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THE SENSITIVITY OF CASH-FLOW ANALYSIS TO THE CHOICE OF STATISTICAL MODEL FOR INTEREST RATE CHANGES

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ABSTRACT

In "Statistical Tests of the Lognormal Distribution as a Basis for Interest Rate Changes" [4], Becker rejects the hypothesis that successive interest rate changes are independently, identically lognormally distributed. This paper explores some of the implications of this conclusion on cash-flow analysis. First, the stable Paretian distribution is described. The reasons for using the normal distribution to model the logarithm of the ratio of consecutive interest rates are explored and found to justify the stable Paretian as well. Stable Paretian parameters are estimated for a series of interest rates, and statistical tests are performed.

Herzog's discussion of Becker's paper points out that a model can be wrong but still useful. The questions he raises are (1) What other models are available? and (2) How robust is the model to misparameterization? Having presented an alternative model, this paper goes on to test the sensitivity of cash-flow analysis to the change from the lognormal model to the best fitting stable Paretian. Two cash-flow tests are performed with identical assumptions except for the distribution of interest rate changes. The results differ drastically.

I. INTRODUCTION

The actuarial profession is developing a consensus on the proper methodology for analyzing the cash flows of an insurance company; this consensus is expressed in [19]. One common assumption in cash-flow analysis is that the change in interest rate from period to period is the result of a stationary lognormal stochastic process (see [8], [19], [20], [21], [25], and [26]). Becker tested four assumptions inherent in this assumption and found strong evidence for rejecting three of them [4]. Specifically, he found that the random variables

$$J_t = \log_e \frac{I_{t+1}}{I_t}, \qquad (1)$$

where I_t is the interest rate at time t, are not stochastically independent, are not normally distributed, and do not have a constant variance over time.

Actuarial Standard of Practice No. 7, entitled "Performing Cash Flow Testing for Insurers" [2], states that "[t]he actuary should consider the sensitivity of the model to the effect of variations in key assumptions. . . ." According to Deakins and Tulin [10], "The interest scenario represents the single most important assumption the actuary will make. . . ." This paper extends the results of Becker by analyzing the sensitivity of the results of cash-flow analysis to this key assumption, the distribution of interest rate changes. The assumption that interest rate changes are the result of a stationary stochastic process, that is, that they are independent and identically distributed, is retained throughout this paper, while the assumption of lognormality is questioned. However, since the independence and identical distribution assumptions are both rejected by Becker's tests, the results presented should not be taken as a prescription for remedying current cash-flow methodology. Rather, they should be taken as an indication of the extreme sensitivity of the results of cash-flow testing and other types of risk analysis to model assumptions that may not reflect reality.

II. DISTRIBUTION OF INTEREST RATE CHANGES

A. *Possible Distributions for Interest Rate Changes*

Bachelier demonstrated that, given certain assumptions, economic time series would follow the Gaussian or normal distribution [3]. His argument has since been modified to the hypothesis that for many types of commodities, the logarithm of the quotient of price changes over successive periods is normally distributed. This hypothesis, which can be called the lognormal hypothesis, is also commonly applied to interest rate changes. That is, if I_t is a random variable representing the interest rate at time t, then the random variables

$$J_t = \log_e \frac{I_{t+1}}{I_t} \tag{2}$$

are assumed to be independently, identically, normally distributed with mean zero. As a corollary,

$$I_{t+1} = I_t e^{sZ_t}, \tag{3}$$

where s is the standard deviation of the stochastic process of interest rate changes and the Z_t are independent random variables with the standard normal distribution.

The assumption that changes in interest rates or prices of securities or commodities are independent, identically distributed lognormal random variables is inherent in a substantial amount of financial theory, including the capital asset pricing model [28] and the Black-Scholes model for pricing options [6]. It is also virtually universally accepted by actuaries for cash-flow analysis, whether used directly or as the basis for the probabilities in a Markov chain of yield curve shift (see, for example, [8], [19], [20], [21], [25], and [26]).

In 1963 Mandelbrot proposed an alternative to the lognormal hypothesis [24]. In analyzing changes in the price relativities (that is, the logarithm of the ratio of successive prices) of cotton, he found that the tails of the empirical distribution were too large to have come from a normal distribution. His proposal, called the stable Paretian hypothesis, is that non-normal members of the family of distributions called stable Paretian, not the normal distribution, are appropriate for describing changes in relativities of commodity prices or interest rates. The stable Paretian distribution, of which the normal distribution is a special case, is described in the next section.

B. Properties of the Stable Paretian Distribution

The stable Paretian distribution, first described in 1925 by Lévy [23] and later described in English by Gnedenko and Kolmogorov [16], is a family of probability distributions with the following characteristic function:

$$\begin{aligned}\phi(t) = E[e^{itX}] &= \exp\left\{ i\delta t - \gamma|t|^{\alpha}\left[1 + i\beta\left(\frac{t}{|t|}\right)\omega(\alpha, t)\right]\right\} \\ &= \exp\left\{ i\delta t - |ct|^{\alpha}\left[1 + i\beta\left(\frac{t}{|t|}\right)\omega(\alpha, t)\right]\right\},\end{aligned} \tag{4}$$

where

$$\omega(\alpha, t) = \begin{Bmatrix} \tan\left(\dfrac{\alpha\pi}{2}\right), & \alpha \neq 1 \\ \dfrac{(2 \ln |t|)}{\pi}, & \alpha = 1 \end{Bmatrix}, \tag{5}$$

$i=\sqrt{-1}$, and $\exp(w)$ means e^w. A particular member of the family is completely specified by four parameters: α, β, $\gamma=c^\alpha$, and δ.

Parameter α is limited by $0<\alpha\leq 2$. When $\alpha=2$, the distribution is normal, with $\mu=\delta$ and $\sigma^2=2\gamma$, since the characteristic function of the normal distribution is

$$\phi(t) = E[e^{itX}] = \exp\left\{ i\mu t - \frac{\sigma^2 t^2}{2} \right\}. \tag{6}$$

Thus, the stable Paretian distribution is a more general category, which includes the normal distribution as a specific case. As α decreases from 2 toward 0, the distribution becomes more "fat-tailed." This property led to its being named after the Pareto distribution, which is also "fat-tailed" and is discussed in [7] and [17]. Parameter α is referred to as the characteristic exponent, since it appears as an exponent in the characteristic function.

Parameter β is a parameter of skewness and is limited by $-1\leq\beta\leq 1$. When $\beta=0$, the distribution is symmetric. When $\alpha=2$, that is, in the normal case, the term involving β vanishes, so the distribution is symmetric.

Parameter γ is equal to c^α, and consequently γ and c are both required to be positive. The variance of a stable Paretian distribution with $\alpha<2$ is infinite. Other measures of dispersion are needed. Parameter γ plays a role similar to that played by σ^2, variance, in the normal case. For instance, the sum of n stable Paretian random variables whose distributions have the same α's and β's has a distribution with γ equal to the sum of the γ's of the component distributions. Parameter c, which is equal to $\gamma^{(1/\alpha)}$, plays a role similar to that played by σ, standard deviation, in the normal case. For instance, a stable Paretian random variable is transformed into a standardized stable Paretian random variable by first subtracting δ and then dividing by c. Thus c is a scaling parameter.

Parameter δ is the location parameter. When $\alpha>1$, as is generally the case in economic and natural phenomena, δ is equal to the mean of the distribution. When $\alpha\leq 1$, the mean of the distribution is infinite.

There are no simple expressions for the probability density function (pdf), except for three cases. First, when $\alpha=2$, the distribution is normal with $\mu=\delta$ and $\sigma^2=2\gamma$. Secondly, when $\alpha=1$ and $\beta=0$, the distribution is Cauchy with c and γ (since $\gamma=c^\alpha=c$) both equal to the semi-interquartile range, that is, half of the difference between the 75th percentile and the 25th percentile. And thirdly, when $\alpha=0.5$, $\beta=1$, $\gamma=1$, and $\delta=0$, the pdf can also be specified [24]. Thanks to Bergstrom, however, there are series expansions that can be numerically evaluated to approximate the pdf and the cdf [5], [13]. These are discussed in Section III-A.

The function

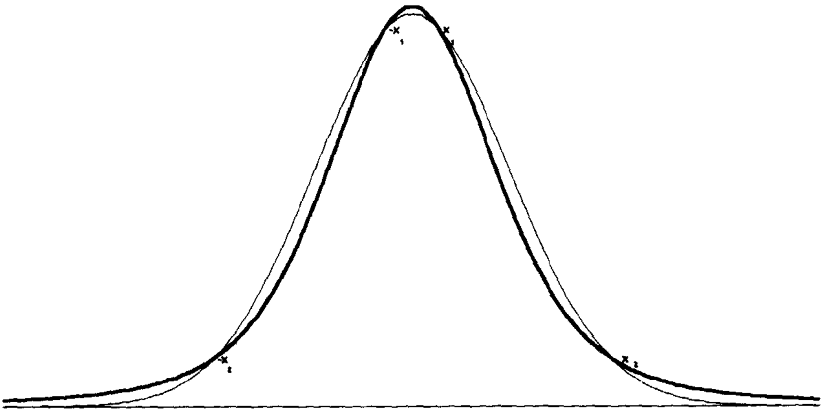
$$U=\frac{X-\delta}{c} \tag{7}$$

transforms a stable Paretian random variable X into a standardized stable Paretian random variable U with $\delta=0$ and $\gamma=c^\alpha=1$. This is analogous to the standardization of normal random variables, but a standardized stable Paretian with $\alpha=2$ is not a standardized normal. It is, in fact, a normal with $\mu=0$ and $\sigma^2=2\gamma=2$.

A graph comparing the pdf's of the standardized stable symmetric ($\beta=0$) Paretian distribution with $\alpha=2$ (normal) versus $\alpha=1.5$ illustrates the differences between the two (see Figure 1). First, for $0\leq x<x_1$, $f_{1.5}(x)>f_{2.0}(x)$, where $f_\alpha(x)$ is the pdf of a standardized symmetric stable Paretian distribution with characteristic exponent of α. Secondly, for $x_1<x<x_2$, $f_{1.5}(x)<f_{2.0}(x)$. And finally, for $x_2<x$, $f_{1.5}(x)>f_{2.0}(x)$. Similar observations can be made about the negative range, since both pdf's are symmetric. Table 1 shows several values from the cumulative distribution functions for the standardized stable Paretian distribution for five values of α. As this table shows, as the value of α decreases, the amount of probability in the tails increases. In other words, the probability of extreme events increases as α decreases.

This property of "fat-tailedness" in Mandelbrot's data led him to this family of distributions as an alternative to the normal. Becker's data also exhibited this property, which he described as high positive kurtosis. Cash-flow analysis and other types of ruin theory are concerned with the probabilities of extreme situations, that is, those in the tails. The use of

FIGURE 1
PROBABILITY DENSITY FUNCTIONS OF STANDARDIZED STABLE PARETIAN DISTRIBUTION FOR TWO VALUES OF ALPHA



a model such as the lognormal hypothesis, which ignores the high positive kurtosis found in the data, will lead to misleading results, since the probability of these extreme situations will be understated by that model.

An instance in which a model reflecting high positive kurtosis could have proved useful is the stock market crash of October 1987. When the stable Paretian distribution is used as a model for stock price changes, large changes in price happen more abruptly than when the lognormal distribution is used as a model. In other words, the stable Paretian model better reflects the existence of stock market crashes by assigning a higher probability to tail events than the lognormal does. Of course, the stable Paretian model merely reflects the higher probability of such a crash—it could not have predicted the timing. Nevertheless, strategies such as portfolio insurance that break down when there are large abrupt changes in security prices may have been avoided by using the stable Paretian hypothesis [27].

An important characteristic of stable Paretian distributions is that they generalize the central limit theorem. The central limit theorem states that, as n grows without bound, the distribution of the sum of n independent random variables—each with finite variance—approaches the normal distribution with mean equal to the sum of the means and variance equal to the sum of the variances of the random variables. A generalization of

TABLE 1

CUMULATIVE DISTRIBUTION FUNCTIONS OF STANDARDIZED SYMMETRIC STABLE DISTRIBUTIONS FOR FIVE VALUES OF ALPHA

u	$\alpha=1.00$	$\alpha=1.25$	$\alpha=1.50$	$\alpha=1.75$	$\alpha=2.00$
0.00	0.5000	0.5000	0.5000	0.5000	0.5000
0.05	0.5159	0.5148	0.5144	0.5142	0.5141
0.10	0.5317	0.5296	0.5287	0.5283	0.5282
0.15	0.5474	0.5443	0.5430	0.5424	0.5422
0.20	0.5628	0.5589	0.5572	0.5565	0.5562
0.25	0.5780	0.5733	0.5713	0.5704	0.5702
0.30	0.5928	0.5875	0.5853	0.5843	0.5840
0.35	0.6072	0.6016	0.5991	0.5881	0.5977
0.40	0.6211	0.6153	0.6127	0.6117	0.6114
0.45	0.6346	0.6288	0.6262	0.6251	0.6248
0.50	0.6476	0.6420	0.6394	0.6384	0.6382
0.55	0.6601	0.6549	0.6524	0.6515	0.6513
0.60	0.6720	0.6674	0.6651	0.6643	0.6643
0.65	0.6835	0.6796	0.6776	0.6770	0.6771
0.70	0.6944	0.6914	0.6898	0.6894	0.6897
0.75	0.7048	0.7028	0.7017	0.7015	0.7021
0.80	0.7148	0.7138	0.7133	0.7135	0.7142
0.85	0.7243	0.7244	0.7245	0.7251	0.7261
0.90	0.7332	0.7347	0.7355	0.7365	0.7377
0.95	0.7419	0.7445	0.7461	0.7476	0.7491
1.00	0.7500	0.7540	0.7563	0.7583	0.7603
1.10	0.7652	0.7717	0.7759	0.7790	0.7817
1.20	0.7788	0.7881	0.7940	0.7984	0.8019
1.30	0.7913	0.8030	0.8108	0.8165	0.8210
1.40	0.8025	0.8166	0.8263	0.8334	0.8389
1.50	0.8128	0.8290	0.8406	0.8491	0.8556
1.60	0.8222	0.8402	0.8536	0.8635	0.8710
1.70	0.8308	0.8505	0.8655	0.8767	0.8853
1.80	0.8386	0.8599	0.8763	0.8888	0.8985
1.90	0.8458	0.8683	0.8861	0.8998	0.9104
2.00	0.8524	0.8761	0.8950	0.9098	0.9213
2.20	0.8642	0.8896	0.9103	0.9269	0.9401
2.40	0.8743	0.9010	0.9228	0.9407	0.9552
2.60	0.8831	0.9105	0.9331	0.9517	0.9670
2.80	0.8908	0.9187	0.9415	0.9604	0.9761
3.00	0.8976	0.9256	0.9484	0.9673	0.9831
3.20	0.9036	0.9316	0.9542	0.9727	0.9882
3.40	0.9089	0.9369	0.9590	0.9970	0.9919
3.60	0.9137	0.9414	0.9631	0.9803	0.9945
3.80	0.9181	0.9455	0.9665	0.9830	0.9964
4.00	0.9220	0.9490	0.9694	0.9852	0.9977
4.40	0.9289	0.9550	0.9742	0.9884	0.9991
4.80	0.9346	0.9599	0.9778	0.9906	0.9997
5.20	0.9395	0.9639	0.9807	0.9922	0.9999
5.60	0.9438	0.9672	0.9830	0.9933	1.0000
6.00	0.9474	0.9701	0.9848	0.9943	1.0000
7.00	0.9548	0.9755	0.9882	0.9958	1.0000
8.00	0.9604	0.9794	0.9905	0.9968	1.0000
9.00	0.9648	0.9823	0.9922	0.9974	1.0000
10.00	0.9683	0.9846	0.9934	0.9979	1.0000
11.00	0.9711	0.9863	0.9943	0.9982	1.0000
12.00	0.9735	0.9878	0.9950	0.9985	1.0000
13.00	0.9756	0.9890	0.9956	0.9987	1.0000
14.00	0.9773	0.9900	0.9961	0.9989	1.0000
15.00	0.9788	0.9908	0.9965	0.9990	1.0000
20.00	0.9841	0.9936	0.9977	0.9994	1.0000

this states that the stable Paretian distributions are the only possible limiting distributions for sums of independent random variables. This generalization does not require that the variance of the random variables be finite [16].

Another important property of the stable Paretian distributions is stability, from which they get the first part of their name. A property of the normal distribution is that the sum of n independent normally distributed random variables is itself normally distributed with a mean equal to the sum of the means of the summands and a variance equal to the sum of the variances of the summands. This property is referred to as stability. A generalization of this property is that the sum of n independent random variables from a stable Paretian distribution with the same α's and β's has a stable Paretian distribution with α and β the same as in the summands, γ equal to the sum of the γ's of the summands, and δ equal to the sum of the δ's of the summands.

Since the stable Paretian distribution shares many of the important traits of the normal, it should certainly be considered as a candidate for modeling the distribution of J_t. In fact, since the normal distribution is a special case of the stable Paretian distribution, it is already being used for this purpose, but with no thought given to the possibility that α may not be equal to 2, that is, that the lognormal hypothesis may be false. The next section discusses why the normal distribution has been used for this purpose in the past and why the entire family of stable symmetric Paretian distributions should be considered.

C. Considerations in Selecting a Distribution for Interest Rate Changes

Four reasons can be given for using the normal distribution to model changes in the relativities of interest rates. This section explores these reasons and discusses the stable Paretian distribution as an alternative.

First, it is often asserted that interest rates change because of many small pieces of information moving through the markets. If it is assumed that these changes have distributions that are mutually independent, then the central limit theorem would encourage us to use the normal distribution for the change in interest rates, which is the sum of these small changes. Likewise, if we assume that interest rates change due to many small multiplicative changes, then the lognormal distribution would be appropriate for modeling these changes, since the logarithm of the product is equal to the sum of the logarithms of the multiplicands.

A generally unstated assumption in this line of reasoning is that the change caused by each of the pieces of information has a distribution that has a finite variance. There is no reason to make that assumption a priori, and since the generalization of the central limit theorem states that any stable Paretian distribution can be a limiting distribution, this argument applies equally well to the lognormal hypothesis or the stable Paretian hypothesis. Empirical tests must be used to choose between the hypotheses. These tests, described in Section D, reject the lognormal hypothesis and generally support the stable Paretian hypothesis. As a corollary, the changes in interest rates caused by some of the pieces of information must have distributions with infinite variance.

Secondly, the normal distribution has the property of stability, or invariance under addition, discussed in Section B above. This is a desirable trait because, for instance, if the distribution of weekly changes in interest rates is known, then the distribution for annual changes is known. Likewise, if the distribution for each different instrument in a portfolio is known and the changes for the different instruments can be modeled by mutually independent random variables, then the distribution for the portfolio as a whole is known. Because this property is not unique to the normal members of the stable Paretian family, this argument of convenience also applies equally well to either the lognormal hyothesis or the stable Paretian hypothesis. Note that this property is unique to the stable Paretian distributions, so that the use of any other distribution sacrifices this property. For example, while a mixture of normals [17, pp. 49–51] has fatter tails than the normal distribution, it does not have the property of stability.

Thirdly, the normal distribution is convenient because the function for the pdf is known and the cdf can be approximated easily by series expansion. This justification for the use of the normal distribution is based on convenience, not empirical evidence. In addition, a similar argument can be made for the non-normal stable Paretian distributions, since Bergstrom has derived series for numerically approximating the pdf and cdf of any of the symmetric members of the family. Today's personal computers have the ability to quickly generate these functions and to perform Monte Carlo simulations using non-normal stable Paretian distributions. Section III-A discusses the methodology involved.

Fourthly, one can argue that the normal distribution is "close enough" to the true underlying distribution and that the extra work involved in using a non-normal member of the stable Paretian family is not justified. This question is raised by Herzog in his discussion of Becker's paper

[4, p. 59]. The questions raised by this justification are (1) Close enough for what purpose? and (2) What evidence is there that the normal distribution is indeed close enough? For many purposes, including cash-flow analysis, statements are made that are highly dependent on what happens in the tails of distributions. Section III of this paper shows that there is a substantial difference between the results of two cash-flow analyses identical in every way except for the assumption of the distribution of interest rate changes, raising serious doubt that the normal distribution is "close enough."

Finally, note that the difference between the lognormal hypothesis and the stable Paretian hypothesis is merely one of estimating α. Proponents of the lognormal hypothesis assert, a priori, that α is equal to 2. The stable Paretian hypothesis states that α is not equal to 2. The next section of this paper mentions some of the empirical work that has been published estimating α for various commodities, concluding with parameter estimates and statistical tests for 30-year Treasury bonds that are used in the cash-flow analysis in Section III.

