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INTEREST RATE FUTURES: AN ALTERNATIVE TO TRADITIONAL IMMUNIZATION IN THE FINANCIAL MANAGEMENT OF GUARANTEED INVESTMENT CONTRACTS

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ABSTRACT

As discussed in this paper, guaranteed investment contracts (GICs) include those agreements that are characterized by a stream of deposits and payouts, the timing and amount of which are known at the inception of the contract. The most common are the compound GIC, similar to a zero-coupon bond, and the simple-interest GIC, with interest payouts occurring annually. The analysis can be extended to recurring deposit GICs and GICs with benefit payments made at book value. The accuracy of the model in these cases necessarily depends upon the accuracy of the estimated stream of deposits and/or benefit payments. This definition does not include the Deposit Administration or immediate participation guarantee (IPG) types of contracts for which crediting rates are determined by an investment generation approach and, thus, change periodically during the unspecified term of the contract.

Financial management encompasses the pricing and ongoing management of assets backing a GIC portfolio. Immunization and dedication, its offspring, have been the traditional approaches used in financial management. Using long-term assets to back short-term liabilities, and thereby assuming the interest rate risk, is also a common strategy. Using interest rate futures in pricing and in keeping the asset portfolio duration in balance is just one more variation of traditional immunization.

This paper assumes that the reader understands immunization theory and the concept of duration. Immunization is used here in the broad sense of duration matching an ongoing block of business rather than a closed block with a fixed time horizon.

I. INTRODUCTION

Some of the confusion concerning the all important concept of duration is addressed in section II. Interest rate futures, particularly treasury bond futures, are explained in section III. The price sensitivity, or duration, of a futures position and the concept of convergence are the most important points to understand when futures are used for immunization purposes. Section III explains the bond futures market more broadly, including the price determination and the delivery process, for those with little previous exposure to futures.

Section IV shows the mechanics of using interest rate futures in conjunction with long-term utility bonds in the pricing and ongoing asset-liability management of GICs. The pricing model takes the form of a "fund analysis." Assets backing a hypothetical GIC, including the futures positions, are traced and accumulated over the GIC term with direct futures–related expenses deducted along the way. After the net asset accumulation at the GIC maturity date is converted into a net effective yield, the direct hedging costs can be quantified by comparing the gross and net yields on assets.

Although the pricing model assumes assets allocated among individual GIC liabilities, this serves no worthwhile purpose when managing a block of assets and liabilities. Interest rate futures are used to keep aggregate asset and liability durations equal by means of a "portfolio hedge." Periodic adjustments are made to the hedge to maintain a duration match over time. This replaces the rebalancing aspect of traditional immunization.

Sections V and VI compare traditional immunization and the interest rate futures approach, outlining the relative advantages and risks involved with each method as they apply to the financial management of GICs. The appendix shows a numerical example of the pricing model outlined in section IV for the general case. This is expected to give the reader a more concrete and understandable demonstration of the process.

II. DURATION DEFINED

While this paper assumes that the reader understands immunization, there is considerable potential for confusion concerning duration—the key concept of immunization theory. There are two commonly used measures labeled duration—Macaulay duration and modified duration. Although they are similar, their slight differences lead to different verbal interpretations. The derivation of duration might provide some insight into these differences and how they arose. It probably is easier to conceptualize duration in terms of an asset, a bond or mortgage loan, for example, but the following development would be just as applicable for a liability (GIC).

Let $P(i^{(n)})$ be the price or present value of an asset or portfolio of assets yielding $i^{(n)}$, compounded $n$thly. Then

$$P(i^{(n)}) = \sum_{t=1}^{mn} B_t \, (1 + i^{(n)}/n)^{-t} \tag{1}$$

where B_t is the cash flow from the asset to be received in t/n years. The

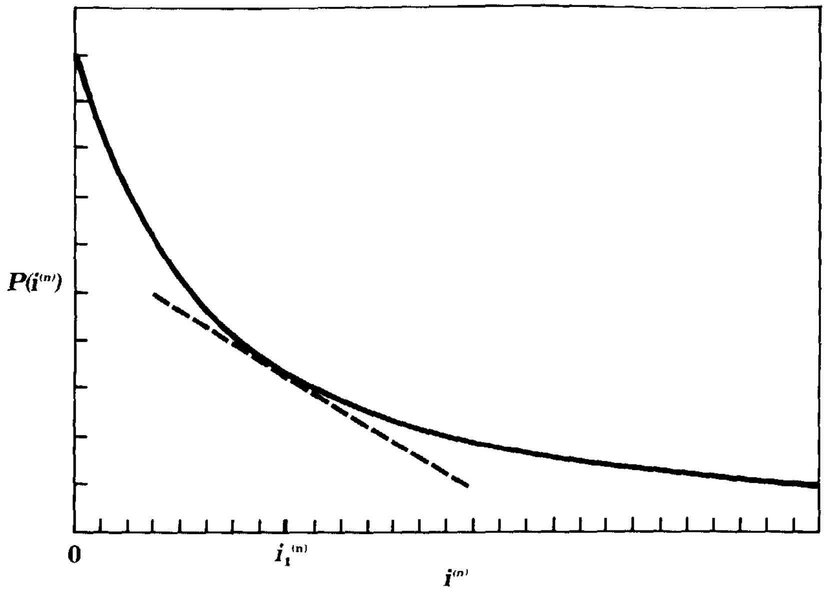


FIGURE 1
Price Curve

asset matures in m years. Equation (1) can be illustrated graphically as in figure 1.

Differentiating $P(i^{(n)})$ with respect to $i^{(n)}$, we have

$$\frac{dP}{di^{(n)}} = -\sum (t/n)\, B_t \cdot \left(1+\frac{i^{(n)}}{n}\right)^{-t-1}. \tag{2}$$

Equation (2) evaluated at some point, $i_1^{(n)}$, will equal the slope of the line tangent to the price curve at that point, as illustrated in figure 1. Equation (2) is, in essence, the price elasticity or sensitivity. This tangent line can be used to estimate changes in price resulting from a change in $i^{(n)}$. Likewise, the quantity $\frac{1}{P}\frac{dP}{di^{(n)}}$ can be interpreted as a proportional measure of price sensitivity. This is "modified duration," after removing the negative sign. If modified duration were equal to 5 for an asset or portfolio of assets, then a 1 percent increase in $i^{(n)}$ would result in a 5 percent reduction in price, using the tangent line at $i_1^{(n)}$ to estimate the change in price. This estimate will be less accurate for large changes in $i^{(n)}$. Modified duration can be calculated using any mode of compounding, n. Whichever mode is chosen, the resulting modified duration must be applied to changes in yield of the same mode to estimate price changes. Convention in the securities world sets n at 2.

If $\frac{-1}{P}\frac{dP}{di^{(n)}}$ is multiplied by the quantity $\left(1+\frac{i^{(n)}}{n}\right)$, the result is

$$-\frac{1}{P}\frac{dP}{di^{(n)}}\left(1+\frac{i^{(n)}}{n}\right) = \frac{\Sigma\, t/n\, B_t\left(1+\frac{i^{(n)}}{n}\right)^{-t}}{\Sigma B_t\left(1+\frac{i^{(n)}}{n}\right)^{-t}}. \tag{3}$$

If examined carefully, this expression is seen to be the present-value-weighted average life. This resulting formula is a modification to modified duration in this development, but it is commonly referred to as "Macaulay" duration.[1] It seems to be used more often in the insurance world, which measures liabilities as well as assets.

Macaulay duration can be derived directly by looking at the textbook definition of elasticity. In this case, the price elasticity with respect to $R = 1 + i$ (assuming $n = 1$) is $(dP/P) \div (dR/R) = (dP/dR) \times (R/P)$. This can be interpreted as a first order estimate of the proportional price change resulting from a proportional (1 percent) change in R. So the difference between Macaulay and modified duration is that the former measures price sensitivity to proportional changes in $(1 + i)$ while the latter deals with absolute changes in i(or $1 + i$). Despite the fact that interest rate changes are almost never measured as proportions, Macaulay duration has gained a large following. This might be explained in part by the weighted average life interpretation it has. It is easier to conceptualize than price sensitivity in the context of insurance company liabilities. It is also intuitively appealing in the sense that a zero-coupon bond or compound GIC has a Macaulay duration equal to the time remaining until maturity. This is not the case with modified duration.

It will be seen in section IV that it makes little difference which of the measures is used as long as the measure chosen is used throughout. They differ only by a factor of $(1 + i^{(n)}/n)$, and, since the key hedging calculations involve ratios of durations, most of that difference will cancel out in those ratios. In an actuarial journal, there is no need to avoid the concept of price sensitivity of assets and liabilities or to manipulate the modified duration

[1]The concept of duration was first introduced by Frederick Macaulay in his book, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1856*, (New York: National Bureau of Economic Research, 1938).

measure for any other reason. A measure of price sensitivity of assets and liabilities as illustrated in figure 1 is the sought after quantity. Its most direct measure is unadjusted modified duration. Following convention, modified duration is used throughout the remainder of the paper with a compounding mode of 2.

III. TREASURY BOND FUTURES

A treasury bond futures contract is an agreement whereby a seller (short) promises to deliver $100,000 of par value in treasury bonds to a buyer (long) at some predetermined date in the future. Contract delivery dates are set at 3-month intervals up to two and a half years into the future. The bonds finally delivered must not be callable for at least fifteen years from the delivery date. In practice, one of the more recently issued 30-year treasury bonds is usually delivered. The futures positions are "marked to market" and settled daily, similar to other commodity futures contracts.

