A Fuzzy Random Variable Primer

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> "it is a fundamental error to base probability theory on bivalent logic ... because ... there is a fundamental conflict between bivalence and reality." Lotfi A. Zadeh (2002)

Abstract

A fuzzy random variable (FRV) has been conceptualized as a vague perception of a crisp but unobservable random variable (RV) and also as a random fuzzy set. Since actuaries seem receptive to these notions of FRV, and there are sources of uncertainty that RVs can not accommodate, but FRVs can, one would expect FRVs to be a component of the actuarial arsenal, but generally this is not the case. Assuming that the reason for this is that potential users are not sufficiently familiar with FRV methodology, the purpose of this article is to help alleviate this situation by presenting a FRV primer.

Keywords: fuzzy random variables, random fuzzy sets

1 Introduction

Consider the following question, which, for some, is reminiscence of questions on the old SOA statistics examination. An urn contains approximately 100 balls of various sizes. Several are large. What is the probability that a ball drawn at random is large?

If the descriptive variables, namely "approximately", "several" and "large", were crisp numbers, the answer to the questions would be a numerical probability. However, since these terms are fuzzy, rather than crisp values, the solution, like the data upon which it is based, is a fuzzy number. Situations of this sort, which involve a function from a probability space to the set of fuzzy variables, give rise to the notion of a FRV.

When the notion of FRVs is explained to actuaries, they seem receptive to the idea. They generally recognize that random variables (RVs) do not capture sources of uncertainty such as modeling choices, parameter choices, application of expertise, boundary conditions, and lack of knowledge, while FRVs can accommodate these things. Consequently, one would expect FRVs to be a component of the actuarial arsenal, but this is not the case.

A plausible explanation of why FRVs are not being implemented more often is that potential users are not sufficiently familiar with FRV methodology, and, consequently, they forego opportunities for implementation. Assuming this to be so, the purpose of this article is to help alleviate this situation by presenting an overview of FRVs.

The topics addressed include: differentiating between fuzziness and randomness, FRVs, and conceptualizing FRVs. For the most part, the discussion is conceptual rather than technical. The goal is to illustrate how naturally compatible and complementary probability theory and fuzzy logic are and to illustrate how the two can be combined.

2 Differentiating Between Fuzziness and Randomness

Before proceeding, it is important to differentiate between fuzziness and randomness. To that end, we present two intuitive examples which emphasis their differences: the first, focuses on distributional differences, while the second focuses on the concept of membership.

2.1 Distributional Differences

Consider first, a simple comparison of a probability distribution with a possibility distribution. The scenario we consider is based on the assumption that a particular attendee at ARC 2007 has two or more eggs for breakfast on the second day, and the question is how many eggs s/he will have.[1]

As indicated in Figure 1(a), a representative probability distribution, based on a sample of attendees who had eggs on the first day of the conference, might assign a 0.8 probability of two eggs and a 0.1 probability of either one or three eggs.

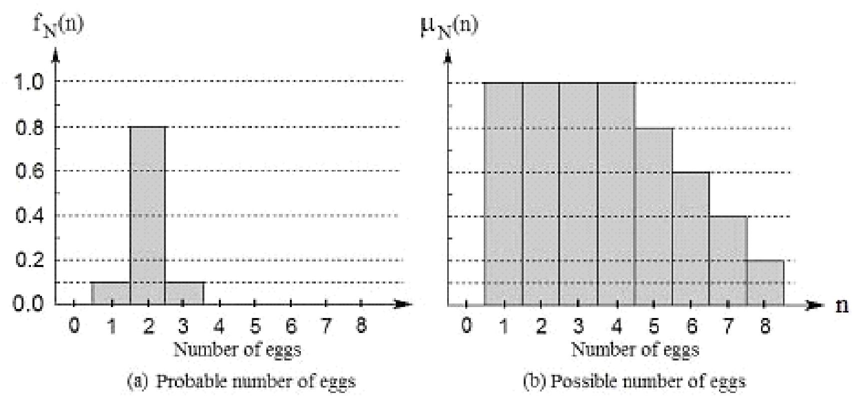

Figure 1: Number of eggs eaten for breakfast on the second day

In contrast, a representative possibility distribution, Figure 1(b), shows that one to four eggs each have a high possibility of occurrence, and there is some possibility that the person will eat as many as eight eggs.[2]

An obvious conclusion is that a probable event is always possible, while a possible event need not be probable.