D. Statistical Tests of the Stable Paretian Hypothesis

In addition to Becker's paper [4], the lognormal hypothesis has been rejected in research on such commodities as common stocks, Treasury bills, and cotton (see [11], [24], [28], [29], [31], and [33]). These papers rejected the lognormal hypothesis based on parameter estimates of α. Roll found values of α generally between 1 and 1.5 for Treasury bills. Teichmoeller found values of α between 1.6 and 1.7 for a combination of 30 common stocks. Mandelbrot, using older, less sophisticated techniques, estimated α at 1.7 for cotton prices. Phenomena outside of economics, such as rainfall and sunspots, have also been shown to have non-normal stable Paretian distributions [28].

Table 2 contains yield rates on 30-year Treasury bonds from 1977 to 1990 [15]. First, these semiannual (bond-equivalent) rates are converted to effective annual rates called i_t (where t is the number of months beyond December 1976). For example, i_1, the effective annual rate for January 1977, is $(1 + 0.0755/2)^2 - 1 = 0.076925$. The realized values of the random variable J_t, namely,

$$j_t = \log_e \frac{i_{t+1}}{i_t}, \tag{8}$$

are shown in Table 3, ordered from smallest to largest. For example, $j_1 = i_2/i_1 = \log_e(0.078586/0.076925) = 0.021363$. This j_1 turns out to be

TABLE 2

AVERAGE YIELD TO MATURITY ON 30-YEAR TREASURY BONDS, 1977–1990

Year and Month	Yield	Year and Month	Yield	Year and Month	Yield	Year and Month	Yield	Year and Month	Yield
1977		1980		1983		1986		1989	
Jan.	7.55	Jan.	10.60	Jan.	10.63	Jan.	9.40	Jan.	8.93
Feb.	7.71	Feb.	12.13	Feb.	10.88	Feb.	8.93	Feb.	9.01
Mar.	7.80	Mar.	12.34	Mar.	10.63	Mar.	7.96	Mar.	9.17
Apr.	7.73	Apr.	11.40	Apr.	10.48	Apr.	7.39	Apr.	9.03
May	7.80	May	10.36	May	10.53	May	7.52	May	8.83
June	7.64	June	9.81	June	10.93	Jun	7.57	June	8.27
July	7.64	July	10.24	July	11.40	July	7.27	July	8.08
Aug.	7.68	Aug.	11.00	Aug.	11.82	Aug.	7.33	Aug.	8.12
Sep.	7.64	Sep.	11.34	Sep.	11.63	Sep.	7.62	Sep.	8.15
Oct.	7.77	Oct.	11.59	Oct.	11.58	Oct.	7.70	Oct.	8.00
Nov.	7.85	Nov.	12.37	Nov.	11.75	Nov.	7.52	Nov.	7.90
Dec.	7.94	Dec.	12.40	Dec.	11.88	Dec.	7.37	Dec.	7.90
1978		1981		1984		1987		1990	
Jan.	8.18	Jan.	12.14	Jan.	11.75	Jan.	7.39	Jan.	8.26
Feb.	8.25	Feb.	12.80	Feb.	11.95	Feb.	7.54	Feb.	8.50
Mar.	8.23	Mar.	12.69	Mar.	12.38	Mar.	7.55	Mar.	8.56
Apr.	8.34	Apr.	13.20	Apr.	12.65	Apr.	8.25	Apr.	8.76
May	8.43	May	13.60	May	13.43	May	8.78	May	8.73
June	8.50	Jun	12.96	June	13.44	June	8.57	June	8.46
July	8.65	July	13.59	July	13.21	July	8.64	July	8.50
Aug.	8.47	Aug.	14.17	Aug.	12.54	Aug.	8.97	Aug.	8.86
Sep.	8.47	Sep.	14.67	Sep.	12.29	Sep.	9.59	Sep.	9.03
Oct.	8.67	Oct.	14.68	Oct.	11.98	Oct.	9.61	Oct.	8.86
Nov.	8.75	Nov.	13.35	Nov.	11.56	Nov.	8.95	Nov.	8.54
Dec.	8.88	Dec.	13.45	Dec.	11.52	Dec.	9.12	Dec.	8.24
1979		1982		1985		1988			
Jan.	8.94	Jan.	14.22	Jan.	11.45	Jan.	8.83		
Feb.	9.00	Feb.	14.22	Feb.	11.47	Feb.	8.43		
Mar.	9.03	Mar.	13.53	Mar.	11.81	Mar.	8.63		
Apr.	9.08	Apr.	13.37	Apr.	11.47	Apr.	8.95		
May	9.19	May	13.24	May	11.05	May	9.23		
June	8.92	June	13.92	June	10.44	June	9.00		
July	8.93	July	13.55	July	10.50	July	9.14		
Aug.	8.98	Aug.	12.77	Aug.	10.56	Aug.	9.32		
Sep.	9.17	Sep.	12.07	Sep.	10.61	Sep.	9.06		
Oct.	9.85	Oct.	11.17	Oct.	10.50	Oct.	8.89		
Nov.	10.30	Nov.	10.54	Nov.	10.06	Nov.	9.02		
Dec.	10.12	Dec.	10.54	Dec.	9.54	Dec.	9.01		

TABLE 3

ORDER STATISTICS OF THE NATURAL LOGARITHMS OF THE RATIOS OF CONSECUTIVE INTEREST RATES ON 30-YEAR TREASURY BONDS SHOWN IN TABLE 2

t	OS_t	t	OS_t	t	OS_t	t	OS_t	t	OS_t	t	OS_t
1	−0.117362	31	−0.027834	61	−0.006265	91	0.005877	121	0.017860	151	0.043245
2	−0.098192	32	−0.026300	62	−0.005320	92	0.006750	122	0.017993	152	0.043948
3	−0.098182	33	−0.025797	63	−0.004430	93	0.006836	123	0.019232	153	0.045444
4	−0.081515	34	−0.024723	64	−0.003563	94	0.006881	124	0.019421	154	0.045770
5	−0.079678	35	−0.024096	65	−0.003504	95	0.007181	125	0.019942	155	0.047510
6	−0.075699	36	−0.023855	66	−0.002476	96	0.007705	126	0.020463	156	0.048991
7	−0.072763	37	−0.023708	67	−0.001134	97	0.008306	127	0.021363	157	0.051728
8	−0.066891	38	−0.022886	68	0.000000	98	0.008367	128	0.021402	158	0.054540
9	−0.061176	39	−0.021821	69	0.000000	99	0.008441	129	0.022229	159	0.057531
10	−0.059587	40	−0.021469	70	0.000000	100	0.008693	130	0.022414	160	0.061722
11	−0.058271	41	−0.021119	71	0.000000	101	0.008722	131	0.023585	161	0.063561
12	−0.058073	42	−0.020744	72	0.000000	102	0.009114	132	0.023828	162	0.067025
13	−0.055891	43	−0.020517	73	0.000706	103	0.009187	133	0.023855	163	0.068350
14	−0.054343	44	−0.019421	74	0.000769	104	0.009381	134	0.023937	164	0.073195
15	−0.053673	45	−0.019358	75	0.001145	105	0.010440	135	0.029229	165	0.073445
16	−0.052442	46	−0.018944	76	0.001350	106	0.010640	136	0.030038	166	0.090382
17	−0.051407	47	−0.018069	77	0.001794	107	0.010954	137	0.030367	167	0.138547
18	−0.049751	48	−0.017818	78	0.002132	108	0.011319	138	0.030821		
19	−0.047337	49	−0.016667	79	0.002495	109	0.011620	139	0.031268		
20	−0.043881	50	−0.015727	80	0.002759	110	0.011826	140	0.031490		
21	−0.041173	51	−0.015126	81	0.003401	111	0.012311	141	0.035884		
22	−0.038326	52	−0.014577	82	0.003761	112	0.013547	142	0.036394		
23	−0.037569	53	−0.012824	83	0.004815	113	0.014835	143	0.037192		
24	−0.036708	54	−0.012283	84	0.004845	114	0.014987	144	0.037200		
25	−0.036495	55	−0.011319	85	0.004881	115	0.015066	145	0.038257		
26	−0.033024	56	−0.010690	86	0.005036	116	0.015778	146	0.038290		
27	−0.032077	57	−0.010085	87	0.005320	117	0.017191	147	0.039513		
28	−0.030480	58	−0.009187	88	0.005644	118	0.017364	148	0.040638		
29	−0.030038	59	−0.009114	89	0.005706	119	0.017674	149	0.042361		
30	−0.028929	60	−0.008897	90	0.005844	120	0.017757	150	0.043194		

the 127th smallest such $i_t (t=1, 2, \ldots, 167)$, so the 127th entry in Table 3 is 0.021363. These are the data that are used in the subsequent analysis.

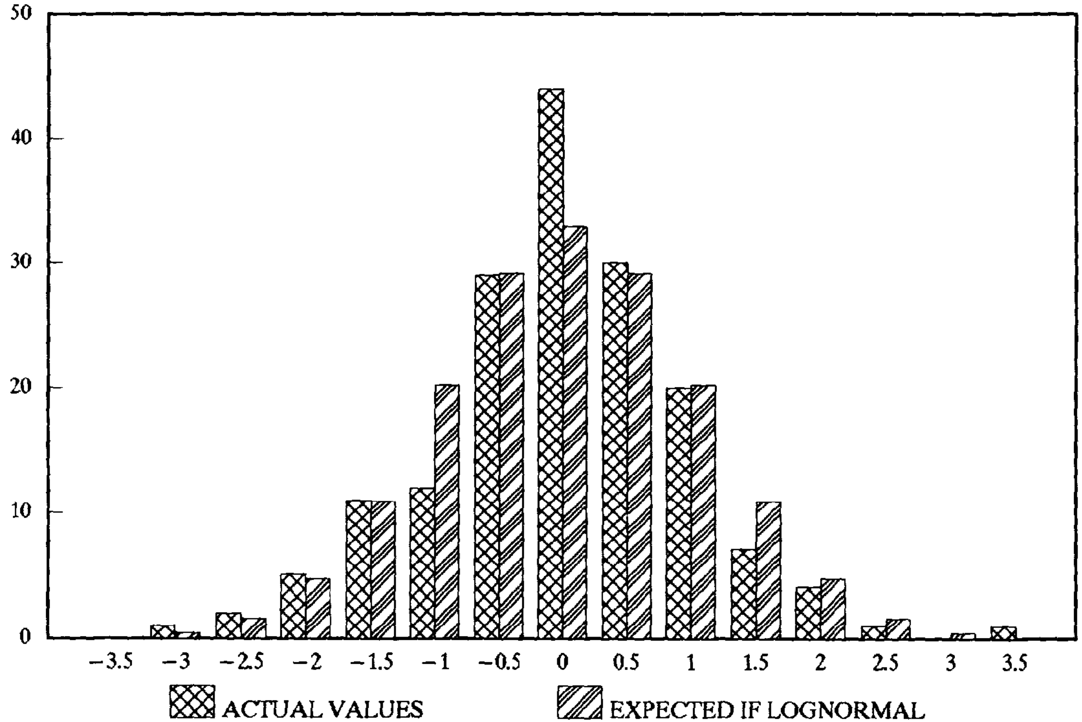
Figure 2 compares the empirical distribution of the j_t from Table 3 with the normal curve. The figures on the horizontal axis are the midpoints of each interval measured in sample standard deviations from the sample mean. Each interval has a width of 0.5 sample standard deviations. For instance, the middle interval ranges from −0.25 sample standard deviations between the sample mean to +0.25 sample standard deviations above the sample mean. A casual look at this graph raises doubt about the lognormal hypothesis. The number of values observed in the center interval, that is, from −0.25 to +0.25 sample standard deviations from the sample mean, is 44, much higher than the 33 that would be expected if the distribution were normal. In the intervals centered at 1 sample standard deviation above and 1 sample standard deviation below the sample mean, there were a total of 32 values, compared with the 40 that would be expected in these two intervals of a normal sample. And finally, in all the intervals centered 3 or more sample standard deviations from the mean, there were a total of 2 observed values compared with an expected number of about 1.

Even without statistical tests and parameter estimation, the data appear to be too "peaked" and "fat-tailed" to have been from a normal distribution. Again, this result is consistent with Becker's findings that "[e]ach [data set] has the same pattern. . . . [T]he frequency of data near the mean is too high [to be normal]; the frequency away from the mean is too low; and, significantly, many points are more than three standard deviations from the mean" [4, p. 17]. Thus, the lognormal hypothesis is rejected, and a good candidate for an alternative appears to be the stable Paretian hypothesis, which would better model the observed peaked middle and fat tails.

The stable Paretian alternative is suggested by Becker as one of three possible explanations for the results he observed, and it is consistent with his three major findings. First, it explains the high positive kurtosis from which he rejected the hypothesis of normality. Secondly, it explains the lack of a constant variance. When the real distribution has infinite variance, the sample variance still exists, but it becomes meaningless. Thirdly, it explains the conclusion of the lack of independence. The autocorrelation coefficients, on which this conclusion was based, also become meaningless if the underlying distribution has infinite variance.

FIGURE 2

ACTUAL VALUES OF *J* COMPARED WITH EXPECTED VALUES BASED ON LOGNORMAL DISTRIBUTION



This informal test of goodness of fit will be formalized with a χ^2 test for both the normal and the stable Paretian distributions after stable Paretian parameters have been estimated, a process much more complicated and less efficient than in the normal case.

1. Parameter Estimates for the Stable Paretian Distribution

The process of parameter estimation is covered in some detail for the monthly changes in 30-year Treasury yields from 1977 to 1990 shown in Table 3, and results are also presented for the average interest rate on long-term Treasury bonds from January 1953 to December 1976, shown in Table 4 [15]. The realized values of j_t for this series of data, ordered from smallest to largest, are shown in Table 5. For both series of data, it was assumed that the distributions were symmetric, that is, that $\beta=0$. This was done largely because the procedure of parameter estimation and simulation has not yet been worked out for skewed distributions. An examination of the data indicates that they might be slightly skewed to the left. Further research needs to be done to determine the effects of assuming a symmetric distribution if, in fact, the distribution is skewed.

a. Estimation of c, *the Scale Parameter*

Fama and Roll describe procedures for estimating c [14], α [14], and δ [13] for a symmetric stable Paretian distribution. Table 1 shows that the cumulative distribution function of a standardized symmetric stable Paretian distribution at $u=0.827$, that is, $F_\alpha\ (0.827)$, is approximately equal to 0.72 regardless of the value of α. For a nonstandardized distribution, $F_\alpha(\delta+0.827c)$ is approximately 0.72. As a result of symmetry, $F_\alpha(\delta-0.827c)$ is approximately equal to $1-0.72=0.28$. Thus, c can be estimated as

$$\hat{c} = \frac{1}{2\,(0.827)}\,[OS_{0.72(n+1)} - OS_{0.28(n+1)}], \tag{9}$$

where $\hat{c}$ is the estimator of c, n is the sample size from which c is being estimated, and the OS_x means the x-th order statistic. For nonintegral values of x, OS_x is determined by linear interpolation.

For the 30-year Treasury bond rates in question, shown in Table 2, there were 168 months of interest rates. There are 167 values of J_t, so $\hat{c}$ is calculated as follows:

$$\hat{c} = \frac{1}{2\,(0.827)}\,[OS_{0.72(167+1)} - OS_{0.28(167+1)}]$$

$$= \frac{1}{1.654}\,[OS_{120.96} - OS_{47.04}]$$

$$= \frac{1}{1.654}\,[0.017856 - (-0.018059)]$$

$$= 0.021714 \qquad (10)$$

b. Estimation of α, the Characteristic Exponent

As can be seen in Table 1 and Figure 1, the effect of decreasing α is that the tails become thicker; that is, for large values of u, $F_{\alpha}(u)$ decreases as α decreases from 2 to 1. Fama and Roll [14] found that a robust procedure for estimating α is to define

$$\hat{z}_{0.96} = \frac{1}{2\hat{c}}\,[OS_{0.96(n+1)} - OS_{0.04(n+1)}], \qquad (11)$$

where $\hat{z}_{0.96}$ is an estimator of the 96th percentile (and the negative of the 4th percentile) of a standardized symmetric stable Paretian distribution. The estimator, $\hat{\alpha}$, is the value of α for which the 96th percentile of the distribution is equal to $\hat{z}_{0.96}$. In symbols, the equation $F_{\hat{\alpha}}(\hat{z}_{0.96})=0.96$ is solved for $\hat{\alpha}$.

For the data being considered, $\hat{z}_{0.96}$ was estimated as follows:

$$\hat{z}_{0.96} = \frac{1}{2\,(0.021714)}\,[OS_{0.96(167+1)} - OS_{0.04(167+1)}]$$

$$= \frac{1}{0.043428}\,[OS_{161.28} - OS_{6.72}]$$

$$= \frac{1}{0.043428}\,[0.064531 - (-0.073585)]$$

$$= 3.1803 \qquad (12)$$

Since the 96th percentile of a standardized symmetric stable Paretian distribution with $\alpha=1.580$ is 3.1803, that is, $F_{1.580}(3.1803)=0.96$, $\hat{\alpha}$, the estimator of α, is equal to 1.580.

TABLE 4

AVERAGE YIELD TO MATURITY ON LONG-TERM TREASURY BONDS, 1953–1976

Year and Month	Yield	Year and Month	Yield	Year and Month	Yield	Year and Month	Yield
1953		1956		1959		1962	
Jan.	2.80	Jan.	2.88	Jan.	3.90	Jan.	4.08
Feb.	2.83	Feb.	2.85	Feb.	3.92	Feb.	4.09
Mar.	2.89	Mar.	2.93	Mar.	3.92	Mar.	4.01
Apr.	2.97	Apr.	3.07	Apr.	4.01	Apr.	3.89
May	3.12	May	2.97	May	4.08	May	3.88
June	3.13	June	2.93	June	4.09	June	3.90
July	3.04	July	3.00	July	4.11	July	4.02
Aug.	3.05	Aug.	3.17	Aug.	4.10	Aug.	3.97
Sep.	3.01	Sep.	3.21	Sep.	4.26	Sep.	3.94
Oct.	2.87	Oct.	3.20	Oct.	4.11	Oct.	3.89
Nov.	2.86	Nov.	3.30	Nov.	4.12	Nov.	3.87
Dec.	2.79	Dec.	3.40	Dec.	4.27	Dec.	3.87
1954		1957		1960		1963	
Jan.	2.69	Jan.	3.34	Jan.	4.37	Jan.	3.88
Feb.	2.62	Feb.	3.22	Feb.	4.22	Feb.	3.92
Mar.	2.53	Mar.	3.26	Mar.	4.08	Mar.	3.93
Apr.	2.48	Apr.	3.32	Apr.	4.17	Apr.	3.97
May	2.54	May	3.40	May	4.16	May	3.97
June	2.55	June	3.58	June	3.99	June	4.00
July	2.47	July	3.60	July	3.86	July	4.01
Aug.	2.48	Aug.	3.63	Aug.	3.79	Aug.	3.99
Sep.	2.52	Sep.	3.66	Sep.	3.82	Sep.	4.04
Oct.	2.54	Oct.	3.73	Oct.	3.91	Oct.	4.07
Nov.	2.57	Nov.	3.57	Nov.	3.93	Nov.	4.10
Dec.	2.59	Dec.	3.30	Dec.	3.88	Dec.	4.14
1955		1958		1961		1964	
Jan.	2.68	Jan.	3.24	Jan.	3.89	Jan.	4.15
Feb.	2.77	Feb.	3.26	Feb.	3.81	Feb.	4.14
Mar.	2.78	Mar.	3.25	Mar.	3.78	Mar.	4.18
Apr.	2.82	Apr.	3.12	Apr.	3.80	Apr.	4.20
May	2.81	May	3.14	May	3.73	May	4.16
June	2.82	June	3.19	June	3.88	June	4.13
July	2.91	July	3.36	July	3.90	July	4.13
Aug.	2.95	Aug.	3.60	Aug.	4.00	Aug.	4.14
Sep.	2.92	Sep.	3.75	Sep.	4.02	Sep.	4.16
Oct.	2.87	Oct.	3.76	Oct.	3.98	Oct.	4.16
Nov.	2.89	Nov.	3.70	Nov.	3.98	Nov.	4.12
Dec.	2.91	Dec.	3.80	Dec.	4.06	Dec.	4.14