It is important to note that most users of interest rate futures never actually intend to take or make delivery of the bonds. The ability to participate in the price volatility of the bonds at a nominal cost while the contract is outstanding, whether used for hedging or speculation, is often more important than eventual bond ownership. This is the case when using futures to immunize GICs. Prior to delivery, the futures position will be closed out, or reversed, by taking an opposite position in the market and replacing it with a more distant futures contract. Some traders actually do take delivery, but rolling back to more distant contracts is a means of avoiding the complicated delivery process.

Table 1 gives the closing treasury bond futures prices on December 2, 1982 at the Chicago Board of Trade. The December 1982 contracts are the "nearby" contracts in this example. They must be delivered within the month. The prices are given in points ($1,000) and 32nds ($31.25). Open interest represents the number of $100,000 contracts outstanding. By buying or selling short one March 1983 contract, an investor acquires the volatility of $75,000 in market value worth of long-term treasury bonds. If the price on that contract rose sixteen 32nds to 75-16 the following day, the short would be required to pay $500 to the clearinghouse corporation which, in turn, would pay the long. Thus, the long position benefits from a drop in yields and the resulting rise in price just as the holder of a bond would. The short, on the other hand, benefits from rising interest rates. For users of interest rate futures with no intention of taking or making delivery, the price is not really price as such. It is merely a point of reference for the determination of all futures gains and losses. The direct cost of using futures

TABLE I

TREASURY BOND FUTURES PRICES
December 2, 1982
$100,000 Par

Delivery	Price (Points-32nds)	Open Interest
December 1982	75-22	29,798
March 1983	75-00	55,444
June	74-20	17,097
September	74-11	18,042
December	74-04	22,185
March 1984	73-31	19,852
June	73-28	7,901
September	73-26	2,193
December	73-25	2,690
March 1985	73-24	380
June	73-23	104

depends on commissions, initial margin requirements, and convergence. These will be discussed in detail later in this section and in section IV.

Any treasury bond meeting the fifteen-year call restriction can be delivered by the short seller. Whichever bond is chosen, $100,000 in face amount must be delivered. The amount the buyer must pay the short is calculated by multiplying the conversion factor for the particular bond delivered by the futures price at delivery. The determination of this conversion factor is straightforward. Suppose that in December of 1982 the 10⅜'s of 2012 T-bond were priced to yield 8 percent. The resulting price would be about $125.40. Dividing by 100 yields the conversion factor of 1.254 for the 10⅜'s of 2012 delivered in December of 1982. The actual price of the 10⅜'s of 2012 in early December was around $97. Similarly, the December 1982 conversion factor for the 14's of 2011 was 1.633, and its price was around $124. If the price of the futures contract then being delivered, the December contract at 75-22, is multiplied by either conversion factor, the resulting formula price is close to the actual market price of the respective bond. This formula price plus accrued interest is the amount the long must pay the short when delivery is made.

All deliverable bonds have a conversion factor for each delivery month and the approximate relationship between the nearby futures price, conversion factor, and market price holds. But the relationship is approximate and the formula price may be slightly higher or lower than the actual market price for any particular bond. The difference between the formula price (futures price times conversion factor) and the actual market price will determine whether a particular bond is "cheapest to deliver," that is, whether it is advantageous for the seller, who has to buy the bond in the market, to deliver that bond or some other. The bond for which the formula price exceeds the market price by the largest amount is generally cheapest. Arbitrage tends to keep the differences fairly small. Supply also plays a role and helps explain why the recent issues are generally cheapest. In any event, the nearby futures price is approximately proportional, by way of the conversion factor, to the market price of the cheapest-to-deliver T-bond.

For the purposes of immunization, understanding the price determination of futures is necessary only in order to understand the price movements or relative price volatility of futures, that is, the duration of futures. Since the price of the futures contract near delivery is approximately proportional to the price of the cheapest-to-deliver T-bond, the price movements also will be proportional. In summary, the duration of a futures contract currently being delivered (nearby) is equal to the duration of the cheapest-to-deliver T-bond.

Knowledge of the cheapest-to-deliver T-bond often is not critical since most long T-bonds that are likely candidates for delivery have very similar durations. With the duration of futures now established, it should be remembered that the short futures position realizes gains and losses opposite to those of the long position but in the same relative magnitude. Since the performance of a short position is directly related to interest rate increases in this way, the duration of a short position in the nearby contract is the negative of the long futures duration, or T-bond duration.

The price of the futures contract currently being delivered, or the nearby contract, is approximately proportional to long treasury bond prices by way of the conversion factor and, thus, a function of T-bond yields. However, this is not entirely true for the more distant contracts, as can be seen in table 1. The price of the distant contracts happens to be a function of the shape of the treasury yield curve as well as prevailing T-bond yields. On December 2, 1982, the yield curve sloped upward. A buyer taking a long position in a distant contract, like that of December 1983, would be required to invest any cash set aside for the future purchase of T-bonds in a lower yielding short-term investment for the year while the contract was outstanding. To compensate the long for this lower yield, the price of the distant contract is lower. The appreciation in price due to "convergence" as the contract nears delivery will enable him to earn the long-term yield prevailing when the contract was established. In other words, as delivery approaches, the futures price on a given contract will increase even if long-term yields do not change. This increase in price will make up for the difference between short-term and long-term yields while the contract is outstanding. This difference, which convergence offsets, is often called "cost of carry." While the long position benefits, the short position incurs losses from convergence when the yield curve is sloping upward.

Figure 2a shows a positive yield curve. Figure 2b shows the corresponding futures price curve and the convergence pattern. The price pattern shown in