[1] Adapted from Zadeh (1978: 8).
[2] Recall that in "Cool Hand Luke (1967)" Luke ate 50 eggs at one sitting.

We can formalize the foregoing observations by comparing a probability space with a possibility space. The key features of these spaces are summarized in Table 1, where they are juxtaposed for comparison purposes, and the discussion that follows.

Table 1: Probability Space v. Possibility Space

Probability Space	Possibility Space
$(\Omega, \mathfrak{F}, \Pr)$ is a probability space	$(\Theta, \wp(\Theta), \mathrm{Pos})$ is a possibility space
Ω, the sample space	Θ: sample space
$\mathfrak{F}$: σ-algebra of subsets of Ω	$\wp(\Theta)$: power set of Θ
Pr: probability measure on Ω	Pos: possibility measure on Θ

As indicated in the table, a probability space is defined as the 3-tuple $(\Omega, \mathfrak{F}, P)$, where $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$ is a sample space, $\mathfrak{F}$ is the σ-algebra of subsets of Ω, that is, the set of all possible potentially interesting events, and P is a probability measure on Ω. P satisfies:

$\Pr(\Omega) = 1$
$\Pr(A) \geq 0$ for any $A \in \mathfrak{F}$,
For every countable sequence of mutually disjoint events $\{A_i\}$, i=1, 2, ...

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}$$

In contrast, a possibility space is defined as the 3-tuple $(\Theta, \wp(\Theta), \mathrm{Pos})$, where $\Theta=\{\theta_1, \theta_2, ..., \theta_N\}$ is a sample space, $\wp(\Theta)$, also denoted as 2^{Θ}, is the power set of Θ, that is, the set of all subsets of Θ, and Pos is a possibility measure on Θ. Pos(A), the possibility that A will occur, satisfies:

$\mathrm{Pos}(\Theta) = 1$
$\mathrm{Pos}(\varnothing) = 0$
$\mathrm{Pos}\{\cup_i A_i\} = \sup_i \mathrm{Pos}\{A_i\}$ for any collection $\{A_i\}$ in $P(\Theta)$

2.2 Membership Differences

Consider next the situation of an individual who has been wandering in the desert for days without food or water.[3] S/he happens on the two bottles shown in Figure 2(a) and is confronted with the uncertainty with respect to their suitability for drinking.

[3] Adapted from Bezdek and Pal (1992 pp 5-6).

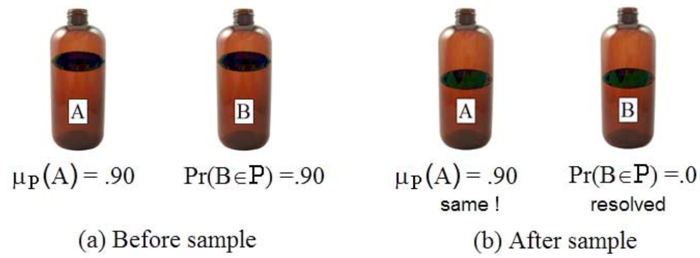

Figure 2: Potable (P) liquid for the thirsty wanderer

As indicated, the bottle labeled "A" has a grade of membership of 0.9 in the set of all potable (suitable for drinking) liquids (P), while the bottle labeled "B" has a probability of 0.9 of containing potable liquid. Assuming the person will choose to drink from one or the other of the bottles, which one should be chosen?

