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COMPUTING THE PROBABILITY OF EVENTUAL RUIN

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ABSTRACT

Shiu derives two formulas for calculating the probability of eventual ruin in a collective risk model. This paper implements one of the formulas by incorporating an algorithm to reduce round-off error due to convolution. It gives the results of the programs for some claim distributions. The usefulness of these two formulas is also discussed.

1. INTRODUCTION

An important problem in the collective risk theory, which has over the years attracted a great deal of interest, is the computation of the probability of ruin. Some recent works include [1], [4]–[6], [9], [11]–[19], [22]–[25], [27]–[29], [31]–[34], [36]–[39], [41], [42], [44], [46]–[52]. In this paper, we implement one of the formulas for probability of ruin recently derived by Shiu [48]–[50], by incorporating an algorithm to reduce round-off error due to convolution.

We follow the notation of Bowers et al. [8]. Let $N(t)$ be a Poisson process that counts the number of claims up to time t, $t \geq 0$ and $E[N(t)] = \lambda t$. Also, let X_1, X_2, X_3 ... be mutually independent and identically distributed random variables for individual claim amounts, with $Pr(X_i \leq x) = P(x)$ and $E(X_i) = p_1 < \infty$ for all i. Assume $N(t)$ is independent of random variables $\{X_i\}$. Then the aggregate claim process $S_{N(t)}$, defined by $S_{N(t)} = X_1 + X_2 + \ldots + X_{N(t)}$, is said to be compound Poisson.

For a given relative security loading θ (where $\theta \geq 0$), let c be the premium rate. Thus $c = (1+\theta)\, p_1 \lambda$.

The ruin function $\psi(u)$ is defined as the probability that the risk reserve, $u + ct - S_{N(t)}$, is ever negative. Here, u is the amount of risk reserve at time 0.

2. THE FORMULAS

In Bowers et al. [8], two formulas for $\psi(u)$ are presented (see p. 352 and p. 363):

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} \mid T < \infty]} \tag{2.1}$$

$$\int_0^\infty e^{xr}[1 - \psi(x)]\,dx = \frac{\theta p_1 r}{1 + (1+\theta)p_1 r - M_x(r)} \tag{2.2}$$

An explicit evaluation of the denominator in (2.1) is generally not possible; exceptions include the case in which the claim amount distribution is exponential and the case in which $u=0$. For Formula (2.2), an inversion is required to evaluate $\psi(u)$, which again only works for some families of claim amount distributions, such as a mixture of exponential distributions (see [3, p. 45] and [17]). Thus, these two formulas are not practical for general claim amount distributions.

For large values of initial risk reserve u, an efficient method for evaluating $\psi(u)$ is the use of Lundberg's asymptotic formula (see [3], [10], [20], [21], and [45]). However, situations involving large values of u are not very interesting, because in these cases the $\psi(u)$'s are quite small in general (see Appendix 2). In practice, insurance companies are constrained by limited capital and resources, and hence small values of u are the more interesting cases. The purpose of this paper is to examine formulas that can be used to calculate $\psi(u)$ efficiently for relatively small values of u.

Let us consider the case in which the individual claim amount random variables X_i take values on positive integers only; that is, $p(n)=Pr(X_i=n)=c_n$ for all i and $n=1, 2, 3, \ldots$, where $c_n \geq 0$ and $\Sigma_{n=1}^{\infty} c_n = 1$.

Using operational calculus methods, Shiu [48]–[50] derives the following two formulas for $\psi(u)$:

$$\psi(u) = 1 - \frac{\theta e^{au}}{1+\theta}\left\{1 + \sum_{k=1}^{\lfloor u \rfloor} e^{-ak} \sum_{j=1}^{k} \frac{c_k^{*j}\,[a(k-u)]^j}{j!}\right\} \tag{2.3}$$

$$\psi(u) = \frac{\theta e^{au}}{1+\theta} \sum_{k=\lfloor u \rfloor + 1}^{\infty} e^{-ak} \sum_{j=1}^{k} \frac{c_k^{*j}\,[a(k-u)]^j}{j!} \tag{2.4}$$

In these formulas,

$$c_k^{*j} = Pr\left(\sum_{i=1}^{j} X_i = k\right)$$

for $j = 1, 2, 3, \ldots$; $\lfloor u \rfloor$ is the largest integer less than or equal to u; and a is the Lundberg security factor, which is given by $a = \lambda c^{-1} = [(1+\theta)p_1]^{-1}$.

We note that both (2.3) and (2.4) are exact, and we can implement them easily on computers.

3. CONVOLUTION PLANNING

The coefficients $\{c_k^{*j}\}$ in (2.3) and (2.4) can be evaluated by the formula

$$c_k^{*j} = \sum_{m+n=k} c_m^{*(j-1)} c_n$$

(see, for example, [26, p. 402]). Using this formula, a total of $(j-1)$ convolution operations are used to compute the j-th convolution c_k^{*j}, where $k=0$, 1, ... This can result in significant buildup of round-off error in c_k^{*j}, especially when the initial risk reserve u is large and when j increases with u. The number of convolution operations can be reduced significantly by resorting to the more general convolution formula:

$$c_k^{*j} = \sum_{m+n=k} c_m^{*g} c_n^{*h},$$

where $g + h = j$.