FIGURE 2a
Positive-Slope Yield Curve

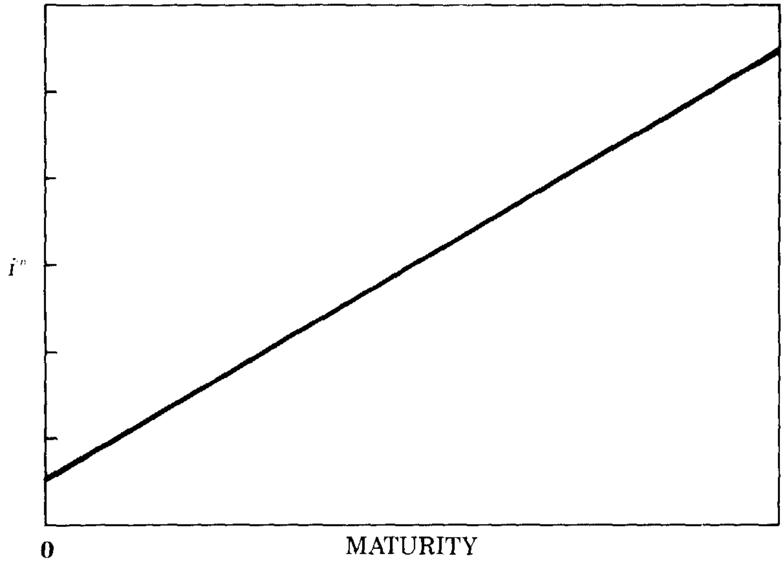


FIGURE 2b
Futures Price Curve and Convergence

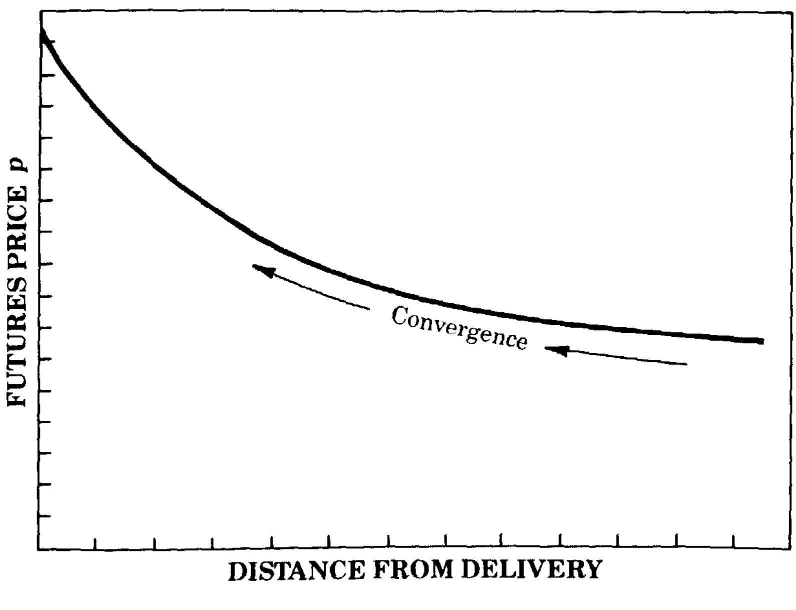


FIGURE 3a
Negative-Slope Yield Curve

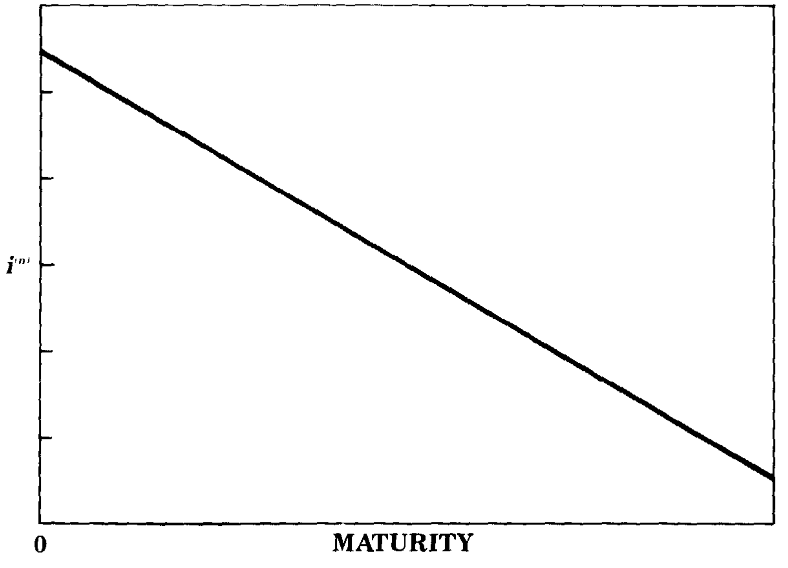


FIGURE 3b
Futures Price Curve and Convergence

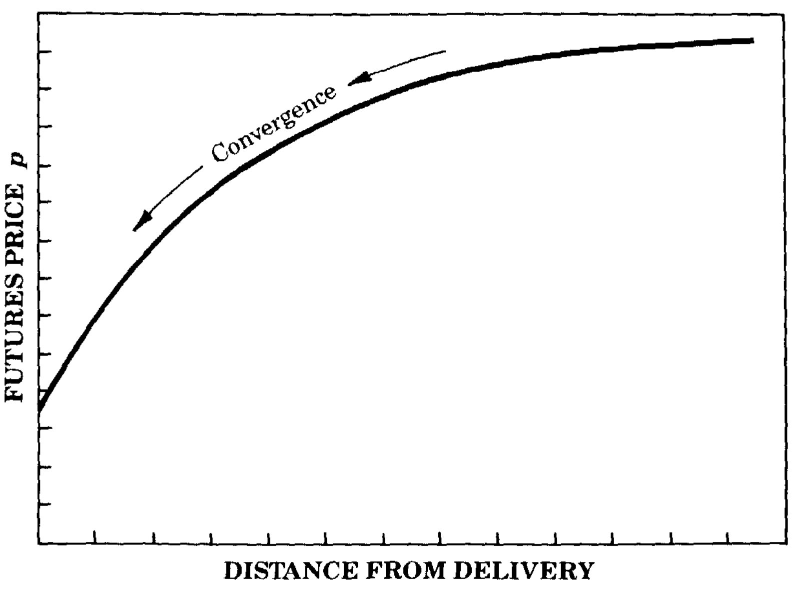


table 1 has a shape similar to figure 2b. If the yield curve were negatively sloped, the futures price curve would also invert. In this case, short-term yields exceed long-term yields. Due to convergence, the futures price reductions would offset the high yield on cash invested by the long while the contract was outstanding to produce a lower long-term yield over the entire investment horizon. The short would benefit from convergence here. The "cost of carry" is negative in this case. This is illustrated in figures 3a and 3b.

As can be seen in figures 2b and 3b, the rate of convergence decreases the more distant the delivery. This is due to the exponential nature of the discount function. Thus, convergence is usually slower at the more distant contracts. The key point is that even if treasury yields remain unchanged, futures prices will increase due to convergence when the yield curve is positively sloped.

In attempting to calculate the duration, or price sensitivity, of distant futures contracts, it must be remembered that duration is a measure of proportional price sensitivity at a point in time. If yield changes are assumed to occur uniformly along the yield curve, then these parallel yield curve shifts will result in parallel futures price shifts as shown in figures 4a and 4b.

While the absolute price change will be about the same for all contracts,

Figure 4a
Parallel Yield Curve Shift

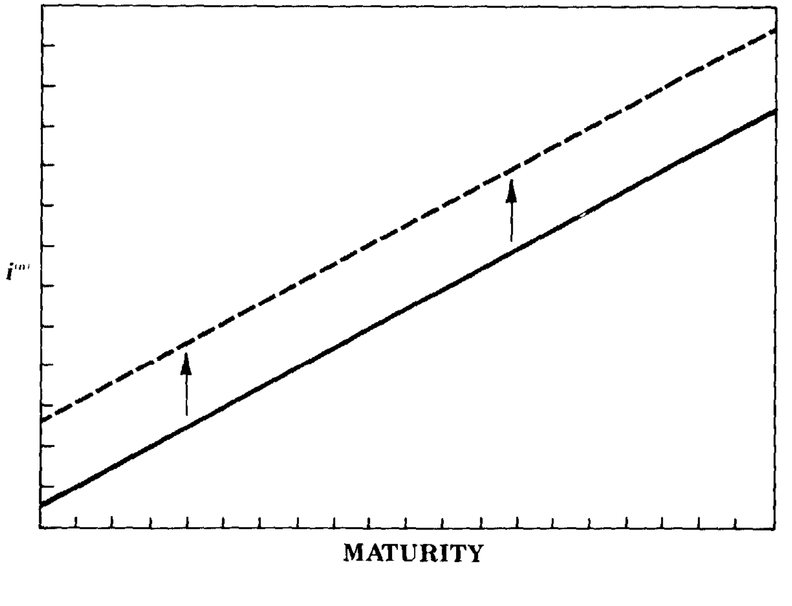
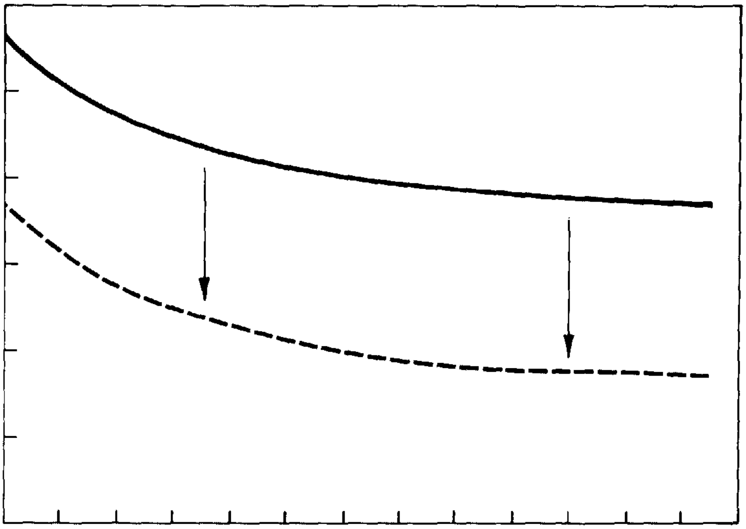


FIGURE 4b
Futures Price Horizontal—Distance from Delivery



the relative price change, that is duration, will be greater for the lower priced, more distant contracts. Specifically, for a given set of futures prices by delivery date (a futures price curve), the duration of any one of the contracts is inversely proportional to the price of that contract. A useful corollary is that the product of the price and duration is the same for all contracts on a given futures price curve. This is convenient when weighting duration by price for all portfolio investments as is done in section IV. The duration of the nearby contract, being that of the long T-bond, is easiest to calculate. The risks associated with the parallel-yield-curve-shift assumption are discussed in section V.

The direct costs of hedging with futures include the initial margin, commission costs, and convergence, which can be positive or negative depending on the shape of the yield curve. The initial margin requirement is about \$2,000 per contract and can be in the form of treasury bills. This is more or less a demonstration of good faith required by the clearinghouse corporation, which must collect and distribute the daily "variation margin" from all parties involved. Commissions are approximately \$30 per round-turn transaction (buy-sell or sell-buy). Convergence depends on the shape of the yield curve. Thus, the price volatility of about \$75,000 worth of T-bonds (market value, 12/2/82) can be acquired with an initial outlay of a small commission and a small interest-bearing security deposit. The ability to acquire substantial interest rate volatility at minimal cost makes the use of futures a potential alternative to traditional immunization in the financial management of GICs.

Two important points in this section are worth remembering as they relate to using futures in GIC financial management. The first is that nearby T-bond futures sold short have a duration equal to the negative of the long T-bond duration. The second is the concept of convergence and the fact that it is a cost to the short position when the yield curve is sloping upward, although this can be alleviated somewhat by shorting longer term, more slowly converging contracts. This approach does present a greater risk of yield curve shifts, which will be discussed later.

IV. PRICING AND PORTFOLIO MANAGEMENT USING FUTURES

The subject of treasury bond futures (section III) is covered in countless publications if a more detailed and accurate description is desired. Section III should provide sufficient background to make the application of futures in GIC financial management, as outlined in this section, understandable.

Establishing a quantifiable relationship between the yield on the hedging instrument (T-bonds) and the yield on the hedged assets as yield levels change is critical any time a crosshedge is employed. Regression analysis often is used to quantify such a relationship. This approach has the advantage of being an objective method of forecasting corporate treasury yield spreads. But while spreads historically have shown a definite correlation with yield levels, results from regression analysis can be far off the mark in predicting the sensitivity of spreads to changes in rates over a given time period, especially during short time periods experiencing large yield changes. This "basis risk" is the major risk encountered when using a crosshedge. It is no doubt responsible for most of the benefits anticipated when such a strategy is used. The success of such an immunization strategy with futures is dependent on the behavior of yield spreads while the hedge is employed.

This section relies on the assumption that, over the long run, there exists a relationship in the form

$$\Delta i_u = \beta(\Delta i_t) \tag{4}$$

where the Δi_t are the first differences of periodically measured long T-bond yield and the Δi_u are the first differences of the long-term Baa-BBB utility yields which are representative insurance company investments in this analysis.

