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CASH-FLOW MATCHING AND LINEAR PROGRAMMING DUALITY

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ABSTRACT

Cash-flow matching, or dedication, is an important and practical tool for managing interest rate risk. This paper applies the duality theory of linear programming to provide insights for generalizing and solving the cash-flow matching problem.

I. INTRODUCTION

Interest rate fluctuations are a major risk for the insurance and pension industry. If assets are invested shorter than the corresponding liabilities, reinvestment risk arises because interest rates can fall. On the other hand, if assets are longer than liabilities, then liquidation risk or market risk exists because interest rates can rise. The concept of cash-flow matching is an important and practical tool for managing interest rate risk (C-3 risk).

Suppose that at time $t=0$, a decision-maker (an insurer or a pension fund manager) has a stream of liability obligations of amount l_t to be paid at time t, $t=1, 2, 3, \ldots$. (For simplicity, we assume all cash flows occur at the end of time periods.) These liability cash flows $\{l_t\}$ are assumed to be fixed and certain. The decision-maker faces the problem of constructing from the currently available universe of noncallable and default-free fixed-income securities an investment portfolio that will meet the future liability payments. With a finite amount of resources, the decision-maker seeks an initial investment portfolio with minimum cost such that its cash flow will at least meet the projected liability payment for each and every period in the planning horizon.

Let p_k denote the current price for one unit of the k-th security and $c_{k,t}$ its cash flow at time t, $t=1, 2, 3, \ldots$. Let n_k denote the number of units of

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the k-th security to be purchased. The decision-maker may seek to find the investment portfolio $\{n_k\}$ by minimizing total cost

$$\sum_k n_k \, p_k \tag{1.1}$$

under the constraints

$$\sum_k n_k \, c_{k,t} \geq l_t \qquad \text{for all } t \tag{1.2}$$

and

$$n_k \geq 0 \qquad \text{for all } k.$$

Thus the decision-maker's problem can be formulated as a linear program.

A main advantage of the cash-flow matching technique is its simplicity. To implement the strategy, the decision-maker needs only to know the prices of the fixed-income securities available in the marketplace and their future cash flows. The decision-maker does not need to worry about the term structure of interest rates, duration, convexity, and so on. However, this paper shows that the term structure of interest rates actually plays an intrinsic role in the method of cash-flow matching. The concept of term structure arises naturally as we consider the dual problem of the linear program above. By studying the dual linear program, we show how an important and useful extension of the classical formulation can be developed.

Discussions on and numerical examples of the method of cash-flow matching and related topics can be found in [1], [3], [4], [5, Chapter 19], [6, Chapter 14], [7], [8, Chapter 6], [10], [11, Chapter 7], [13], [15], [16], [17], [18], [19], [21], [22], [23], and [24].

II. DUALITY THEORY OF LINEAR PROGRAMMING

The basic tool for this paper is the duality theory of linear programming, which we now briefly review. Let m and n be positive integers and let $A=(a_{i,j})$ be a given real m by n matrix. Let $\mathbf{b}$ and $\mathbf{c}$ be given (column) vectors in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. (Zero vectors are denoted by $\mathbf{0}$, the dimension of which is to be determined by the context.) The standard (primal) linear programming problem seeks to determine a vector $\mathbf{x}\geq\mathbf{0}$ in $\mathbf{R}^n$ which satisfies the system of m linear inequalities

$$A\mathbf{x} \leq \mathbf{b} \tag{2.1}$$

(such a vector $\mathbf{x}$ is called feasible) and maximizes the so-called objective function

$$\mathbf{c}^T\mathbf{x} = c_1x_1 + c_2x_2 + \dots + c_nx_n. \tag{2.2}$$

The dual of this problem is to find a vector $\mathbf{y}\geq\mathbf{0}$ in $\mathbf{R}^m$ that satisfies the system of n linear inequalities

$$A^T\mathbf{y} \geq \mathbf{c} \tag{2.3}$$

(such a vector $\mathbf{y}$ is called a dual feasible vector) and minimizes the objective function

$$\mathbf{b}^T\mathbf{y} = b_1y_1 + b_2y_2 + \dots + b_my_m. \tag{2.4}$$

It is not difficult to verify that the dual of a dual problem is the primal problem.

