485.92	515.64	7,218.98	567.53	12,076.08
25	51	50	24,026.96	1,430.78	37,200.15	866.33	48,059.36
25	64	10	24,000.00	4,733.37	184,601.43	1,647.37	187,687.43
35	39	60	23,321.60	493.57	1,974.28	754.02	3,448.07
35	51	80	16,094.54	1,277.88	20,446.07	1,073.98	26,966.33
45	51	30	9,662.11	1,150.73	6,904.40	1,297.30	9,280.40

TABLE A-4

PERCENTILE VALUATION RESULTS, t=0

				PUC		EAN	
w	x	# ee's	$B_0^j(65)$	${}_{\alpha}NC_0^j$	${}_{\alpha}AL_0^j$	${}_{\alpha}NC_0^j$	${}_{\alpha}AL_0^j$
25	27	90	56,872.64	183.13	366.25	379.34	814.10
25	39	40	32,485.92	562.99	7,881.81	619.63	13,184.88
25	51	50	24,026.96	1,562.15	40,615.78	945.87	52,472.05
25	64	10	24,000.00	5,167.98	201,551.08	1,798.63	204,920.43
35	39	60	23,321.60	538.89	2,155.56	823.25	3,764.66
35	51	80	16,094.54	1,395.21	22,323.37	1,172.59	29,442.32
45	51	30	9,662.11	1,256.39	7,538.35	1,426.41	10,132.50

TABLE A-5

TOTALS, $t=0$

	PUC	EAN
NC_0	320,960.50	283,839.60
AL_0	5,986,245.90	7,472,595.70
$_\alpha NC_0$	350,430.50	309,900.10
$_\alpha AL_0$	6,535,888.40	8,158,711.20

Tables 2 and 3 give the mean, standard deviation and skewness needed to compute $_\alpha PVFB_t$ [Equation (75)], $_\alpha PVFBW_0$ [Equation (76)] and $_\alpha PVAB$ [Equation (78)].

TABLE A-6

GROUP PERCENTILE INFORMATION FOR PFVB

	$t=0$		$t=1$	
	$L_0^{(act)}$	$L_0^{(ret)}$	$L_1^{(act)}$	$L_1^{(ret)}$
μ_t	10,829,519.97	1,066,954.85	10,326,868.78	2,807,363.44
σ_t	269,523.31	112,344.46	210,390.84	228,711.73
γ_t	−0.152726	−0.244681	−0.079806	−0.266540

This yields the following:

$$_\alpha PVFB_0 = 11,902,988.78$$
$$PVFB_0 = 11,896,469.66$$
$$_\alpha\psi_0 = 1.000547988$$
$$_\alpha PVFB_1 = 13,141,008.87$$
$$PVFB_1 = 13,134,232.22$$
$$_\alpha\psi_1 = 1.000515953$$

In addition, the following moments are needed to evaluate $_\alpha PVFBW_0$ and $_\alpha PVAB_0$ in Equations (76) and (78). Note the latter is found for the entire plan.

TABLE A-7

MOMENTS AT $t=0$

	PVFBW	PVAB
μ_t	2,059,782.76	7,053,301.27
σ_t	39,607.70	244,343.58
γ_t	−0.073918	−0.187857

which gives

$$_{\alpha}PVFBW_0 = 2{,}060{,}270.57$$

$$_{\alpha}PVAB_0 = 7{,}060{,}830.36$$

The normal cost under the FIL, AAN, and AGG methods is now computed for each of the three forms of retiree accrued liabilities. Berin's tabular arrangement is followed. Here the prefix α is dropped. It should be understood that the percentile approaches must use the corresponding α quantities. $PVFNC_t$ denotes the present value of future normal costs; it is the numerator in the expressions for $_{\alpha}U_t$ in Section 3.3.

TABLE A-8

FROZEN INITIAL LIABILITY, $t=0$

		Percentile	
	Traditional	Individual	Group
$AL_0^{(ACT)}$	7,472,595.70	8,158,711.20	7,388,350.98
$AL_0^{(RET)}$	1,066,954.85	1,160,654.36	1,067,539.53
Total AL_0	8,539,550.55	9,319,365.56	8,455,890.51
F_0	−2,950,000.00	−2,950,000.00	−2,950,000.00
$U\ AL_0$	5,589,550.55	6,369,365.56	5,505,890.51
$PVFB_0^{(ACT)}$	10,829,514.81	11,823,859.50	10,835,449.25
$AL_0^{(RET)}$	1,066,954.85	1,160,654.36	1,067,539.53
Total $PVFB_0$	11,896,469.66	12,984,513.86	11,902,988.78
$U\ AL_0$	−5,589,550.55	−6,369,365.56	−5,505,890.51
F_0	−2,950,000.00	−2,950,000.00	−2,950,000.00
$PVFNC_0$	3,356,919.11	3,665,148.30	3,447,098.27
$PVFS_0$	123,845,275.00	123,845,275.00	123,845,275.00
U_0	2.710575%	2.959458%	2.783391%
ΣS_0^j	9,800,000.00	9,800,000.00	9,800,000.00
NC_0	265,636.35	290,026.84	272,772.30