In this equation, β measures the relative sensitivity of utility bond yields to changes in treasury bond yields. Since duration measures the price sensitivity of each security to the respective yield changes, the treasury bond duration divided by β produces a measure of treasury bond price sensitivity to changes in utility bond yields.[2] Assuming, for now, the existence of the relationship specified in equation (4), let us examine why the futures approach might compare favorably to traditional immunization in most interest rate environments.

The basis for the model to be outlined involves the use of interest rate futures sold short to hedge long-term investments an insurance company might make, utility bonds, for example. The excess price volatility, or long duration, of the utility bond is offset by the gains and losses on futures in the opposite direction, to match the price volatility of the GIC liability. It is presumed that the duration of the cash assets, utility bonds in this case, is longer than the duration of the GIC. But why might this approach be more advantageous than the traditional method of buying the cash asset with the appropriate duration and avoiding the complexities of futures?

Figure 5 shows representative yield curves of both treasury and utility bonds. The horizontal axis represents duration rather than maturity as is customary. This has little effect on the shape of the curves but makes the following explanation easier to understand.

The notation, ${}_{\alpha}i_X$ represents the yield of an α-duration bond of quality X. The letters u and t in figure 5 stand for utility and treasury, respectively. Notice that the BBB curve is steeper than the treasury curve. This is generally true due to the higher default and liquidity premiums charged on long duration, lower quality bonds. Interest rate futures make this premium available on the shorter "synthetic" asset, composed of the cash asset and futures, subject to the additional risks that will be discussed in section V. These include the extra default and liquidity risk assumed and, most importantly, the basis risk. In essence, futures allow the users to rotate the BBB yield curve clockwise as indicated in figure 5.

To understand this rotation of the BBB yield curve, suppose an investor wanted to construct a treasury bond of duration c out of a d-duration T-bond. It could be done easily by selling short the appropriate number of T-bond futures contracts. Due to the efficiency of the market, however, this would result in little or no advantage. The costs of hedging—convergence, commission, and lost interest on margin—would be at least equal to the excess yield on longer term bonds, ${}_{d}i_t - {}_{c}i_t = m$ in figure 5. The investor would likely earn at least as high a yield by buying the shorter T-bond in the first place. But if the investor were willing to move down in quality to

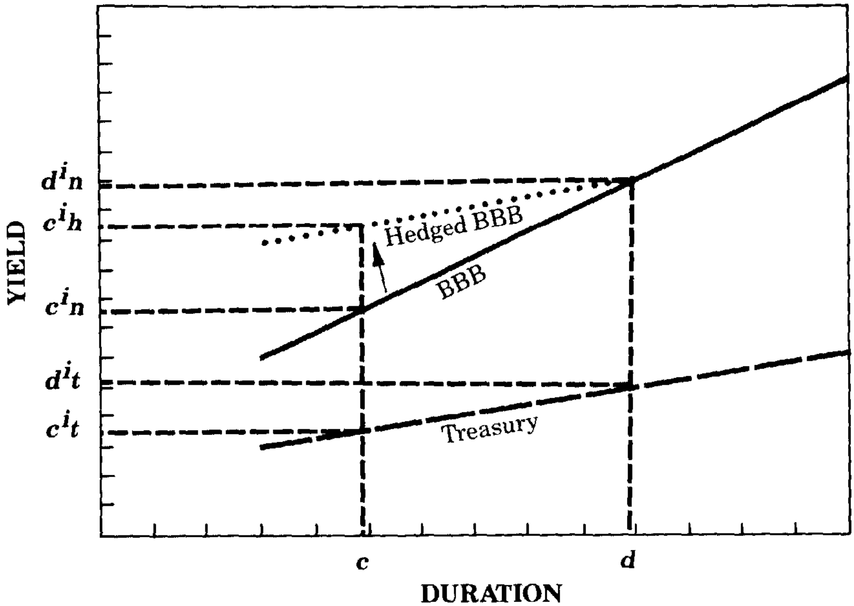
[2] If $P(i_t)$ and $P(i_u)$ are the price curves of long treasury and utility bonds, respectively, then

$$\Delta i_u = \beta(\Delta i_t) \text{ implies } di_u = \beta di_t.$$

Therefore

$$\frac{1}{\beta} \times \frac{1}{P(i_t)} \times \frac{dP(i_t)}{di_t} = \frac{1}{P(i_t)} \frac{dP(i_t)}{di_u}.$$

FIGURE 5
Yield Curves



BBB bonds, and if he or she had confidence in the relationship between the two types of bonds as in equation (4), then a higher yielding synthetic c-duration bond could be constructed out of a d-duration BBB bond and short treasury bond futures. In general, such an approach will allow the clockwise rotation of the yield curve if

$$ {}_d i_u - {}_c i_u > \beta m. \tag{5} $$

Here βm is the expected cost of hedging based on the current treasury yield, curve. The difference in inequality (5) must cover at least the extra default, liquidity, and basis risks mentioned earlier. Any excess could be used for competitive advantage or surplus augmentation.

A special case of inequality (5) is when $\beta = 1$. In this case, long BBB utility yields move basis point for basis point with long treasury yields, i.e., the spread is constant. On the other hand, the investor might believe that, in the long run, narrowing and widening spreads and the resulting gains and losses will balance out over time. Whatever the rationalization, inequality (5) becomes

$$ {}_d i_u - {}_c i_u > m. \tag{6} $$

in such a case.

Since the BBB yield curve almost always has a steeper positive slope than the treasury curve, the rotation to the BBB hedged yield curve in figure 5 will be possible to some extent, subject to the hedging risks. An estimate of the yield in the case where $\beta = 1$ is ${}_c i_h$ where

$${}_d i_u - {}_c i_h = {}_d i_t - {}_c i_t = m.$$

Again, this is prior to any default, liquidity, or basis risk deductions. It should be pointed out that the treasury yield curve in figure 5 technically should be the yield curve implied by the T-bond futures price curve. This usually will be similar to the cash market treasury yield curve, but there will be differences.

The GIC pricing illustration will be shown for the general case. In the appendix, actual numbers are used to provide an illustration of how the method might work in practice. In both cases, expenses and risk charges other than direct futures hedging costs will be ignored. In practice, of course, other expenses probably would be deducted from the asset accumulation at least once a year. The model will take the form of a fund analysis with quarterly deductions for hedging costs. When the GIC matures, the hedging costs can be quantified in basis points by comparing the gross and net yields on assets.

Assume the following definitions applied at the time of a particular new GIC deposit. The goal is to determine the appropriate crediting rate on this money that will allow the insurer to recoup all hedging costs.

GIC	– α years compound; duration c_0; A_0 dollars.	
30-year BBB utility bonds	– duration b_0; nominal yield equals $${}_{b_0} i_u = {}_{b_j} i_u = i_u,\ 0 \leq j \leq \alpha.$$	
Long treasury bonds	– duration e_0; nominal yield equals ${}_{e_0} i_t = {}_{e_j} i_t = i_t,\ 0 \leq j \leq \alpha.$	
Futures	– *Convergence*	– k points per quarter (eg. 5/32).
	Price (proportion of par)	– $p_0 = p_j = p,\ 0 \leq j \leq \alpha.$
	Commission	– h dollars per round-turn
	Initial margin	– r dollars per contract in treasury bills with duration g_0 and yielding ${}_{g_0} i_n = {}_{g_j} i_n = i_n,\ 0 \leq j \leq \alpha.$

Reinvestment	– Immediate reinvestment of coupons and futures gains in the same BBB utility bonds. Futures losses paid by bond liquidation.
β	– as in equation (4).

Notice the t subscript refers to treasury while the j refers to time.

In this model, duration is modified duration rather than Macaulay duration. This explains why c_0 does not equal α. Convergence cost is dictated by the distance from delivery of the futures contracts used. Limited volume at extremely distant contracts tends to preclude their use. It is assumed for now that the most distant contracts with sufficient volume, whatever that may be, are used in order to minimize convergence. The choice of the particular contract used is discussed in section V.

As is evident from the definitions, there is an implicit assumption that yield levels remain unchanged throughout the term of the GIC. This produces a futures price pattern that is identical from quarter to quarter and, thus, a constant value for p. The level rate assumption was made merely for simplicity. Indeed, interest rate changes would have had little effect on hedging costs as long as convergence, that is, the shape of the treasury yield curve, remained the same. This is similar to the yield curve assumptions made in traditional immunization. Dynamic interest rates could be incorporated to verify that the portfolio is immunized. (See the end of the appendix.) This section's main concern will be the determination of hedging costs. This is also the typical approach to determine the cost of rebalancing, or bouncing down the yield curve when traditional immunization is being used.

The big leap of faith, aside from equation (1), is not the level yield assumption but the assumption that the yield curve retains its shape throughout the term of the GIC. This produces constant convergence costs of k per quarter, assuming futures contracts are rolled back each quarter. Different scenarios involving a shifting yield curve will generate varying hedging costs—not unlike traditional immunization. Discussion of all of these risks and a comparison of the futures approach versus traditional immunization also is given in section V.

Proceeding with the GIC analysis, if d_j is the duration of cash assets at time j then

$$d_0 = \frac{b_0(A_0 - rF_0) + rF_0 g_0}{A_0} \tag{7}$$

where F_j is the number of \$100,000 futures contracts outstanding at time j. Equation (7) uses the linear properties of duration to account for the effect the margin requirement has on the asset portfolio duration.