Before sampling, the rational decision is to choose bottle "A", since its contents is fairly close (90%) to perfectly potable liquids. For example, it might contain swamp water rather than pure water. With bottle "B" there is a 10% chance that the contents are lethal.

Suppose that after sampling, the situation is as shown in Figure 2(b). Now the choice is more obvious since it turns out that bottle "B" contained non-potable water. Note that sampling does not change the grade of membership associated with the bottle labeled "A".

An obvious conclusion is that fuzzy degrees are not the same as probability percentages. That is, grade of membership is different than probability of membership.

2.3 Comment

The foregoing generic examples can easily be extended to actuarial applications. See, for example, Shapiro (2007), where distributional differences are discussed in the context of accidents involving insured drivers and membership differences are discussed in the context of classification for life insurance.

3 Fuzzy Random Variables

This section give an overview of the two views of FRVs that have emerged as the ones most often cited in the literature. In each instance, intuitive descriptions are first presented, followed by a formal definition.

3.1 α-cuts

Before preceding to formal definitions, we first need to discuss a fundamental function in fuzzy logic, the α-cut, since the definition of a FRV is often based on the measurability of these functions. An example of an α-cut is depicted in Figure 3.[4]

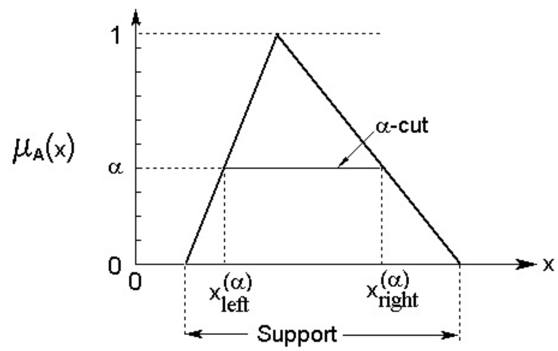

Figure 3: Alpha Cut

In the figure, $\mu_A(x)$ denotes a membership function (MF) or grade of membership of x in the fuzzy set A. As indicated, the essence of the α-cut is that it limits the domain under consideration to the set of elements with degree of membership of at least alpha, that is, the α-level set. Thus, while the support of fuzzy set A is its entire base, its α-cut is from $x_{left}^{(\alpha)}$ to $x_{right}^{(\alpha)}$. Values outside that interval will be considered to have a level of membership that is too insignificant to be relevant and should be excluded from consideration, that is, cut out.

3.2 The Kwakernaak FRV

The term fuzzy random variable was coined by Kwakernaak (1978), who introduced FRVs as "random variables whose values are not real, but fuzzy numbers," and conceptualized a FRV as a vague perception of a crisp but unobservable RV. [5] As an example of his view, consider the subconscious task of assigning an age to people we pass on the street. Their actual age, X say, is an ordinary random variable on the positive real line. However, we can only perceive a random variable ξ through a set of "windows" like "young", "middle age" and "old". That is, we perceive fuzzy sets as observation results since the original X is not observable.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $F(\mathbb{R})$ denote the set of all fuzzy numbers in $\mathbb{R}$, the set of real numbers. Formally, $F(\mathbb{R})$ denotes the class of the normal convex fuzzy subsets[6] of the

[4] Adapted from Sinha and Gupta (2000), Figure 7.13.

[5] It should be mentioned that Kwakernaak was not the first one to explore the relationship between fuzzy variables and random variables. That honor goes to Goodman (1976), who, in a correspondence to Zadeh, used randomized level sets to develop an explicit connection between fuzzy set theory and probability theory.

[6] A fuzzy number is said to be normal if there is a $t \in A$ such that $\mu_A(t) = 1$. It is a convex fuzzy subset of the real line if $\mu_A(\lambda x_1 + (1 - \lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$, for $\lambda \in [0, 1]$.