TABLE A-9

ATTAINED AGE NORMAL, $t=0$

	Traditional	Percentile: Individual	Percentile: Group
$AL_0^{(ACT)}$	5,986,245.90	6,535,888.40	5,993,290.83
$AL_0^{(RET)}$	1,066,954.85	1,160,654.36	1,067,539.53
Total AL_0	7,053,200.75	7,696,542.76	7,060,830.36
F_0	−2,950,000.00	−2,950,000.00	−2,950,000.00
$U\ AL_0$	4,103,200.75	4,746,542.76	4,110,830.36
$PVFB_0^{(ACT)}$	10,829,514.81	11,823,859.50	10,835,449.25
$AL_0^{(RET)}$	1,066,954.85	1,160,654.36	1,067,539.53
Total $PVFB_0$	11,896,469.66	12,984,513.86	11,902,988.78
$U\ AL_0$	−4,103,200.75	−4,756,542.76	−4,110,830.36
F_0	−2,950,000.00	−2,950,000.00	−2,950,000.00
$PVFNC_0$	4,843,268.91	5,287,971.10	4,842,158.42
$PVFS_0$	123,845,275.00	123,845,275.00	123,845,275.00
U_0	3.910742%	4.269821%	3.909845%
ΣS_0^j	9,800,000.00	9,800,000.00	9,800,000.00
NC_0	383,252.72	418,442.46	383,164.82

TABLE A-10

AGGREGATE, $t=0$

	Traditional	Percentile: Individual	Percentile: Group
$PVFB_0^{(ACT)}$	10,829,514.81	11,823,859.50	10,835,449.25
$AL_0^{(RET)}$	1,066,954.85	1,160,654.36	1,067,539.53
Total $PVFB_0$	11,896,469.66	12,984,513.86	11,902,988.78
F_0	−2,950,000.00	−2,950,000.00	−2,950,000.00
$PVFNC_0$	8,946,469.66	10,034,513.86	8,952,988.78
$PVFS_0$	123,845,275.00	123,845,275.00	123,845,275.00
U_0	7.223909%	8.102460%	7.229713%
ΣS_0^j	9,800,000.00	9,800,000.00	9,800,000.00
NC_0	707,943.08	794,041.08	708,458.92

2.2 Valuation on January 1, 1992 ($t=1$)

This valuation uses the actual salaries.

TABLE A-11

TRADITIONAL VALUATION RESULTS, $t=1$

				PUC		EAN	
w	x	# ee's	$B_1^j(65)$	${}_\alpha NC_1^j$	${}_\alpha AL_1^j$	${}_\alpha NC_1^j$	${}_\alpha AL_1^j$
25	25	20	71,542.20	152.36	0.00	346.86	0.00
25	28	89	61,172.76	211.21	633.64	416.46	1,376.48
25	40	40	33,154.06	593.96	8,909.44	621.00	14,564.06
25	52	49	23,787.94	1,569.61	42,379.59	904.15	53,673.32
35	40	59	25,878.69	618.16	3,090.82	897.08	5,262.83
35	52	80	16,728.89	1,471.78	25,020.20	1,176.75	32,294.98
45	52	30	10,101.73	1,333.10	9,331.68	1,429.76	12,254.01

TABLE A-12

PERCENTILE VALUATION RESULTS, $t=1$

				PUC		EAN	
w	x	# ee's	$B_1^j(65)$	${}_\alpha NC_1^j$	${}_\alpha AL_1^j$	${}_\alpha NC_1^j$	${}_\alpha AL_1^j$
25	25	20	71,542.20	166.35	0.00	378.71	0.00
25	28	89	61,172.76	230.61	691.82	454.69	1,502.86
25	40	40	33,154.06	648.50	9,727.49	678.01	15,901.30
25	52	49	23,787.94	1,713.73	46,270.78	987.17	58,601.47
35	40	59	25,878.69	674.92	3,374.61	979.44	5,746.05
35	52	80	16,728.89	1,606.91	27,317.50	1,284.80	35,260.22
45	52	30	10,101.73	1,455.50	10,188.50	1,561.04	13,379.14

TABLE A-13

TOTALS, $t=1$

	PUC	EAN
NC_1	316,721.63	303,105.56
AL_1	4,953,296.58	6,596,787.33
${}_\alpha NC_1$	345,802.24	330,935.96
${}_\alpha AL_1$	5,408,096.10	7,202,488.14

TABLE A-14

FROZEN INITIAL LIABILITY, $t=1$

	Traditional	Percentile: Individual	Percentile: Group
$U\ AL_0$	5,589,550.55	6,369,365.56	5,505,890.51
NC_0	265,636.35	290,026.84	272,772.30
$(U\ AL_0 + NC_0)(1 + i)$	6,323,601.82	7,192,143.79	6,240,955.83
$C_0 + I_0^c$	−301,376.84	−301,376.84	−301,376.84
$U\ AL_1$	6,022,224.98	6,890,766.95	5,939,578.99
$PVFB_1^{(ACT)}$	10,326,868.78	11,275,056.50	10,332,196.96
$AL_1^{(RET)}$	2,807,363.44	3,058,498.88	2,808,811.91
Total $PVFB_1$	13,134,232.22	14,333,555.38	13,141,008.87
$U\ AL_1$	−6,022,224.98	−6,890,766.95	−5,939,578.99
F_1	−3,350,000.00	−3,350,000.00	−3,350,000.00
$PVFNC_1$	3,762,007.24	4,092,788.43	3,851,429.88
$PVFS_1$	142,702,092.32	142,702,092.32	142,702,092.32
U_1	2.636266%	2.868065%	2.698930%
ΣS_1^j	10,950,000.00	10,950,000.00	10,950,000.00
NC_1	288,671.17	314,053.09	295,532.86