Using the fact that nearby short futures contracts have a duration equal to the negative of the long treasury bond duration and incorporating futures into the asset portfolio, we see that

$$A_0 d_0 + 100{,}000\, pF_0(-e_0)/\beta = A_0 c_0. \tag{8}$$

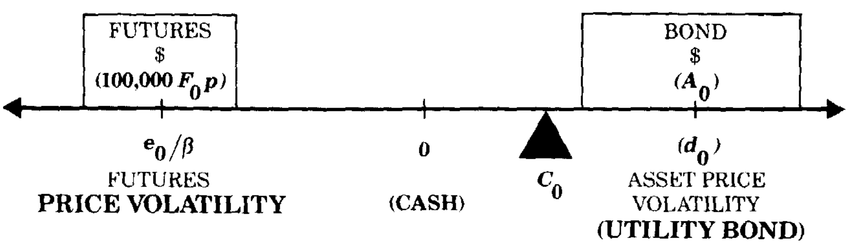
Equation (8) shows how futures weighted by their duration are used to shorten the asset duration from d_0 to c_0, the duration of liabilities. It can also be seen that the price of the nearby futures contract, p, is used rather than the price of the more distant contract actually sold short. This is because the duration of the nearby contract already is known to be the negative of the long T-bond duration, e_0. Since the product of price and duration is the same for any contract on a given futures price curve, as discussed in section III, the most convenient combination might as well be used. Recalling an earlier discussion, using β as a divisor puts the duration of T-bonds, e_0, on the same basis as d_0, reflecting price sensitivity to changes in utility bond yields.

Rearranging terms to solve for F_0,

$$F_0 = \frac{(d_0 - c_0)\, A_0\beta}{100{,}000\, e_0\, p} \tag{9}$$

Thus, F_0 is the number of contracts required to immunize the c_0 duration GIC at $j = 0$. Equations (8) and (9) can also be illustrated with the "teeter-totter" type figure 6.

FIGURE 6
Asset-Liability Balance



The required weighting of $-e_0/\beta$ to bring the assets into balance is 100,000 $F_0\, p$. Substituting equation (7) for d_0 in equation (9) and solving again for F_0 we have

$$rF_0 = \frac{(b_0 - c_0)\, A_0\beta}{100{,}000\, pe_0 + r(b_0 - g_0)\,\beta}\,. \tag{10}$$

For simplicity, it is assumed that commission costs and convergence are incurred at the end of each quarter and deducted at that time. Commission, therefore, will be $F_0\,h$ at the end of the first quarter. Convergence will be $F_0\,k$. Interest lost on the low yielding margin will be

$$rF_0\left[\left(1+\frac{i_u}{2}\right)^{.5}-\left(1+\frac{i_n}{2}\right)^{.5}\right].$$

We are now ready to begin moving through the fund progression. After one quarter, the asset accumulation net of hedging costs is

$$A_{.25}=A_0\left(1+\frac{i_u}{2}\right)^{.5}-F_0\left\{h+k+r\left[\left(1+\frac{i_u}{2}\right)^{.5}-\left(1+\frac{i_n}{2}\right)^{.5}\right]\right\} \quad (11)$$

where A_j is the net asset accumulation at the end of j years. Likewise, the hedge calculation is

$$F_{.25}=\frac{(b_{.25}-c_{.25})\,A_{.25}\,\beta}{100{,}000\,e_{.25}\,p+r(b_{.25}-g_{.25})\,\beta} \quad (12)$$

similar to equation (10). The liability duration c_j, will shorten quickly as maturity approaches. The asset durations (utility, T-bonds, bills) will change little over the course of the GIC due to the long maturities and reinvestment assumptions made. It is assumed that the futures contracts are "rolled back" three months every quarter to maintain a constant time to delivery. This is done to keep convergence from exceeding k. It also has the effect of incurring commission expense on all contracts, not just the new ones. It is assumed that this tradeoff favors the lower convergence, higher commission alternative.

Continuing the process for each quarter during the term of the GIC, the progression of assets can be determined by applying the following equations in succession:

$$F_j=\frac{(b_j-c_j)\,A_j\,\beta}{100{,}000\,e_j\,p+r(b_j-g_j)\,\beta} \quad (13)$$

and

$$A_{j+.25}=A_j\left(1+\frac{i_u}{2}\right)^{.5}-F_j\left\{\left(h+k+r\left[\left(1+\frac{i_u}{2}\right)^{.5}-\left(1+\frac{i_n}{2}\right)^{.5}\right]\right\}. \quad (14)$$

Continuing the process until $j = \alpha$, the value of A_α is finally determined after 4α recursions. The yield on assets net of direct hedging costs is then

$$(A_\alpha/A_0)^{1/\alpha} - 1 = {}_\alpha i_{\text{net}}.$$

The hedge cost expressed as a percent is $q = i_u - {}_\alpha i_{\text{net}}$.

Expenses related to administration, profit, investments, and so on, could easily have been incorporated into equation (14). There would likely be additional deductions for default and liquidity. These are risks with which insurance companies historically have been successful, so the charge probably would reflect that. A charge for basis risk is much more difficult to quantify. Also, the assumption of immediate reinvestment of futures gains and bond liquidations to pay for losses would be very cumbersome, if not impossible, in practice. A short-term cash balance likely would be necessary to handle variation margin without actual bond liquidation. There would be a cost of carry associated with such a balance.

In the ongoing management of assets, the portfolio hedge calculation is almost identical to equation (14). Since there is no apparent need to match assets and liabilities on an individual contract basis, the hedge calculation is done for the aggregate portfolios of assets and liabilities. All durations, yields, and asset market values used in equation (14) would be the values attributable to the portfolio as a whole. New GIC sales or maturities would have a marginal effect on all of these values.

In practice, the required hedge would be recalculated each time a GIC is sold or matures. In any event, the hedge computation should be done periodically to reflect the effects of changing yields, coupon payments, and the passage of time. All of these affect the durations and values of the assets and liabilities used in the hedge computation. Weekly and even daily readjustment might be made. This periodic adjustment of the hedge is analogous to rebalancing in traditional immunization, but much easier to carry out since no asset has to be liquidated.

The flexibility to manipulate the duration of the asset portfolio easily also can be used in conjunction with more aggressive investment strategies. This would be classified as active management rather than immunization. The futures hedge would be adjusted and possibly removed altogether, at times, to shorten or lengthen the asset duration to reflect the portfolio manager's outlook for interest rates. The interest rate risk involved in this deliberate mismatching could be quantified by examining the difference between asset and liability durations. Some basis risk also would be present unless the hedge were removed altogether.

Expectations about yield curve shifts can also be incorporated into the pricing of GICs by examining the effect on convergence costs. The assumption in the pricing model was that the yield curve retained its shape throughout the GIC term, resulting in the constant convergence cost k. That assumption can be modified if, for example, the present yield curve is thought to be "abnormal." Convergence costs consistent with some average or equilibrium yield curve might be used in such a case. Another possibility would be yield curve scenarios and the accompanying convergence costs, which then would vary over the GIC term but could, nevertheless, be expressed in terms of a level annual basis point charge.

V. COMPARISON OF THE INTEREST RATE FUTURES APPROACH WITH TRADITIONAL IMMUNIZATION

Whether immunization is employed in the traditional manner, that is, using only cash assets, or with the aid of bond futures, the purpose is the same—eliminating the pure interest rate risk of mismatching. Insurance companies and other financial institutions have become reluctant to assume this risk because of its unstable nature in recent years and the resulting adverse experience. The elimination of interest rate risk generates additional costs and increased exposure to other types of risks, presumably of a lesser magnitude. Costs and increased exposure to risk will vary depending on whether immunization is using cash assets or bond futures. This section discusses these considerations for each of the two approaches.

As with traditional immunization, parallel yield curve shifts have little effect on the final results when futures are used, since convergence costs are virtually unchanged by such a shift. But also like traditional immunization, problems can arise when the yield curve—in this case, the treasury curve—shifts in a nonparallel fashion. When the traditional approach is used, the counterclockwise shift (steepening) results in market value losses not being offset by future reinvestment gains. Alternatively, this shift may result in lower than assumed reinvestment rates not offset by portfolio appreciation, depending on the nature of the shift. A clockwise shift (flattening) generally would result in gains.

This risk is minimized if the degree of dispersion of asset and liability flows around the duration is similar. This condition is very difficult to meet, however, although cash-flow matching is attempted by some companies. If the futures contracts used to hedge an asset were those contracts whose time to delivery matches the remaining term of the GIC, the results would be very similar to the traditional approach. To demonstrate, assume long-term treasury yields remained constant while short-term rates dropped over the GIC term. This would produce lower prices on distant futures contracts, but the price of the contract being delivered (nearby) would not change. In other words, the futures price curve would steepen but converge to the same point.

The total convergence cost of the initial short futures position would be unaffected by such a shift, although the pattern of convergence would be somewhat U-shaped rather than steadily increasing. The additional contracts needed during the course of the GIC to keep durations matched, however, would incur higher convergence costs than the futures price curve at inception would have indicated. This additional cost on the incremental contracts is analogous to the lower reinvestment rate effect of such a shift when traditional immunization is used. Conversely, convergence costs less than projected at the outset, perhaps even negative, will be incurred on the additional contracts when the yield curve flattens or inverts.