Euclidean space $\mathbb{R}$ having compact[7] α levels for $\alpha \in [0, 1]$. This is the class of mappings U: $\mathbb{R} \rightarrow [0, 1]$, such that U_α= is a nonempty compact interval, where

$$U_\alpha = \{x \in \mathbb{R} \mid U(x) \geq \alpha\} \quad \text{if } \alpha \in (0,1]$$
$$= \text{cl(supp U)} \quad \text{if } \alpha = 0.$$[8]

Then, as envisioned by Kwakernaak (1978), enhanced by Kruse and Meyer (1987: 64-65), and interpreted by Gil (2004: 11), a FRV is a mapping ξ: $\Omega \rightarrow F(\mathbb{R})$ such that for any $\alpha \in [0, 1]$ and all $\omega \in \Omega$, the real valued mapping

$\inf \xi_\alpha: \Omega \rightarrow \mathbb{R}$, satisfying $\inf \xi_\alpha(\omega) = \inf (\xi(\omega))_\alpha$, and
$\sup \xi_\alpha: \Omega \rightarrow \mathbb{R}$, satisfying $\sup \xi_\alpha(\omega) = \sup (\xi(\omega))_\alpha$,

are real valued random variables, that is, Borel-measurable real-valued functions.

3.3 The Puri and Ralescu FRV

Puri and Ralescu (1986) conceptualized a FRV as a fuzzification of a random set. They expressed concern about two limitation of the Kwakernaak model: the first was the mapping to the real line, rather than an Euclidean n-space, and the second was the concept of measurability, since it could not be extended beyond fuzzy numbers in $\mathbb{R}$. To overcome both these problems, they defined and studied the concept of FRVs whose values are fuzzy subsets of $\mathbb{R}^n$ (or, more generally, of a Banach space[9]) and, in so doing, they linked FRVs to the well-developed concept of random sets.

To set the context, we follow Puri and Ralescu (1986), Colubi et al. (2001) and Gil (2004), and let (B, $|\cdot|$) be a separable Banach space, $\mathcal{K}$(B) be a nonempty compact subsets of B, and

$$\mathcal{F}(B) = \{U : B \rightarrow [0,1] \mid U_\alpha \in \mathcal{K}(B) \text{ for all } \alpha \in [0,1]\},$$

where

$$U_\alpha = \{x \in B \mid U(x) \geq \alpha\} \quad \text{if } \alpha \in (0,1]$$
$$= \text{cl(suppU)} \quad \text{if } \alpha = 0.$$

In other words, $\mathcal{F}$(B) is the class of the normal upper semicontinuous[10] [0,1]-valued functions defined on B with bounded closure of the support.

Puri and Ralescu (1986) formalized a FRV as a mapping $\mathcal{X}$: $\Omega \rightarrow \mathcal{F}(B)$ such that for any $\alpha \in [0, 1]$ the set-valued mapping $\mathcal{X}_\alpha: \Omega \rightarrow \mathcal{K}(B)$ (with $\mathcal{X}_\alpha(\omega) = (\mathcal{X}(\omega))_\alpha$ for all $\omega \in \Omega$) is a compact

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[7] "A subset of Euclidean space $\mathbb{R}^n$ is called compact if it is closed and bounded. For example, in $\mathbb{R}$, the closed unit interval [0, 1] is compact.

[8] cl(supp U) denotes the closure of the support of U.

[9] A Banach space is a real or complex normed vector space that is complete as a metric space under the metric $d(x, y) = \|x-y\|$ induced by the norm. Completeness means that Cauchy sequences in Banach spaces converge.

[10] The α-level sets are closed for $0<\alpha\leq 1$.

random set,[11] that is, it is Borel-measurable with the Borel σ-field generated by the topology associated with the Hausdorff metric[12] on $\mathcal{K}$(B), [Gil (2004: 12)]

$$d_H(A,B) = \max\left\{ \sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b| \right\}.$$

There has been some tweaking of the original definition of Puri and Ralescu (1986). However, Colubi et al. (2001) concluded, based on measurability considerations, that their original definition is the most suitable one.