For example, to compute c_k^{*8}, one can compute c_k^{*2}, c_k^{*4}, and c_k^{*8} in that order, by using a total of three convolution operations (instead of seven). It turns out that in this case, three is the minimum number of convolution operations required for $j=8$.

The problem of finding the minimum number of convolution operations required for c_k^{*j} is equivalent to finding the minimum number of multiplications for computing x^j, where j is integral. In fact, the problem has been well studied and is known as an "addition chain" in computer literature [30, pp. 441–53]. For example, to calculate x^8 (or c_k^{*8}) using the minimum number of multiplications (or convolution operations) is equivalent to finding the smallest number of additions to produce 8 from 1 $(2=1+1,\ 4=2+2,\ 8=4+4)$.

Finding an optimal "addition chain" for an arbitrary integer j can be tricky and time-consuming, especially when j is large. To avoid the problem of enumerating the optimal "addition chain," we use the following heuristic (see also [35]).

The heuristic stems from the observation that if x is the number of binary digits and y is the total number of 1's in the binary representation of an integer j, then j can be obtained from 1 using exactly $x+y-2$ additions. We first perform $(x-1)$ additions to get 2, 2^2, 2^3, ..., 2^{x-1}, and then use $(y-1)$ additions, combining those powers of 2 (there are y of them) that have a corresponding 1 in the binary representation of j. We illustrate this by using a larger value of j as an example. Let $j=1{,}000=1{,}111{,}101{,}000_2$, then $x=10$, $y=6$, and the number of additions is $10+6-2=14$ $(2=1+1,$

$4=2+2$, $8=4+4$, $16=8+8$, $32=16+16$, $64=32+32$, $128=64+64$, $256=128+128$, $512=256+256$, $768=256+512$, $896=128+768$, $960=64+896$, $992 = 32+960$, $1{,}000=8+992$). Thus, we first use nine additions to obtain 2, 4, ..., 512, and then five operations to add 8, 32, 64, 128, 256, and 512.

Denote the number of convolution operations required for c_k^{*j} using this heuristic by H_j, and the minimum number of convolution operations required for c_k^{*j} by O_j. The following table gives the frequency distribution of the difference, $D=H_j-O_j$, for $j=1$ to 1,024 inclusive (the largest difference of 5 occurs when $j=1{,}023$). We note that although H_j is greater than O_j in most cases, it is of the order $2 \cdot \log_2 j$, which compares favorably to $(j-1)$.

D	Cardinality of $\{j : H_j - O_j = D\}$
0	307
1	385
2	260
3	49
4	22
5	1
Total	1,024

We implement the heuristic for c_k^{*j}, $j=2, 3, \ldots$, in an orderly manner, by calculating c_k^{*j} as a convolution of c_k^{*g} and c_k^{*h}, where $g+h=j$, and g and h are chosen such that $H_g+H_h+1=H_j$. The following algorithm is used to determine g and h from j.

1. If j is a power of 2, set g and h to $j/2$.
2. If j is not a power of 2, find the last 1 in the base 2 representation of j, and let k be its position counted from the right-most digit. Set $g=2^{k-1}$ and $h=j-g$.

We illustrate this algorithm by the following two examples:

1. c_k^{*4}: Since 4 is a power of 2, we set both g and h to 2, and convolute c_k^{*2} and c_k^{*2} to determine c_k^{*4}.
2. c_k^{*5}: Since $5=101_2$, we have $k=1$, $g=1$, and $h=4$. Hence, c_k^{*1} and c_k^{*4} are used to calculate c_k^{*5}.

4. AN IMPLEMENTATION OF FORMULA (2.3)

We implement Formula (2.3) by using the programming language APL, incorporating the heuristic discussed in the last section. A listing of the programs is given in Appendix 1.

The main function to invoke is called RUIN. Other functions include CALCPROB, CREATECSTAR, CREATEP, and SPLIT. Function SPLIT accepts an integer j and determines g and h such that $g+h=j$ and $H_g+H_h+1=H_j$, as described above. Function CREATECSTAR calculates $\{c_k^{*j}\}$ for $j=1, 2, \ldots, \lfloor u \rfloor$ and $k=0, 1, \ldots, \lfloor u \rfloor$. Function CREATEP sets up a vector for claim probabilities, and function CALCPROB sums the series given by Formula (2.3).

We ran the programs for various values of u and θ against the following three claim distributions, which have been referred to in various earlier works. The results are listed in Appendix 2.

(1) $p(x)=1$ for $x=1$ and 0 otherwise (see, for example, [2]). We calculate $\psi(u)$ for $\theta=0.01, 0.02, \ldots, 0.06$ and $u=1, 2, \ldots, 10$.