Practically speaking, no futures contracts are available more than two and a half years from delivery, and there is very little volume beyond one year out. This makes the approach of using contracts with the same time to delivery as the remaining term of the GIC impossible for any GIC longer than two and a half years. The approach used in section IV and the appendix is to maintain a constant time to delivery by rolling back the contracts every quarter. This strategy has the effect of increasing the risk associated with a steepening yield curve while the remaining GIC term is longer than the time to delivery of the futures used. As the GIC term shortens, this risk also lessens.

In either case, traditional immunization or immunization using futures, the risk of yield curve shifts is somewhat symmetric, with potential for gains as well as losses. While this alone does not justify taking the risk, it does seem to be less threatening than the pure interest rate risk of mismatching, considering the historical range of yield curve shifts. This may lead to the conclusion that, over time, gains and losses from yield curve shifts tend to balance one another out. That conclusion may explain why companies tend to make a very small charge if any at all for such a risk. The costs of eliminating it are high and involve sales and investment inflexibility associated with cash-flow matching (dedication) as well as some inevitable mismatching.

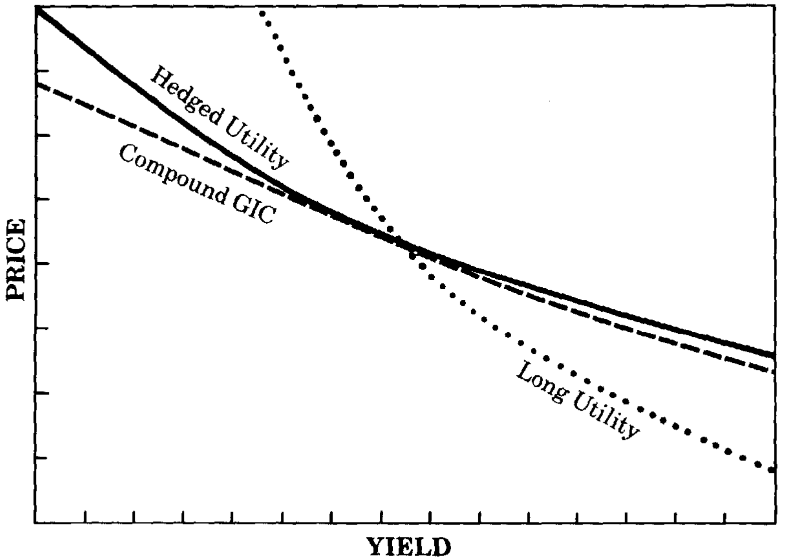
An ideal situation to encounter when futures are used is a BBB yield curve sloping upward, as it almost always is for intermediate and long maturities, while the treasury yield curve is inverted, as it occasionally has been in recent years. The higher long yields can be obtained on the cash assets while the convergence cost on short futures will be negative, as in figure 3b. In this case, the nearby contracts will be most desirable from the standpoint of convergence, but there will be more exposure to counterclockwise shifts (returning to positive). The flat or negative yield curve is also advantageous when traditional immunization is used since the effect of "bouncing down" the yield curve when rebalancing would be mitigated to some extent. In fact, it would be "bouncing up" over the inverted portions of the curve.

As mentioned in section IV, the yield advantage, when futures are used, results from the availability, on the shorter synthetic assets, of the extra liquidity and default premiums derived from the longer assets. While the holder of such assets presumably is exposed to these additional risks, this risk-reward tradeoff (unlike pure interest rate risk) is the type historically so often sought and successfully underwritten by insurance companies. With a growing or stable block of GIC business, the liquidity premium, in particular, seems like a "free lunch." When traditional immunization is used, the insurance company pays for the extra liquidity and security on short and intermediate term assets whether it is needed or not.

The other risk resulting from the use of futures is the basis risk. This is the risk that the correlation of the yields on the hedging instrument (treasury bonds) and the cash assets (BBB utility bonds) is quantified incorrectly. There is no such risk under the traditional approach. Although not as volatile as yields themselves, the BBB treasury yield spreads have shown considerable variation in recent years and over the short run. The assumed relationship in equation (1) is bound to be wrong. If the yield spreads widen more than anticipated, either portfolio gains will not be sufficient to offset futures losses, or portfolio losses will not be offset by futures gains. As in traditional immunization with the yield curve risk, some comfort may be taken in a long-run view emphasizing the symmetrical nature of the risk. One possible strategy would be to vary the risk margin depending on existing yield spreads relative to historical spreads. Wide spreads would require less risk margin and narrow spreads, more. Whatever view is taken, the basis risk cannot be rationalized away. Its magnitude should be understood before it is undertaken. It is the major drawback to immunization with bond futures.

Another consideration when using futures is the effect on the price (market value) curve of the asset portfolio, as shown in figure 1. The shape of the curve is generally concave up since $\frac{d^2P}{di^{(n)}}$ always will be positive for cash assets. Thus, the price curve is always above the tangent line at any point. Estimates of price changes in response to yield change using the tangent line always will be conservative. This effect is more pronounced for an asset whose payments are widely dispersed around the duration, resulting in a larger second derivative. This applies to treasury bonds as well. But treasury bond futures sold short have the opposite effect since the second derivative will in effect be negative. When included in the asset portfolio, they would have a tendency to flatten out the the portfolio price curve. The net effect of this will probably still be satisfactory since the price curve of long cash assets will have a very large positive second derivative. Whether or not the synthetic asset has a larger second derivative than the appropriate duration

FIGURE 7
Price Curves



cash asset, either should compare favorably to a zero-coupon bond or compound GIC, both of which have relatively low second derivatives due to the absence of any dispersion in the cash flows. Figure 7 illustrates the effect futures have on a price curve.

The futures have the effect of straightening the asset price curve as well as dampening the price volatility by reducing the slope. The liability price curve is still less concave in figure 7, indicating that some of the conservatism due to the higher second derivative associated with the greater dispersion of asset flows is retained.

While a high value for the second derivative of the asset price curve is desirable from the standpoint mentioned, it also does expose the portfolio to risks associated with nonparallel yield curve shifts. Minimizing this risk would require an equal dispersion of asset and liability flows about the duration, as would be the case when the cash flows are matched. Of course this would eliminate the benefits of the greater asset flow dispersion just discussed.

On a related issue, which may be of special interest to those familiar with options pricing, the effect of the call provision on bonds can be analyzed in the same manner. This discussion applies to both approaches to immunization. Similar to bonds themselves, fixed-income call options have a price curve that is concave up, as illustrated in figure 8.

FIGURE 8
Call Option Price Curve

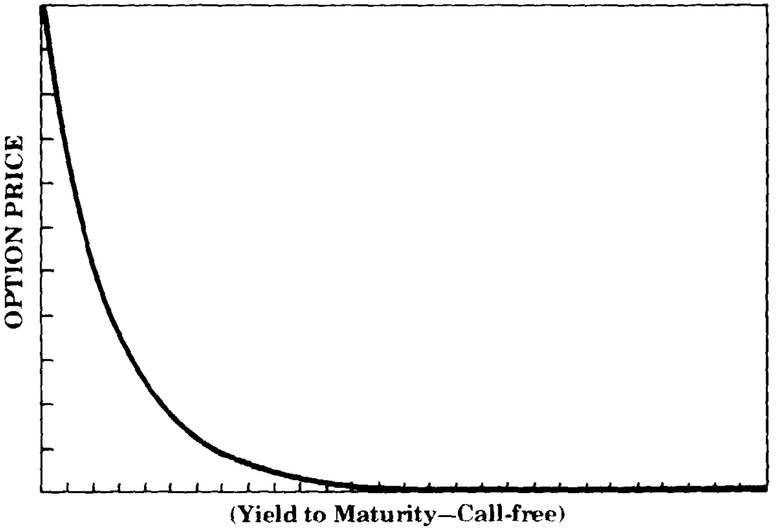
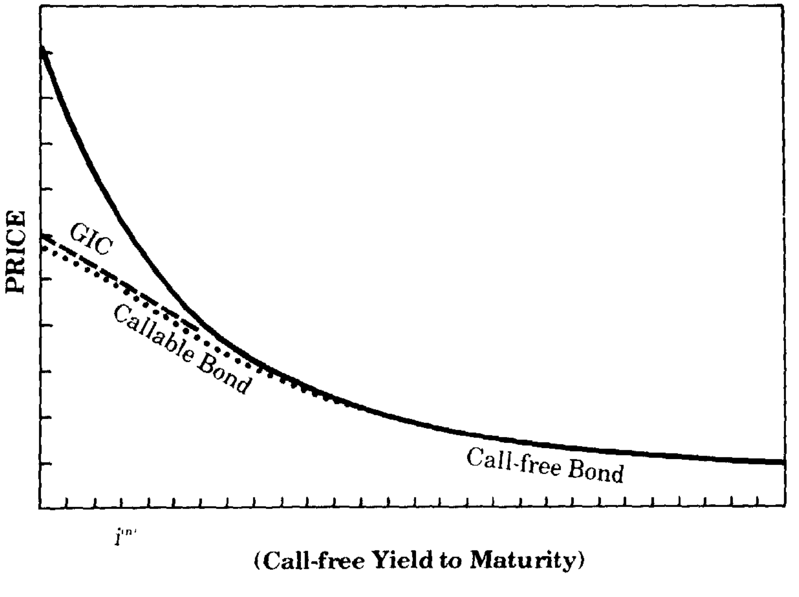