TABLE 4—*Continued*

Year and Month	Yield	Year and Month	Yield	Year and Month	Yield	Year and Month	Yield
1965		1968		1971		1974	
Jan.	4.14	Jan.	5.18	Jan.	5.91	Jan.	6.56
Feb.	4.16	Feb.	5.16	Feb.	5.84	Feb.	6.54
Mar.	4.15	Mar.	5.39	Mar.	5.71	Mar.	6.81
Apr.	4.15	Apr.	5.28	Apr.	5.75	Apr.	7.04
May	4.14	May	5.40	May	5.96	May	7.07
June	4.14	June	5.23	June	5.94	June	7.03
July	4.15	July	5.09	July	5.91	July	7.18
Aug.	4.19	Aug.	5.04	Aug.	5.78	Aug.	7.33
Sep.	4.25	Sep.	5.09	Sep.	5.56	Sep.	7.30
Oct.	4.27	Oct.	5.24	Oct.	5.46	Oct.	7.22
Nov.	4.34	Nov.	5.36	Nov.	5.44	Nov.	6.93
Dec.	4.43	Dec.	5.65	Dec.	5.62	Dec.	6.78
1966		1969		1972		1975	
Jan.	4.43	Jan.	5.74	Jan.	5.62	Jan.	6.68
Feb.	4.61	Feb.	5.86	Feb.	5.67	Feb.	6.61
Mar.	4.63	Mar.	6.05	Mar.	5.66	Mar.	6.73
Apr.	4.55	Apr.	5.84	Apr.	5.74	Apr.	7.03
May	4.57	May	5.85	May	5.64	May	6.99
June	4.63	June	6.06	June	5.59	June	6.86
July	4.74	July	6.07	July	5.57	July	6.89
Aug.	4.80	Aug.	6.02	Aug.	5.54	Aug.	7.06
Sep.	4.79	Sep.	6.32	Sep.	5.70	Sep.	7.29
Oct.	4.70	Oct.	6.27	Oct.	5.69	Oct.	7.29
Nov.	4.74	Nov.	6.51	Nov.	5.50	Nov.	7.21
Dec.	4.65	Dec.	6.81	Dec.	5.63	Dec.	7.17
1967		1970		1973		1976	
Jan.	4.40	Jan.	6.86	Jan.	5.94	Jan.	6.94
Feb.	4.47	Feb.	6.44	Feb.	6.14	Feb.	6.92
Mar.	4.45	Mar.	6.39	Mar.	6.20	Mar.	6.87
Apr.	4.51	Apr.	6.53	Apr.	6.11	Apr.	6.73
May	4.76	May	6.94	May	6.22	May	6.99
June	4.86	June	6.99	June	6.32	June	6.92
July	4.86	July	6.57	July	6.53	July	6.85
Aug.	4.95	Aug.	6.75	Aug.	6.81	Aug.	6.79
Sep.	4.99	Sep.	6.63	Sep.	6.42	Sep.	6.70
Oct.	5.18	Oct.	6.59	Oct.	6.26	Oct.	6.65
Nov.	5.44	Nov.	6.24	Nov.	6.31	Nov.	6.62
Dec.	5.36	Dec.	5.97	Dec.	6.35	Dec.	6.39

TABLE 5

ORDER STATISTICS OF THE NATURAL LOGARITHMS OF THE RATIOS OF CONSECUTIVE INTEREST RATES ON LONG-TERM TREASURY BONDS SHOWN IN TABLE 4

t	OS_t	t	OS_t	t	OS_t	t	OS_t	t	OS_t
1	−0.079312	31	−0.027479	61	−0.013301	91	−0.005772	121	0.000000
2	−0.064212	32	−0.026541	62	−0.012928	92	−0.005662	122	0.000000
3	−0.062999	33	−0.025632	63	−0.012895	93	−0.005475	123	0.000000
4	−0.059933	34	−0.024954	64	−0.012640	94	−0.005204	124	0.000000
5	−0.055881	35	−0.022832	65	−0.012088	95	−0.005137	125	0.000000
6	−0.055434	36	−0.022562	66	−0.011231	96	−0.005050	126	0.000000
7	−0.047975	37	−0.022251	67	−0.011216	97	−0.004595	127	0.000000
8	−0.044898	38	−0.020978	68	−0.010706	98	−0.004534	128	0.000000
9	−0.044239	39	−0.020933	69	−0.010546	99	−0.004175	129	0.000000
10	−0.042145	40	−0.020891	70	−0.010437	100	−0.003918	130	0.000000
11	−0.041708	41	−0.020085	71	−0.010339	101	−0.003719	131	0.000000
12	−0.041144	42	−0.019952	72	−0.010296	102	−0.003634	132	0.001673
13	−0.039348	43	−0.019392	73	−0.010249	103	−0.003577	133	0.001736
14	−0.036887	44	−0.019190	74	−0.010237	104	−0.003515	134	0.002437
15	−0.036749	45	−0.019093	75	−0.010099	105	−0.003411	135	0.002437
16	−0.036217	46	−0.018766	76	−0.009995	106	−0.003145	136	0.002443
17	−0.035927	47	−0.018498	77	−0.009761	107	−0.003103	137	0.002455
18	−0.035845	48	−0.018475	78	−0.009668	108	−0.003097	138	0.002473
19	−0.035299	49	−0.018396	79	−0.009028	109	−0.002935	139	0.002473
20	−0.035179	50	−0.018233	80	−0.008945	110	−0.002599	140	0.002522
21	−0.034603	51	−0.017953	81	−0.008394	111	−0.002461	141	0.002573
22	−0.034431	52	−0.017822	82	−0.008066	112	−0.002437	142	0.002599
23	−0.034085	53	−0.017627	83	−0.007979	113	−0.002437	143	0.002605
24	−0.033446	54	−0.017396	84	−0.007917	114	−0.002431	144	0.002688
25	−0.033364	55	−0.016235	85	−0.007660	115	−0.002426	145	0.003225
26	−0.033169	56	−0.015105	86	−0.007614	116	−0.002110	146	0.003309
27	−0.032407	57	−0.015012	87	−0.007375	117	−0.001790	147	0.003577
28	−0.032074	58	−0.014844	88	−0.007312	118	−0.001781	148	0.003628
29	−0.030679	59	−0.013659	89	−0.006150	119	0.000000	149	0.003954
30	−0.029399	60	−0.013565	90	−0.005804	120	0.000000	150	0.004065

TABLE 5—*Continued*

t	OS_t	t	OS_t	t	OS_t	t	OS_t	t	OS_t
151	0.004326	181	0.007955	211	0.016100	241	0.026223	271	0.041194
152	0.004378	182	0.007959	212	0.016195	242	0.026916	272	0.042674
153	0.004435	183	0.008079	213	0.016434	243	0.027471	273	0.044176
154	0.004437	184	0.008147	214	0.017479	244	0.027504	274	0.044349
155	0.004744	185	0.008305	215	0.018114	245	0.027882	275	0.045790
156	0.004823	186	0.008373	216	0.018287	246	0.028866	276	0.047023
157	0.004869	187	0.008573	217	0.018386	247	0.029414	277	0.049371
158	0.004869	188	0.008981	218	0.018571	248	0.030101	278	0.049615
159	0.004892	189	0.009691	219	0.019118	249	0.030602	279	0.049643
160	0.004928	190	0.009714	220	0.020099	250	0.031020	280	0.052033
161	0.004964	191	0.009808	221	0.020748	251	0.031640	281	0.052342
162	0.005037	192	0.009872	222	0.020986	252	0.032377	282	0.053407
163	0.005152	193	0.009995	223	0.021038	253	0.032623	283	0.054364
164	0.005165	194	0.010226	224	0.021044	254	0.032996	284	0.054568
165	0.005191	195	0.010356	225	0.021129	255	0.033052	285	0.055541
166	0.005191	196	0.010732	226	0.021481	256	0.033204	286	0.061903
167	0.005327	197	0.011816	227	0.022017	257	0.033254	287	0.069588
168	0.005621	198	0.012445	228	0.022042	258	0.033608		
169	0.006203	199	0.012577	229	0.022769	259	0.033781		
170	0.006418	200	0.012639	230	0.022922	260	0.034382		
171	0.006439	201	0.012727	231	0.022939	261	0.035785		
172	0.006946	202	0.013192	232	0.023397	262	0.036132		
173	0.006994	203	0.013541	233	0.023510	263	0.036388		
174	0.007079	204	0.013751	234	0.023682	264	0.037838		
175	0.007302	205	0.014233	235	0.023752	265	0.038154		
176	0.007418	206	0.014367	236	0.023784	266	0.038544		
177	0.007438	207	0.014385	237	0.024009	267	0.038678		
178	0.007473	208	0.015922	238	0.024055	268	0.039798		
179	0.007603	209	0.015957	239	0.024792	269	0.040273		
180	0.007802	210	0.016026	240	0.025565	270	0.041119		

The estimation of α serves not only as a parameter estimate, but also as a test of the lognormal hypothesis, since a value far from 2 would not be expected from data generated from a normal distribution. If the lognormal hypothesis were true, the statistic $\hat{z}_{0.96}$ would be approximately normally distributed with mean of about 2.5 and standard deviation of about 0.28. Since the actual value was 3.1803, the lognormal hypothesis can be rejected at the 1% significance level. Put another way, the critical region of this test in terms of $\hat{\alpha}$ is $\{\hat{\alpha}:\hat{\alpha}<1.59\}$. Since $\hat{\alpha}=1.580$, the lognormal hypothesis is rejected at the 1% significance level.

c. *Estimation of δ, the Location Parameter*

Fama and Roll [13] found that the 50% truncated mean, that is, the arithmetic average of the middle 50% of the order statistics, of the sample is a more robust estimator of δ than the sample mean. This is because the sample mean is more influenced by the "outlying" values in the "fat tails." The 50% truncated mean can be thought of as a compromise between the mean, which is too influenced by the outliers, and the median, which fails to incorporate the magnitude of the values around it. For the data in question, the 50% truncated mean is 0.0024613. In the cash-flow analysis that follows, a value of 0 was used for δ. In the past the mean of the lognormal distribution for interest rate changes has generally been assumed to be 0, with the justification that the expected value at any time in the future was equal to the starting value. This lack of bias in either direction can be a desirable property and will be preserved by the assumption that $\delta=0$. Note that the assumption of a mean equal to 0 was not rejected by Becker's statistical tests [4, p. 29].

2. *Parameter Estimates for Long-Term Interest Rates 1953–1976*

The data shown in Tables 4 and 5 yielded the following estimates for the three stable Paretian parameters:

$$c = 0.014227$$
$$\alpha = 1.592$$
$$\delta = 0.0027875$$

It is noteworthy that both samples yielded quite similar values of α. The fact that these values are significantly different from 2 is strong evidence that the lognormal hypothesis should be rejected in the case of interest rate changes on long-term Treasury bonds. The large discrepancy in c's between the two sets of data can be described as an increase in volatility. This increase in volatility manifests itself to users of the lognormal hypothesis as an increase in the sample standard deviation between the two sets of data. It is important to realize that c, not α, is a measure of volatility. The absence of a finite variance requires the use of this measure of volatility. Parameter α is a measure of "fat-tailedness." From this analysis, α appears to be more stable over time than c.

3. χ^2 Goodness-of-Fit Test for Lognormal and Stable Paretian Hypotheses

Hsu, Miller, and Wichern [18] propose two statistical tests, in addition to one suggested by Fama and Roll [14], which a series of data should be able to pass if it is generated by a stable Paretian stochastic process. The first of these is a χ^2 test, which tests goodness of fit and also provides a means for refining the parameter estimates. Note that one test with which Becker rejects the lognormal hypothesis is a χ^2 test [4]. The purpose of the test described here is not to duplicate his work, but to compare the fit of the stable Paretian distribution with that of the normal.

The test consists of dividing the data from Table 3 into 13 categories with the following borders: $(-\infty, -5.5\hat{c}+\hat{\delta}]$, $(-5.5\hat{c}+\hat{\delta}, -4.5\hat{c}+\hat{\delta}]$, $(-4.5\hat{c}+\hat{\delta}, -3.5\hat{c}+\hat{\delta}]$, ... $(+4.5\hat{c}+\hat{\delta}, +5.5\hat{c}+\hat{\delta}]$, $(+5.5\hat{c}+\hat{\delta}, +\infty)$. Then, the following statistic is defined

$$X^2 = \sum_{i=1}^{13} \frac{(n_i - n_i^*)^2}{n_i^*}, \qquad (13)$$

where n_i and n_i^* are the actual and expected number of observations, respectively, in each interval. If the data are from a stable Paretian distribution, then for large n, X^2 is distributed approximately as a χ^2 random variable with 10 degrees of freedom. Hsu, Miller, and Wichern recommend the use of 12 degrees of freedom for conservatism, since the boundaries are themselves dependent on parameter estimates.

The results of this test for the lognormal hupothesis and for the stable Paretian parameters estimated from the 1977–1990 data are as follows:

	c	X^2
Lognormal	0.02619	22.08
Stable Paretian	0.02171	11.08

The stable Paretian hypothesis passes this test at the 5% significance level. In fact, the value of X^2 is lower than the expected value of the statistic if the data fit the distribution. The lognormal hypothesis is rejected at the 5% significance level. This is consistent with the results of Becker's χ^2 tests.

Hsu, Miller, and Wichern discuss the inefficiency of Fama and Roll's method of parameter estimation and propose a minimum χ^2 refinement. They suggest that the X^2 statistic be calculated for a number of values of c and α around $\hat{c}$ and $\hat{\alpha}$. Then the values that produce the minimum value of X^2 are taken as refined estimates of c and α. This procedure was attempted, but since only slightly smaller values of X^2 were attained and since the resulting values of α and c were quite close to the original estimates, the original estimates were retained for the cash-flow analysis of Section III.

4. Tests I and II

Fama and Roll [14] suggest a test for stability which Hsu, Miller, and Wichern [18] dub Test I. In this test, nonoverlapping groups of k observations are summed, and α is estimated for the resulting sums. In other words, a new series K_s is defined by

$$K_s = \sum_{i=k(s-1)+1}^{sk} J_i, \quad s = 1, 2, \ldots, m \tag{14}$$

where m is the greatest integer in n/k. Parameter α is then estimated for increasing values of k by the procedure described above. If the underlying data are from a stable Paretian distribution, then the estimates of α should be about the same as k increases. However, the converse is not true; that is, nonstable Paretian data can pass Test I.

A series of data generated partly from one normal distribution and partly from another normal distribution, with unequal variances, can pass Test I. It is sometimes asserted that the variance of interest rate changes shifted in the late 1970s because of a shift in Federal Reserve policy. If this were true and the underlying processes were normal, then the resulting series would have $\hat{\alpha}$ significantly lower than 2 and might pass Test I. Test II, proposed by Hsu, Miller, and Wichern, is identical to Test I, except that the data are randomized before they are summed. The randomization decreases the likelihood that data from a normal process with parameters shifting over time will pass the test.

These tests were performed on the 1977–1990 30-year Treasury data shown in Table 3, with $k=4$. The preliminary estimate of α in Test I was 1.647. Fama and Roll point out that the procedure described above for estimating α is biased for small sample sizes [14]. Since the sample size for Tests I and II is 41, the preliminary estimate of α must be increased by 0.09 to eliminate this bias. The revised estimate of α for Test I is 1.737. For Test II, the preliminary estimate was 1.829 and the revised estimate was 1.919. The original estimate of α, that is, when $k=1$, was 1.580. Since the estimates increased fairly substantially, these tests suggest that the data may not be generated by a stationary stable Paretian stochastic process. Note, however, that these estimates are based on a sample size of only 41. Neither of the papers referenced quantifies the critical region for either of these tests.

The results of Test II suggest that the data may not be generated by a stationary stochastic process. In other words, the distribution of interest rate changes may change over time. Hsu, Miller, and Wichern propose an alternative to the stable Paretian hypothesis that states that there are "subperiods of homogeneous behavior" during which the normal distribution or a mixture of normals should be adequate. Shifts in the underlying reality occur periodically and result in shifts in the model parameters. Although these authors prefer a normal distribution or a mixture of normals, any stable Paretian distribution could be used. These authors prefer the notion that the shifts in parameters are discrete. As an alternative, interest rates could be modeled by a distribution with continuously shifting parameters.

In either case, several problems are raised that have not been adequately solved. First, there is no procedure for estimating how frequently the parameters of the distribution shift. If they shift continuously, then it is easy to have a perfect fit with the data by simply using a mixture of degenerate distributions (that is, normal distributions with zero variance), although the results are not very meaningful. Secondly, it is not clear what probability distribution should be used to model the parameter shift. Thirdly, it is not clear how to estimate the parameters of the distribution that is chosen to model the shifts. This problem is tied in with the first problem of parameter estimation. Fourthly and most importantly, the use of a combination of normal distributions or a combination of stable Paretian distributions eliminates the property of stability, or invariance under addition. For instance, if daily changes in interest rates can be modeled by a combination of normals, that gives no easy way to model monthly or annual changes [17]. This last consideration was very important to Mandelbrot.

More research needs to be done on the nature of changes in interest rates. While the stable Paretian distribution fits the data much better than the lognormal, it appears that an even better model may be a stable Paretian with parameters changing discretely or continuously with the possibility of stochastic dependence. Because of the difficulties mentioned above, the remainder of this paper conforms with the currently common actuarial practice of assuming stationarity of parameters and independence.

E. *Reasons Why the Stable Paretian Hypothesis Has Been Slow to Gain Acceptance*

Although the stable Paretian hypothesis was proposed nearly 30 years ago, it has not gained widespread acceptance. This section examines some of the reasons for this and comments on their validity.

The most obvious reasons for the lack of acceptance of the stable Paretian hypothesis when Mandelbrot's paper [24] was first published in 1963 are those mentioned by Fama [11]. He notes "[t]he absence of explicit expressions for the density functions . . ." and states that "[t]he statistical intractability of these distributions is, at this point, probably the most important shortcoming of the Stable Paretian Hypothesis." In Fama and Miller's 1972 finance textbook [12], the evidence for the stable Paretian hypothesis is presented, but most of the book is developed using the lognormal hypothesis, apparently for the reason stated above. Bergstrom's numeric expansions for the pdf and the cdf of the stable Paretian distributions had been published in 1952, but computers capable of using them were not widely available.

A related reason is the relative inefficiency of the procedure of parameter estimation. While computers speed up the process, the parameters are not estimable with the precision that normal parameters are. Nevertheless, this is no reason to use the normal distribution if the data do not fit it.

A third reason why the stable Paretian distributions are infrequently used is that they have infinite variance. Many people are averse to the idea of using a distribution with an infinite variance. They assume that there are some upper and lower bounds on the underlying variable and that therefore the variance exists. However, Fama and Roll [14] demonstrate that variables with a very large but finite variance behave much like ones with infinite variance.

Peters [27] states another reason why lognormality has been used in spite of the evidence: "[D]uring the 1970s, and particularly during the 1980s, the EMH [Efficient Market Hypothesis, which assumes lognormally distributed returns] was generally taught as fact. Because of the large number of MBAs earned during the 1980s, a perception that the EMH is a proven truth has resulted." In other words, many of the theoretical results in the field of finance were taught as facts with insufficient emphasis on the underlying assumptions. At the same time, evidence mounted that these assumptions were not true.

In summary, while the stable Paretian hypothesis has not gained wide acceptance, the evidence supporting it is stronger than the evidence supporting the lognormal hypothesis. To justify the continued use of the lognormal hypothesis, it would be necessary to assert that, while interest rate changes are not exactly independent and identically, lognormally distributed, the lognormal distribution is "close enough" for modeling purposes. The remainder of this paper tests this assertion by presenting a comparison of the results of a cash-flow analysis under both the lognormal hypothesis and the stable Paretian hypothesis, in other words, a test of the sensitivity of cash-flow analysis to changes in the characteristic exponent.

III. A CASH-FLOW ANALYSIS COMPARING THE STABLE PARETIAN HYPOTHESIS WITH THE LOGNORMAL HYPOTHESIS

A. *Procedure for Monte Carlo Simulation*

In general, Monte Carlo simulation begins with the generation of a sequence of random numbers Y_t from the uniform distribution on (0,1). These numbers are then mapped onto the domain of the cdf of the given distribution using the inverse of the cdf. That is, the sequence of random numbers, U_t, from the given distribution is generated as $U_t=F^{-1}(Y_t)$.

In general, simple formulas are not known for the cdf of the stable Paretian distribution. However, Bergstrom derived series expansions for the pdf, which Fama and Roll [13] integrated to arrive at the cdf. The following formulas can be used to numerically approximate the cdf of a standardized, symmetric stable Paretian distribution.

$$F_\alpha(u) = F_\alpha\left(\frac{x-\delta}{c}\right) = \frac{1}{2} + \frac{1}{\pi\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma\left(\frac{2k-1}{\alpha}\right)}{(2k-1)!}u^{2k-1} \qquad (15)$$

$$F_\alpha(u) = F_\alpha\left(\frac{x-\delta}{c}\right)$$
$$= 1 + \frac{1}{\pi}\sum_{k=1}^{n}(-1)^k\frac{\Gamma(\alpha k)}{k!u^{\alpha k}}\sin\left(\frac{k\pi\alpha}{2}\right) - \int_u^\infty R(u)du, \qquad (16)$$

where, for some positive constant M,

$$\left|\int_u^\infty R(u)du\right| < M\left(\frac{u^{-\alpha(n+1)}}{\alpha(n+1)}\right). \qquad (17)$$

Formula (15) converges for small absolute values of u, while Formula (16) is asymptotic for large values of u. Fama and Roll recommend using Formula (15) for values of u where $|u| \leq -4+5\alpha$ and Formula (16) for values of u where $|u| > -4+5\alpha$. Since these formulas are for standardized variables, the inverse of the transformation in Formula (7) must be made. Thus, we have

$$X_t = \delta + cU_t = \delta + cF^{-1}(Y_t). \qquad (18)$$

This is analogous to the procedure for generating nonstandardized normal random variables.