(2)

x	$p(x)$
4	0.15304533960
6	0.07882237436
8	0.11199119040
10	0.10432698260
12	0.09432769021
14	0.10925807990
16	0.09727308107
20	0.18073466720
25	0.07022059474

This example can be found in [7], [34], [40], and [43].
We calculate $\psi(u)$ for $\theta=0.25, 0.50, 0.75, 1.00$ and $u=0, 25, 50, 75, 100$.

(3)

x	$p(x)$
1	0.5141
2	0.3099
3	0.0639
4	0.0220
5	0.0194
7	0.0096
8	0.0276
10	0.0036
12	0.0041
13	0.0019
15	0.0013
16	0.0226

This example can be found in [5].
We calculate $\psi(u)$ for $\theta=0.10$, 0.20, 0.30, 0.40, 0.50 and $u=100$, 200, 300, 400.

Formula (2.3) involves summing a finite alternating series and works well for relatively small u. However, as the initial risk reserve, u, increases, we run into two problems: run-off error and overflow. The computer and the APL system on which the programs are run maintain approximately 15 decimal digits of precision. The run-off error problem is caused by the fact that as u gets larger, $\exp(u)$ increases rapidly, and consequently an increasing number of decimal digits have to be chopped off. The overflow problem is caused by the fact that each number is stored by using a fixed amount of memory in the computer. Thus there is an upper limit for the magnitude of numbers that can be represented.

Note that the convolution planning that we employ can only delay the buildup of round-off error; it cannot solve the problem.

5. FORMULA (2.4)

Formula (2.4) requires summing an infinite series. Because the series is not alternating, this might be expected to be a good formula for computing $\psi(u)$. Unfortunately, in implementing this formula, we found that the convergence is very slow. Using Example (1) in Section 3, with $\theta=0.01$ and $u=1$, we need to let k go up to about 185,000 before we obtain seven decimal digits of accuracy for $\psi(u)$. In this case, we also encounter run-off error and overflow problems. Therefore, Formula (2.4) is not practical for computing.

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APPENDIX 1

LISTING OF APL PROGRAMS

```
    ∇RUIN[⎕]∇

[0]  X RUIN C;THETA;U;A;P1;P;CSTAR
[1]  ⍝ X is the vector of claim amounts, and C is the associated probabilities
[2]  →(∨/X≤0)/EXIT1
[3]  →(1E¯4≤|1-+/C)/EXIT2
[4]  LP:'Initial risk reserve? (To quit the program, enter a negative number)'
[5]  →(0>U←⎕)/0
[6]  'Enter relative security loading'
[7]  THETA←⎕
[8]  A←÷(1+THETA)×P1←+/X×C
[9]  CREATEP
[10]  CREATECSTAR
[11]  CALCPROB
[12]  3 1 ⍴' '
[13]  →LP
[14]  EXIT1:'Nonpositive claim amounts'
[15]  →0
[16]  EXIT2:'Probabilities do not add up to one'
```

```
    ∇CREATEP[⎕]∇

[0]  CREATEP
[1]  P←(1+⌈/X)⍴0
[2]  P[1+X]←C
```

```
    ∇CALCPROB[⎕]∇

[0]  CALCPROB;X;SUM;T
[1]  SUM←0
[2]  X←0
[3]  LP:→((⌊U)<K←K+1)/END
[4]  T←+/(K↑1↓CSTAR[K+1;])××\(K⍴A×K-U)÷⍳K
[5]  SUM←SUM+T×*-A×K
[6]  →LP
[7]  END:'Claim amounts ',⍕C
[8]  'Probabilities ',⍕X
[9]  'Initial risk reserve ',⍕U
[10]  'Relative security loading ',⍕THETA
[11]  'Probability of ruin ',⍕1-(*A×U)×(1+SUM)×THETA÷1+THETA
```

```
    ∇CREATECSTAR[⎕]∇

[0]  CREATECSTAR;R;I;V;V1;T
[1]  R←1+⌊U
[2]  CSTAR←((R,1)⍴1,(¯1+R)⍴0),R↑P
[3]  I←2
[4]  LP:V←SPLIT I
[5]  J←1
[6]  V1←⍴0
[7]  LP1:V1←V1,CSTAR[;V[2]+1]+.×R↑(¯1+J)↓⌽CSTAR[;V[1]+1]
[8]  →(R≥J←J+1)/LP1
[9]  CSTAR←CSTAR,⌽V1
[10]  →((R-1)≥I←I+1)/LP
[11]
[12]  →0
[13]
[14]  ⍝ In implementing these programs on computers with larger memory
[15]  ⍝ (such as mainframes or mini-computers), the following 2 lines
[16]  ⍝ of code are much more efficient, and can replace lines [5] to [9]
[17]  ⍝ above.
[18]  T←(R,R)↑(¯1+⍳R)⌽((R,R)⍴⌽CSTAR[;V[1]+1]),(R,R)⍴0
[19]  CSTAR←CSTAR,⍉CSTAR[;V[2]+1]+.×T
```

```
    ∇SPLIT[⎕]∇

[0]  Z←SPLIT N;D;T;V
[1]  V←((D←1+⌈2⍟N)⍴2)⊤N
[2]  →(1=+/V)/END
[3]  Z←2⊥(T←D-(⌽V)⍳1)↑V
[4]  V[T+1]←0
[5]  Z←Z,2⊥V
[6]  →0
[7]  END:Z←2⍴N÷2
```