TABLE A-15

ATTAINED AGE NORMAL, $t=1$

	Traditional	Percentile: Individual	Percentile: Group
$U\ AL_0$	4,103,200.75	4,746,542.76	4,110,830.36
NC_0	383,252.72	418,442.46	383,164.82
$(U\ AL_0 + NC_0)(1 + i)$	4,845,369.75	5,578,184.04	5,481,941.64
$C_0 + I_0^c$	−301,376.84	−301,376.84	−301,376.84
$U\ AL_1$	4,543,992.91	5,276,807.20	4,552,137.95
$PVFB_1^{(ACT)}$	10,326,868.78	11,275,056.50	10,332,196.96
$AL_1^{(RET)}$	2,807,363.44	3,058,498.88	2,808,811.91
Total $PVFB_1$	13,134,232.22	14,333,555.38	13,141,008.87
$U\ AL_1$	−4,845,369.75	−5,276,807.20	−4,552,137.95
F_1	−3,350,000.00	−3,350,000.00	−3,350,000.00
$PVFNC_1$	5,240,239.31	5,706,748.18	5,238,870.92
$PVFS_1$	142,702,092.32	142,702,092.32	142,702,092.32
U_1	3.672153%	3.999064%	3.671194%
ΣS_1^j	10,950,000.00	10,950,000.00	10,950,000.00
NC_1	416,136.09	437,897.52	401,995.76

TABLE A-16

AGGREGATE, $t=1$

		Percentile	
	Traditional	Individual	Group
$PVFB_1^{(ACT)}$	10,326,868.78	11,275,056.50	10,332,196.96
$AL_1^{(RET)}$	2,807,363.44	3,058,498.88	2,808,811.91
Total $PVFB_1$	13,134,232.22	14,333,555.38	13,141,008.87
F_1	−3,350,000.00	−3,350,000.00	−3,350,000.00
$PVFNC_1$	9,784,232.22	10,983,555.36	9,791,008.87
$PVFS_1$	142,702,092.32	142,702,092.32	142,702,092.32
U_1	6.856404%	7.696843%	6.861153%
ΣS_1^j	10,950,000.00	10,950,000.00	10,950,000.00
NC_1	750,776.26	842,804.26	751,296.25

2.3 Plan's Gains for 1991

TABLE A-17

PROJECTED UNIT CREDIT METHOD

		Percentile	
	Traditional	Individual	Group
$U\ AL_0$	4,103,200.75	4,746,542.76	4,657,429.43
NC_0	320,960.50	350,430.50	350,430.50
$(U\ AL_0 + NC_0)(1 + i)$	4,778,094.15	5,504,731.12	5,408,488.72
$C_0 + I_0^c$	−301,376.84	−301,376.84	−301,376.84
$U\ AL_1$	−4,410,660.02	−5,116,594.88	−4,875,591.71
Gain	66,057.29	86,759.40	231,520.17

TABLE A-18

ENTRY AGE NORMAL METHOD

		Percentile	
	Traditional	Individual	Group
$U\ AL_0$	5,589,550.55	6,369,365.56	6,280,252.23
NC_0	283,839.60	309,900.10	309,490.10
$(U\ AL_0 + NC_0)(1 + i)$	6,343,261.36	7,213,606.91	7,117,364.52
$C_0 + I_0^c$	−301,376.84	−301,376.84	−301,376.84
$U\ AL_1$	−6,054,150.77	−6,910,987.02	−6,669,983.61
Gain	−12,266.25	1,243.05	146,004.07

TABLE A-19

SPREAD GAIN METHODS

		Percentile	
	Traditional	Individual	Group
FIL	106,040.50	130,419.72	120,527.61
AAN	340,471.35	386,375.50	340,559.97
AGG	524,437.32	578,823.96	525,172.24

Notice that, as expected, the individual percentile approach consistently yields the largest gains.

2.4 Analysis of 1991's Gains

- **Interest:**

$$G_0^{(i)} = 3{,}350{,}000 - [2{,}450{,}000(1.08) + 290{,}000(1.08)^{1/2} - 134{,}000\ddot{s}_{\overline{1}|}^{(12)}]$$

$$= 2{,}361.56$$

- **Pension Payments:** The traditional and percentile approaches are dealt with separately.