The Appendix provides a very accurate and easily programmed routine for generating values of the Γ (gamma) function.

To calculate values of $F^{-1}(y)$, a table of values of $F(u)$ was created. For a given value of y, this table was searched until two consecutive values of $F(u)$ were found that surround y. Then, the interval between the two $F(u)$'s was searched using the bisection algorithm until $F^{-1}(y)$ was found to the desired degree of accuracy. If a value of y was greater (less) than the largest (smallest) value of $F(u)$ in the table, then the largest (smallest) value of u was used as $F^{-1}(y)$.

B. Assumptions for the Cash-Flow Analysis

A model company was created for the cash-flow analysis comparison. As of December 31, 1990, this company has one product, a single-premium deferred annuity with the following traits:

Number of Policies	1,000
Fund Value of Each Policy	$10,000
Total Reserve	$10,000,000
Surrender Charge	None
Interest Guarantee	4%
Reserve Method	Equal to Account Balance

The asset portfolio backing this product contains the following investments:

30-Year 9.5% GNMAs	$ 8,000,000
1-Year Treasuries	2,000,000
Total Assets	10,000,000

While an extremely poor match between assets and liabilities was deliberately chosen, it is certainly not unusual for companies to back annuities with long-term assets.

The following yields to maturity are assumed to be in effect on December 31, 1990, at the beginning of the projection:

1-Year Treasuries	7.00%
5-Year Treasuries	7.50
20- to 30-Year Treasuries	8.25
Current Coupon GNMAs	9.50

The Treasury yields are stated on a bond-equivalent basis. The GNMA yields are nominal rates, compounded monthly. The following interest rates are assumed on 5-year Treasuries at the end of the indicated years.

December 31, 1989	7.75%
December 31, 1988	9.09
December 31, 1987	8.45
December 31, 1986	6.67

It is assumed that $J_t = \log_e(I_{t+1}/I_t)$, where I_t is the annualized yield to maturity of 30-year Treasuries at time t (in years), is governed by a symmetric stable Paretian stochastic process with the following parameters, based on the data for 30-year Treasuries for 1977–1990 discussed above.

	Case 1 (Lognormal)	Case 2 (Stable Paretian)
α	2.000	1.580
c	0.09074	0.1046
δ	0	0

The normal distribution is, of course, a subset of the stable Paretian class, and the stable Paretian case could more appropriately be referred to as logstable Paretian. However, for convenience and clarity, case 1 is referred to as lognormal and case 2 as stable Paretian.

The value of c for the lognormal case was derived as follows. The sample variance for monthly changes in J_t is

$$s^2 = 0.0013725. \tag{19}$$

As stated in Section II-A, if $\alpha=2$, then the distribution is normal with $\sigma^2=2\gamma$. So $\gamma=0.5\sigma^2$. In this case, using s^2 in place of σ^2,

$$\gamma = 0.5(0.0013725) = 0.0006862. \tag{20}$$

The annual change in the logarithm of I_t is the sum of the 12 monthly changes. As a result of the stability property discussed in Section II-B, the annual change will have the same form of distribution with α the same, γ equal to the sum of the 12 γ's, and δ equal to the sum of the 12 δ's. So the distribution of the annual change has

$$\gamma = 12(0.0006862) = 0.008234 = c^2. \tag{21}$$

Therefore,

$$c = \sqrt{0.008234} = 0.09074. \tag{22}$$

In other words, a distribution of annual changes that is symmetric stable Paretian with $\alpha=2$, $c=0.09074$, $\delta=0$ is the same as a distribution of independent, identically distributed monthly changes that is normal with $\mu=0$ and $\sigma^2=0.0013725$.

The value of c for the stable Paretian case was derived as follows. The distribution of monthly changes in the J_t was estimated to have $c=0.021714$. This implies that

$$\gamma = c^\alpha = (0.021714)^{1.580} = 0.002355. \tag{23}$$

The annual change in the logarithm of I_t is the sum of the 12 monthly changes. Again, the property of stability results in an annual change that has the same form of distribution as the monthly change, with the same

α, γ equal to the sum of the 12 γ's, and δ equal to the sum of the 12 δ's. So the distribution of the annual change has

$$\gamma = 12(0.002355) = 0.02826. \quad (24)$$

Therefore, for the distribution of annual changes in J_t,

$$c = \gamma^{1/\alpha} = 0.02826^{1/1.580} = 0.1046. \quad (25)$$

It is further assumed that the yield to maturity on 30-year Treasuries never goes below 1.25%. This assumption is necessary for both cases to prevent the yield to maturity on 1-year Treasuries from being negative. It is also assumed that this same yield never goes above 50%. This is done in part to avoid interest rates that may be considered extreme and in part to avoid the criticism that the extreme results obtained in the stable Paretian case are due to impossibly high interest rates. Note that capping the range of possible interest rates also results in a finite variance for the distribution of J_t.

Other assumptions are as shown in the following table.

Lapse Rate ($q_t^{(w)}$)	$0.05+0.05[100(i_{comp}-i_{cred})]^2$ if $i_{comp}-i_{cred}>0$ 0.05 if $i_{comp}-i_{cred}\leq 0$ In all cases, $q_t^{(w)}$ is limited to 0.50. This is a commonly used functional form, not based on empirical data.
Credited Interest (i_{cred})	Currently anticipated portfolio yield rate for the coming year on a book basis less 150 basis points (1.5%). The minimum guaranteed interest rate is 4%, and the company will always credit a minimum of $i_{comp}-2\%$.
Competition Interest Rate (i_{comp})	The greater of (a) and (b) less 50 basis points, where (a) = 1-year Treasury nominal yield to maturity; and (b) = 5-year average of 5-year Treasury nominal yields to maturity.
Expenses and Taxes	None
Mortality	Included in lapses
Annuitization	None
Mortgage Prepayment Rate ($rate_t$)	The rate of prepayment is calculated separately for each year's purchases of GNMAs as follows: $0.05 + 0.03[100(i_{coup\,t} - i_{curr})] + 0.02[100(i_{coup\,t} - i_{curr})]^2$ if $i_{coup\,t} - i_{curr} > 0$ 0.05 if $i_{coup\,t} - i_{curr} \leq 0$ In all cases, $rate_t$ is limited to a maximum of 0.40.

Coupon Rate at Purchase ($i_{coup\ t}$)	The coupon on newly issued 30-year GNMAs issued in year t
Current Coupon Rate (i_{curr})	The coupon rate on currently issued 30-year GNMAs. These are assumed to shift in parallel with 30-year Treasury rates, that is, the change in yield to maturity on 30-year Treasuries is added to the coupon rate to get the new coupon rate.
Interest Rates on 1-Year and 5-Year Treasuries	Assumed to shift in parallel with 30-year Treasury rates; that is, the change in yield to maturity on 30-year Treasuries is added to the yield to maturity on each instrument to get the new yield to maturity.
Reinvestment Strategy	If the net cash flow in a given year is positive, any loans are paid off first, then 1-year Treasuries are purchased with any additional funds until their book value is equal to 20 percent of the total book value of the assets, and finally, newly issued, current coupon GNMAs are purchased with any remaining cash flows.
Loans	If the net cash flow in a given near is negative, money is borrowed internally at the same terms as those on the 1-year Treasuries.
Book Value of GNMA	Equal to the present value of the future principal and interest payments, discounted at the coupon interest rate at which they were purchased, assuming no prepayments.
Market Value of GNMA	Equal to the present value of the future principal and interest payments, discounted at the current coupon interest rate, and using the prepayment schedule based on the same rate, assuming that the current coupon rate for all future periods is level.
Market Value of Annuities	Equal to account value.
Timing of Cash Flows	It is assumed that all cash flows occur at the end of the year.

For each of the two cases, 100 scenarios were run. The projection period was 10 years. At the end of 10 years, the assets and liabilities were valued on a market-value basis. Tables 6 and 7 show the tenth-year surplus on a market-value basis ordered from smallest to largest for the lognormal case and the stable Paretian case, respectively. Figures 3, 4, and 5 present the same information graphically. Figure 3 shows the data from Table 6. Figures 4 and 5 are both based on the data in Table 7, the only difference being one of scale. The magnitude of the most

TABLE 6

100 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE LOGNORMAL CASE

t	OS_t	t	OS_t	t	OS_t
1	(3,935,523)	41	1,653,456	71	2,355,672
2	(1,669,056)	42	1,655,185	72	2,379,694
3	(1,573,561)	43	1,664,288	73	2,390,921
4	(820,183)	44	1,762,839	74	2,394,020
5	(353,858)	45	1,765,942	75	2,443,931
6	(221,643)	46	1,807,418	76	2,480,899
7	(169,424)	47	1,813,668	77	2,484,730
8	(133,671)	48	1,838,789	78	2,492,091
9	(33,429)	49	1,850,946	79	2,502,697
10	19,452	50	1,886,076	80	2,522,071
11	73,653	51	1,914,971	81	2,548,917
12	81,165	52	1,917,268	82	2,561,432
13	90,034	53	1,928,550	83	2,576,288
14	107,789	54	1,983,319	84	2,609,795
15	273,784	55	2,005,929	85	2,631,667
16	440,493	56	2,014,959	86	2,632,096
17	523,963	57	2,048,876	87	2,685,044
18	544,884	58	2,073,869	88	2,700,908
19	563,234	59	2,103,727	89	2,726,955
20	676,511	60	2,152,675	90	2,750,020
21	823,056	61	2,175,269	91	2,828,278
22	833,762	62	2,190,039	92	2,835,635
23	911,935	63	2,201,091	93	2,850,040
24	956,295	64	2,203,264	94	2,900,917
25	957,929	65	2,224,343	95	2,941,225
26	1,070,222	66	2,241,418	96	3,085,131
27	1,140,108	67	2,242,624	97	3,197,324
28	1,259,043	68	2,253,579	98	3,215,420
29	1,287,237	69	2,263,852	99	3,249,634
30	1,338,988	70	2,288,596	100	3,457,466
31	1,346,303				
32	1,395,566				
33	1,442,120				
34	1,478,390				
35	1,486,547				
36	1,488,913				
37	1,496,903				
38	1,602,665				
39	1,607,273				
40	1,634,789				
Sample Mean:					1,616,004
Sample Standard Deviation:					1,194,937

TABLE 7

100 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE STABLE PARETIAN CASE

t	OS_t	t	OS_t	t	OS_t
1	(121,993,000)	41	993,838	71	2,031,191
2	(53,000,540)	42	1,019,117	72	2,080,799
3	(38,609,200)	43	1,089,538	73	2,119,110
4	(10,555,610)	44	1,113,836	74	2,119,991
5	(6,178,578)	45	1,231,521	75	2,122,132
6	(5,981,680)	46	1,256,956	76	2,136,687
7	(4,820,628)	47	1,321,327	77	2,186,615
8	(4,546,422)	48	1,349,667	78	2,282,669
9	(3,875,923)	49	1,359,857	79	2,308,413
10	(3,552,448)	50	1,376,576	80	2,313,259
11	(3,313,565)	51	1,457,943	81	2,339,239
12	(2,298,297)	52	1,470,926	82	2,340,308
13	(2,183,848)	53	1,496,642	83	2,384,956
14	(1,553,025)	54	1,499,971	84	2,388,649
15	(1,420,309)	55	1,516,970	85	2,447,761
16	(1,415,358)	56	1,583,856	86	2,471,754
17	(1,347,743)	57	1,589,302	87	2,500,889
18	(1,121,014)	58	1,635,374	88	2,518,084
19	(1,028,966)	59	1,638,909	89	2,523,765
20	(916,352)	60	1,642,424	90	2,558,197
21	(561,895)	61	1,681,499	91	2,606,791
22	(480,189)	62	1,755,726	92	2,618,745
23	(436,869)	63	1,815,831	93	2,641,468
24	(344,403)	64	1,885,251	94	2,665,568
25	(273,309)	65	1,888,703	95	2,812,686
26	(105,880)	66	1,948,011	96	2,918,104
27	(15,508)	67	1,954,664	97	3,091,850
28	151,809	68	1,974,142	98	3,232,074
29	196,355	69	1,977,154	99	3,254,004
30	428,123	70	2,024,769	100	3,383,891
31	552,660				
32	559,877				
33	562,296				
34	563,053				
35	648,420				
36	700,347				
37	815,332				
38	871,732				
39	904,513				
40	958,118				
Sample Mean:					(1,420,680)
Sample Standard Deviation:					14,050,517

FIGURE 3

100 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE LOGNORMAL CASE

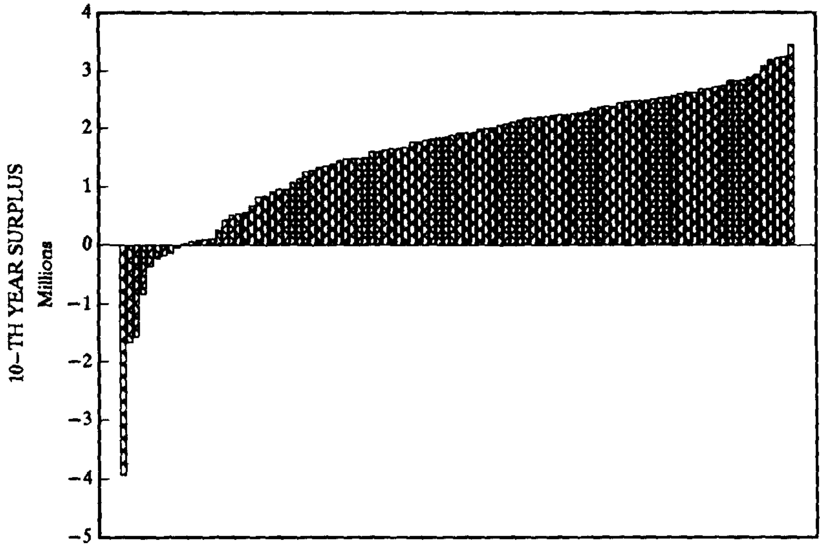


FIGURE 4

100 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE STABLE PARETIAN CASE

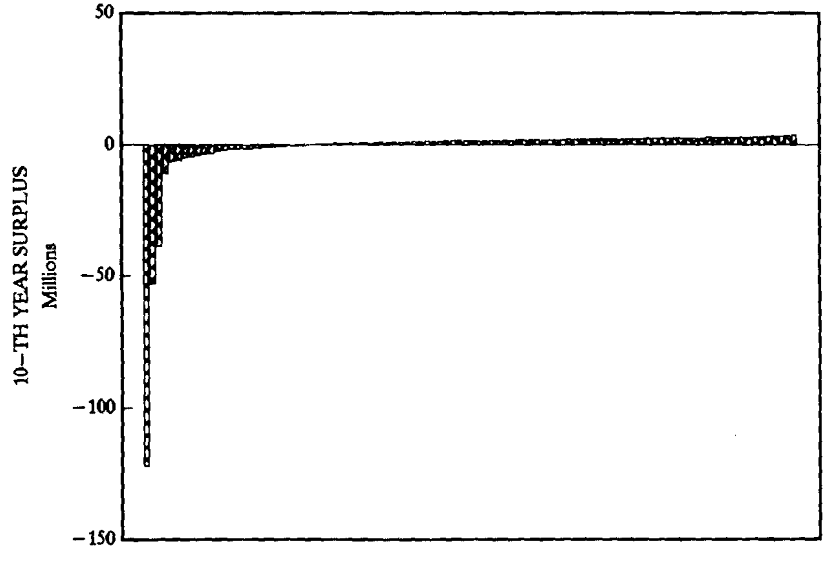
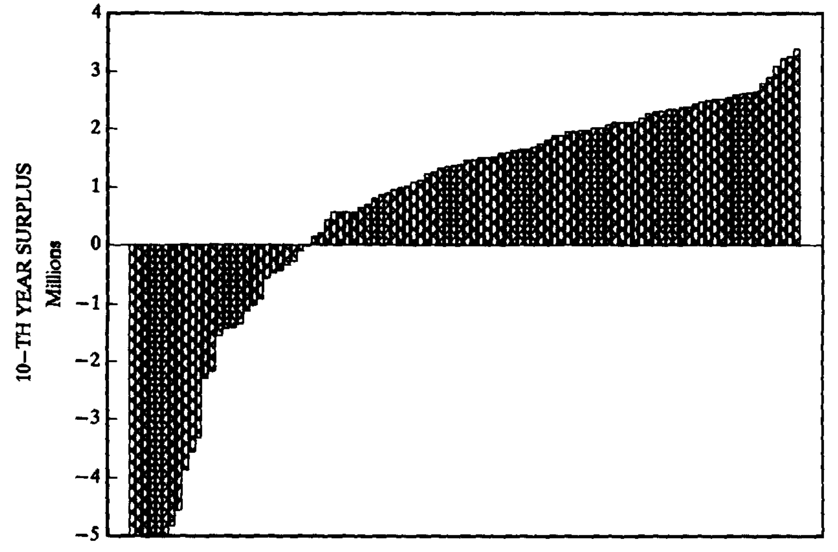


FIGURE 5

100 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE STABLE PARETIAN CASE



extreme result dwarfs the others, so that the scale necessary to include all the final surpluses is very large. This is shown in Figure 4. Figure 5 has the same scale as Figure 3.

Currently accepted actuarial methodology advocates calculating the sample mean and standard deviation based on the simulated sample and using the normal approximation to make probability statements concerning the distribution of all possible ending surpluses (see [8], [19] and [25]). Doll [19, p. 99] points out that this is generally not valid since the distribution of final surpluses is often skewed to the left. The next section discusses the flaw in this line of reasoning and quantifies the error that results from the use of this faulty assumption by comparing an estimator based on it to an unbiased estimator.

C. Probability Distribution of Final Surplus

An analysis of the final surpluses in the stable Paretian case, shown in Table 7 and Figure 4, shows that the distribution of the final surpluses is skewed to the left. The highest value is 0.3419 standard deviations above the mean. The four smallest values deviate from the sample mean by more than 0.3419 sample standard deviations. Most of the sample mean and standard deviation are due to a few extreme values. For instance, the smallest value is more than 8.5 standard deviations below the sample mean. Thus, the distribution of the surpluses from the stable Paretian case is obviously highly skewed and it is therefore not the normal distribution.

The same sort of informal analysis can be applied to the final surpluses from the lognormal case, shown in Table 6. The largest value is 1.541 sample standard deviations above the sample mean, while the smallest value is 4.646 sample standard deviations below the sample mean. From these results, the distribution of final surpluses is obviously skewed to the left in this case, too. Thus, it is not the normal distribution.

It is often asserted that the normal approximation is "close enough" to the actual distribution of the final surplus. This is similar to the claim that the lognormal hypothesis is close enough. Like that claim, it should be empirically tested and abandoned if it is found to be untrue.

This claim is based partly on a misapplication of the central limit theorem and partly on the desire to simplify as much as possible. The central limit theorem can be used to make probability statements about the mean of an unknown distribution based on a random sample. It says nothing about a particular percentile of the distribution, however. If one wants to know the first percentile of a non-normal distribution, the normal approximation will not give an accurate answer no matter how large a sample is used.