APPENDIX 2

PROBABILITIES OF RUIN

EXAMPLE 1

u	θ					
	0.01	0.02	0.03	0.04	0.05	0.06
1	0.973351	0.947735	0.923100	0.899395	0.876577	0.854602
2	0.954660	0.911928	0.871624	0.833582	0.797650	0.763686
3	0.935920	0.876707	0.821935	0.771222	0.724222	0.680622
4	0.917509	0.842772	0.774975	0.713398	0.657403	0.606423
5	0.899459	0.810151	0.730698	0.659909	0.596747	0.540311
6	0.881765	0.778792	0.688951	0.610432	0.541690	0.481409
7	0.864420	0.748648	0.649590	0.564664	0.491712	0.428928
8	0.847415	0.719671	0.612478	0.522328	0.446346	0.382169
9	0.830745	0.691815	0.577485	0.483166	0.405165	0.340507
10	0.814403	0.665037	0.544492	0.446940	0.367784	0.303386

EXAMPLE 2

u	θ			
	0.25	0.50	0.75	1.00
0	0.800000	0.666667	0.571429	0.500000
25	0.433995	0.232316	0.141606	0.094198
50	0.222739	0.072766	0.030113	0.014607
75	0.114114	0.022685	0.006349	0.002236
100	0.058463	0.007072	0.001339	0.000342

EXAMPLE 3

u	θ				
	0.1	0.2	0.3	0.4	0.5
100	0.522132	0.308428	0.199082	0.137226	0.099443
200	0.307110	0.118771	0.054149	0.027873	0.015734
300	0.180699	0.045752	0.014725	0.005654	0.002482
400	0.091223	0.016231	0.003896	0.001105	0.000383

DISCUSSION OF PRECEDING PAPER

BEDA CHAN:

We congratulate Dr. Seah for his contributions towards the implementation of exact calculation of ruin probabilities. Although many exact formulas are known to theorists, it is algorithms such as the APL programs in the paper that deliver applications to practitioners.

Another well-known group of exact calculations is that ruin probabilities for a combination of exponential claims can be computed exactly as a combination of exponentials. In this discussion, we use a combination of two exponentials to approximate the claim distributions in the three examples in the paper.

In a compound Poisson claim process with claim amounts distributed as a mixture of exponentials

$$p(x) = \sum_{i=1}^{n} A_i \, \beta_i \, e^{-\beta_i x} \tag{1}$$

for $x>0$ where all $A_i>0$ and $\Sigma_{i=1}^{n} A_i = 1$, the ruin probability is also a linear combination of exponentials

$$\psi(u) = \sum_{i=1}^{n} C_i \, e^{-r_i u} \tag{2}$$

where $\{r_1, \ldots , r_n\}$ are solutions to the adjustment coefficient equation

$$(1+\theta)p_1 = \frac{M_X(r) - 1}{r}$$

and $\{C_1, \ldots , C_n\}$ are determined by the partial fractions of

$$\sum_{i=1}^{n} \frac{C_i r_i}{r_i - r} = \frac{\theta}{1 + \theta} \cdot \frac{\dfrac{M_X(r) - 1}{r}}{(1 + \theta) p_1 - \dfrac{M_X(r) - 1}{r}}. \tag{3}$$

See Bowers et al. [1, §12.6] for details. This result was later extended by Dufresne and Gerber [3] to the case in which the claim distribution is a translated (density function moved by τ to the left) combination (where the

A_i's need not be positive of exponentials. They found that the coefficients C_i's are the solution to the system:

$$\sum_{k=1}^{n} \frac{\beta_i}{\beta_i - r_k} C_k = 1, \qquad i = 1, \dots, n, \tag{4}$$

and gave C_k explicitly. An alternative expression for the solution for (1) was given in Chan [2]:

$$C_k = \prod_{\substack{i \neq k \\ i=1}}^{n} \frac{r_i}{r_i - r_k} \prod_{i=1}^{n} \frac{\beta_i - r_k}{\beta_i}. \tag{5}$$

Following Shiu [4] we consider $r \to \beta_k$ in (3) to obtain

$$\sum_{i=1}^{n} \frac{C_i r_i}{\beta_k - r_i} = \frac{\theta}{1 + \theta} \qquad k = 1, \dots, n. \tag{6}$$