  — Traditional: Using Equation (104) with $B_{Old}+I_{Old}=134{,}000\ddot{s}_{\overline{1}|}^{(12)}$ gives

$$G_0^{(p)} = 7 \times 12{,}000(1.0428239 - 0.47132q_{67})$$
$$+ 5 \times 10{,}000(1.0428239 - 0.47132q_{70}) - 134{,}000\ddot{s}_{\overline{1}|}^{(12)}$$
$$= 138{,}305.57 - 139{,}738.40 = -1{,}432.83$$

  — INDV Percentile: From Equation (100) and Table 7

$$G_0^{(p)} = 7 \times 12{,}000(\ddot{s}_{\overline{1}|}^{(12)} + (1.08)_{0.5}\theta_{67}^{(12)})$$
$$+ 5 \times 10{,}000(\ddot{s}_{\overline{1}|}^{(12)} + (1.08)_{0.5}\theta_{70}^{(12)} - 134{,}000\ddot{s}_{\overline{1}|}^{(12)}$$
$$= 13{,}401.72$$

- **Retiree Mortality Gains:**

  — Traditional: Using Equation (101) and Table A-1.

$$G_0^{(d)} = 12{,}000\ddot{a}_{68}^{(12)} - (7 \times 12{,}000q_{67}\ddot{a}_{68}^{(12)} + 5 \times 10{,}000q_{70}\ddot{a}_{71}^{(12)})$$
$$= 95{,}886.10 - 13{,}292.50 - 10{,}080.84 = 72{,}512.76$$

— INDV Percentile: Using Equation (98)

$$G^{(d)} = 12{,}000_{0.5}\ddot{\xi}_{68}^{(12)} - (7 \times 12{,}000 q_{67}\ {}_{0.5}\ddot{\xi}_{68}^{(12)} + 5 \times 10{,}000 q_{70}\ {}_{0.5}\ddot{\xi}_{71}^{(12)})$$

$$= 104{,}403.92 - (14{,}473.31 + 10{,}871.19) = 79{,}059.42$$

The other components of the gain depend on the accrued liabilities calculated at age $x+1$ and at $t=1$ assuming salaries had increased according to the salary scale. These quantities are called expected accrued liabilities. Table A-20 displays the individual expected accrued liabilities for the PUC and EAN.

TABLE A-20

EXPECTED ACCRUED LIABILITIES

				PUC		EAN	
w	x	# ee's	$B_0^j(65)$	$\widetilde{AL}_1^j$	${}_\alpha\widetilde{AL}_1^j$	$\widetilde{AL}_1^j$	${}_\alpha\widetilde{AL}_1^j$
25	28	89	56,872.65	589.10	643.19	1,279.72	1,397.22
25	40	40	32,485.92	8,729.89	9,531.45	14,270.56	15,580.85
25	52	49	24,026.96	42,805.41	46,735.70	54,212.62	59,190.29
35	40	59	23,321.59	2,785.41	3,041.16	4,742.80	5,178.28
35	52	80	16,094.54	24,071.45	26,281.63	31,070.37	33,923.17
45	52	30	9,662.11	8,925.58	9,745.10	11,720.73	12,796.89

This gives the following total accrued liabilities, $\widetilde{AL}_1$, using the expected salaries at $t=1$ over the set $A_0 \cap A_1$.

TABLE A-21

TOTALS, $\widetilde{AL}_1$

	PUC	EAN
Traditional	4,856,913.13	6,458,212.41
Percentile	5,302,862.96	7,051,189.60

From Tables A-11 and A-19

TABLE A-22

SALARY SCALE GAINS, $G_0^{(s)}$

	PUC	EAN
Traditional	−96,383.45	−138,574.92
Percentile	−105,233.14	−151,298.54

The components of the gain can now be determined.

TABLE A-23

PROJECTED UNIT CREDIT METHOD

Gains	Traditional	Percentile Individual
Interest	2,361.56	2,361.56
Pension Payments	−1,432.83	13,401.72
Retiree Mortality	72,512.76	79,059.42
Active Mortality	202,401.93	220,985.99
Terminations	−113,402.45	−123,814.55
Salary Changes	−96,383.45	−105,233.14
New Entrants	0.00	0.00
Explained	66,057.52	86,761.00
Unexplained	−0.23	−1.60
Total	66,057.29	86,759.40

TABLE A-24

ENTRY AGE NORMAL METHOD

Gains	Traditional	Percentile Individual
Interest	2,361.56	2,361.56
Pension Payments	−1,432.83	13,401.72
Retiree Mortality	72,512.76	79,059.42
Active Mortality	208,144.89	227,256.25
Terminations	−155,277.87	−169,535.09
Salary Changes	−138,574.92	−151,298.54
New Entrants	0.00	0.00
Explained	−12,266.40	1,245.61
Unexplained	0.15	−2.56
Total	−12,266.25	1,243.05

The following are needed to determine the various components of the gain for spread gain methods as described in Section 3.3.4.

$$\text{TRAD } \widetilde{PVFB}_1^{(ACT)} = 9{,}887{,}015.76$$

$$\text{INDV } {}_{\alpha}\widetilde{PVFB}_1^{(ACT)} = 10{,}794{,}817.26$$

$$\text{GROUP } {}_{\alpha}\widetilde{PVFB}_1^{(ACT)} = 10{,}210{,}239.27$$

$$\widetilde{PVFS}_1 = 127{,}828{,}775.79$$

$$\Sigma\, \tilde{S}_1^j = 9{,}980{,}070.23 \text{ total expected salaries}$$
$$PVFB_1^{(n)} = 121{,}894.80 \text{ for new entrants}$$
$$\text{INDV }_{\alpha}PVFB_1^{(n)} = 133{,}086.87 \text{ for new entrants}$$
$$\text{GROUP }_{\alpha}PVFB_1^{(n)} = 121{,}957.69 \text{ for new entrants}$$
$$PVFS_1^{(n)} = 7{,}028{,}273.60 \text{ for new entrants}$$
$$\text{Total new entrant salaries} = 400{,}000.00$$

Note that the "tilde" symbol over a quantity suggests the summation is taken over the lives in the set $A_0 \cap A_1$ with salaries assumed to increase as expected.