Obviously, whenever $\mathbf{x}$ and $\mathbf{y}$ are feasible,

$$\mathbf{c}^T\mathbf{x} \leq (\mathbf{y}^TA)\mathbf{x} \leq \mathbf{y}^T(A\mathbf{x}) \leq \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}.$$

Consequently,

$$\sup\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ in } \mathbf{R}^n \text{ and } A\mathbf{x} \leq \mathbf{b}\} \leq \inf\{\mathbf{b}^T\mathbf{y} \mid \mathbf{y} \geq \mathbf{0} \text{ in } \mathbf{R}^m \text{ and } A^T\mathbf{y} \geq \mathbf{c}\}, \tag{2.5}$$

where an empty supremum equals $-\infty$ and an empty infimum equals $+\infty$. The celebrated *fundamental theorem of linear programming* ([12, p. 138], [9, p. 62]) states that inequality (2.5) is in fact an equality:

$$\sup\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ in } \mathbf{R}^n \text{ and } A\mathbf{x} \leq \mathbf{b}\} = \inf\{\mathbf{b}^T\mathbf{y} \mid \mathbf{y} \geq \mathbf{0} \text{ in } \mathbf{R}^m \text{ and } A^T\mathbf{y} \geq \mathbf{c}\}, \tag{2.6}$$

unless both the primal and dual problems are infeasible.

An equality $r=s$ is equivalent to the pair of simultaneous inequalities: $r\leq s$ and $-r\leq -s$. Each real number r can be written as the difference of two non-negative numbers, $r=r^+-r^-$ with $r^+\geq 0$, $r^-\geq 0$. Hence the fundamental theorem of linear programming can be modified as:

$$\sup\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \text{ unconstrained in sign in } \mathbf{R}^n \text{ and } A\mathbf{x} \leq \mathbf{b}\} = \inf\{\mathbf{b}^T\mathbf{y} \mid \mathbf{y} \geq \mathbf{0} \text{ in } \mathbf{R}^m \text{ and } A^T\mathbf{y} = \mathbf{c}\}, \tag{2.7}$$

unless both the primal and dual problems are infeasible. For more detail, see Section 6.4 of [12] or Section 1.8 of [9].

III. TERM STRUCTURE OF INTEREST RATES

The linear program formulated in Section I seeks a non-negative vector $\mathbf{n}$ that minimizes

$$\mathbf{p}^T \mathbf{n} \tag{3.1}$$

under the constraint

$$C^T\mathbf{n} \geq \mathbf{l}, \tag{3.2}$$

where

$$\begin{aligned} \mathbf{p} &= (p_1, p_2, \ldots)^T, \\ \mathbf{n} &= (n_1, n_2, \ldots)^T, \\ \mathbf{l} &= (l_1, l_2, \ldots)^T \\ C &= (c_{i,j}). \end{aligned}$$

We call this linear program **LP1**.

The problem dual to **LP1** is:

$$\begin{gathered} \text{Maximize } \mathbf{l}^T\mathbf{v} \\ \mathbf{v} \geq \mathbf{0} \end{gathered} \tag{3.3}$$

subject to

$$C\mathbf{v} \leq \mathbf{p}. \tag{3.4}$$

We call this linear program **LP*1**. How is $\mathbf{v}$ interpreted?

For $t=1, 2, 3, \ldots$, let i_t denote the t-period spot rate, that is, $(1+i_t)^{-t}$ is the (present) value at time 0 for \$1 to be paid at time t [8, p. 282]. The shape of the graph of i_t versus t, $t>0$, is known as the term structure of interest rates ([2, p. 220], [8, p. 282], [14, p. 154]). In a perfect capital market, in which there are no taxes, no transaction costs, no arbitrage opportunities, and so on, each noncallable and default-free fixed-income security is priced by the spot rates $\{i_t\}$; that is, for each k,

$$p_k = \sum_t \frac{c_{k,t}}{(1 + i_t)^t}. \tag{3.5}$$

(See also Section II of [20].) Since the vector $\mathbf{u}=[(1+i_1)^{-1}, (1+i_2)^{-2}, \ldots]^T$ satisfies the equation $C\mathbf{u}=\mathbf{p}$, it is a feasible vector, and by inequality (2.5), the sum

$$\mathbf{l}^T\mathbf{u} = \sum_t \frac{l_t}{(1 + i_t)^t} \tag{3.6}$$

is a lower bound for the minimum cost (3.1) of **LP1**. This is hardly surprising since the sum (3.6) is merely the present value of the liability cash flows.