To solve (6) consider

$$\sum_{i=1}^{n} \frac{C_i r_i}{x - r_i} = \frac{\theta}{1 + \theta} - \frac{\theta}{1 + \theta} \prod_{i=1}^{n} \frac{(x - \beta_i)}{(x - r_i)}$$

where the two sides are different expressions for the same rational function of (degree $n-1$/degree n) with simple poles $\{r_1, \dots, r_n\}$ and takes the value $\theta/1+\theta$ at $x = \beta_1, \dots, \beta_n$. Multiply by $x - r_k$ and let $x = r_k$ to obtain

$$C_k = \frac{\theta}{1 + \theta} \frac{\prod_{i=1}^{n} (\beta_i - r_k)}{r_k \prod_{\substack{i \neq k \\ i=1}}^{n} (r_i - r_k)} \tag{7}$$

Comparing the two explicit solutions for C_k, (5) and (7), we obtain the following equality, which can be used for checking the roots $\{r_i\}$ of the adjustment coefficient equation.

$$\prod_{i=1}^{n} \frac{r_i}{\beta_i} = \frac{\theta}{1 + \theta} \tag{8}$$

For approximating the three examples in the paper, consider the family:

$$p(x) = \frac{\beta\gamma}{\gamma - \beta}(e^{-\beta x} - e^{-\gamma x}) \qquad \text{for } x \geq 0$$

$$= \frac{\gamma}{\gamma - \beta}\beta e^{-\beta x} - \frac{\beta}{\gamma - \beta}\gamma e^{-\gamma x} \text{ for } x \geq 0 \qquad (9)$$

where $0<\beta<\gamma$. From

$$E(X) = \frac{1}{\beta} + \frac{1}{\gamma}$$

$$\text{Var}(X) = \frac{1}{\beta^2} + \frac{1}{\gamma^2},$$

we notice that for a fixed value of $E(X)$, $\text{Var}(X)$ falls in the range of

$$\frac{E(X)^2}{2} < \text{Var}(X) \leq E(X)^2$$

where the lower bound is not attained—but approached by $\beta \approx \gamma$—and the upper bound is attained by $\gamma = \infty$. In the three examples,

$$E(X) = 1 \qquad \text{Var}(X) = 0$$
$$E(X) = 12.61243786 \qquad \text{Var}(X) = 39.89429478$$
$$E(X) = 2.2896 \qquad \text{Var}(X) = 7.50993184$$

where the variances are all out of range, fitting the first two moments with a member of this class, as described by (9), is not possible. The closest fit is provided by matching $E(X)$ and getting the closest $\text{Var}(X)$:

matching $E(X)$	minimum $\text{Var}(X)$
matching $E(X)$	minimum $\text{Var}(X)$
matching $E(X)$	maximum $\text{Var}(X)$

In Example 1, with $(\beta,\gamma)=(1.999, 2.001)$, the four corners in the ruin probability table are

$$\begin{bmatrix} 0.978130 & 0.879896 \\ 0.868490 & 0.444304 \end{bmatrix}$$

Further squeezing (β,γ) to $(2,2)$ does not improve the four corners towards the exact values in Example 1. Our values are higher because the approximating claim distribution has the same mean but variance $\approx 1/2$, whereas the original $X \equiv 1$ has variance $= 0$.

In Example 2, with $(\beta,\gamma) = (0.15758, 0.15958)$ the four corners of the table are

$$\begin{bmatrix} 0.8 & 0.5 \\ 0.0941233 & 0.00184648 \end{bmatrix}$$

In Example 3, with $(\beta,\gamma) = (1/2.2896, \infty)$, the four corners of the table are

$$\begin{bmatrix} 0.0171486 & 3.171\ 10^{-7} \\ 1.15107\ 10^{-7} & 3.41238\ 10^{-26} \end{bmatrix}$$

The class (9) is the sum of two independent exponentials. In general, these are known as Erlang distributions. We found that the sum of four independent exponentials of parameters (0.29442, 0.31022, 0.32602, 0.34182) has the same mean and variance as Example 2. Formulas (5) or (7) can then be used to compute the exact probability of ruin. We would then have discrete and continuous models for which exact expressions for ruin probabilities are readily computable for contrast and comparison.

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HANS U. GERBER:

Recently Shiu derived Formula (2.3), which gives the probability of ruin for the case in which the claim amount distribution is discrete. The great merit of Seah's paper is that it shows that this formula can be successfully applied.

In his program Seah calculates C_k^{*j} recursively with respect to j, which is the traditional way to calculate convolutions. An alternative possibility is to calculate C_k^{*j} recursively with respect to k (for each j), according to the formulas

$$C_j^{*j} = (C_1)^j$$

$$C_k^{*j} = \frac{1}{C_1} \sum_{n=1}^{k-j} \left(n \frac{j+1}{k-j} - 1 \right) C_{n+1} C_{k-n}^{*j} \quad \text{for } k = j+1, j+2, \ldots$$

See De Pril [1], who also discusses the case in which $C_1 = 0$. I do not know how useful this alternative method might be for computing.

In general, a claim amount distribution is not discrete. Then two methods of discretization are possible. To describe them, we recall Beekman's convolution formula:

$$1 - \psi(u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} (1+\theta)^{-n} H^{*n}(u);$$

here

$$1 - H(u) = \Pi(u)/\Pi(0),$$

with

$$\Pi(x) = \int_x^\infty [1 - P(y)]\, dy = \int_x^\infty (y - x)\, dP(y),$$

which can be interpreted as the stop-loss premium, considered a function of the deductible x.