TABLE A-25

VALUES AT AGE $x + 1$ AND $t = 1$

w	x	$\widetilde{PVFB}_1^j$	INDV $_{\alpha}\widetilde{PVFB}_1^j$	$\widetilde{PVFS}_1^j$
25	27	7,854.62	8,575.81	378,480.26
25	39	23,279.63	25,417.11	476,233.52
25	51	63,415.39	69,238.03	371,797.62
25	64	207,318.96	226,354.46	0.00
35	39	16,712.41	18,246.90	396,861.27
35	51	42,479.01	46,379.33	316,683.67
40	51	25,501.62	27,843.11	265,569.73

For the spread gain methods the salary scale gains and new entrant gains are

$$G_0^{(s)} = -(317{,}958.20 - 7{,}845{,}042.98U_0) \text{ TRAD}$$
$$G_0^{(s)} = -(347{,}152.37 - 7{,}845{,}042.98U_0) \text{ INDV}$$
$$G_0^{(s)} = -(318{,}132.30 - 7{,}845{,}042.98U_0) \text{ GROUP}$$
$$G_0^{(n)} = -(121{,}894.80 - 7{,}028{,}273.60U_0) \text{ TRAD}$$
$$G_0^{(n)} = -(133{,}086.87 - 7{,}028{,}273.60U_0) \text{ INDV}$$
$$G_0^{(n)} = -(121{,}957.69 - 7{,}028{,}273.60U_0) \text{ GROUP}$$

TABLE A-26

FROZEN INITIAL LIABILITY

Gains	Traditional	Percentile Individual	Percentile Group
Interest	2,361.56	2,361.56	2,361.56
Pension Payments	−1,432.83	13,401.72	−1,432.83
Retiree Mortality	72,512.76	79,059.42	72,552.46
Active Mortality	205,910.74	224,816.97	205,946.54
Terminations	−136,565.72	−149,148.28	−133,391.85
Salary Changes	−105,312.45	−114,981.62	−99,774.08
New Entrants	68,611.83	74,911.94	73,662.79
Retirement Benefit Changes	0.00	0.00	0.00
Proportional Adjustment Factor (ψ)	0.00	0.00	416.85
Explained	106,085.89	130,421.71	120,341.41
Unexplained	−45.39	−1.99	186.20
Total	106,040.50	130,419.72	120,527.61

TABLE A-27

ATTAINED AGE NORMAL METHOD

Gains	Traditional	Percentile Individual	Percentile Group
Interest	2,361.56	2,361.56	2,361.56
Pension Payments	−1,432.83	13,401.72	−1,432.83
Retiree Mortaity	72,512.76	79,059.42	72,552.46
Active Mortality	204,939.76	223,756.83	205,027.39
Terminations	−79,708.49	−87,026.92	−80,027.06
Salary Changes	−11,158.81	−12.183.08	−11,403.28
New Entrants	152,962.85	167,007.83	152,836.91
Retirement Benefit Changes	0.00	0.00	0.00
Proportional Adjustment Factor (ψ)	0.00	0.00	416.85
Explained	340,446.80	386,377.36	340,331.97
Unexplained	24.55	−1.86	228.00
Total	340,471.35	386,375.50	340,559.97

TABLE A-28

AGGREGATE METHOD

Gains	Traditional	Percentile	
		Individual	Group
Excess Contributions	−463,201.69	−556,187.53	−463,758.79
Interest	2,361.56	2,361.56	2,361.56
Pension Payments	−1,432.83	13,401.72	−1,432.83
Retiree Mortality	72,512.76	79,059.42	72,552.46
Active Mortality	202,259.27	220,655.89	202,318.94
Terminations	77,361.28	94,669.84	77,233.26
Salary Changes	248,760.56	288,489.10	248,999.43
New Entrants	385,821.29	436,376.19	386.128.37
Retirement Benefit Changes	0.00	0.00	0.00
Proportional Adjustment Factor (ψ)	0.00	0.00	416.85
Total Explained	524,442.20	578,826.19	524,809.22
Unexplained	−4.88	−2.23	363.02
Total	524,437.32	578,823.96	525,172.24

Excess contributions are calculated from Equation (60). The unexplained gains in the group percentile FIL, AAN and AGG methods may appear to be high in absolute value. However, they are less than 0.5 percent of the total gain. This is very good given that an actual theoretical breakdown of the gain into its components would be extremely complicated.

DISCUSSION OF PRECEDING PAPER

WILLIAM A. BAILEY:

It is a pleasure to read such a well-constructed paper. The author's willingness to move ahead with approximate methods, when the problem is otherwise intractable, is to be commended. Dr. Ramsay's paper provides various insights for pension actuaries who are, or may become, interested in dealing with distributions of financial functions connected with pension valuations.

First, a few minor comments.

It would be helpful to identify (early on) the postretirement mortality table used in Tables 1 to 7. Table 5 could have been calculated more accurately.