To illustrate this, the above simulations were repeated with sample sizes of 6,000 for each case. Tables 8 and 9 show the results of these simulations. To quantify the error of assuming that the resulting surpluses are normally distributed, two estimators of the first percentile of the distribution of final surpluses are compared. An estimator of the first percentile of the distribution of final surpluses is often used in making the statement that there is a 99% probability that the tenth-year surplus will exceed this value. The 60th-order statistic from the lognormal case with a sample size of 6,000 was used as the first estimator of the first percentile of the underlying distribution of surpluses. An alternative estimator, which is frequently used though based on the faulty assumption of normality, is $x-2.3s$. In the stable Paretian case, the sample mean and sample standard deviation were very heavily influenced by the very

TABLE 8

6000 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE LOGNORMAL CASE

t	OS_t	t	OS_t	t	OS_t
1	(7,799,011)	41	(2,658,344)	5961	3,607,196
2	(6,666,244)	42	(2,654,256)	5962	3,607,487
3	(5,970,016)	43	(2,653,839)	5963	3,611,086
4	(5,946,807)	44	(2,566,318)	5964	3,619,469
5	(5,854,456)	45	(2,561,330)	5965	3,633,630
6	(5,549,356)	46	(2,508,854)	5966	3,635,851
7	(4,840,982)	47	(2,480,143)	5967	3,637,566
8	(4,833,958)	48	(2,467,913)	5968	3,638,792
9	(4,826,955)	49	(2,452,505)	5969	3,640,204
10	(4,710,439)	50	(2,445,718)	5970	3,651,101
11	(4,593,918)	51	(2,442,453)	5971	3,651,117
12	(4,563,361)	52	(2,418,294)	5972	3,658,623
13	(4,465,991)	53	(2,410,762)	5973	3,661,447
14	(4,370,141)	54	(2,401,123)	5974	3,665,963
15	(4,165,350)	55	(2,398,479)	5975	3,673,348
16	(4,054,478)	56	(2,391,440)	5976	3,694,082
17	(3,971,333)	57	(2,350,858)	5977	3,704,228
18	(3,935,523)	58	(2,350,034)	5978	3,714,761
19	(3,859,245)	59	(2,346,127)	5979	3,721,975
20	(3,839,542)	60	(2,321,051)	5980	3,723,872
21	(3,836,310)	·	·	5981	3,729,356
22	(3,653,188)	·	·	5982	3,736,607
23	(3,563,914)	·	·	5983	3,746,882
24	(3,496,210)	2994	2,064,335	5984	3,781,358
25	(3,459,134)	2995	2,064,386	5985	3,796,661
26	(3,251,604)	2996	2,064,453	5986	3,798,898
27	(3,249,248)	2997	2,064,513	5987	3,799,267
28	(3,177,316)	2998	2,064,647	5988	3,810,334
29	(3,159,414)	2999	2,064,659	5989	3,831,823
30	(3,141,848)	3000	2,064,772	5990	3,834,731
31	(3,011,483)	3001	2,065,048	5991	3,853,009
32	(2,974,589)	3002	2,065,050	5992	3,873,613
33	(2,941,473)	3003	2,065,470	5993	3,875,597
34	(2,938,696)	3004	2,065,734	5994	3,942,996
35	(2,921,827)	3005	2,065,762	5995	3,981,141
36	(2,783,883)	3006	2,065,953	5996	3,983,800
37	(2,741,466)	3007	2,066,282	5997	4,011,578
38	(2,698,107)	·	·	5998	4,052,147
39	(2,697,097)	·	·	5999	4,159,897
40	(2,691,754)	·	·	6000	4,319,538
Sample Mean:					1,787,899
Sample Standard Deviation:					1,195,950

TABLE 9

6000 SCENARIOS OF 10TH-YEAR SURPLUS FOR THE STABLE PARETIAN CASE

t	OS_t	t	OS_t	t	OS_t
1	(1.9986×10^9)	41	(116,525,900)	5961	4,226,545
2	(285,710,100)	42	(116,504,300)	5962	4,244,382
3	(239,549,200)	43	(116,311,700)	5963	4,247,622
4	(239,549,200)	44	(114,565,200)	5964	4,256,108
5	(239,549,200)	45	(113,200,100)	5965	4,268,834
6	(239,549,200)	46	(112,777,500)	5966	4,272,829
7	(239,549,200)	47	(111,917,100)	5967	4,313,339
8	(239,549,200)	48	(111,908,100)	5968	4,340,391
9	(239,549,200)	49	(104,938,300)	5969	4,404,904
10	(238,434,200)	50	(95,691,720)	5970	4,427,105
11	(228,153,700)	51	(95,178,400)	5971	4,433,411
12	(207,329,200)	52	(94,649,190)	5972	4,449,286
13	(186,122,600)	53	(91,936,580)	5973	4,484,608
14	(184,304,100)	54	(91,921,510)	5974	4,485,332
15	(183,237,100)	55	(90,043,470)	5975	4,497,738
16	(176,149,100)	56	(83,074,330)	5976	4,519,914
17	(172,694,900)	57	(81,231,930)	5977	4,553,557
18	(169,926,900)	58	(80,910,190)	5978	4,563,169
19	(167,961,100)	59	(80,283,180)	5979	4,593,807
20	(167,597,600)	60	(80,102,800)	5980	4,633,062
21	(167,460,000)	·	·	5981	4,669,444
22	(167,414,200)	·	·	5982	4,829,623
23	(167,169,000)	·	·	5983	4,866,440
24	(165,987,700)	2994	1,637,419	5984	4,874,628
25	(160,818,100)	2995	1,638,862	5985	4,957,546
26	(150,455,900)	2996	1,638,909	5986	4,981,374
27	(147,504,500)	2997	1,639,606	5987	5,029,171
28	(139,393,100)	2998	1,639,940	5988	5,286,632
29	(134,818,300)	2999	1,639,955	5989	5,296,121
30	(130,241,500)	3000	1,641,659	5990	5,692,282
31	(124,420,400)	3001	1,642,396	5991	5,773,226
32	(122,010,700)	3002	1,642,424	5992	5,830,193
33	(121,993,000)	3003	1,642,550	5993	5,838,900
34	(119,776,300)	3004	1,642,650	5994	5,888,690
35	(119,387,300)	3005	1,643,752	5995	5,998,347
36	(119,285,800)	3006	1,644,652	5996	6,045,817
37	(118,078,700)	3007	1,645,309	5997	6,451,161
38	(117,470,700)	·	·	5998	7,229,951
39	(117,327,400)	·	·	5999	7,473,759
40	(117,106,900)	·	·	6000	9,020,304
Sample Mean:					(1,541,756)
Sample Standard Deviation:					30,743,645

low values of the first few order statistics. Therefore, only the lognormal case is used in comparing the two estimators of the first percentile.

The first estimator, the 60th-order statistic from the lognormal case, is equal to −2,321,051. The second estimator, $x-2.3s$, is equal to 1,787,889−2.3(1,195,950)=−962,796. Thus, in this case, the improper use of the normal approximation results in an overstatement of 1,358,255 in the value that has a 99% chance of being exceeded by the tenth-year surplus. This is approximately 13.6% of the initial assets. Since the distribution of final surpluses in the stable Paretian case is even farther from normality, the normal assumption in both cases is statistically invalid and not close enough to the actual distribution to provide meaningful probability statements about values in the tails of the distribution. This result should not be surprising, because the distribution of the final surplus is highly skewed to the left.

If a skewed distribution could be found that was a close fit to the distribution of final surplus, its parameters could be estimated from a moderate number of scenarios, and it could be used to make probability statements. Otherwise, it is necessary to run a large number of scenarios. The "order statistics method" can then be used to make probability statements without biasing the results. Statistical methods can be used to determine how large a sample must be used to estimate a given percentile of the distribution to a desired degree of accuracy.

D. Application to the Valuation Actuary Concept

A comparison of the results from the lognormal and stable Paretian cases produces the following, rather startling results:

Case	Estimator of 1st Percentile of 10th-Year Surplus
1. Lognormal	−2,321,051
2. Stable Paretian	−80,102,800

It is clear that the results of the lognormal hypothesis are not "close enough" to those from the more justifiable stable Paretian hypothesis.

Current valuation actuary methodology requires the selection of an acceptable probability of negative surplus at the end of the projection period. A figure that has been used is 1%. Without digressing into a discussion about what an acceptable level for this figure is, this figure is used in the following analysis.

First, a simulation was run using the lognormal hypothesis to find the amount of assets that must be set aside initially to have only a 1% chance of the tenth-year surplus being negative. Using the "order statistics method" described above with a sample size of 6,000, a value of $10,850,000 was arrived at. That is, an initial asset $850,000 greater than the initial liability of $10,000,000 is necessary to provide a positive surplus in 99% of scenarios under the lognormal hypothesis.

Next, to test the sensitivity of this result to the value of α used, 6,000 scenarios were generated using the same initial asset, $10,850,000, but using the stable Paretian assumptions of $\alpha=1.580$ and $c=0.1046$. Of the 6,000 trials, 719 yielded a negative tenth-year surplus. Thus, the probability of a negative surplus is approximately 719/6,000, or 12%. This is significantly higher than the 1% probability obtained by using the lognormal hypothesis. This difference is due entirely to the different stochastic process for interest rate generation.

Looked at another way, using the stable Paretian assumptions, it was found that an initial asset of approximately $15,850,000 is necessary to arrive at only a 1% probability of a negative tenth-year surplus. This is 58.5% of the initial liability, as compared with 8.5% under the lognormal hypothesis. Again, the results clearly demonstrate the extreme sensitivity of cash-flow analysis to the choice of α.

The following table summarizes these results:

	Probability of Negative 10th-Year Surplus	
Initial Asset	Lognormal Case	Stable Paretian Case
$10,000,000	0.078	0.234
$10,850,000	0.010	0.120
$15,850,000	0.000	0.010

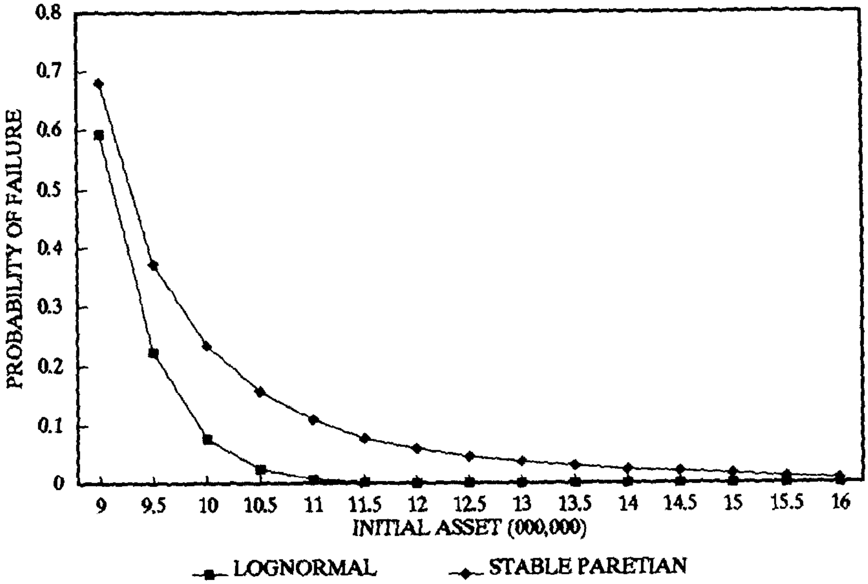
Figure 6 shows the probability of negative tenth-year surplus as a function of the initial asset under both the lognormal case and the stable Paretian case.

IV. CONCLUSION

In any attempt at mathematically modeling reality, simplification is a necessity. But when the simplification causes the model to yield drastically different results, the value of the model is compromised. The

FIGURE 6

PROBABILITY OF FAILURE AS A FUNCTION OF INITIAL ASSETS



assumptions and simplifications must provide results that are close to those of the underlying reality.

Levy and Salvadori [22] describe the disastrous consequences of using models that diverge greatly from reality. One of their examples described the collapse of the roof of the Hartford Civic Center. The architects and structural engineers had used a model that assumed that the supporting structure of the roof was made of a homogeneous material. In reality, it was made of a grid of steel tubing, a material nowhere near homogeneous. The choice of a relatively simple model, one of whose underlying premises was blatantly false, over a more complicated model that had more connection with the real structure led to the collapse of the roof. Prior to the collapse, it might have been argued that the model was close enough to reality. The model was not necessarily wrong because it had a false premise. It was wrong because its conclusions were not closely related to reality.

The reality of interest rate changes is that they are not lognormally distributed. The simplifying assumption of lognormality may not result in a large departure from reality in certain applications, particularly those concerned with the expected values or statements of probability where the tails are unimportant. On the other hand, the lognormal assumption is inappropriate for making probabilitity statements about extreme events when the tails of the true distribution are significantly fatter than those of the lognormal. Claire [8] illustrates the difference between a cash-flow analysis based on the lognormal hypothesis and one based on reality. She performed a cash-flow analysis using 99 stochastically generated scenarios. Then she ran a final scenario based on actual interest rates from the 1980s. The final scenario was as bad as the worst of the 99 stochastic ones. This is evidence enough that the lognormal model is too far from reality to be useful.

The point is not that independent, identically distributed stable Paretian random variables should be substituted for independent, identically distributed normal random variables currently used. A different fat-tailed distribution such as a combination of normals might be used, although this creates problems, which were discussed at the end of Section II-D. A stable Paretian model with c changing over time also might be used, although this creates other problems, which were also discussed at the end of Section II-D. The assumption of independence may need to be abandoned. The point is that the nature of changes in interest rates is far more complex than the lognormal model. It is not possible to use the lognormal model to make credible probability statements about the tails of distributions that are functions of those changes.

Currently accepted actuarial methodology of cash-flow analysis is an attempt to apply a technique that has proved useful in the past, the normal distribution, to an area in which actuaries have only recently become involved, modeling of the capital markets.

If actuaries are seriously understating the probability of insolvency, then the gravity of the problem is further masked by the assumption that failures of insurance companies are stochastically independent. If 100 companies are all selling annuities backed with long-term bonds or mortgages, their future levels of surplus are highly correlated. In other words, even if each of the 100 companies has a 1 percent chance of failure in the next 10 years, the scenarios in which they fail will be essentially the same, so that dozens of them could fail in one year.

V. AREAS FOR FURTHER RESEARCH

A few areas related to these topics need further research. First, the distribution of changes in interest rates may not be a member of the family of symmetric stable Paretian distributions. If it is a limiting distribution, it must be a member of the stable Paretian family, but it does not necessarily have to be symmetric. This paper demonstrates that the best choice out of the symmetric members of the family is a non-normal distribution, but there might be better choices for the distribution.

Secondly, the assumption that interest rate changes are independent of each other also is questioned by Becker [4]. His tests may not be accurate, since they assume that the variance of the distribution exists, but Peters [28] makes a similar claim using a technique called rescaled range analysis, which, unlike autocorrelation, does not assume that the variance exists. Peters' claim is that changes in stock and bond prices are "persistent," that is, that the changes tend to be in the same direction as previous changes.

Thirdly, as mentioned above, the possibility should be investigated that interest rate changes can be modeled by a nonstationary stochastic process. This could mean either an occasional shift in parameters or parameters that shift continuously.

Fourthly, the simplifying assumption of parallel shifts in interest rates should be replaced by an understanding of the interrelations of the yields on various instruments. The analysis in this paper had no possibility of yield curve inversions. The concept of covariance of yields will prove problematic if the variances are infinite.

VI. ACKNOWLEDGMENTS

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APPENDIX

The Γ (gamma) function is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{26}$$

The *Handbook of Mathematical Functions* [1] contains a formula for accurately evaluating this function, which is necessary for evaluating Formulas (15) and (16). If $0 \leq x < 1$, then

$$\Gamma(x + 1) = 1 + \sum_{i=1}^{8} b_i x^i + \epsilon(x), \tag{27}$$

where the following table gives the coefficients, b_i:

i	b_i
1	−0.577191652
2	+0.988205891
3	−0.897056937
4	+0.918206857
5	−0.756704078
6	+0.482199394
7	−0.193527818
8	+0.035868343

The error term is bounded by $|\epsilon(x)| < 3 \times 10^{-7}$.

For $x < 0$ or $x \geq 1$, the following recursive formula should be used:

$$\Gamma(x + 1) = x\Gamma(x). \tag{28}$$

DISCUSSION OF PRECEDING PAPER

WILLIAM A. BAILEY:

Mr. Klein has demonstrated that the data can be fit better by means of a stable Paretian distribution than by a lognormal distribution. There remain the questions of whether the assumption of symmetry is reasonable and also whether the stable Paretian distribution does the job.

I applied Kolmogorov-Smirnov's two-sample test to determine whether the assumption of symmetry is reasonable, and Komogorov's goodness-of-fit test to determine whether the stable Paretian distribution derived by Mr. Klein can be considered to be a reasonable choice of the population distribution underlying the Table 3 data.

For the two-sample test, I used Table 3 as the first sample, say x_i's for $i=1, 2, \ldots, 167$. For the second sample, I used $x_i'=\bar{x}-(x_i-\bar{x})$, where the x_i's and $\bar{x}$ are from the first sample. The null hypothesis is that the population distribution underlying the data in Table 3 is symmetric about the population mean. The Kolmogorov-Smirnov two-sample test statistic is

$$D_{n_1,n_2} = \text{max over all } x \text{ of } |F_{n_1}(x) - F_{n_2}(x)|$$

and the value of D_{n_1,n_2} large enough to call for rejection is

$$k_\alpha \times \sqrt{\frac{n_1 + n_2}{n_1 \times n_2}}$$

where α is the confidence level, $k_{0.05}=1.48$ and $k_{0.025}=1.36$, and n_1 and n_2 are the sizes of the first and second samples, respectively. In our case $n_1=n_2=167$, and $D_{167,167}$ turned out to be 0.1557. It follows that the null hypothesis (symmetry) is rejected at a p value of about 3.7 percent. The Kolmogorov-Smirnov two-sample test is not limited to testing for symmetry; for further information about this test, see, for example, Lehmann [2].

In the goodness-of-fit test, the null hypothesis is that Mr. Klein's fitted stable Paretian distribution is the population distribution underlying the data in Table 3. The Kolmogorov goodness-of-fit test statistic is

$$D_n = \text{max over all } x \text{ of } |F_n(x) - F_0(x)|$$

where F_n is the empirical distribution implied by the data in Mr. Klein's Table 3, F_0 is Mr. Klein's fitted stable Paretian distribution, and the value of D_n large enough to call for rejection is

$$\frac{k_\alpha}{\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}}}$$

where α is the degree of confidence, n is the size of the sample, and

$$(\text{for } n > 80)\ k_{0.01} = 1.628 \text{ and } k_{0.05} = 1.358.$$

D_{167} turned out to be 0.0605 and 0.0928, where the stable Paretian distribution's parameters are

$$(\alpha, \beta, c, \delta) = (1.58, 0, 0.021714, 0.0024613)$$

and

$$(\alpha, \beta, c, \delta) = (1.58, 0, 0.021714, 0),$$

respectively. It follows that the null hypothesis is not rejected (p value is 56.1 percent) where $\delta = 0.0024613$, but is rejected at a p value of 7.2 percent where $\delta = 0$. Although the fitting process produced $\delta = 0.0024613$, the cash-flow projections used $\delta = 0$; so this latter assumption may be questionable. For further information about the Kolmogorov goodness-of-fit test, see, for example, Bickel and Doksum [1].

Mr. Klein fits a stable Paretian distribution to the data and uses a χ^2 test to include this distribution but exclude the lognormal distribution. Lehmann states [2, p. 480] that "reduction of the data through grouping results [a verb!] in tests [for example, the χ^2-test] that tend to be less efficient than those based on the Kolmogorov . . . statistic. . . ." The stable Paretian distribution involves the selection of four parameters, whereas the lognormal involves the selection of only two parameters. (A model involving 162 parameters could fit the data exactly.) We have no evidence that the tail of the resulting stable Paretian distribution (that is, beyond the largest value in the data) is, or is not, representative of what can be expected there.

It would be interesting to see some Monte Carlo results corresponding to Mr. Klein's Tables 6, 7, 8, and 9, but where the empirical distribution formed from Table 3 is used in place of the stable Paretian distribution. Such runs would reflect the lack of symmetry and would not involve the appending of any tail. I happen to believe that the appending of tails is often unnecessary and perhaps even misleading.

Further results, of theoretical interest, would be where the assets are assumed to be invested in stocks rather than in bonds and mortgages. Presumably, this would require taking account of any correlation between the stock market indexes and the Treasury bond rates.

Mr. Klein is to be commended for leading actuaries toward a fuller understanding of the stable Paretian family of distributions and its possible use as a desirable replacement for the lognormal family, which is commonly used in asset/liability testing. My own preference is to use empirical distributions formed from the data, but for some purposes it is desirable to add tails to those distributions.

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TIMOTHY C. CARDINAL:

It is always a difficult task to question the validity of beliefs that have achieved widespread use. Mr. Klein is commended on considering an alternative to the traditional lognormal model. The lognormal model has been used due to its theoretical properties, not empirical substantiation. I have produced results [1] that corroborate a logstable model. This discussion focuses on the development of the logstable model. My goal is to clarify and strengthen several statements and results in the paper. For example, parameter estimates are improved and symmetry is not assumed; standard errors are calculated; and the first and fourth areas for further research are resolved.

1. Definitions and Properties

The stability-under-addition *property* is a misnomer. Stable distributions are *defined* as the class of distributions that are invariant under convolution. F is *stable*, if for every a_1, $a_2>0$ and real b_1, b_2 there exists an $a>0$ and real b such that

$$F\left(\frac{x-b_1}{a_1}\right) \times F\left(\frac{x-b_2}{a_2}\right) = F\left(\frac{x-b}{a}\right).$$

Section II.B indicates that the term "Paretian" is used since stable and Pareto distributions are both "fat-tailed." Use of this term is much stronger; otherwise terms such as "studentian" may be applicable. When $\alpha \neq 2$ and $|\beta| \neq 1$, both tails are *asymptotically Pareto*, specifically, for $x>0$, $1-F(x) \rightarrow c_1/x^a$ and $F(-x) \rightarrow c_2/x^a$. However, the Pareto distribution is not a member of the stable laws. Being asymptotically Pareto is a property that led Mandelbrot to use stable distributions as an alternative. He defines the strong Pareto law as the Pareto distribution and the weak Pareto law as a law that is asymptotically Pareto. Stable laws are weak Pareto. He uses the adjective Paretian in conjunction with strong and weak Pareto laws; thus he uses the term stable Paretian synonymously with stable.