The *first method* consists of discretizing the function H, which leads to an approximation that can be calculated recursively; see references [24] and [41] in Seah's paper. Dufresne and Gerber [2], [3] carry out this discretization systematically to obtain lower and upper bounds for the probability of ruin.

A *second method* would now be to discretize the claim amount distribution, P, so that Shiu's formula and Seah's program can be applied. If the discretization is carried out appropriately, one can also get lower and upper bounds for the probability of ruin. For example, one might replace the original distribution P by the distribution $\bar{P}$ that is obtained by *dispersing* the probability mass to the adjacent integers without changing the mean; see Gerber and Jones [4]. Then one can prove the following inequalities:

$$\tilde{\Pi}(x) \geq \Pi(x), \text{ but } \tilde{\Pi}(0) = \Pi(0),$$

$$\tilde{H}(u) \leq H(u),$$

$$\tilde{H}^{*n}(u) \leq H^{*n}(u),$$

$$\tilde{\psi}(u) \geq \psi(u).$$

In a similar fashion we can obtain lower bounds for the probability of ruin, for example, by modifying P by *concentrating* the probability mass appropriately.

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ELIAS S.W. SHIU:

Dr. Seah is most kind in giving me credit for Formulas (2.3) and (2.4). These two formulas are easy consequences of the formulas

$$\psi(u) = 1 - \frac{\theta}{1+\theta}\sum_{j=0}^{\infty}\frac{(-a)^j}{j!}E[(u - S_j)_+^j\, e^{a(u-S_j)}], \qquad u \geq 0, \qquad (D.1)$$

and

$$\psi(u) = \frac{\theta}{1+\theta}\sum_{j=1}^{\infty}\frac{a^j}{j!}E[(S_j - u)_+^j\, e^{-a(S_j-u)}], \qquad u \geq 0, \qquad (D.2)$$

respectively. For $\alpha \geq 0$,

$$x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Formulas (D.1) and (D.2) are equivalent to [12, p. 61, formula (17)] and [7, formula (5.55)], respectively. I would like to supplement this paper by proving (D.1) and (D.2). Alternative proofs can be found in [4], [5], [10] and [11].

Define the non-ruin function

$$\phi(u) = 1 - \psi(u).$$

The function ϕ is monotonic increasing; it takes the value 0 on the negative axis and

$$\lim_{u \to \infty} \phi(u) = 1. \tag{D.3}$$

Consider a small time interval $(0, s)$. Under the Poisson hypothesis, the probability that a claim will occur in that interval is $\lambda s + o(s)$. Hence, for $u \geq 0$, the non-ruin function $\phi(u)$ satisfies the relation:

$$\phi(u) = \lambda s E[\phi(u + cs - X)] + (1 - \lambda s)\phi(u + cs) + o(s). \tag{D.4}$$

Dividing (D.4) by s, rearranging and letting s tend to $0+$, we obtain the integro-differential equation

$$0 = \lambda E[\phi(u - X)] + c\phi'(u) - \lambda\phi(u), \quad u > 0,$$

or

$$\phi'(u) = a\{\phi(u) - E[\phi(u - X)]\}, \qquad u > 0. \tag{D.5}$$

The convolution of two functions g and h is defined by

$$(g * h)(x) = \int_{-\infty}^{\infty} g(x - t)\, h(t)\, dt. \tag{D.6}$$

Note that

$$g * h = h * g.$$

If g and h are zero on the negaive axis, then (D.6) becomes

$$(g * h)(x) = \int_0^x g(x - t)\, h(t)\, dt.$$

Let $p(x)$ denote the derivative of the probability distribution function $P(x)$. If $P(x)$ is not differentiable, then $p(x)$ is a *generalized function* [13]. Let $\delta(x)$ denote the Dirac delta function. Then Equation (D.5) becomes

$$\begin{aligned}\phi'(u) &= a[\phi(u) - (\phi * p)(u)] \\ &= a\{[\delta(u) - p(u)] * \phi(u)\}, \quad u > 0. \end{aligned} \tag{D.7}$$

With the definition

$$f(u) = a[\delta(u) - p(u)], \tag{D.8}$$

Equation (D.7) can be written as

$$\phi'(u) = (f * \phi)(u), u > 0. \tag{D.9}$$

We are to seek a function ϕ, which is zero on the negative axis and satisfies conditions (D.3) and (D.9).