I wondered whether the symbol ${}_{\alpha}\xi_x^{(m)}$ could perhaps better be labeled ${}_{\alpha}\mathbf{Y}_x^{(m)}$, so as not to introduce a new unnecessary symbol.

On page 368, the author states that "However, to fully enjoy the benefits of the group approach, the plan must be large. The group percentile method is *not* suited for use by small plans." The paper does not make clear why this is so, although perhaps it is obvious to pension actuaries.

In dealing with ${}_{\alpha}PVFB_t$ I would probably have used his Equation (63), because it completely separates the probabilities from the financial functions such as v^{y-x}, $I_t^j(x)$, B_t^j, and $\mathbf{Y}_y^{(12)}$; whereas his Equation (64) incorporates the preretirement survival probabilities. But on page 380 he gives reasons for his having selected Equation (64) instead of (63). If Equation (63) were used, care would have to be taken that dependent random variables (for example, liabilities and normal costs) were not treated as being independent when they are, in fact, dependent.

The symbols δ on page 383 should perhaps be γ.

On page 380 Dr. Ramsay states that ". . . the distribution of $L_t^{(act)}$ must be approximated because $L_t^{(act)}$ is a convolution of independent (though not necessarily identically distributed) random variables." The purist may object to the wording that "$L_t^{(act)}$ is a convolution," although the meaning is perhaps clear.

The main purpose of my discussion is to suggest a more accurate computer-intensive alternative to the use of "Approximation 1 (Haldane's Type A)" and "Approximation 2 (Normal)."

The Appendix in my paper* describes a univariate numerical generalized convolutions algorithm, which can be used to perform the convolution for sums (or other functions) of two independent (not necessarily identical) distributions. This algorithm is incorporated in a C language computer program, which is named COCONUT™, available from MathWare. I used this algorithm to generate the distribution of $L_0^{(ACT)}$ and the distribution of $L_0^{(RET)}$ for the example in Appendix A of Dr. Ramsay's paper. The resulting percentiles are shown in the column labeled "Bailey" in Table 1.

TABLE 1

$L_0^{(ACT)}$

	Haldane	Bailey	Haldane Relative Error
0.000001	9,365,635	9,430,169	−0.006843
0.000010	9,538,991	9,582,187	−0.004508
0.000100	9,725,483	9,751,106	−0.002628
0.001000	9,931,504	9,943,608	−0.001217
0.010000	10,171,011	10,173,640	−0.000258
0.025000	10,281,461	10,282,182	−0.000070
0.050000	10,375,011	10,374,390	0.000060
0.100000	10,480,464	10,479,395	0.000102
0.200000	10,605,427	10,604,423	0.000095
0.300000	10,693,962	10,693,057	0.000085
0.400000	10,767,734	10,767,677	0.000005
0.500000	10,836,269[a]	10,836,510	−0.000022
0.600000	10,904,121	10,904,469	−0.000032
0.700000	10,975,437	10,976,087	−0.000059
0.800000	11,057,557	11,058,632	−0.000097
0.900000	11,170,127	11,170,466	−0.000030
0.950000	11,260,628	11,260,671	−0.000004
0.975000	11,338,201	11,337,300	0.000080
0.990000	11,427,499	11,424,568	0.000257
0.995000	11,486,921	11,482,756	0.000363
0.999900	11,752,979	11,738,334	0.001248
0.999990	11,875,926	11,854,272	0.001827
0.999999	11,983,941	11,955,235	0.002401
Mean	10,829,519.86	10,829,520.83	
Std Dev	269,528.11	269,523.31	
Abs Dev	214,745.53	215,147.32	
Min	8,521,717.94	8,906,200.00	
Max	12,430,675.87	12,307,556.41	

[a]I am treating the active and retired groups separately here. The effect on $L_0^{(ACT)}$ of combining these two groups is discussed below.

*BAILEY, W. A. "A Method for Determining Confidence Intervals for Trend," *TSA* XLIV (1993): 1–29.

The figures shown in the column labeled "Haldane" below were calculated by using the formula

$$x_0 = (x_1 \times \sigma(h, s) + \mu(h, s))^{1/h} \times \mu_X$$

to translate the amounts, x_1, in a normal (0, 1) distribution to the amounts, x_0, described in the Approximation 1 (Haldane's Type A) on pages 380–381 of Dr. Ramsay's paper.

Bailey's 50th percentile figure for the combined active and retired groups would be 11,889,475, derived by convoluting together the distributions for these two separate groups. The sum of $L_0^{(ACT)} = 10{,}836{,}510$ from Table 1 for the active group and $L_0^{(RET)} = 1{,}071{,}696$ from Table 2 for the retired