It seems that an economic interpretation for a vector $\mathbf{v}$, which is feasible with respect to **LP*1**, is that it is a vector of discount factors and the objective function $\mathbf{l}^T\mathbf{v}$ gives the "present value" of the liability cash flows. However, this is not quite correct. Let

$$\mathbf{v} = (v_1, v_2, \ldots, v_t, \ldots)^T;$$

if $\{v_t\}$ are discount factors, then we should have the monotonicity condition

$$v_1 \geq v_2 \geq \ldots \geq v_t \ldots \geq 0. \tag{3.7}$$

But (3.7) is nowhere to be found in **LP*1**.

For example, let $\mathbf{l}=(1, 12)^T$, $\mathbf{p}=(1, 1)^T$ and

$$C^T = \begin{pmatrix} 1.1 & 0.11 \\ 0 & 1.11 \end{pmatrix}.$$

The optimal feasible vector $\mathbf{v}$ for **LP*1** is $\mathbf{v}=(0, 0.9009)^T$, which does not satisfy (3.7).

IV. CARRY-FORWARD ALLOWED

The absence of condition (3.7) is a symptom of a deficiency in the classical cash-flow matching model. The requirement that $C^T\mathbf{n}\geq\mathbf{l}$ is unnecessarily restrictive. The model should at least allow for the carry-forward of positive cash balances at zero interest rate. In this section we show that, if this feature is included in the model, condition (3.7) is automatically satisfied. It then follows from the fundamental theorem of linear programming that the minimum cost of the asset portfolio is the same as the maximum "present value" of the liability cash flows.

We now generalize the model by allowing the carry-forward of positive cash balances at zero or low interest rate. The cost of the optimal investment portfolio of the new model should be at least as low as that of the old model. Let τ be the dimension of the liability vector $\mathbf{l}$ and assume that the asset cash-flow matrix C consists of τ columns. Define

$$\mathbf{g} = C^T\mathbf{n} - \mathbf{l} \tag{4.1}$$

and let $\mathbf{g}=(g_1, g_2, \ldots, g_\tau)^T$. For $t=1, 2, 3, \ldots, \tau-1$, let r_t denote a conservative estimate of the one-period reinvestment interest rate at time t. For $t=1, 2, 3, \ldots, \tau$, let b_t denote the (cumulative) cash balance at time t; that is, $b_1=g_1$ and for $t=1, 2, 3, \ldots, \tau-1$,

$$b_{t+1} = g_{t+1} + (1 + r_t)\, b_t. \tag{4.2}$$

The decision-maker is to seek an investment portfolio $\{n_k | n_k \geq 0\}$, which minimizes the cost

$$\mathbf{p}^T\mathbf{n} = \sum_k p_k n_k$$

while subject to the condition that the cash balances $\{b_t\}$ are to be nonnegative. (The formulation given in Section I is a special case of the present model with $r_1=r_2= \ldots = -1$.)

Define

$$R = \begin{pmatrix} -1 & 0 & 0 & . & . & 0 & 0 \\ 1+r_1 & -1 & 0 & . & . & 0 & 0 \\ 0 & 1+r_2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1+r_{\tau-1} & -1 \end{pmatrix} \tag{4.3}$$

and $\mathbf{b}=(b_1, b_2, \ldots, b_\tau)^T$. The generalized problem is:

$$\underset{\mathbf{n} \geq \mathbf{0},\, \mathbf{b} \geq \mathbf{0}}{\text{Minimize}} \quad \mathbf{p}^T\mathbf{n} \tag{4.4}$$

subject to

$$(C^T R) \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \mathbf{l}. \tag{4.5}$$

We call this linear program **LP2**. Note that the objective function (4.4) can be expressed as

$$(\mathbf{p}^T\, \mathbf{0}^T) \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \tag{4.6}$$