FIGURE 9
Callable Bond Price Curve

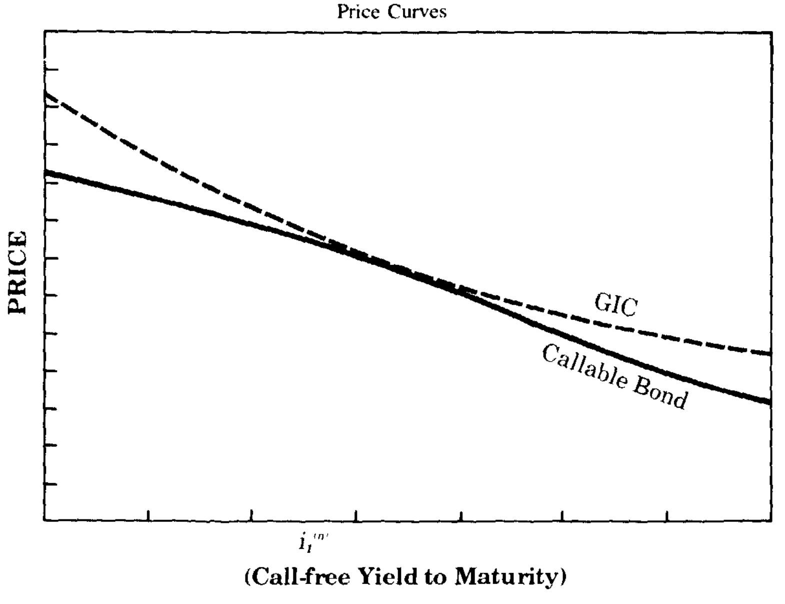


Like bond futures sold short, a written call option such as that acquired with a callable bond will have a straightening effect on the bond price curve. The price curve for a callable bond might look like that illustrated in figure 9.

As can be seen in figure 9, the price of the callable bond is less than that of the call-free bond due to the option premium the purchaser of the bond receives. This premium, represented by the difference between the two curves in figure 9, is never stated explicitly in dollars but is reflected in a higher yield to maturity. If one believes the market prices these call options efficiently, the extra yield attributable to them probably should not be credited to the GIC contracts. Another way to look at it is that the GIC liabilities are generally call-free by the insurer so the insurer should credit only the yield remaining after the effect of the call provision has been removed, to reflect the risk the insurer is assuming.

The effect of the call option is most pronounced at the lower yields where the option price is most volatile. The option may even influence the price curve to become concave downward at the very low yields. If this occurs, the bond price curve will become less steep at those lower yields. This can be interpreted as a shortening of duration. But because the price curve is concave downward at those yields, the estimated price resulting from a yield change using the tangent line always will be too high—just the opposite of what happens over the portions of the curve that are concave up. So, while

FIGURE 10
Price Curves



a callable premium bond may at first appear to be the high-yielding short-duration asset so difficult to find, its concave downward price curve would cause problems when used to back a GIC, which has a concave upward price curve. Figure 10 illustrates this situation. While durations technically are matched at rate $i_1^{(n)}$, a change in rates would result in a shortfall of assets.

The adverse effect of a concave downward price curve can be offset somewhat by frequent rebalancing. The futures approach to immunization would be more suitable for frequent rebalancing from the standpoint of cost and time required. In short, the effect of the call option on duration and yield should be taken into account. A good options pricing model for fixed income investments is invaluable for this purpose. All references to yield should be considered net of the call premium in this paper, however.

The choice between traditional immunization and immunization with futures obviously depends on the advantages of each method and the weight given to each of these advantages by the user. When futures are used, the need to actually turn over cash assets for rebalancing purposes is eliminated. The hedge merely is adjusted instead. This flexibility is attributable to the superior liquidity of treasury bond futures and is convenient any time the asset duration needs modification, whether it be as a result of changing yields, the effect of call provisions, or just the passage of time. The use of futures also offers the potential to flatten the yield curve at the shorter durations as shown in figure 4. On the other hand, traditional immunization does not have to rely on the correlation of BBB yields, for example, with treasury yields. This basis risk is without a doubt the most serious drawback of the futures approach. For the investor with a short time horizon, this consideration assumes increasing importance. Both methods have risks associated with rotating yield curves.

VI. CONCLUSION

This section will deal briefly with some additional considerations regarding the use of interest rate futures. The example using BBB utility bonds as representative insurance company investments can be extended to any investment likely to have a high degree of correlation with long treasury bonds. Commercial mortgages and direct placements may meet this requirement although a suitable method to estimate market yields and resulting prices would have to be found in the absence of an active secondary market. Although such investments are relatively illiquid, that may not be a problem since disposing of assets for rebalancing purposes is not required with a stable or growing block of GIC business and perhaps some marketable securities in the portfolio.

On the liability side, the illustrations presented in section IV and the appendix deal with the easiest case, that of a lump sum deposit, compound interest GIC. If the payout were, in fact, a stream of payments—for example, a simple interest GIC—the futures hedge calculation would have to incorporate a different calculation for the liability duration. The periodic interest payouts also would be reflected in the asset value factor in the hedge calculation. If the payouts are of an uncertain nature, perhaps benefit payments, estimates will have to be made and revised in order to calculate a liability duration. Any error in the estimated benefit payments will, of course, show up in the hedge calculation.

There are many other applications of futures in addition to the one covered in section IV. The short hedge also can be used to keep the yields current on an unused inventory of assets. This would involve creating a synthetic asset of zero duration to protect against rising interest rates prior to using the assets. This applies to unfunded forward commitments as well as those that already have been taken down. If the GIC is not a lump sum deposit but, rather, recurring installments, the risk is that interest rates may drop prior to receipt and investment of those deposits. In such a case, a long hedge is necessary to provide the desired protection. The hedge calculation is similar to equation (8) except the futures duration is positive. A long hedge can also be used when a liability longer than any available cash asset is sold. These might include a long compound GIC or deferred annuity. The hedge will, in effect, lengthen the asset to the necessary duration. In summary, futures can be used any time the manipulation of duration is required to keep assets and liabilities in balance.

Some practical limitations to the use of the futures should be mentioned. The New York legal situation is far from resolved with regard to futures. The most recent legislation there seems to allow the use of the futures in a very limited manner although there are different interpretations. In general, companies not licensed in New York appear to have considerably more flexibility at the present time. An important corporate consideration is whether management will tolerate the large margin calls that will occur in a falling interest rate environment when a short hedge is used. These losses likely will be offset by unrealized bond appreciation, but large negative cash flows of any kind tend to be looked upon unfavorably by management. At least one company has taken the position that futures gains and losses can be amortized over a period of up to thirty years for both GAAP and statutory accounting purposes. Such treatment may make the large cash-flow fluctuations more palatable. Tax treatment is unclear.

Using interest rate futures to shorten the duration of treasury bonds has been widely written about and discussed. This paper extends that concept to lower quality investments while describing the benefits and the risks involved. Immunization is also incorporated into the analysis by outlining an approach using long utility bonds in conjunction with futures in the management of GIC liabilities. The approach could be extended to the asset-liability management of other insurance products and fixed income management as well.

Whether traditional immunization or the futures approach is used, the risks involved should be well understood. These include nonparallel yield curve shifts, call provisions, and the basis risk, to name a few. These are the costs required to lessen or eliminate the pure interest rate risk that is encountered when duration is ignored. The reader is referred once again to the appendix, which applies the principles outlined in section IV using a hypothetical 3-year GIC.

APPENDIX

GIC PRICING—AN ILLUSTRATIVE EXAMPLE

This appendix provides an example of the determination of hedging costs for a hypothetical GIC. It is similar to the development in section III with the exception that illustrative numbers are used for duration, yield, and so on.

The assumptions are as follows:

1. The sale of a $5 million, 3-year, compound interest GIC in December 1982.
2. Immediate investment of the proceeds in 13.5 percent 30-year BBB-rated utility bonds priced at par to yield 13.96 percent effective and having a duration of 7.26 years initially.
3. The "cheapest to deliver" treasury bonds are the 10⅜'s of 2012 priced at par to yield 10.64 percent effective and having a duration of 9.13 years initially. The yield stated is net of any option premium implicit in the yield to maturity quoted in the market.
4. The treasury bond futures sold short initially are those of March of 1984, fifteen months out. Delivery is never taken. Each quarter all outstanding contracts are "rolled back" and, thus, never get closer than twelve months from delivery. This maintains convergence at 5/32nds ($156.25) per quarter per contract.
5. The nearby futures contracts are priced at 75 percent of par, that is, 75 points and no thirty-seconds.
6. Commission on a round-turn futures transaction is $30. Margin requirements are $2,000 face amount per contract in 6-month treasury bills earning 8 percent (8.16 percent annual effective) with a duration of 0.5 years initially.

7. The yield curve does not change shape during the 3-year period. This means convergence cost will not change during the period.

A few other simplifying assumptions are made. All bond maturities are assumed to be in December, just before the anniversary of the GIC sale. The β-value discussed in section IV will be assumed equal to 1.0. That is, the long-run perspective will be taken whereby long treasury yields and BBB utility yields are assumed to move basis point for basis point. Yield levels for each security will be assumed to remain unchanged over the period. Finally, it is assumed that hedging costs are incurred at the end of the quarter when, in reality, they are incurred throughout the quarter.

These last few assumptions can easily be refined and reincorporated into the analysis. It was felt that doing so in this case would create unnecessary detail and divert attention from the main point, namely, the determination of hedging costs. In actual practice, further refinement probably would be made to some or all of these.