4 Conceptualizing Fuzzy Random Variables

We turn now to the conceptualization of FRVs in order to provide the reader with some visual benchmarks for further exploration. The topics addressed are a representation of FRVs, a simple application, an example of a pdf with a fuzzy parameter, and a fuzzy cumulative distribution function.

Operationally, a FRV is a RV taking fuzzy values, so as an example of a FRV[13], we start with the probability space (Ω, $\mathfrak{F}$, P) and let $u_1, u_2, ..., u_N$ be fuzzy variables. Then $\xi(\omega)$ is a FRV, where $\xi(\omega) = u_i$ if $\omega=\omega_i$, i=1, ..., N. A representation of $\xi(\omega)$ is show in Figure 4.[14]

[11] "A compact random set M is a random variable for which for any compact set K all "hit-events" M ∩ K ≥ ∅ and all "miss-events" M ∩ K= ∅ are measurable. [Nather (2001: 71)]

[12] The Hausdorff metric can be conceptualized as the distance between A and B, defined as the minimum value ϵ such that $B \subseteq A+[-\epsilon, \epsilon]$ and $A \subseteq B +[-\epsilon, \epsilon]$, where A+B = {a + b: a ∈A, b ∈B}. [Kruse and Meyer (1987: 50]

[13] Adapted from Liu (2004: 194)

[14] Adapted from Möller et al (2002)

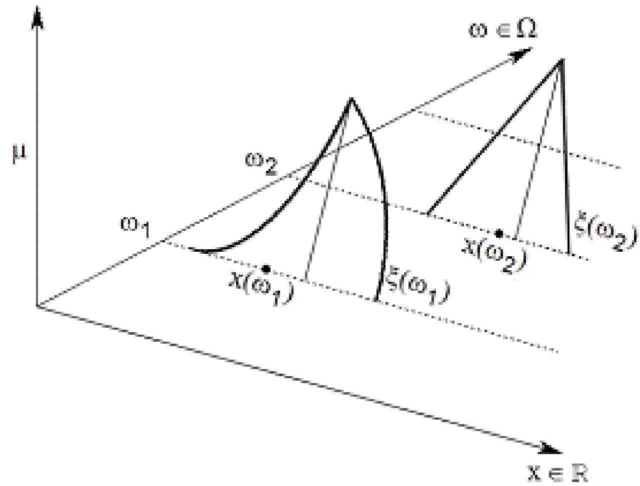

Figure 4: Model of a FRV

As indicated, instead of a real-valued realization, a fuzzy realization is assigned to each event ω in Ω.

A simple application[15] would be the addition of the two FRVs ξ_1 and ξ_2, each of which can be represented by the following triangular membership functions and their associated probabilities:

$$\xi_1 = \begin{cases} (a_1, a_2, a_3) \text{ with probability } 0.2 \\ (b_1, b_2, b_3) \text{ with probability } 0.8 \end{cases}$$

$$\xi_2 = \begin{cases} (c_1, c_2, c_3) \text{ with probability } 0.4 \\ (d_1, d_2, d_3) \text{ with probability } 0.6 \end{cases}$$

Adding these two FRVs together, we get:

$$\xi_1 + \xi_2 = \begin{cases} (a_1 + c_1; a_2 + c_2; a_3 + c_3) \text{ with probability } 0.08 \\ (a_1 + d_1; a_2 + d_2; a_3 + d_3) \text{ with probability } 0.12 \\ (b_1 + c_1; b_2 + c_2; b_3 + c_3) \text{ with probability } 0.32 \\ (b_1 + d_1; b_2 + d_2; b_3 + d_3) \text{ with probability } 0.48 \end{cases}$$

The potential for FRVs becomes more apparent when we explore probability distributions. An example[16] is the distribution of a random variable $X \sim N(\mu, \sigma^2)$, with fuzzy μ and crisp σ^2.[17] The situation is depicted in Figure 5.

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[15] Adapted from Liu (2004), Example 5.5.