Since (4.5) is an equality constraint, we apply the fundamental theorem of linear programming in the form of (2.7). The problem dual to **LP2** is:

$$\underset{\mathbf{v}}{\text{Maximize}} \quad \mathbf{l}^T\mathbf{v} \tag{4.7}$$

subject to

$$\begin{pmatrix} C \\ R^T \end{pmatrix} \mathbf{v} \leq \begin{pmatrix} \mathbf{p} \\ \mathbf{0} \end{pmatrix}. \tag{4.8}$$

We call this linear program **LP*2**. Although there is no explicit sign restriction on **v**, in the next paragraph we show that **v** has to be non-negative because the reinvestment interest rates are non-negative.

Inequality (4.8) is equivalent to the pair of matrix inequalities:

$$C\mathbf{v} \leq \mathbf{p},$$

which is the same as (3.4), and

$$R^T\mathbf{v} \leq \mathbf{0}, \tag{4.9}$$

which, in turn, is equivalent to the system of linear inequalities:

$$(1 + r_t)\, v_{t+1} \leq v_t,\ t = 1, 2, 3, \dots, \tau - 1, \tag{4.10}$$

and

$$0 \leq v_\tau. \tag{4.11}$$

Since the reinvestment interest rates $\{r_t\}$ should be non-negative, we have the monotonicity condition (3.7), $v_1 \geq v_2 \geq \dots \geq v_\tau \geq 0$. For the example at the end of Section III, with the additional condition $r_1 = 0.05$, the optimal feasible vector **v** is $\mathbf{v} = (0.8568, 0.8160)^T$, which satisfies the monotonicity condition.

For $\mathbf{v} = (v_1, v_2, \dots, v_\tau)^T$, define

$$\phi_t = (v_t/v_{t+1}) - 1,\ t = 1, 2, 3, \dots, \tau - 1. \tag{4.12}$$

Then (4.10) can be written as

$$r_t \leq \phi_t,\ t = 1, 2, 3, \dots, \tau - 1. \tag{4.13}$$

Recall that i_t denotes the t-period spot rate. For $t = 1, 2, 3, \dots,$

$$f_t = \frac{(1 + i_{t+1})^{t+1}}{(1 + i_t)^t} - 1 \tag{4.14}$$

is known as the one-period forward rate at time t ([2, p. 221], [8, p. 283], [14, p. 155]); it may be interpreted as the market forecast, at time 0, of the one-period interest rate at time t. Thus, in choosing a value for the reinvestment rate r_t, it might be prudent to ensure that

$$0 \le r_t \le f_t. \tag{4.15}$$

Note that condition (4.15) implies that vector $[(1+i_1)^{-1}, (1+i_2)^{-2}, \dots, (1+i_\tau)^{-\tau}]^T$ is a feasible vector with respect to **LP*2** and that the present value of the liability cash flows (with respect to the current term structure of interest rates) is a lower bound for the cost of the optimal investment portfolio.

The optimal value of the objective function in the example at the end of Section III is 10.81. By allowing the (positive) cash balance at time 1 to be carried forward to time 2, that is, by switching from **LP1** to **LP2**, we expect the optimal value to be lowered. Indeed, with $r_1 = 0.05$, the optimal value is lowered to 10.65. The fundamental theorem of linear programming provides an alternative explanation, because condition (4.13) in **LP*2** represents an extra set of constraints not present in **LP*1**. Now, if we impose further constraints on **LP*2**, the optimal value should come down even further. Motivated by (4.13), we might impose the extra conditions that

$$s_t \ge \phi_t,\ t = 1, 2, 3, \dots, \tau - 1, \tag{4.16}$$

where $\{s_t\}$ are constants; that is, we consider the formulation:

$$\underset{\mathbf{v}}{\text{Maximize}}\ \mathbf{l}^T\mathbf{v}$$

subject to

$$C\mathbf{v} \le \mathbf{p},$$

$$r_t \le \phi_t \le s_t,\ t = 1, 2, 3, \dots, \tau - 1, \tag{4.17}$$

and

$$0 \le v_\tau.$$

Is there an economic interpretation for s_t? Recall that r_t is a one-period reinvestment rate at time t. Hence, s_t might be conjectured as a one-period borrowing rate at time t. This turns out to be true, as we show in Section V by means of the fundamental theorem of linear programming. We note that (4.17) can be found on p. 249 of [13].