Different representations of the characteristic function shown in Klein's paper as Equation (4) have been used. The "accepted" representation reverses the sign of β in Equation (4) for $\alpha \neq 1$. For this representation, the distribution is skewed left (right) for $\beta<0$ ($\beta>0$), and c_1 and c_2 are related by $\beta=(c_1-c_2)/(c_1+c_2)$. For the representation used in Equation (4), the distribution is skewed left (right) for $\beta>0$ ($\beta<0$) for $\alpha \neq 1$ and vice versa for $\alpha=1$. Thus, β does not have a consistent interpretation for the direction of skewness or for the relationship between β and the area in the tails. This fact has not always been recognized in the literature.

Another representation for $\alpha \neq 1$ used in analytical expressions (for example, Bergström, Klein's ref. [5]) is

$$\ln \phi(t) = i\delta t - \gamma^{*}|t|^{\alpha} \exp[-i\beta^{*} sgn(t)].$$

By equating, squaring, and adding real and imaginary parts, one can show that

$$\tan(\beta^{*}) = \beta \tan \frac{\pi\alpha}{2} \quad \text{and} \quad |c^{*}|^{\alpha} = |c|^{a}\left(1 + \beta^{2} \tan^{2} \frac{\pi\alpha}{2}\right)^{1/2}.$$

A slightly different expression used by Chambers et al. [2] has $\beta_c=2\beta^{*}/\pi \min(\alpha, 2-\alpha)$.

2. Empirical Techniques

Klein states ". . . it was assumed that the distributions were symmetric. . . . This was done largely because the procedure of parameter estimation and simulation has not yet been worked out for skewed distributions." Fortunately this is not true. The most recent articles on empirical techniques referenced in the paper were published in 1972 and 1974. A substantial amount of research has been carried out since then.

Simulation

The expansions for standard stable pdf's derived by Bergström (Klein's ref. [5]) are for the general asymmetric case. For $\beta^*=0$, these reduce to the pdf's implied by the cdf's given in Equations (15) and (16). For $\alpha>1$, the following converges rapidly for small $x>0$,

$$f(x) = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma\left(\frac{k+1}{\alpha}\right)}{\alpha}x^k\cos\left(k\left(\frac{\pi}{2}+\frac{\beta^*}{\alpha}\right)+\frac{\beta^*}{\alpha}\right).$$

For $\alpha>1$, the following converges rapidly for large $x>0$,

$$f(x) = \frac{1}{\pi}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}\frac{\Gamma(\alpha k+1)}{x|x|^{\alpha k}}\sin\left(k\left(\frac{\alpha\pi}{2}+\beta^*-\alpha\arg x\right)\right)+R(x),$$

where the remainder is of order $x^{-\alpha(n+1)-1}$.

Thus simulation can be done in the asymmetric case using the inverse cdf technique. However, this technique is inefficient. A faster, exact method was developed by Chambers et al. [2]. By using transformations, two random uniform numbers are used to generate a standard stable deviate. Let W be standard exponential, Φ be uniform on $(-\pi/2, \pi/2)$, and $\Phi_0=-\beta^*/\alpha=-\pi\beta^c\min(\alpha, 2-\alpha)/2\alpha$. The standard stable cdf $(\alpha\neq 1)$ can be represented as

$$F = \frac{\sin\alpha(\Phi-\Phi_0)}{(\cos\Phi)^{1/\alpha}}\left(\frac{\cos(\Phi-\alpha(\Phi-\Phi_0))}{W}\right)^{(1-\alpha)/\alpha}.$$

For $\alpha=2$, this method is equivalent to the Box-Muller method.

Estimation Techniques

Fama and Roll estimators work reasonably well when the distribution is known to be symmetric. However, their observations of fractile tables do not hold for asymmetric distributions. Sampling error and possible asymmetry result in inferior estimates with large standard errors. Bias in

$\hat{c}$ is compounded in $\hat{\alpha}$. The process of adjusting for bias is often arbitrary and unsatisfactory.

McCulloch [6] uses fractile methods to simultaneously provide asymptotically unbiased estimators of all four parameters. Allowing for asymmetry and unbiased estimators is a substantial improvement. Furthermore, the interval of estimation for α is increased to [0.6, 2].

Since the characteristic function is the only simple expression for stable laws, it is logical to use it for empirical fitting. Given n independent observations $x_1 \ldots, x_n$ of a random variable X, the *sample characteristic function* is

$$\hat{\phi}(t) = \frac{1}{n}\sum_{k=1}^{n} \exp(itx_k),$$

where $\hat{\phi}(t)$ is a stochastic process, $|\hat{\phi}(t)| \leq 1$, all moments of $\hat{\phi}(t)$ are finite, and $\hat{\phi}(t)$ is a consistent estimator of $\phi(t)$. Using $\hat{\phi}$ also allows estimates of all the parameters.

Paulson and Leitch [7] suggest estimation of the parameters by minimizing the function

$$I(\alpha, \beta, c, \delta) = \int |\hat{\phi}(t) - \phi(t)|^2 \exp(-t^2)dt.$$

DuMouchel [3] uses maximum likelihood estimators requiring a fast-Fourier transform. Press [8] suggests the method of moments. Koutrouvelis [5] generalizes Press by using regression rather than equality. These estimators are consistent, asymptotically unbiased, and more efficient than most other estimators.

Statistical Inference

The standard errors of estimates for parameters of a normal distribution can be given in terms of the estimates and number of data points. Unfortunately, this is not the case for stable parameters. A simulation method known as bootstrapping provides the distribution for the estimated parameters. Suppose that, given a data set S_0 of size n, one wishes to obtain an estimate $\hat{\theta}$ for a parameter θ. For $1 \leq i \leq m-1$, form a sample data set S_i by randomly selecting with replacement, n points from S_0. Estimate θ for each S_i, labeling the estimate $\hat{\theta}_i$. The set of $\hat{\theta}_i$ provides a sample distribution for $\hat{\theta}$. That is,

$$\hat{\theta} = \frac{1}{m} \sum_{i=0}^{m-1} \theta_i$$

and

$$\hat{\sigma}_{\theta}^2 = \frac{1}{m-1} \sum_{i=0}^{m-1} (\theta_i - \hat{\theta})^2.$$

An estimate of the bias is $\hat{\theta}-\hat{\theta}_0$. Note that α and β are bounded; hence estimates are asymptotically normal. See Effron [4] for a detailed description of bootstrapping.

Bootstrapping does not improve or demonstrate that an estimation technique is faulty, inferior, or biased. Bootstrapping provides an estimate of the bias due to sampling error, not bias inherent in the estimation technique: each $\hat{\theta}_i$ has on average the same inherent bias. It is necessary to verify that the estimation technique will accurately estimate parameters from known distributions.

3. Statistical Results

The results of Tests I and II are not conclusive since:

1. Comparisons were made by visual inspection
2. Biases for the original α estimates were not corrected
3. Only one sum size was examined
4. A small data set was used.

As indicated in the paper, visual inspection was necessary since the referenced papers did not quantify the critical regions. Much of the existing literature, including Hsu et al. (Klein's ref. [18]), does not adequately resolve the stability-under-addition test for the same reasons. Similarly, the statements "[t]he large discrepancy in c's between the two sets of data . . ." and ". . . α appears to be more stable over time than c" are opinions, not statistical demonstrations.

Parameters for the data given in Tables 2 and 4 in Klein's paper were estimated by using both the Fama and Roll and the regression techniques. Since stability-under-addition holds only for independent identically distributed variables, any effects of serial correlation and dependency should be removed in the randomized data (Test II). The stability-under-addition test can be formalized as:

$$\text{Hypothesis 1: } \alpha_k = \hat{\alpha}_1 \quad \text{or} \quad \text{Hypothesis 2: } \alpha_k = \alpha_1.$$

Tables 1 and 2 contain α estimates for both time periods, both chronological and randomized data, and for various sum sizes, and Table 3 contains estimates of c and δ. The analysis is still limited due to the effects of a small data set. For α, regression estimates are higher than Fama and Roll estimates with smaller standard errors. Errors for estimates greater than 1.9 are small since estimates are right truncated at 2.

For the 1977–1990 data, normality is rejected; hypothesis 1 is not rejected using either Fama and Roll or regression estimates; hypothesis 2 is rejected using a two-sided t test; and the hypotheses that $\delta=0$ and $\beta=0$ cannot be rejected ($\beta=-0.16$). For the 1953–1976 data, *normality cannot be rejected*; hypothesis 1 is not rejected using either Fama and Roll or regression estimates; hypothesis 2 is rejected using a two-sided t-test; and the hypotheses that $\delta=0$ and $\beta=0$ cannot be rejected ($\beta=0.13$). Estimates of α and c for the two data sets are within one and three standard errors, respectively, of each other.

4. Conclusion

The data sets are too small to make definitive conclusions. I have examined [1] daily, weekly, and quarterly changes for 0.25, 0.5, 1, 2, 3, 5, 7, 10, and 30 year U.S. Treasuries using a larger data set. A logstable model is demonstrated to be appropriate, while a lognormal model is not.

Several reasons why stable distributions have not gained widespread acceptance are given in Section II.E. I would add two additional misconceptions that appear in the literature:

1. Statistical inference statements are not possible.
2. Logstable models are economically intractable.

Bootstrapping refutes the first concern. The second arises from the fact that the only stochastic process with a continuous path of price changes is normal. Continuity is often incorrectly thought to be a condition necessary to a tractable arbitrage pricing theory (that is, form hedge portfolios and derive stochastic differential equations). In a logstable model, hedge portfolios can be formed and Monte Carlo methods circumvent the need to solve for prices, that is, expected present values, by first deriving differential equations.

Finally, the fourth area for further research suggests, "The concept of covariance of yields will prove problematic if the variances are infinite."

TABLE 1

α ESTIMATES: 1977–1990 DATA

	Chronological Data				Randomized Data			
	Sum 1	Sum 2	Sum 3	Sum 4	Sum 1	Sum 2	Sum 3	Sum 4
Fama and Roll Estimates	1.590	1.437	1.538	1.686	1.585	1.798	1.700	1.953*
	(0.138)	(0.170)	(0.234)	(0.215)	(0.134)	(0.159)	(0.231)	(0.090)
Regression Estimates	1.780	1.699	1.721	1.907	1.775	1.944	1.861	1.964*
	(0.107)	(0.136)	(0.181)	(0.141)	(0.099)	(0.103)	(0.163)	(0.084)

*Indicates $\hat{\alpha}_k$ is more than two standard errors from $\hat{\alpha}_1$.

TABLE 2

α ESTIMATES: 1953–1976 DATA

	Chronological Data					Randomized Data				
	Sum 1	Sum 2	Sum 3	Sum 4	Sum 5	Sum 1	Sum 2	Sum 3	Sum 4	Sum 5
Fama and Roll Estimates	1.649	1.784	1.586	1.429	1.631	1.657	1.810	1.808	1.828	1.803
	(0.107)	(0.104)	(0.205)	(0.262)	(0.199)	(0.127)	(0.122)	(0.162)	(0.155)	(0.159)
Regression Estimates	1.879	1.948	1.797	1.676	1.833	1.882	1.904	1.906	1.953	1.923
	(0.066)	(0.049)	(0.152)	(0.268)	(0.164)	(0.068)	(0.066)	(0.088)	(0.115)	(0.089)

TABLE 3

	1953–1976 Data		1977–1990 Data	
	$10^2\hat{c}$	$10^2\hat{\delta}$	$10^2\hat{c}$	$10^2\hat{\delta}$
Fama and Roll Estimates	1.445	0.312	2.082	0.190
	(0.14)	(0.12)	(0.22)	(0.29)
Regression Estimates	1.573	0.319	2.172	0.200
	(0.11)	(0.12)	(0.20)	(0.29)

Prior empirical use of stable distributions has been limited to one dimension. However, just as standard deviation may be generalized to the scale parameter, the covariance matrix may be generalized to a codispersion matrix. The characteristic function for a symmetric multivariate stable distribution is

$$\ln \phi(\mathbf{t}) = i\boldsymbol{\delta}'\mathbf{t} - \frac{1}{2}\sum_{j=1}^{m} (\mathbf{t}'\Omega_j\mathbf{t})^{\alpha/2},$$

where each Ω_j, is a positive semidefinite matrix of order p and rank r_j, $1 \leq r_j \leq p$.

For a normal distribution, $\alpha=2$, $\boldsymbol{\delta}=\boldsymbol{\mu}$, and

$$\sum_{j=1}^{m} \Omega_j = \Sigma.$$

I have developed [1] an estimation technique for multivariate parameters, and estimates are given for a nine-dimensional interest rate vector for daily, weekly, and quarterly changes.

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BEDA CHAN:

I have two brief comments on Section II of Mr. Klein's excellent paper on the deep and important topic of interest rate models and its cash-flow consequences.

First, Becker (Klein's ref. [4]) refers to three characteristics of the changes in rates [4, p. 71]: non-zero autocorrelations, positive kurtosis, and nonconstant standard deviation. The middle one is directly addressed by the stable Paretian model in this paper; the other two become undefined under the infinite variance implied by the stable Paretian model. As suggested in the second paragraph in Section V, there are tests for dependence of J_t. One such test is the runs test. The Minitab outputs (Outputs 1 and 2) indicate that the J_t's are not independent. Also included are the histograms and normal scores plots for J_t that indicate too high kurtoses to be normal, as Figure 2 of this paper also indicates.

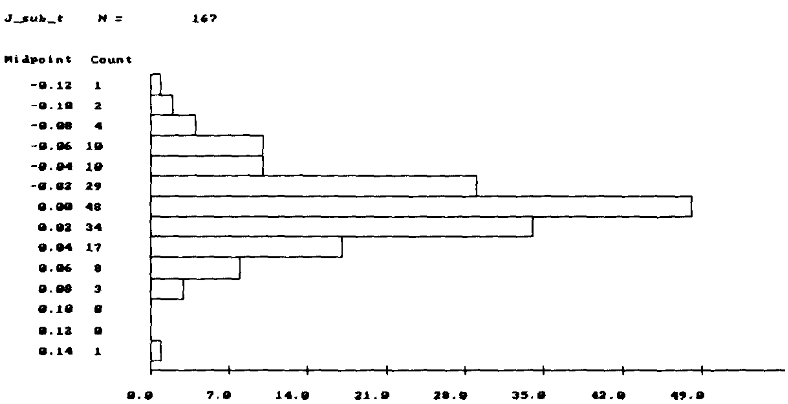
Second, the estimates for c, α, and δ by Fama and Roll (Klein's refs. [14], [13]) are computationally simple. For maximum likelihood estimates, see the papers by DuMouchel [1], [2], where likelihood contours are used to distinguish $\alpha=2$ and $\alpha<2$.

Outputs 1 and 2 were produced using the Student Edition of Minitab, Release 8, a $50 software. Output 3 was produced using S-plus, a graphical statistics environment evolving since 1984 and thus two decades younger than Minitab.

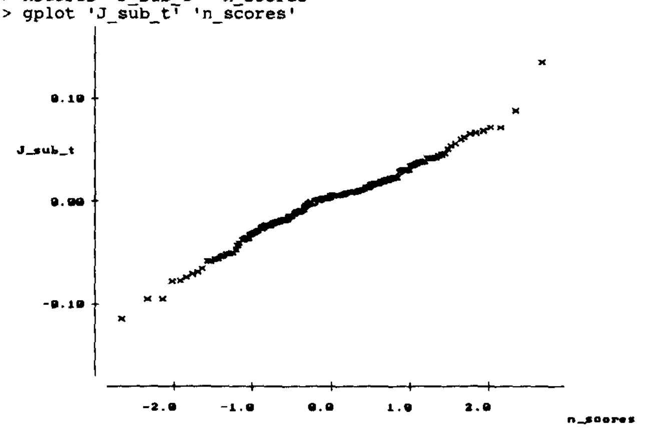
OUTPUT 1

30-YEAR TREASURY BONDS, 1977–1990

```
MTB > runs 'J_sub_t'

    J_sub_t

    K =      0.0005


    THE OBSERVED NO. OF RUNS =   64
    THE EXPECTED NO. OF RUNS =  82.9162
    95 OBSERVATIONS ABOVE K    72 BELOW
              THE TEST IS SIGNIFICANT AT  0.0028

MTB > ghistogram 'J_sub_t'
```



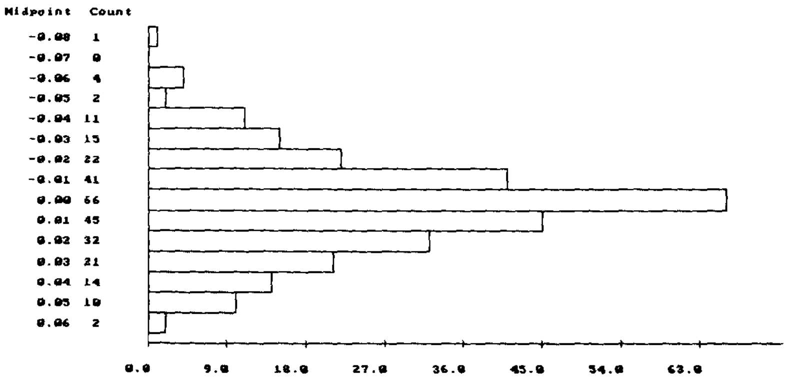

```
MTB > nscores 'J_sub_t' 'n_scores'
MTB > gplot 'J_sub_t' 'n_scores'
```



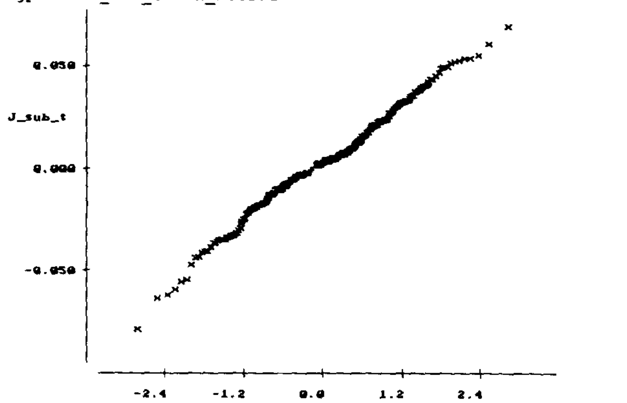
OUTPUT 2

LONG-TERM TREASURY BONDS, 1953–1976

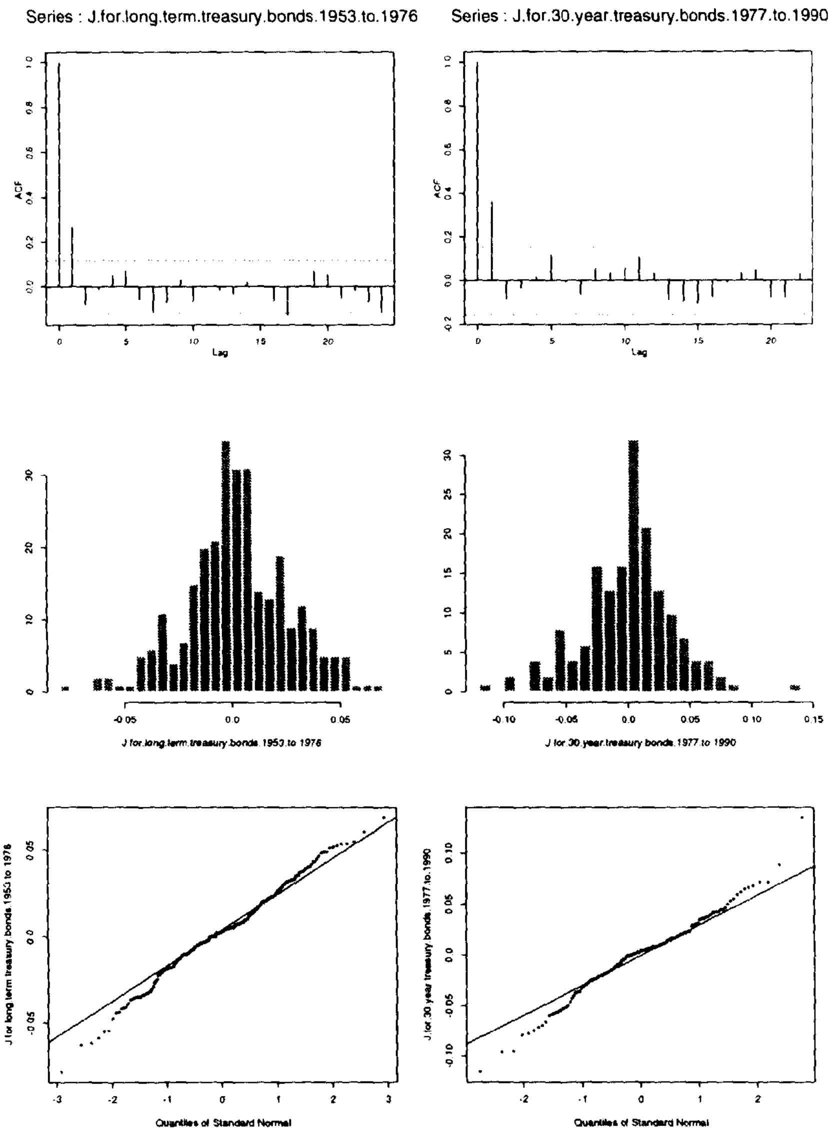
```
MTB > runs 'J_sub_t'

    J_sub_t

    K =      0.0029


    THE OBSERVED NO. OF RUNS = 118
    THE EXPECTED NO. OF RUNS = 144.4983
   143 OBSERVATIONS ABOVE K  144 BELOW
              THE TEST IS SIGNIFICANT AT  0.0018

MTB > ghistogram 'J_sub_t'
J_sub_t     N =        287
       1   Observations are above the last class
```




```
MTB > nscores 'J_sub_t' 'n_scores'
MTB > gplot 'J_sub_t' 'n_scores'
```



OUTPUT 3


Series : J.for.long.term.treasury.bonds.1953.to.1976
Series : J.for.30.year.treasury.bonds.1977.to.1990
ACF
Lag
J.for.long.term.treasury.bonds.1953.to.1976
J.for.30.year.treasury.bonds.1977.to.1990
J.for.long.term.treasury.bonds.1953.to.1976
J.for.30.year.treasury.bonds.1977.to.1990
Quantiles of Standard Normal
Quantiles of Standard Normal

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STEVE CRAIGHEAD:

Mr. Klein's article was very enjoyable and fascinating in his use of the stable Paretian distribution in modeling long-term interest rate changes. However, I would like to raise the following questions.