All of the data thus far outlined can be summarized in terms of section IV values as follows:

c_0 = 2.81 = 3/1.0675 years
A_0 = \$5,000,000
b_0 = 7.26 years
e_0 = 9.13 years
g_0 = .5 years
k = 5/32 points per quarter (\$156.25)
r = \$2,000
i_n = 13.50 percent (13.96% effective)
i_t = 10.375 percent (10.64% effective)
i_h = 8.00 percent (8.16% effective)
p = .75
β = 1.0

Once again, duration here is modified duration. Using bond convention, modified duration is Macaulay duration divided by one plus one-half the bond yield.

The interest rates and futures-related data were chosen to approximate actual market conditions on December 2, 1982, with some rounding. Table I of section I indicates that convergence between the December 1983 and the March 1984 contracts was, in fact, 5/32 points.

Exhibit IA shows the quarterly progression of the fund. Interest earned is added while hedging costs, including convergence, commission, and interest lost on margin, are deducted using equations (13) and (14) in section III.

At the end of three years, the fund has reached a value of \$7,250,961. This represents a 13.19 percent annual effective return on the initial \$5,000,000

EXHIBIT IA

(Dollars and contracts rounded to nearest unit)

Time in years (j)	(1) Beginning Balance A_j	(2) Interest Earned $(1)\times[1.1396^{.25}-1]$	(3) Duration Liability (c_j)	(4) Duration Utility Bonds (b_j)	(5) Duration Treasury Bills (g_j)	(6) Duration Treasury Bonds (e_j)	(7) Futures Contracts Sold Short $F_j = \frac{(b_j - c_j)A_j}{100{,}000pe_j + 2{,}000(b_j - g_j)}$	(8) Convergence Cost $156.25F_j$	(9) Commission Cost $30F_j$	(10) Interest Lost On Margin $2{,}000F_j[(1.1396)^{.25} - (1.0816)^{.25}]$	(11) Ending Balance $(1)+(2)-(8)-(9)-(10) = A_{j+.25}$
0	5,000,000	166,044	2.81	7.26	.50	9.13	32	5,000	960	858	5,159,226
.25	5,159,226	171,332	2.58	7.03	.25	8.89	34	5,313	1,020	912	5,323,313
.50	5,323,313	176,781	2.34	7.25	.50	9.10	38	5,938	1,140	1,019	5,491,997
.75	5,491,997	182,383	2.11	7.02	.25	8.86	40	6,250	1,200	1,072	5,665,858
1.00	5,665,858	188,157	1.87	7.24	.50	9.07	44	6,875	1,320	1,180	5,844,640
1.25	5,844,640	194,094	1.64	7.01	.25	8.83	46	7,188	1,380	1,233	6,028,933
1.50	6,028,933	200,214	1.41	7.23	.50	9.04	51	7,969	1,530	1,367	6,218,281
1.75	6,218,281	206,502	1.17	6.99	.25	8.80	54	8,438	1,620	1,448	6,413,277
2.00	6,413,277	212,977	.94	7.22	.50	9.01	58	9,063	1,740	1,555	6,613,896
2.25	6,613,896	219,640	.70	6.98	.25	8.77	62	9,688	1,860	1,662	6,820,326
2.50	6,822,326	226,495	.47	7.20	.50	8.98	67	10,469	2,010	1,796	7,032,546
2.75	7,032,546	233,543	.23	6.97	.25	8.74	71	11,094	2,130	1,904	7,250,961
3.00	7,250,961										

deposit. Thus, direct hedging costs expressed in basis points are 13.96% − 13.19% = 0.77% or 77 basis points. After deducting a charge for basis risk, this could be compared with the results from investing in a three-year duration asset and bouncing down the yield curve in the traditional immunization case.

The 77-basis-point hedging cost can be categorized as follows:

convergence	56
commission	11
interest lost	10
total	77.

Hedging costs will be higher for shorter GICs and lower for longer ones if the same cash investments are made.

To demonstrate that the assets and liabilities are duration-matched or immunized, at least at the outset, it will be assumed that yield levels increase 1 percent immediately after the GIC sale. When using duration to estimate price changes, the mode of the yield to which duration is applied must be the same as that used in the duration calculation. Thus the 1 percent increase refers to bond yields rather than annual effective yields. In other words, the BBB and treasury bond yields rise to 14.5 percent and 11.375 percent, respectively. The treasury bill yields also rise. Table IA uses the product of duration and the yield change to estimate price changes of the asset and liability portfolios.

The cash assets include both the utility bonds and the treasury bills used for the initial margin requirement. The initial value shown for the futures position is not an asset as such but a benchmark used to determine the subsequent gains and losses. The gain in this case, however, does become a cash asset. The initial futures value is determined by taking the product of the par value of one contract ($100,000), the price as a proportion of par (.75), and the number of outstanding contracts (32). The futures duration, $-e_0$, is negative since a short hedge is being used. The slight change in

TABLE IA

(1) Assets	(2) Initial Value	(3) Duration	(4) = −(2)x(3) Gain (Loss)	(5) Value after Change
Cash assets	A_0 = 5,000,000	d_0 = 7.17	(358,500)	4,641,500
Futures position	$100{,}000pF_0$ = 2,400,000	$-e_0 = -9.13$	219,120	219,120
Asset totals	5,000,000	2.79	(139,380)	4,860,620
Liability	5,000,000	c_0 = 2.81	(140,500)	4,859,500
Net Change in Financial Position				1,120

financial position results from the fact that only an integral number of contracts can be used, requiring the hedge calculation to be rounded to the nearest contract.

Table IA shows that the tangent lines to the asset and liability curves are approximately the same. A more exact approach would be to calculate the new bond and futures prices based on the new yields rather than using the tangent-line estimate of price sensitivity. This would show a larger positive change in financial position resulting from the effects of the second derivative of the price curves as shown in figure 7 of section V.

The basis risk can be demonstrated in the extreme case by assuming that only BBB utility yields increase while treasury yields do not change. This widening of the spread will result in no futures gain to offset the $358,500 unrealized loss in table IA. Of course, narrowing spreads will have a positive effect on financial position.

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DISCUSSION OF PRECEDING PAPER

ELIAS S. W. SHIU:

I have two comments on this interesting and insightful paper.

The Macaulay duration and modified duration can be defined as

$$-\frac{d}{d\delta}\log_e P$$

and

$$-\frac{d}{di^{(n)}}\log_e P ,$$

respectively. Since

$$\frac{di^{(n)}}{d\delta} = 1 + \frac{i^{(n)}}{n} ,$$

an alternative way to express equation (3) is

$$\left(-\frac{d}{di^{(n)}}\log_e P\right)\left(\frac{di^{(n)}}{d\delta}\right) = -\frac{d}{d\delta}\log_e P .$$

To see the price-elasticity definition of the Macaulay duration, note that

$$\log_e R = \log_e(1 + i) = \delta .$$

In the Appendix, the modified duration c_0 of the GIC is calculated with the factor 1.0675, where 6.75 percent is the effective semiannual interest rate of the utility bond. On the other hand, it is shown that 13.19 percent is the annual effective return on the initial deposit. Should one use the factor $1.1319^{.5}$ instead of 1.0675 and reiterate the calculations? Although there is not much difference in this example, the correct discounting factor may have a significant effect in other situations.

(AUTHOR'S REVIEW OF DISCUSSION)

ALLAN MING FEN:

I thank Elias S. W. Shiu for his discussion of my paper.

In academic work on immunization, duration, and interest rate theory, yield is often expressed as a continuously compounded rate of interest. I would guess that the reason for this is that the resulting functions lend themselves better to integration, differentiation, and the use of logarithms (the last of which Professor Shiu demonstrates). Another fortunate consequence is that there is no ambiguity in the meaning of duration since modified duration and Macaulay duration are one in the same as can be seen in Professor Shiu's first two equations. The analysis with logarithms brings to mind an article along the same lines in the December 1984 *Actuary* by C. L. Trowbridge called "Interest Sensitivity" (pages 5-6).

In practice, yields are almost never expressed as continuously compounded rates of interest. Generally, the compounding mode is less than or equal to 12 for intermediate and long-term investments. That being the case, Macaulay duration will overstate the price sensitivity of a cash-flow stream. The less frequent the compounding mode, the greater the overstatement.

I agree that the correct representation of the modified duration for the liability at time j is

$$C_j = (3 - j)/x$$

where $x = (1.1319)^{.5}$ rather than $(1.1396)^{.5} = 1.0675$. This can be shown by first remembering that, under the assumptions made, the 77 basis point cost of hedging, which is 72 basis points compounded semiannually, will be constant no matter how rates change. Then the present value of the liability "stream" is

$$L_j = 7{,}250{,}961[1 + (i_u - .0072)/2]^{-(6 - 2j)}$$

where i_u is the normal yield on long BBB utility bonds. If $i_u = 13.50$ percent, then L_0 equals 5,000,000. The modified duration of the liability is then

$$-\frac{dL_j}{di_u}\frac{1}{L_j} = (3 - j)/[1 + (i_u - .0072)/2].$$

When $i_u = 13.50$ percent, the denominator equals $1.0639 = (1.1319)^{.5}$ as Professor Shiu suggests. In the example in the Appendix, this adjustment does not result in any change in Exhibit IA, column 7, "Futures Contracts Sold Short," when rounded to the nearest contract. As a result, none of the dollar amounts will change, but this, of course, will not be the case in all situations.