V. BORROWING ALSO ALLOWED

In Section IV, we improved the classical cash-flow matching model by allowing the carry-forward of positive cash balances. If borrowing is also allowed, that is, if negative cash balances are allowed and carried forward, the cost would be lowered further. The interest rate for borrowing should be at least as high as that for investing; otherwise, there would be arbitrage opportunities.

As in Section IV, let $C^T\mathbf{n}-\mathbf{l}=(g_1, g_2, \ldots, g_\tau)^T$. For $t=1, 2, 3, \ldots, \tau-1$, let r_t and s_t denote the one-period reinvestment and borrowing rates, respectively, at time t. (The $\{s_t\}$ will turn out to be the same as those in (4.16).) For $t=1, 2, 3, \ldots, \tau$, let b_t denote the cumulative cash balance at time t. Thus $b_1=g_1$, and for $t=1, 2, 3, \ldots, \tau-1$,

$$b_{t+1} = g_{t+1} + (1 + r_t)b_t \qquad \text{if } b_t \geq 0 \tag{5.1a}$$

and

$$b_{t+1} = g_{t+1} + (1 + s_t)b_t \qquad \text{if } b_t < 0. \tag{5.1b}$$

The decision-maker is to seek an investment portfolio $\{n_k|n_k\geq 0\}$, which minimizes the cost

$$\mathbf{p}^T\mathbf{n} = \sum_k p_k n_k$$

while subject to the condition that the final cash balance, b_τ, is non-negative. (The formulation in Section IV is a special case of the present one with $s_1=s_2= \ldots =+\infty$.)

Because the decision-maker should not have riskless arbitrage opportunities, we impose the condition that $r_t\leq s_t$ for each t. However, unless $r_t=s_t$ for all t, the mathematical program thus formulated is nonlinear. The formulation given in Section III is quite well-known; it is frequently used for the management of pension fund assets. The present formulation is a very obvious generalization, which would lower the portfolio cost. However, it is nonlinear. The paper [15] presented two methods to convert the problem to a linear one. We now demonstrate that the linearization given in Section IV of [15] is a direct consequence of the duality theory of linear programming.

We extend **LP*2.** Let S denote the τ by $(\tau-1)$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & . & . & 0 \\ -1-s_1 & 1 & 0 & . & . & 0 \\ 0 & -1-s_2 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & -1-s_{\tau-1} \end{pmatrix}. \tag{5.2}$$

Note that the inequality $\mathbf{v}^T S \le \mathbf{0}^T$ is equivalent to (4.16). Consider the following linear program, which we call **LP*3.**

$$\underset{\mathbf{v}}{\text{Maximize}} \quad \mathbf{1}^T\mathbf{v}$$

subject to

$$\begin{pmatrix} C \\ R^T \\ S^T \end{pmatrix} \mathbf{v} \le \begin{pmatrix} \mathbf{p} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{5.3}$$

The problem dual to **LP*3** is:

$$\underset{\mathbf{y} \ge \mathbf{0}}{\text{Minimize}} \ (\mathbf{p}^T \ \mathbf{0}^T \ \mathbf{0}^T)\mathbf{y} \tag{5.4}$$

subject to

$$(C^T \ R \ S)\mathbf{y} = \mathbf{1}. \tag{5.5}$$

The vector $\mathbf{y}$ has three obvious components,

$$\mathbf{y} = \begin{pmatrix} \mathbf{n} \\ \mathbf{b}^+ \\ \mathbf{b}^- \end{pmatrix}, \tag{5.6}$$

where

$$\mathbf{b}^+ = (b_1^+, b_2^+, \ldots, b_\tau^+)^T \tag{5.7}$$

and

$$\mathbf{b}^- = (b_1^-, b_2^-, \dots, b_{\tau-1}^-)^T. \tag{5.8}$$

We can rewrite (5.4) as

$$\underset{\mathbf{n} \ge \mathbf{0},\, \mathbf{b}^+ \ge \mathbf{0},\, \mathbf{b}^- \geqslant \mathbf{0}}{\text{Minimize}} \quad \mathbf{n}^T\mathbf{p} \tag{5.9}$$