All data analysis below uses daily 30-year interest rate data from April 1, 1981, to May 29, 1992, excluding July 31, 1991, September 16, 1991, and October 31, 1991 due to missing data; see [11] and [12]. There are a total of 2741 log ratios from these data.

Using Klein's Formula (9), n=2741, I calculated

$$c = 1/(2 * 0.827) * [OS_{1974.24} - OS_{767.76}] = 0.0043702.$$

Using Formula (11)

$$z_{0.96} = 1/(2c)[OS_{2632.32} - OS_{109.68}] = 3.195489$$

Determining α such that F_α (3.1955)=0.96, α=1.575. That δ= −0.00022535 was determined by 50 percent truncated mean estimation. The parameter γ was determined to be 0.0001922. Mr. Klein's γ was determined, for the 1977–1990 data, to be 0.002355.

The average number of days of security quotation for each month in the 11 years and 2 months was 20.5 days.

My observations with these additional interest values is that α appears to be fairly consistent in the daily and monthly analysis. Also, δ is very close to 0. This is due to $z_{0.96}$ being very close to Mr. Klein's. However, Mr. Klein noted that the sum of n stable Paretian random variables whose distributions have the same α and β has a distribution with γ equal to the sum of the γ's of the component distributions. If this was the case, Mr. Klein's γ divided by the daily γ above should be approximately equal to the average days of issue in a month. However, this result is 12.25, not 20.5. This raises the following questions:

1. Are Formulas (9) and (11) accurate estimators? Since the value of α is consistent with the two different monthly data sets and the daily data set, could the formula for c be faulty?

2. Could c drift where α does not? Mr. Klein discusses a possible drift in α, which is in his Tests I and II section. Could the daily γ versus the monthly γ not be statistically significant? How could one study this?
3. Should we not model interest rates from a stochastic method at all? Why not use a persistent fractal noise generation process based on rescaled range analysis? See [1] through [10]. These techniques, which also assume an infinite variance, may produce results as dramatic as those in Mr. Klein's ruin analysis. The fractal noise techniques do not require the analysis needed to handle drifting parameters. In fact, these techniques could very well be used in modeling nearly efficient markets; see [1].

I have two other concerns:

1. Mr. Klein was not consistent in the use of the J_t. Formula (8) differs from the formula in the last paragraph of page 28. Did Mr. Klein use Formula (8) in his generation of the Paretian stochastic process or the one he quotes? If he changed formulas in the ruin theory analysis, I am concerned that the results could be tainted.
2. In the use of the Paretian distribution, $0<\alpha=2$. In fact, when $\alpha=2$, the Paretian distribution is a normal distribution. Since $\alpha=2$ is an endpoint, I am concerned that the estimate of α would naturally be away from 2. Of course Mr. Klein addressed this with his significance tests, but I wish, for elegance sake, that there was a distribution, similar to the Paretian, that had the normal distribution embedded in the continuum, not at an endpoint.

In all, I thought that Mr. Klein's paper was well written and very well researched. I was very pleased with the demonstrations that he carried out. I am looking forward to meeting him and having a long discussion on the behavior of interest rates.

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MARTIN DEN HEYER:

The author deserves our congratulations for writing an excellent paper. Over the next several years, I suspect it will become a reference of increasing importance, as actuaries attempt to incorporate the discontinuities observed in real world events into their risk models.

In my view, the major achievements of this paper are as follows:

1. Evidence is provided that the stable Paretian distributions produce significantly different results than the normal distribution for typical actuarial problems. This is surprising because the shapes of the respective probability density and cumulative distribution functions are quite similar. The rationale that a normal approximation is "close enough" now seems very tenuous indeed. In this sense, this paper substitutes a convincing demonstration for a seemingly plausible but misinformed impression!
2. One of the barriers that prevented actuaries from applying the Paretian distributions was the difficulty and inaccessibility of the underlying mathematics. The descriptions, examples, and algorithms disseminated in this paper help remove the barrier.

3. The author has critically evaluated some important limitations of his analysis. These are:
   (a) The two sets of sample interest rate data (1953–76 and 1977–90) produced different values of c, the "width" parameter, which indicates nonstationarity. The stable Paretian hypothesis requires stationary parameters.
   (b) The Hsu, Miller, and Wichern tests (Klein's ref. [18]) for stability (Test I and Test II) were not satisfied.

   Both limitations weaken the case for assuming stable Paretian distributions or, in my opinion, indicate the hypothesis needs modification.
4. The last achievement consists of the author's review of possible explanations for the limitations. These are as follows:
   (a) Interest rate changes may be dependent over time.
   (b) The behavior is nonstationary due to shifts in the "underlying reality."
   (c) The stable Paretian distribution may be asymmetric.

Of these three possible explanations, I believe the *dependence* issue to be pivotal, for the following reasons.

1. In addition to Becker's (Klein's ref. [4]) conclusions on dependence, the rescaled range analysis for 30-year Treasury bonds by Peters (Klein's ref. [27]) indicates long-term dependence as measured by the Hurst statistic (H). The results are highly credible because the Hurst statistic is robust to the underlying distribution. In particular, the Hurst statistic does not require the underlying distribution to have a finite variance or mean.
2. Both limitations (a) and (b) are symptomatic of dependence over time increments. For example, dependent ARIMA models exhibit divergent "width" parameter estimates from different data subsets, similar to limitation (a). If dependence exists, Tests I and II would tend to fail because the α parameterization is based on the generalized central limit theorem, with independence as a necessary condition. It is interesting that Test II was performed on scrambled data and produced $\alpha \approx 1.9$. Any dependence would likely have been eliminated by scrambling; therefore, $\alpha \approx 1.9$ might be a better estimate than $\alpha \approx 1.6$.

A practical method for estimating α from a dependent time series and a test for confirming the result is outlined later in this discussion. The theoretical aspects are considered first.

Theory

Consider the stable Paretian hypothesis for interest changes

$$\Delta \log I_t = -\mathscr{E}_t$$

where $1<\alpha<2$, $0<t<\infty$, and ${}^{\alpha}\mathscr{E}_t$ is a stable Paretian increment with appropriate width parameter c. Since time is infinitely divisible, we can utilize the integral expression for

$$\log I_t = u + \int_0^t dP(y),$$

where u is a constant of integration equal to the sample mean of the log I_t, and $P(t)$ is a stochastic process with increments ${}^{\alpha}\mathscr{E}_t$.

If $\alpha=2$ (and $u=0$), this describes the traditional Brownian motion, tracing a *continuous*, but nondifferentiable path in time. The increments, ${}^{\alpha=2}\mathscr{E}_t$, are independent and determined by the normal distribution.

If $0<\alpha<2$, the integral traces a *discontinuous* path in time, with independent Paretian increments, ${}^{\alpha\neq 2}\mathscr{E}_t$. Clearly, α determines the continuity properties of the stochastic process. As α decreases, the quantum and frequency of "tail events" increases, and discontinuity increases.

The Brownian motion path of a particle suspended in a liquid must be continuous to avoid contravening a physical law. The requirement for continuity and independent increments and the central limit theorem ensure this mathematical solution is *unique*.

The paths traced by prices and interest rates are discontinuous, as confirmed by observation and supported by general reasoning; so $\alpha\neq 2$ and the variance of the increments is infinite. The implications for $\alpha\neq 2$ are severe. Traditional time series analysis techniques, such as ARIMA, do not apply because they are based on the normal distribution, implying a continuous path process, and the autocorrelation function, which requires a finite variance. Therefore, the autocorrelation function is not a viable mechanism for introducing dependence into a process with $\alpha\neq 2$.

The Hurst statistic (H) is robust, so it is a good candidate to introduce dependence into the integral referred to earlier. The method of fractional integration proposed by Mandelbrot and Van Ness [1] based on an earlier theorem attributed to Liouville and Reimann [2] produces:

	Example 1 Brownian Motion	Example 2 Fractional Brownian Motion	Example 3 Paretian Motion	Example 4 Fractional Paretian Motion
(In)dependence	$H=0.5$	$H\neq 0.5,\ 0<H\leq 1$	$H=0.5$	$H\neq 0.5,\ 0<H\leq 1$
(Dis)continuity	$\alpha=2$	$\alpha=2$	$0<\alpha<2$	$0<\alpha<2$
Scaling	$S=0.5$	$S=H$	$S=1/\alpha$	$S=H-0.5+1/\alpha$
Description of stochastic process	Continuous random process with independent increments	Continuous ordered random process	Discontinuous random process with independent increments	Discontinuous ordered random process

The stable Paretian hypothesis is a special case of example 3 with $1<\alpha<2$ instead of $0<\alpha<2$. The modified stable Paretian hypothesis is based on the introduction of dependence (or ordering) as in example 4, with the same restriction, $1<\alpha<2$. The consequence of this restriction is that the increments have a finite mean. The rationale is explained in the comments about the impact of discontinuity on pooling and diversification.

Parameterization

Based on the modified hypothesis, c, α and H can be estimated from the combined data series (1953–1990) using the techniques described in Klein's paper and Peters (Klein's ref. [27]) as follows:

1. Perform a rescaled range analysis to determine H. Presumably H is about 0.68, as estimated by Peters (Klein's ref. [27]) for the period 1950–1989.
2. Scramble the monthly data and confirm independence by performing another rescaled range analysis. This time, H should be about 0.50.
3. Convert the scrambled monthly data to four-month (non-overlapping) increments and determine C_{4M} and α_{4M} as outlined in Klein's paper.

   Convert the four-month data to 16 months (non-overlapping) and determine C_{16M} and α_{16M}. Hopefully, the following hold: $\alpha_{16M}\approx\alpha_{4M}$, and $C_{16M}\approx 4^{1/\alpha}C_{4M}$.
4. Now estimate C_{4M} and C_M from the original data (before scrambling). In accordance with the scaling property, confirmation of the fractional Paretian motion model would require that

$$C_{4M} \approx 4^{H-.5+1/\alpha}C_M, \text{ rather than } C_{4M} \approx 4^{1/\alpha}C_M.$$

If the relationships in steps 3 and 4 hold, it would seem to confirm that the underlying distribution is symmetric. If not, it would suggest the need for further research on asymmetry or testing on a longer data series.

Of course, this approach may be used to analyze other economic time series such as stock returns and inflation rates. This may require further generalization by allowing $H>1$ and replacing u with a polynomial trend of higher degree.

To conclude this discussion, I would like to make some general comments about the dilemma of non-stationarity, the rationale for dependence and ordering, and the implications of discontinuity for pooling and diversification strategies.

Nonstationarity

The conjecture that the behavior of economic variables is nonstationary seems very plausible. We can examine stock price, interest rate and inflation records over some 70 years and reasonably conclude that the parameters of the related stochastic variables seem to shift over time. The conventional explanation is that the "underlying reality" changes, and therefore the stochastic "rules" do too. Economists such as Keith Ambachtsheer discern eras, consisting of periods lasting about 10 years during which the rules remain stationary, and shifts, during which new "rules" emerge. The implications of this observed behavior for long-term models based on the normal distribution are unfortunate.

If the parameters shift every 10 years or so, we do not have sufficient data to model the shifts with any credibility. If we do not model the shifts, the exercise is incomplete. If we do model the shifts, the exercise is bound to be subjective. This appears to be a scientific dead end.

Mandelbrot's vision, the Paretian hypothesis, attempts to escape this dilemma by changing the frame of reference from finite variance nonstationary normal models to infinite variance stationary Paretian models.

Paretian-based models seem plausible, because the dependence (H) parameter appears to explain the eras and the discontinuity (α) parameter appears to explain the shifts. If we can derive stationary H and α parameters from data covering the last 70 years, longer term models would have more scientific credibility for quantifying risk. The challenge is to provide evidence that H and α are stationary over multiple-era time periods. For both technical and theoretical reasons, it would appear desirable to estimate the parameters from the longest available time series data. Of course, it would be a cruel joke if we had to deal with both infinite variance distributions *and* nonstationery parameters.

Dependence and Ordering

I prefer to use the term "ordering" rather than dependence because it is more accurate and descriptive. Consider the "underlying reality" (that is, history) as having a sequential order in time.

Under this conjecture, the present is determined not only by the quantum of past (stochastic) events but also by their sequential order. Likewise, the future "n" years from now will be determined by both the quantum and order of events during the intervening period. Since time is infinitely divisible, the ordering necessarily spans across all sizes of discrete time increments—from seconds to centuries, as it were. If economic variables are manifestations of the "underlying reality," it would appear plausible that they are sequentially ordered as well. This calls into question the (unmodified) stable Paretian hypothesis, the efficient market hypothesis and all other random walk models with independent increments.

The mathematical technique for introducing dependence and ordering is integration or its discrete equivalent, summation. Consider the successive summation of a stochastic process

$$_0Y_t = \mathscr{E}_t$$

$$_1Y_t = \sum_{i=0}^{\infty} \mathscr{E}_{t-i} = \mathscr{E}_t + \mathscr{E}_{t-1} + \mathscr{E}_{t-2} \cdots\cdots\cdots$$

$$_2Y_t = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \mathscr{E}_{t-i-j} = \sum_{i=0}^{\infty} (1+i)\,\mathscr{E}_{t-i} = \mathscr{E}_t + 2\mathscr{E}_{t-1} + 3\mathscr{E}_{t-2} \cdots\cdots\cdots$$

$$\vdots$$

$$nY_t = \sum_{i=0}^{\infty} \frac{(i+n)!\,\mathscr{E}_{t-i}}{i!n!}.$$

The $_0Y_t$ process is completely independent of the past.

The $_1Y_t$ process is dependent on past events but is not ordered. If this model described the "underlying reality," the order of World Wars I and II could be reversed, without affecting the present.

In contrast, ${}_2Yt$ is an ordered process. Any change in the order of past events affects the present. Both the process ${}_2Yt$ *and* its increments are dependent on the past, which is the sufficient condition for a process to be "ordered."

The problem is that ${}_2Yt$ does not fit economic time series data; it has "too much" ordering. Clearly, other ordered processes may be obtained by interpolating between the various ${}_nYt$. For example, for $0.5 \leq H \leq 1.5$, the fractional integration $V(t)$ interpolates between ${}_1Yt$ and ${}_2Yt$, which results in the increments $\Delta V(t)$ being an interpolation between ${}_0Yt$ and ${}_1Yt$.

The rescaled range analysis technique has been tested on simulated data series and found to be a credible estimator for H, the "level" of integration.

There are of course an infinite number of interpolation functions that might apply. The distinguishing feature of fractional integration is that it is the *unique* interpolation function that preserves the scaling property of Brownian and Paretian motion processes. A consequence of preserving the scaling property is that the ordering spans all size time increments.

Discontinuity, Pooling and Diversification

We have already reviewed how the scaling property determines volatility over multiple time increments. The additive property of stable Paretian distributions also has significant implications *within* each time increment.

The additive property for stable Paretian distributions (${}^{\alpha}X_i$) with width parameters C_i is:

$$\sum_{i=1}^{n} c_i({}^{\alpha}X_i) = \left(\sum_{i=1}^{n} c_i\right)^{1/\alpha} ({}^{\alpha}X)$$

when the $C_i = 1$, this reduces to

$$\sum_{i=1}^{n} {}^{\alpha}X_i = n^{1/\alpha}({}^{\alpha}X).$$

Within any time increment, the implications for pooling and diversification strategies are as follows.

1. Pooling and diversification mitigate volatility risk for $1 < \alpha \leq 2$. The strategy produces maximum results at $\alpha = 2$, becomes ineffective at

$\alpha=1$ and counterproductive for $0<\alpha<1$. This appears to explain the restriction ($1<\alpha<2$) for the stable Paretian hypotheses.

2. Both distribution width (c) and discontinuity (α) contribute to volatility. For $1<\alpha<2$, pooling and diversifications strategies reduce the width parameter by the scaling exponent $1/\alpha$. However, the discontinuity property of the aggregate distribution remains unchanged. Therefore, pooling and diversification do not mitigate the portion of the risk relating to discontinuity. This implies hedging and exclusion are more effective strategies for mitigating discontinuity risk. Hedging is of course widely used for managing investment related risk. An example of exclusion is the treatment of deaths caused by acts of war under most insurance arrangements.

Although the above conclusions apply only when the "individual elements" share the same α, it demonstrates that stable Paretian models distinguish between different types of volatility while maintaining the utility of a central limit theorem and other useful properties. This has intriguing implications and seems to provide a valuable tool for diverse actuarial problems.

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JOHN DUTEMPLE:

As a student who has recently succeeded in passing Examination V-380, I found Mr. Klein's paper particularly informative—not only for its conclusions but also for its approach.

It's easy for students caught in the midst of focused study to accept the material presented to them as fact rather than as a hypotheses subject to examination and verification or even as simply currently accepted practice. The conclusion to be drawn by the student (and indeed by all practicing actuaries) from this paper is not that the broader family of stable Paretian distributions is a better choice for interest-rate modeling than the narrower set of normal distributions (although, for this particular set of data and many others, this is the case). Rather, the point that should be emphasized is that the actuary needs to look beyond current practice and the SOA syllabus when current techniques do not measure up to the task at hand. This is especially true in the finance arena where the study track is so relatively new.

Because the issue of modeling cash flows has become a major one for many insurance companies and promises to continue to grow in importance, I must echo the call for further research in this area. Not only should further research be done on the modeling process itself, but also research or development of statistical techniques (most notably measures of the notion of correlation in distributions with infinite variances) that would further the modeling research is needed. I hope this paper becomes a starting point for such investigation.

PAUL P. HUBER*:

The paper is an important contribution to the literature because it highlights the extreme sensitivity of cash-flow analyses to the model of interest rate changes. However, the paper fails to assess whether the lognormal hypothesis is close enough for modeling purposes. This is because the paper ignores the dynamic structure of the interest rate time series by assuming that the interest rate changes are independent and identically distributed. This discussion presents additional reasons why the stable Paretian hypothesis has failed to gain acceptance and analyzes the interest rate time series in more detail. This analysis questions the validity of the paper's results and suggests that the paper presents a biased perspective in favor of the stable Paretian hypothesis.

Additional Reasons Why the Stable Paretian Hypothesis Has Failed To Gain Acceptance

An additional reason to those given in the paper (see Section II-E) on why the stable Paretian hypothesis has failed to gain acceptance is, as stated by Sennett [9], due to ". . . the poor descriptive ability of the symmetric stable models." In modeling stock prices, Officer [7], Barnea and Downes [1], Upton and Shannon [10], and Fielitz and Rozelle [4] have all found results similar to those reported in Section II-D-4, in that the characteristic exponent increases for sums of individual observations. Officer [8] reports that the standard deviation of returns appears to be a well-behaved measure of dispersion. Blattberg and Gonedes [2] find that the student (or *t*) distribution is more appropriate than the symmetric

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stable distribution for daily rates of stock returns. Merton [6] concludes that ". . . there is little empirical evidence to support adoption of the stable Paretian hypothesis over that of any leptokurtotic distribution."