Let $f^{*0}(x) = \delta(x)$; for $n = 1, 2, 3, \ldots$, define

$$f^{*n}(x) = f(x) * f^{*(n-1)}(x). \tag{D.10}$$

As

$$(g * h)' = g * (h'), \tag{D.11}$$

$$\frac{d}{du} u_+^0 = \delta(u)$$

and

$$\frac{d}{du} \frac{u_+^n}{n!} = \frac{u_+^{n-1}}{(n-1)!}, \qquad n = 1, 2, 3, \ldots,$$

one can check that (any scalar multiple of) the function

$$\sum_{n=0}^{\infty} \frac{f^{*n}(u) * u_+^n}{n!} \tag{D.12}$$

satisfies Equation (D.9) and is zero on the negative axis. Hence, we have the formula

$$\phi(u) = \phi(0) \sum_{n=0}^{\infty} \frac{f^{*n}(u) * u_+^n}{n!}. \tag{D.13}$$

The value of $\phi(0)$ will be determined later, using condition (D.3).

Substituting

$$f^{*n}(u) = a^n[\delta(u) - p(u)]^{*n}$$

$$= a^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} p^{*j}(u)$$

into (D.13) and interchanging the order of summation yields

$$\phi(u) = \phi(0) \sum_{j=0}^{\infty} \frac{(-1)^j p^{*j}(u)}{j!} * \left[\sum_{n=j}^{\infty} \frac{a^n\, u_+^n}{(n-j)!}\right]$$

$$= \phi(0) \sum_{j=0}^{\infty} \frac{(-a)^j\, p^{*j}(u) * (u_+^j\, e^{au_+})}{j!}$$

$$= \phi(0) \sum_{j=0}^{\infty} \frac{(-a)^j\, E[(u - S_j)_+^j\, e^{a(u-S_j)_+}]}{j!},$$

which is (D.1), provided that

$$\phi(0) = \theta(1 + \theta)^{-1}. \tag{D.14}$$

To prove (D.14), we integrate (D.7) with respect to u from 0 to w:

$$\phi(w) - \phi(0) = a \int_0^w \phi(w - x)\,[1 - P(x)]\,dx. \tag{D.15}$$

Letting w tend to $+\infty$ in (D.15) and applying the Lebesgue dominated convergence theorem, we obtain

$$1 - \phi(0) = a \int_0^\infty [1 - P(x)]dx = ap_1 = \frac{1}{1 + \theta},$$

which gives (D.14).

The simplest way to prove (D.2) seems to be the approach given in [4]; see also [6]. Let $U(t)$ denote the risk reserve at time t,

$$U(t) = u + ct - S_{N(t)}.$$

Since $\theta > 0$,

$$Pr[\lim_{t\to\infty} U(t) = \infty] = 1.$$

If ruin occurs, there is necessarily a last upcrossing of the risk reserve at the level 0. Hence, the probability of ruin can be expressed in terms of the probability of such a last upcrossing of the risk reserve at the level 0. By considering the number of claims n, prior to this last upcrossing, and the time t at which this last upcrossing occurs, we have

$$\psi(u) = \left\{ \int_0^\infty \left[\sum_{n=1}^\infty \frac{e^{-\lambda t}\,(\lambda t)^n}{n!} \right] d_t P^{*n}(u + ct) \right\} [1 - \psi(0)]. \quad \text{(D.16)}$$

(I thank Dr. Beda Chan for helping me figure out this formula.) Let $s = u + ct$. As $\lambda t = a(s-u)$, Formula (D.16) becomes

$$\psi(u) = \left\{ \int_u^\infty \sum_{n=1}^\infty \frac{e^{-a(s-u)}\,[a(s-u)]^n}{n!} d_s P^{*n}(s) \right\} [1 - \psi(0)]$$

$$= \left\{ \int_u^\infty \sum_{n=1}^\infty \frac{e^{-a(s-u)}\,[a(s-u)]^n}{n!} d_s P^{*n}(s) \right\} \frac{\theta}{1 + \theta},$$

which is (D.2).

Consider the special case in which all claim amounts are one, that is,

$$Pr(X_i = 1) = c_1 = 1.$$

Then Formulas (2.3) and (2.4) in the paper become

$$\psi(u) = 1 - \frac{\theta}{1 + \theta} \sum_{j=0}^{\lfloor u \rfloor} \frac{(-1)^j}{j!} \left(\frac{u - j}{1 + \theta}\right)^j \exp\left(\frac{u - j}{1 + \theta}\right) \quad \text{(D.17)}$$

and

$$\psi(u) = \frac{\theta}{1 + \theta} \sum_{j=\lfloor u \rfloor + 1}^{\infty} \frac{1}{j!} \left(\frac{j - u}{1 + \theta}\right)^j \exp\left[-\left(\frac{j - u}{1 + \theta}\right)\right], \quad \text{(D.18)}$$

respectively. Formula (D.17) was given by Erlang in 1909 for a telephone delay problem and by Feller in 1934 in the context of collective risk theory ([1, (5)], [3, XIV.(2.11)], [8, (4.12)]); this result has been rediscovered many times. Formula (D.18) can be found in [1, (10)], [9, 3.3.2.1.(2)], [7, (5.58)], [8, p. 108] and [4, (29)]. Segerdahl [9, p. 283] wrote that a proof of (D.18) had so far only been published for integral values of u, but he had a general proof in manuscript. Arfwedson [1] also gave a formula for the probability of ruin within a fixed finite time period under the assumptions that the claim amounts are unity and that u is an integer. Arfwedson's formula for the finite-time ruin probability can be found in [2, (135)], [9, section 3.8.2.2.4] and [8, (4.20)]. Seal [8, (4.20)] did not require u to be an integer.