Unscrambling (5.5) in terms of $\{g_t\}$, we have

$$b_1^+ - b_1^- = g_1,$$

$$b_{t+1}^+ - b_{t+1}^- = g_{t+1} + (1 + r_t)b_t^+ - (1 + s_t)b_t^-,$$

$$t = 1, 2, 3, \dots, \tau - 2, \tag{5.10}$$

and

$$b_\tau^+ = g_\tau + (1 + r_{\tau-1})b_{\tau-1}^+ - (1 + s_{\tau-1})b_{\tau-1}^-.$$

This is the desired linearization of the nonlinear problem posed at the beginning of this section. We call this linear program **LP3**.

In general, (5.10) is not equivalent to (5.1a) and (5.1b). However, because each borrowing rate is greater than the corresponding reinvestment rate, when the linear program is solved, at most one of b_t^+ and b_t^- is nonzero for each t, $t=1, 2, 3, \dots, \tau-1$. Therefore, at optimality (5.10) is equivalent to (5.1a) and (5.1b).

VI. EXAMPLE

In the formulations above, there is no restriction on the sign of the liability cash flows $\{l_t\}$; that is, we do not require $\mathbf{l} \ge \mathbf{0}$. Cash-flow matching models can be used for the rebalancing of an existing portfolio; negative "liability" cash flows may be due to currently owned assets that cannot be or are not to be traded. (For the purpose of trading, the models can further be extended with the inclusion of bid-ask prices; for a related model that explicitly allows for the transaction costs involved in the bid-ask spread, see [4] and [22].)

As an illustration of all the models presented above, consider the simple example: $\mathbf{l} = (7, -4, 6, 8, -5)^T$, $\mathbf{p} = (1, 1, 1, 1, 1)^T$ and

$$C^T = \begin{pmatrix} 1.08 & 0.085 & 0.09 & 0.0925 & 0.095 \\ 0 & 1.085 & 0.09 & 0.0925 & 0.095 \\ 0 & 0 & 1.09 & 0.0925 & 0.095 \\ 0 & 0 & 0 & 1.0925 & 0.095 \\ 0 & 0 & 0 & 0 & 1.095 \end{pmatrix}. \tag{6.1}$$

The present value of the liability cash flows under the current term structure of interest rates is

$$\mathbf{1}^T C^{-1}\mathbf{p} = 10.1501. \tag{6.2}$$

The optimal value of the objective function in **LP1** or **LP*1** is 17.6532. The optimal value of the objective function in **LP2** or **LP*2**, with $r_1 = r_2 = r_3 = r_4 = 0.05$, is 13.4954. The optimal value of the objective function in **LP3** or **LP*3**, with $s_1 = s_2 = s_3 = s_4 = 0.14$, is 10.41374, which is very close to the present value of the liability cash flows given in (6.2). For a larger example, see [7].

VII. CONCLUSION

In terms of real-world applications, the theory of linear programming is one of the most valuable advances in mathematics in this century. From a computational point of view, either the simplex algorithm or Karmarkar's new method would provide an effective technique for solving large problems. From a theoretical point of view, duality provides valuable insights into the nature of the underlying problem. By formulating the dual of a linear program, a problem can be "turned inside out" and viewed from a different perspective. In this paper, the dual formulation shows that the dual variables should be interpreted as discount factors. The ratios of the discount factors are related to the forward rates. Bounding these rates from above and below leads to a new dual formulation, which, in turn, gives rise to a much improved primal formulation for cash-flow matching.

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REFERENCES

1. BENJAMIN, S. "The Theory of Games and Its Application to Rate of Interest," *Journal of the Institute of Actuaries* 85 (1959): 373–412; Discussion 413–37.
2. BIERWAG, G.O. *Duration Analysis: Managing Interest Rate Risk.* Cambridge, Mass.: Ballinger, 1987.
3. CHRISTENSEN, P.E., AND FABOZZI, F.J. "Dedicated Bond Portfolios." In *The Handbook of Fixed Income Securities.* 2nd ed. Edited by F.J. Fabozzi and I.M. Pollack. Homewood, Ill.: Dow Jones-Irwin, 1987, 704–17.
4. EHRHARDT, M.C. "A New Linear Programming Approach to Bond Portfolio Management: A Comment," *Journal of Financial and Quantitative Analysis* 24 (1989): 533–37.