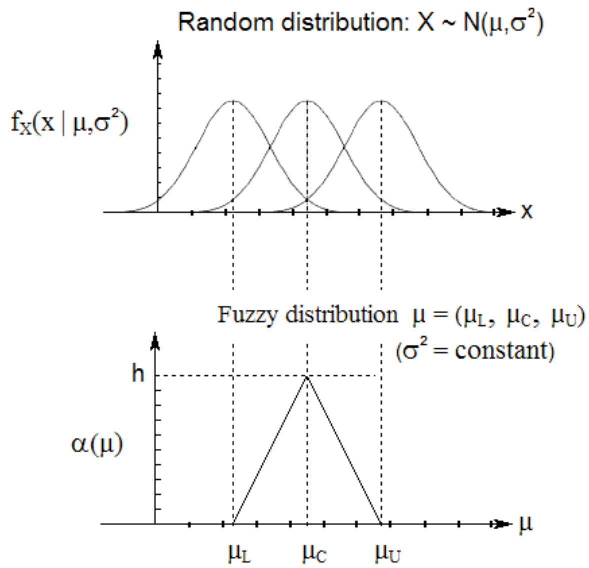

Figure 5: A FRV

As indicated, the mean of the distribution of X is the TFN (μ_L, μ_C, μ_U). As a consequence, μ may fall as far to the left as "μ_L", resulting in the left-hand distribution for X, and as far to the right as "μ_U", resulting in the right-hand distribution for X, although the highest possibility is associated with "μ_C".

A consideration of Figure 5 suggests that a Bayesian approach may provide one vehicle for implementing FRVs, in the sense that Bayes' theorem and the associated Bayesian methods can serve as a conduit between probability theory and fuzzy logic. See, for example, Chao and Ayyub (1995) and Viertl and Hareter (2004). To explore this, we start with the observation that membership functions can serve as priors for the parameters of conditional distributions. Thus, while ordinarily, membership functions, when used as fuzzy numbers, are normalized, that is, h=1 in Figure 5, following Chao and Ayyub (1995: 669-70) we can interpret the membership function, $\alpha(\mu)$, as a probability density function for the random variable μ if $\int_{-\infty}^{\infty} \alpha(\mu)\, d\mu = 1$, in which case $h=2/(\mu_U - \mu_L)$. Then, given the conditional pdf $g(x|\mu)$, where $X \sim N(\mu, \sigma^2)$, the marginal of X is given by:

$$f_X(x) = \int_a^c \alpha(\mu \mid m)\ g(x \mid \mu)\ d\mu .$$

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[16] Adapted from Chao and Ayyub (1995) Figure 1, and Buckley (2003), Section 8.3.

[17] There is not universal agreement as to which parameters should be fuzzified. For example, some researchers, like Buckley (2003: 98), allow that the variance can be a fuzzy variable. Others, such as Feng et al (2001), contend that the variance and covariance of FRVs should have no fuzziness.

From this, assuming that μ is a symmetrical TFN (STFN), with mean m and variance σ_{μ}^{2}, a constant, it follows that E(X) = m and Var(X) = $\sigma^2 + \sigma_{\mu}^{2}$.

In this example, the computation is straightforward. However, ordinarily, when integrating the product or two arbitrary distributions, some approximation method must be used. Candidates for this include finite difference approximations and the method of moments.

Another way a distribution can be fuzzified is exemplified by the fuzzy cumulative distribution function shown in Figure 6.[18]