TABLE 2

$L_0^{(RET)}$

	Haldane	Bailey	Haldane Relative Error
0.000001	334,605	475,162	−0.295808
0.000010	452,406	537,230	−0.157891
0.000100	560,661	607,588	−0.077235
0.001000	667,988	689,107	−0.030648
0.010000	782,625	787,745	−0.006500
0.025000	833,199	834,471	−0.001525
0.050000	874,913	874,256	0.000751
0.100000	921,106	919,442	0.001810
0.200000	974,739	973,112	0.001672
0.300000	1,012,091	1,010,978	0.001100
0.400000	1,043,239	1,042,663	0.000552
0.500000	1,071,502[a]	1,071,696	−0.000181
0.600000	1,099,313	1,100,113	−0.000727
0.700000	1,128,509	1,129,804	−0.001146
0.800000	1,161,979	1,163,527	−0.001331
0.900000	1,207,086	1,208,371	−0.001064
0.950000	1,243,352	1,243,533	−0.000146
0.975000	1,274,216	1,272,628	0.001248
0.990000	1,309,562	1,304,746	0.003692
0.995000	1,333,096	1,325,450	0.005768
0.999900	1,436,988	1,408,836	0.019982
0.999990	1,484,671	1,442,164	0.029474
0.999999	1,526,256	1,468,863	0.039073
Mean	1,066,948.79[b]	1,066,954.84	
Std Dev	112,405.98	112,344.45	
Abs Dev	89,246.67	89,893.52	
Min	0.00	243,528.09	
Max	1,697,971.92	1,563,529.74	

[a] I am treating the active and retired groups separately here. The effect on $L_0^{(RET)}$ of combining these two groups is discussed above.

[b] The mean value turned out to be 1,066,948.79, although I used $\mu = 1{,}066{,}954.85$, $\sigma = 112{,}344.46$ and $\gamma = -0.244681$ in generating x_i as described above.

group amounts to 11,908,206. Applying the ratio 11,889,475/11,908,206=1.998427 to Bailey's figures of 10,836,510 and 1,071,696 produces 10,819,465 and 1,070,010, compared to Ramsay's 10,835,449.25 and 1,067,539.53 shown in Table A-8 for $L_0^{(ACT)}$ and $L_0^{(RET)}$, respectively.

Of course, percentiles other than the 50th could be used. So the caption "percentile" in Tables A-8 through A-28 should be amended to read "50th percentile."

The parameters used in my computer runs of COCONUT™ were nax=4,000, mesh=1 and epsilon=10^{-15}, where nax is the number of output intervals, mesh=1 indicates equal-lengthed intervals, and epsilon is the probability less than which probability products are discarded in the convolution process.

CECIL J. NESBITT:

To start with, I commend Dr. Ramsay for the skill and energy with which he has carried out the statistical theory of his paper and related his work to the pension funding texts of Anderson and Berin. Even though in these turbulent times I might question the actuarial sense of his paper, I welcome the development of his theory, which can then be evaluated in the crucible of experience. Who knows what responses it may invoke, and what uses it may find?

This discussion is in three parts. The first indicates some ideas from a handwritten, unpublished thesis prepared in the 1950s by Robert W. Butcher, FSA, who died on December 7, 1993. The second part indicates a radically changed theory that differs from classical pension theory, but may be more appropriate for large benefit systems for public employees, and for Old-Age, Survivors and Disability Insurance (OASDI, often called Social Security). Finally, I mention a few practical problems in applying the author's theory and some qualifications that I think should be made in the presentation of precise probabilities in relation to pension funding.

Butcher's thesis was prepared in a period when risk assessment was more on a judgment than a theoretical basis in North America and statistical and computer sciences were at much earlier stages of development. The scope of the thesis is indicated by the following abbreviated list of contents, and the appended remarks thereon.

Chapter 1 A General Introduction
Chapter 2 Individual Theory of Risk
2.1 Introduction
2.2 Discrete Theory for an Insured Individual
2.3 Discrete Theory for Insured Groups
2.4 Continuous Theory for Insured Individuals
2.5 Continuous Theory for Insured Groups
Chapter 3 Application to Retired Members
3.1 Introduction
3.2 The Discrete Case
3.3 The Continuous Case
Chapter 4 Application to Active Group
4.1 Introduction
4.2 The General Procedure
4.3 The Total Loss Variable
4.4 Pay-As-You-Go Funding
4.5 Terminal Funding
4.6 Unit Credit Funding
4.7 Entry Age Normal Funding
4.8 Initial Funding
4.9 Bases for Loading Mortality Table Groups
4.10 Some Illustrative Numerical Values of Standard Deviations of Losses and Loading Factors
Appendix
Bibliography

Chapter 1 indicates the various sources of risk for pension funding but clearly states that it will deal only with the mortality risks, namely, from changing mortality and from random fluctuations, mainly the latter. It also indicates the approaches to collective risk theory and to individual risk theory. As in the present paper, the thesis develops by the individual risk theory route.

Chapter 2 provides a very general development of Hattendorf theory (expressing the variance over an insurance term as a weighted sum of year-by-year variances). This development has been enshrined in less general form in Section 7.10 of *Actuarial Mathematics* [1]. Chapter 2 introduces the usual loss variable (referred to as the fund loss), but also considers two other loss variables, namely, the contributor's loss and the total loss. Butcher went on to define functions to aid in the systematic computation of the variances of these loss variables in the context of insurance and annuity situations.

The contents of Chapters 3 and 4 may be those more closely related to the present paper, but again are based on year-by-year variances and Hattendorf theory. In Chapter 4, a fixed age at entry and a fixed retirement age are utilized, as in Dr. Ramsay's paper. Both Butcher's thesis and the present paper provide reviews of the main pension funding methods, but do so by different means. As an additional note, I add that the paper by McCrory [3] provides background material for both the thesis and the current paper.