5. ELTON, E.J., AND GRUBER, M.J. *Modern Portfolio Theory and Investment Analysis*. 3rd ed. New York: Wiley, 1987.
6. FABOZZI, F.J., AND FABOZZI, T.D. *Bond Markets, Analysis and Strategies*. Englewood Cliffs, N.J.: Prentice-Hall, 1989.
7. FABOZZI, T.D., TONG, T., AND ZHU, Y. "Beyond Cash Matching." In *Fixed-Income Portfolio Strategies*. Edited by F.J. Fabozzi. Chicago: Probus, 1989, 189–212.
8. FONG, H.G., AND FABOZZI, F.J. *Fixed Income Portfolio Management*. Homewood, Ill.: Dow Jones-Irwin, 1985.
9. FRANKLIN, J. *Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed-Point Theorems*. New York: Springer, 1980.
10. GRANITO, M.R. "Managing Bonds with Dedicated Portfolios." In *The Revolution in Techniques in Managing Bond Portfolios*. Edited by D.L. Tuttle. Charlottesville, Va.: The Institute of Chartered Financial Analysts, 1983, 66–76.
11. GRANITO, M.R. *Bond Portfolio Immunization*. Lexington, Mass.: D.C. Heath, 1984.
12. HILLIER, F.S., AND LIEBERMAN, G.J. *Introduction to Operations Research*. 4th ed. Oakland, Calif.: Holden-Day, 1986.
13. HODGES, S.D., AND SCHAEFER, S.M. "A Model for Bond Portfolio Improvement," *Journal of Financial and Quantitative Analysis* 12 (1977): 243–60.
14. JARROW, R.A. *Finance Theory*. Englewood Cliffs, N.J.: Prentice-Hall, 1988.
15. KOCHERLAKOTA, R., ROSENBLOOM, E.S., AND SHIU, E.S.W. "Algorithms for Cash-Flow Matching," *TSA* XL, Part 1 (1988): 477–84.
16. LEIBOWITZ, M.L. *Matched-Funding Techniques: The Dedicated Bond Portfolio in Pension Funds*. New York: Salomon Brothers Inc., February 1985.
17. LEIBOWITZ, M.L. "The Dedicated Bond Portfolio in Pension Funds—Part I: Motivations and Basics," *Financial Analysts Journal* 42 (January-February 1986): 68–75.
18. LEIBOWITZ, M.L., AND WEINBERGER, A. *Optimal Cash Flow Matching: Minimum Risk Bond Portfolios for Fulfilling Prescribed Schedules of Liabilities*. New York: Salomon Brothers Inc., August 1981. Reprinted as Society of Actuaries Study Notes 8-203-85 and 220-21-85.
19. MALONEY, K.J., AND LOGUE, D.E. "Neglected Complexities in Structured Bond Portfolios," *Journal of Portfolio Management* (Winter 1989): 59–68.
20. MILGROM, P.R. "Measuring the Interest Rate Risk," *TSA* XXXVII (1985): 241–257; Discussion, 259–302.
21. PRISMAN, E.Z. "A Unified Approach to Term Structure Estimation: A Methodology for Estimating the Term Structure in a Market with Frictions," *Journal of Financial and Quantitative Analysis* 25 (1990): 127–42.
22. RONN, E.I. "A New Linear Programming Approach to Bond Portfolio Management," *Journal of Financial and Quantitative Analysis* 22 (1987): 439–66.
23. ROSENBLOOM, E.S., AND SHIU, E.S.W. "The Matching of Assets with Liabilities by Goal Programming," *Managerial Finance* 16, no. 1 (1990): 23–6.
24. SCHAEFER, S.M. "Tax-Induced Clientele Effects in the Market for British Government Securities," *Journal of Financial Economics* 10 (1982): 121–59.