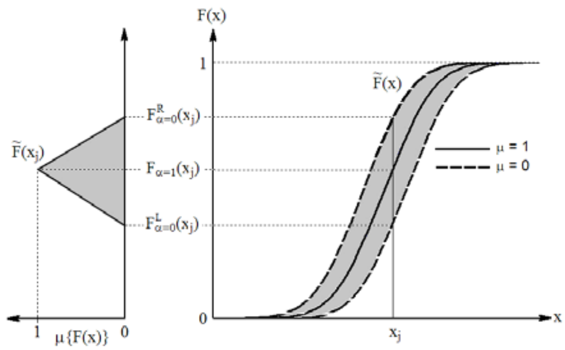

Figure 6: Fuzzy cdf

On the right side of the figure, F(x) is represented by the solid line labeled μ=1, and serves as the mode of the MF. $\widetilde{F}(x)$ represents the fuzzy version of F(x), whose spread is shown by the dashed lines labeled μ=0. For a given x_j, the MF associated with $\widetilde{F}(x_j)$ is shown on the left side of the figure, which represents the grade of membership, $\mu\{F(x)\}$. Here, $\widetilde{F}_{\alpha=1}(x_j)$, $F^L_{\alpha=0}(x_j)$, and $F^R_{\alpha=0}(x_j)$, represents the mode, left spread and right spread, respectively.

5 Computational Methods for Implementing FRVs

This section briefly discusses computational methods for implementing of FRVs. The two main methods have been discussed in the literature, other than simulation, are Zadeh's extension principle and the α-cut and interval arithmetic approach.

5.1 Zadeh's extension principle

Using the extension principle (Zadeh, 1975), the nonfuzzy arithmetic operations can be extended to incorporate fuzzy sets and fuzzy numbers[19]. Briefly, if * is a binary operation such as addition (+), min ($\wedge$), or max ($\vee$), the fuzzy number z, defined by z = x * y, is given as a fuzzy set by

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[18] Adapted from Kato et al (1999) and Möller et al (2002).

$$\mu_z(w) = \vee_{u,v}\ \mu_x(u) \wedge \mu_y(v), \quad u,v,w \in \Re,$$

subject to the constraint that w = u * v, where μ_x , μ_y, and μ_z denote the membership functions of x, y, and z, respectively, and $\vee_{u,v}$ denotes the supremum over u,v.[20]

A simple application of the extension principle is the sum of the fuzzy numbers A and B, denoted by $A \oplus B = C$, which has the membership function:

$$\mu_C(z) = \max\ \{\min\ [\mu_A(x), \mu_B(y)]: x+y=z\}$$

5.2 The α-cut and interval arithmetic approach

As the name suggests, this approach combines α-cuts and interval arithmetic. A brief explanation of interval arithmetic is as follows (Moore 1966 Chapter 3). Let $[a_1, b_1]$ and $[a_2, b_2]$ be two compact intervals of real numbers. If * is a binary operation, then interval arithmetic on those intervals may be represented as:

$$[a_1, b_1] * [a_2, b_2] = [\alpha, \beta]$$

where

$$[\alpha, \beta] = \{a*b|\ a_1 \le a \le b_1 \text{ and } a_2 \le a \le b_2\}.$$

If the operation is division, zero cannot belong to $[a_2, b_2]$.

Then, for any value of α, the fuzzy computation is accomplished by substituting the right and left α-cuts, $0 \le \alpha \le 1$, for $[a_j, b_j]$, j= 1,2, in the previous equation.

6 Comments

A basic problem in actuarial science is how to account for intrinsic features of uncertainty, including imprecision and vagueness. This has both theoretical and practical implications. The traditional analytical approach has developed tools and procedures assuming that uncertainty is basically due to randomness, which is appropriately handled by means of probability. Fuzzy

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[19] Fuzzy arithmetic is related to interval arithmetic or categorical calculus, where the operations use intervals, consisting of the range of numbers bounded by the interval endpoints, as the basic data objects. The primary difference between the two is that interval arithmetic involves crisp (rather than overlapping) boundaries at the extremes of each interval and it provides no intrinsic measure (like membership functions) of the degree to which a value belongs to a given interval. Babad and Berliner (1995) discussed the use interval arithmetic in an insurance context.

[20] See Zimmermann (2001), Chapter 5, for a detailed discussion of the extension principle.

random variables have widened the scope of study, enabling us to deal with more general sources of uncertainty in empirical data and/or models for actuarial analysis.

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