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R.J. VERRALL:*

Seah, in common with many authors, considers calculating the probability of eventual ruin by evaluating an infinite sum. Verrall [3] takes a different

*Dr. Verrall, not a member of the Society, is Lecturer, Department of Actuarial Science and Statistics, City University, London, England.

approach that appears to offer a high degree of accuracy and efficiency. This approach, and the direction of some current research, is outlined below.

Consider Equation (2.2) in Seah. This is the moment generating function (m.g.f.) of $1-\psi(x)$ and can therefore be inverted by using Laplace theory. An accurate approximation, called the saddlepoint approximation in statistical applications, was developed by Esscher [1] and Daniels [2]. This can be applied to Equation (2.2) (which is the m.g.f. of a random sum), whenever $M_X(r)$ exists.

A further development also contained in Verrall [3] is the use of the empirical m.g.f. to give a nonparametric or bootstrap approximation to the probability of eventual ruin.

A focus of current research work is to derive an approximation based on Edgeworth expansions that does not rely on the existence of the m.g.f. I hope to be able to report on this in the near future.

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(AUTHOR'S REVIEW OF DISCUSSION)

ERIC S. SEAH:

I thank Dr. Chan, Dr. Gerber, Dr. Shiu, and Dr. Verrall for their discussions, which add immeasurably to the value of the paper.

Dr. Chan points out that, for a claim amount distribution that is a combination of exponentials, there is an exact formula for ruin probabilities, which involves a combination of exponentials. He illustrates how a combination of exponentials can approximate a claim amount distribution. One situation in which this technique can be extremely useful is when only the (sample) mean and variance of the claim amounts are available.

Dr. Gerber comments that there is an alternative way to calculate c_k^{*j}, which is to proceed recursively with respect to k (for each j). This approach is useful if we are interested in c_k^{*j}, for some fixed j. However, if we need to compute c_k^{*j} for all $k=1, 2, \ldots \lfloor u \rfloor$ and $j=1, 2, \ldots, k$, the recursive method does not seem to offer any significant computational advantage over the method used in the paper (note that u is the initial risk reserve).

Dr. Gerber and Dr. Dufresne demonstrated their computer software, "Risky Business—An Eduational Software," at the University of Manitoba, and it has proved to be a very useful teaching tool.

Dr. Shiu points out that Formulas (2.3) and (2.4) can be obtained easily from Formulas (D.1) and (D.2). It is interesting to note that the summation terms in (D.1) and (D.2) are closely related to the following two results (see [3]):

$$\sum_{j=0}^{\infty} E\left[\frac{a^j(S_j + x)^{j-1}}{j!} e^{-a(S_j+x)}\right] = \frac{1}{x} \tag{1}$$

$$\sum_{j=0}^{\infty} E\left[\frac{a^j\,(S_j + x)^j}{j!} e^{-a(S_j+x)}\right] = \frac{1}{1 - a\mu}. \tag{2}$$

Here, $x \neq 0$, $E[X_i] = \mu$, $S_j = X_1 + X_2 + \ldots + X_j$, and $|a\mu| < 1$.

Both Dr. Gerber and Dr. Shiu refer to the non-ruin function

$$\phi(u) = 1 - \psi(u).$$

Consider the aggregate loss process $\{S(t) - ct,\ t \geq 0\}$, which measures the excess of aggregate claims over the premiums received. It turns out that the non-ruin function is the distribution function of the maximum of this aggregate loss process; that is, $F_L(u) = \phi(u)$, where

$$L = \max_{t \geq 0} \{S(t) - ct\};$$

see [1, pp. 361–62]. By considering the times when the aggregate loss process assumes new record highs, the maximal aggregate loss L can be decomposed as

$$L = L_1 + L_2 + \ldots + L_N,$$

where N is the number of record highs, and L_i is the difference of aggregate loss between the $(i-1)$-th and i-th record highs (the 0-th record high is 0). Since N has a geometric distribution and $L_1, L_2, \ldots$ are mutually independent, identically distributed, and independent of N, Beekman's convolution formula as referred to by Dr. Gerber can be easily derived. Note that the distribution function $H(u)$ in Beekman's convolution formula is the distribution function of L_i; see also formula (12.5.3) on p. 360 of [1].

As pointed out by Dr. Gerber, in the case of a non-discrete claim amount distribution, one can use the method of discretization before applying Formula (2.3). In addition to the references given by Dr. Gerber, interested readers may wish to refer to [2], [4], [5].

Dr. Verrall is correct to point out that many formulas for ruin probabilities involve evaluating an infinite sum and hence are approximate in nature. However, Formula (2.3) has only a finite number of terms and is an exact formula. Note that Formula (2.4), which does involve an infinite series, has been found to be impractical for computing.

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