Turning now to the new system of funding we are considering here for large systems such as OASDI, note that a first prerequisite is the year-by-year projected outgo and income flows for a considerable term of years. One then has the means for applying n-year roll-forward reserve financing. This amounts to pay-as-you go funding but projected n-years ahead. For OASDI, $n=1$ or 2 seems the more appropriate choice. For other large public systems, $n=5$, 10, 15, or even 20 might be considered. As opposed to classical funding for the closed group of participants on the valuation date, n-year roll-forward reserve financing is in regard to the open group of present and future participants for a given term.

In classical pension funding, there could be a long period of years between the date of contribution and the date of withdrawal of the accumulated contribution for benefit payment (see Bowers, Hickman and Nesbitt [2, p. 117]). This was a guarantee in former less turbulent years of individual equity. In n-year roll-forward reserve financing, individual equity rests on the legal enforcement of benefits rather than on accumulated contributions, which are generally used much earlier for benefit payment than is the case for classical funding. The communication problem for n year roll-forward reserve financing should be less than is the case for classical pension funding. For n-year roll-forward reserve financing there is the problem of smoothing the contribution rates over terms longer than n-years. This can be achieved by an amortization process over a term of m years, $m>n$, but this may entail jumps at the junction points of m-year terms. Study of this problem will go ahead here when the 1994 OASDI Trustees' Report becomes available.

Finally, let us consider some practical problems in extending the author's methods to actuarial practice. Here, I am biased by my studies of OASDI financing and by my experience as trustee of a medium-sized public employee retirement system that by some measures is overfunded.

That the author's percentile methods would normally require higher contributions and accrued liabilities is an obvious difficulty for both private and public plans. Also, in large public employee retirement systems, an important assumption is a set of retirement rates extending perhaps from age 50 to age 70.

It may then be simpler to work prospectively, rather than retrospectively, in the calculation of items such as α-accrued liability. I am also reluctant to express a precise degree of confidence, such as an α-probability, that benefits will be paid as stated in the system's documents. Rather, I prefer a careful analysis of the benefit ramifications of the system, a clear statement of the various sets of assumptions used to calculate the funding of those benefits, repeated valuations that permit observation of funding trends, and appropriate gain-and-loss analysis. Will percentile methods aid in the communication and funding of the system's costs? Or is it better to present cost figures somewhat humbly, knowing that economic, demographic and political changes may require substantial reassessment of costs at the next valuation date?

As more specific comments, I would like to see some further exposition of the formula for k_m below the author's Formula (2). Also, in Table A-10 of Appendix A, $PVFNC_0$ has been misprinted, it should be \$8,946,469.66.

It will be interesting to see what adaptations of percentile pension funding occur.

REFERENCES

1. Bowers, N.L., Gerber, H.U., Hickman, J.D., Jones, D.A., and Nesbitt, C.J. *Actuarial Mathematics*. Itasca, Ill: Society of Actuaries, 1986.
2. Bowers, N.L., Hickman, J.C., and Nesbitt, C.J. "The Dynamics of Pension Funding: Contribution Theory," *TSA* XXXI (1979): 93–136.
3. McCrory, R.T. "Mortality Rise in Annuities," *TSA* XXXVI (1984): 309–38.

(AUTHOR'S REVIEW OF DISCUSSIONS)

COLIN M. RAMSAY:

I thank Mr. Bailey and Dr. Nesbitt for their comments.

Mr. Bailey states, without any clarification, that Table 5 could have been calculated more accurately. Since Table 5 is based on Equation (13) with the common assumption of a uniform distribution of deaths (UDD) between integral ages, I am not sure where greater accuracy could have been achieved.

To clarify my comment that the group approach is not ideally suited for small plans, I must add that, theoretically, the group approach can be used for any plan, whatever its size. However, for small plans, with a large degree of skewness in the distribution of the present value of its future benefits, the normal approximation and the Haldane approximation cannot be used. Rather "exact" computations of the necessary convolutions are required. Mr. Bailey appears to have access to software that can compute such convolutions. I must add that the Haldane approximation is, for practical purposes, accurate, fast and easy to use. It requires only a modest amount of computing power to yield results.

I am pleased to see that, for Dr. Nesbitt, my paper invoked memories from many sources. Dr. Nesbitt has done us a great service by bringing to light the unpublished thesis by Robert Butcher. Though 30 to 40 years old, Butcher's ideas are quite modern and sophisticated. The Society of Actuaries should make every effort to publish Butcher's work so that all actuaries can benefit from it.

Dr. Nesbitt correctly asserts that my percentile cost methods would normally require higher contributions and accrued liabilities and that this is an obvious difficulty for both private and public plans. I wholeheartedly agree with him. These cost methods have been founded on the premise that the traditional expected value approach to pension plan valuation actually understates the plan's liabilities. He also raises an important question: Will percentile cost methods aid in the communication and funding of a plan's costs? The answer depends on how the α-percentiles are interpreted. They are not guarantees! They are subject to the same inaccuracies and other limitations that are inherent in the quantities currently calculated by using traditional (expected value) cost methods. That is, they are valid only as long as the valuation assumptions match actual experience. Thus the actuary should present *all* cost figures (including probabilities) somewhat humbly, "knowing that economic, demographic and political changes may require substantial reassessment of costs at the next valuation date." Finally, I thank Dr. Nesbitt for pointing out McCrory's paper, which I had overlooked.