In practice, other types of models have been found to be more appropriate than the stable Paretian model. Merton [6] reports that finite-moment processes whose distributions are nonstationary provide a promising means of modeling prices. In particular, conditionally heteroscedastic models have been found to be suitable for modeling investment return data [3]. The following section of this discussion shows that the goodness-of-fit of the stable Paretian hypothesis deteriorates considerably once the dynamic structure of the interest rate time series is taken into account.

Another reason for not using the stable Paretian model to describe the log of interest rate changes (J_t) is that this implies that the expected value of interest rate changes (I_t/I_{t-1}) does not exist [6].

A Time Series Analysis of the Interest Rate Data

As i_t (see Section II-D) represents the average daily interest rate on 30-year Treasury bonds in month t, $j_t = \log_e(i_t/i_{t-1})$, is likely to have a spurious first-order moving average correlation structure [12]. This can be verified by comparing the correlation structure of j_t to the correlation structure of $j'_t = \log_e(i'_t/i'_{t-1})$, where i'_t represents the interest rate on 30-year Treasury bonds on the last working day of month t. (The values of i'_t [11] are given in the Appendix to this discussion. The yield on January 31, 1977 is not available because the Federal Reserve did not publish this time series prior to February 1977. As the *Federal Reserve Bulletins* for 1977 do not give values for i_t in January 1977 and February 1977, it would be interesting to know the reference for the values given in Table 2 for these two months.)

Over the period February 1977 to December 1990, a first-order moving average model of the following form appears to provide an appropriate description of j_t:

$$j_t = \log_e(i_t/i_{t-1}) = \epsilon_t + \theta\epsilon_{t-1},$$

where $\epsilon_t \sim i.i.d.N(\mu,\sigma^2)$. The estimate of the parameter θ is $\hat{\theta}=0.5920$. As $\hat{\theta}$ has a standard error of 0.0627, a significant moving average component is present in the series j_t.

Over the same period, February 1977 to December 1990, Box-Ljung Q-statistics for j'_t are $Q(6)=6.3$ and $Q(12)=11.6$. These statistics do not indicate first-order serial dependence. Therefore, the moving average component in j_t appears to be spurious. This is simply as a result of using monthly average interest rates.

A stable Paretian distribution was fitted to j_t' (over the period February 1977 to December 1990), using the method given in the paper, and the following parameter estimates were obtained:

$$\hat{c} = 0.0261,\ z_{0.96} = 2.9286,\ \hat{\delta} = 0.0012, \text{ and } \hat{\alpha} = 1.68.$$

The standardized kurtosis (b_2-3) and skewness $(\sqrt{b_1})$ of the series j_t' are 0.4945 and -0.2735, respectively, with standard errors of approximately 0.3802 and 0.1901, respectively.

The statistics a (the ratio of mean deviation to standard deviation [8]) is 0.7706, which is not significant at the 5 percent level.

The statistic w/s (the ratio of range to standard deviation [8]) is 5.4569, which is not significant at the 10 percent level.

Therefore, there is far less evidence to reject the lognormal hypothesis for i_t' (over the period February 1977 to December 1990) than there is to reject it for i_t.

Additional tests reveal that a significant nonlinear effect is present in j_t'. A Box-Ljung Q-statistic for j_t' (over the period February 1977 to December 1990) is $Q^2(12)=28.9$, which suggests the presence of second-order serial dependence. Casual observation of j_t' indicates a sharp increase in the variance of j_t' after July 1979 and a subsequent decrease in the variance of j_t' after July 1987. (The sample standard deviation of j_t' is 0.0160 between February 1977 and July 1979, 0.0487 between July 1979 and July 1987, and 0.0332 between July 1987 and December 1993.)

These changes suggest that j_t' is not independent and identically distributed over the entire time period. Therefore, j_t' should be modeled separately over each subperiod rather than over the entire time period. If these changes can be explained in economic terms or in terms of a change of policy and are found to be more or less permanent, then it is not appropriate to model future interest rates on the basis of the data that occurred before the changes. (The sharp increase in variance in July 1979 is possibly due to the shift in Federal Reserve policy referred to in the paper in Section II-D.4.)

The following three sections test the appropriateness of the lognormal hypothesis over each of the three subperiods.

February 1977–July 1979

Box-Ljung Q-statistics for j_t' are $Q(6)=2.5$ and $Q^2(6)=8.4$. These statistics do not indicate first- or second-order serial dependence.

The following parameter estimates were obtained for a stable Paretian distribution:

$$\hat{c} = 0.0157,\ z_{0.96} = 1.8042,\ \hat{\delta} = 0.0067 \text{ and } \hat{\alpha} > 2.$$

The standardized kurtosis and skewness of the series j_t' are -1.0116 and -0.3130, respectively, with standard errors of approximately 0.9097 and 0.4549, respectively.

The statistic a is 0.8816, which is not significant at the 1 percent level.

The statistic w/s is 3.6846, which is not significant at the 10 percent level.

Therefore, although these statistics may be unreliable for small samples, there appears to be little evidence to reject the lognormal hypothesis for i_t' between February 1977 and July 1979.

July 1979–July 1987

Box-Ljung Q-statistics for j_t' are $Q(12)=9.5$ and $Q^2(12)=17.8$. These statistics do not indicate first- or second-order serial dependence.

The following parameter estimates were obtained for a stable Paretian distribution:

$$\hat{c} = 0.0351,\ z_{0.96} = 2.7049,\ \hat{\delta} = 0.0005, \text{ and } \hat{\alpha} = 1.80.$$

The standardized kurtosis and skewness of the series j_t' are -0.0443 and -0.2404, respectively, with standard errors of approximately 0.5000 and 0.2500, respectively.

The statistic a is 0.7935, which is not significant at the 10 percent level.

The statistic w/s is 4.6845, which is not significant at the 10 percent level.

Therefore, there appears to be no evidence to reject the lognormal hypothesis for i_t' between July 1979 and July 1987.

July 1987–December 1993

Box-Ljung Q-statistics for j_t' are $Q(12)=11.1$ and $Q^2(12)=6.3$. These statistics do not indicate first- or second-order serial dependence.

The following parameter estimates were obtained for a stable Paretian distribution:

$$\hat{c} = 0.0186,\ z_{0.96} = 3.4658,\ \hat{\delta} = -0.0041, \text{ and } \hat{\alpha} = 1.50.$$

The standardized kurtosis and skewness of the series j_t' are −0.1570 and −0.0405, respectively, with standard errors of approximately 0.5583 and 0.2791, respectively.

The statistic a is 0.7696, which is not significant at the 10 percent level.

The statistic w/s is 4.5061, which is not significant at the 10 percent level.

A χ^2 goodness-fit-test, using a similar method to that given in the paper, was performed for the lognormal and stable Paretian hypotheses. (I was unable to exactly reproduce the results given in Section II-D-3. Following Hsu, Miller, and Wichern (Klein's ref. [18]), I first centered the data by subtracting $\hat{\delta}$ and the following borders were used: $(-\infty, -5.5\hat{c}]$, $(-5.5\hat{c}, -4.5\hat{c}]$, ... $(+4.5\hat{c}, +5.5\hat{c}]$, $(+5.5\hat{c}, +\infty)$.) The above parameter values were used for the stable Paretian hypothesis and the following parameter values were used for the lognormal hypothesis:

$$\hat{c} = 0.0235,\ \hat{\delta} = -0.0041,\ \hat{\mu} = 0 \text{ and } \hat{\sigma} = 0.0332.$$

The results of these tests are X^2=5.76 for the lognormal hypothesis and X^2=12.62 for the stable Paretian hypothesis.

Therefore, although the parameter estimate for α suggests that the stable Paretian hypothesis may be more appropriate ($\hat{\alpha}$ is significantly less than 2 at approximately the 2.5 percent significance level), on the basis of the kurtosis, the skewness, the statistic α, the statistic w/s and the X^2 statistic of j_t', there is insufficient evidence to reject the lognormal hypothesis for j_t' between July 1979 and June 1987.

Conclusion and Areas for Future Research

The above results suggest that the too "peaked" and "fat-tailed" features of the original data can be accounted for in terms of a spurious first-order moving average correlation structure induced by the use of daily average data and in terms of two structural changes in the data. Over each of the three subperiods considered, there is insufficient evidence to reject the lognormal hypothesis in favor of the stable Paretian hypothesis. This raises considerable doubt about the validity of the results reported in the paper.

Progress in interest rate modeling is unlikely to be made if the dynamic structure of the time series is ignored. Unless this aspect is taken into account, it is dangerous to arrive at any conclusions on the suitability or otherwise of the lognormal or any other hypothesis for cash-flow analyses.

Additional interest rate data to those presented in this discussion should be examined within a time series framework. In particular, the changing structure of the interest rate time series should be researched in more detail in conjunction with other economic information. Other economic time series, such as the yield on three-month Treasury bills, may provide important additional information for the modeling of long-term interest rates. At the same time, these series may also be useful for cash-flow analyses.

The factors that led to the possible structural changes in j_t' require further examination to establish whether they were permanent and to determine the conditions that are likely to cause a future shift to occur.

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APPENDIX

END-OF-MONTH YIELD TO MATURITY ON 30-YEAR TREASURY BONDS, 1977–1986
(Source: *Federal Reserve Statistical Release*)

Date	Yield	Date	Yield	Date	Yield	Date	Yield	Date	Yield
1977		1981		1985		1988		1991	
Jan.	n.a.	Jan.	12.28	Jan.	11.21	Jan.	8.42	Jan.	8.21
Feb.	7.80	Feb.	12.97	Feb.	11.90	Feb.	8.39	Feb.	8.19
Mar.	7.79	Mar.	12.65	Mar.	11.64	Mar.	8.82	Mar.	8.24
Apr.	7.80	Apr.	13.65	Apr.	11.48	Apr.	9.11	Apr.	8.20
May	7.74	May	13.06	May	10.58	May	9.30	May	8.26
June	7.58	June	13.30	June	10.47	June	8.87	June	8.42
July	7.72	July	13.96	July	10.70	July	9.23	July	8.36
Aug.	7.60	Aug.	14.78	Aug.	10.48	Aug.	9.31	Aug.	8.06
Sep.	7.68	Sep.	15.19	Sep.	10.57	Sep.	8.98	Sep.	7.82
Oct.	7.83	Oct.	14.36	Oct.	10.28	Oct.	8.74	Oct.	7.91
Nov.	7.83	Nov.	12.91	Nov.	9.86	Nov.	9.07	Nov.	7.94
Dec.	8.03	Dec.	13.65	Dec.	9.27	Dec.	9.00	Dec.	7.41
1978		1982		1986		1989		1992	
Jan.	8.18	Jan.	13.91	Jan.	9.34	Jan.	8.84	Jan.	7.77
Feb.	8.25	Feb.	13.83	Feb.	8.27	Feb.	9.14	Feb.	7.80
Mar.	8.33	Mar.	13.68	Mar.	7.44	Mar.	9.11	Mar.	7.96
Apr.	8.39	Apr.	13.39	Apr.	7.47	Apr.	8.91	Apr.	8.06
May	8.50	May	13.39	May	7.74	May	8.60	May	7.84
June	8.62	June	13.91	June	7.24	June	8.05	June	7.79
July	8.56	July	13.42	July	7.46	July	7.92	July	7.46
Aug.	8.46	Aug.	12.50	Aug.	7.21	Aug.	8.21	Aug.	7.42
Sep.	8.61	Sep.	11.79	Sep.	7.60	Sep.	8.24	Sep.	7.38
Oct.	8.87	Oct.	11.01	Oct.	7.61	Oct.	7.92	Oct.	7.63
Nov.	8.80	Nov.	10.70	Nov.	7.41	Nov.	7.90	Nov.	7.59
Dec.	8.96	Dec.	10.43	Dec.	7.49	Dec.	7.98	Dec.	7.40
1979		1983		1987		1990		1993	
Jan.	8.85	Jan.	10.99	Jan.	7.48	Jan.	8.46	Jan.	7.21
Feb.	9.08	Feb.	10.51	Feb.	7.48	Feb.	8.54	Feb.	6.90
Mar.	9.02	Mar.	10.69	Mar.	7.81	Mar.	8.63	Mar.	6.93
Apr.	9.22	Apr.	10.38	Apr.	8.45	Apr.	9.00	Apr.	6.95
May	9.08	May	10.97	May	8.65	May	8.58	May	6.98
June	8.83	June	11.01	June	8.51	June	8.41	June	6.68
July	8.99	July	11.80	July	8.89	July	8.42	July	6.57
Aug.	9.09	Aug.	11.96	Aug.	9.17	Aug.	8.99	Aug.	6.09
Sep.	9.25	Sep.	11.44	Sep.	9.79	Sep.	8.96	Sep.	6.04
Oct.	10.19	Oct.	11.78	Oct.	9.03	Oct.	8.78	Oct.	5.96
Nov.	10.09	Nov.	11.67	Nov.	9.10	Nov.	8.40	Nov.	6.29
Dec.	10.11	Dec.	11.87	Dec.	8.95	Dec.	8.26	Dec.	6.35
1980		1984							
Jan.	11.09	Jan.	11.78						
Feb.	12.25	Feb.	12.14						
Mar.	12.31	Mar.	12.52						
Apr.	10.89	Apr.	12.86						
May	10.37	May	13.84						
June	9.99	June	13.64						
July	10.80	July	12.87						
Aug.	11.27	Aug.	12.51						
Sep.	11.70	Sep.	12.28						
Oct.	12.23	Oct.	11.64						
Nov.	12.32	Nov.	11.58						
Dec.	11.98	Dec.	11.54						

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MERLIN JETTON:

I enjoyed reading Mr. Klein's well-written paper. He discusses the lognormal hypothesis versus the stable Paretian hypothesis for building a model for interest rates. He compares distributions of changes in interest rates output by models using such hypotheses and the consequences for cash-flow analysis. He criticizes the lognormal model because its distribution of changes in interest rates is not sufficiently peaked nor fat-tailed. Historical changes in interest rates have been found to be too peaked and fat-tailed to have been from a normal distribution. He demonstrates that a stable Paretian model, with parameters he develops, does not show this weakness.

Mr. Klein takes care to compare his models to reality. I believe he is right to claim that the stable Paretian model better represents reality than does the lognormal model, when the basis of comparison is how well *monthly* changes in interest rates output by the model represent *monthly* changes in historical interest rates. However, I believe this is an unduly limited comparison. Are monthly changes in interest rates the only important criteria on which to compare a model to reality when the model's purpose is to project rates over several years? How about changes in interest rates on a quarterly or annual basis and even changes over multiple years? While the historical data are more scanty for longer intervals, to ignore it and these questions seems myopic. Yet Mr. Klein's paper does not address these questions.

Mr. Klein chides lognormal model users with the following: "This justification for the use of the normal distribution is based on convenience, not empirical evidence." "One can argue that the normal distribution is 'close enough' to the true underlying distribution and that the extra work involved in using a non-normal member of the stable Paretian family is not justified." But I ask the following: What does the empirical evidence say about changes in interest rates over intervals longer than a month? Is it right to believe that a model that fits the monthly data must fit the data regarding longer intervals as well?

I will not claim that the lognormal model inherently fits reality any better than the stable Paretian model on the criteria offered by these questions. It would depend on the parameters used and what other features, such as mean reversion or regression formulas, might be included in the model. Incidentally, I believe few model builders use formulas as simple as the ones shown by Mr. Klein. For example, virtually all the models described by Christiansen [1] contain a mean reversion feature for very good reasons. But given the specific models presented by Mr. Klein in his paper, I suspect that the lognormal model outperforms the stable Paretian model over longer periods, despite its inferior performance considering monthly changes only. It would be interesting to see how the interest rates put out every third month by Mr. Klein's monthly stable Paretian model compare with historical interest rates at three-month intervals. The same goes for even longer intervals.

It would be interesting to see the performance of a quarterly stable Paretian model developed in the same manner as Mr. Klein's monthly stable Paretian model. Such a model would have parameters calibrated to best fit historical changes in interest rates over quarterly intervals. Similarly, an annual model might be built. I suspect that such models would give results quite different from Mr. Klein's monthly model when changes in interest rate output by the models are compared over the same intervals. (The additive property of the stable Paretian function implies that a quarterly or annual model could be derived from the monthly model, but I suspect the derived model would be quite different from the calibrated model.) Obviously such models would not produce monthly changes in rates for comparison with those from the monthly model, but the models could be compared with one another and with historical interest rates over three-month, one-year and even longer intervals. So which model would be the "right" or best one? The answer would seem to depend on what the model is used for. In testing the model against reality, greater weight should be given to the one or more aspects of reality deemed most relevant.

I believe either model he presents more widely disperses interest rates in the long run than what the historical evidence suggests, and the monthly stable Paretian model does so to an even greater degree. If this is the case, the cash-flow sensitivity analysis is distorted. Although the magnitude of the surpluses he shows in Table 9 (and Tables 6–8) assume questionable investment and crediting rate strategies, imagine a pricing actuary proposing an SPDA to a chief financial officer with possible outcomes like those in Table 9. I suspect the chief financial officer's response to chances of losing $80 million or $240 million or more on $10 million of premium, when the best case is only a $9 million profit, would be "go back to the drawing board!"

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G. THOMAS MITCHELL:

Thanks are due the author for developing the important subject of stable distributions in actuarial literature. It is clear that real world financial risks may well be greater than those indicated using normal distributions.

Why might this be so? And what plausible reasons do we have to wander from orthodox adherence to our old friend, the central limit theorem?

Dethroning the Central Limit Theorem

The central limit theorem requires that the limiting distribution be the mean of a large number of independent variables, each with finite means and finite variances. If the variables are not all identical, then the contribution of any one of them should still vanish as the "large number" increases.

Fluctuations in economic prices and indices are the result of myriad and diverse competing factors. Some are in the form of the sum of a large number of plausibly independent individual situations, but others are global to the economy or marketplace in question and reflect the inevitable interconnections in a market, an economy, a nation, or the world. The assumption that none of the factors has a significant influence on the whole result is doubtful.

However, we do expect conventional time scaling (after seasonal effects are removed). That is, there is no reasonable expectancy of a preferred time scale for operation of this host of factors. Thus, the distribution for a month is expected to be the convolution of 30 daily distributions, a calendar quarter of three monthlies, and so on.

This time scaling expectation leads us directly to the stable Paretian distribution family as a paradigm for price distributions. The normal distribution is a member of the family, but is at one extreme of the possibilities; that is, it is potentially an exception, rather than the norm.

Another way at this is to postulate many diverse "atomic" economic factors acting on the shortest meaningful ("atomic") time frame. Price changes on this atomic time scale are the mean (or some functionally equivalent interaction) of the outcomes of the factors. Price changes over larger time frames are the convolutions of the results for shorter time frames.

The individual factors, I speculate, may tend to be normally inactive, but make a significant difference when they are activated, hence the good possibility for fat-tailed atomic distributions. From a formal viewpoint, the normal distribution will pertain only if the variances and means of the atomic distributions are finite. And such fat-tailed distributions can easily have theoretically infinite variances.

Are Infinite Variances Pathological?

But how can there be, even theoretically, an infinite variance? This can be modeled by any number of real-life, realizable examples. Suppose I put my dollar up against your dollar, and we flip a coin, double or nothing. If I win, I flip a coin to determine whether I stop or not. If I continue, we flip for double or nothing again, and so on. My expectations of ending wealth are 0 with probability 2/3; 2 with probability 1/4; 4 with probability 1/16; 8 with probability 1/64; and so on. The mean is 1. However, each subsequent term adds 1 to the variance, so it grows without limit and is formally infinite.

In real life (and certainly in this example), there are practical outer cutoffs on any distribution. A billionaire banker would have to quit after about 30 losing rounds. *Any* outer cutoff removes the infinity.

With the infinite variance thus cured, convolutions of a fat-tailed distribution *will* converge to a normal distribution. But if the variance is very large, the convergence may be very slow. The distributions for short and intermediate time frames will resemble stable distributions, but with extremely rare events missing. For very long time frames, a normal distribution will eventually be obtained. However, the fat-tailed stable distribution shape will prevail over mortal time frames.

Further Research Indicated

The paper should be helpful in stimulating further research. The following questions come to mind: What are the practical effects on real-life asset/liability risk management studies? What effect does this distribution have on durations, convexity and the target point for a good match? Are risk levels increased, even after rebalancing assets and liabilities?

Further study is in order to determine applicability of this approach to other time frames, eras, and price series (such as short-term interest rates). Presumably patterns and tendencies in the distribution parameters will emerge.

There are techniques to be developed in using these distributions. What outer cutoff points make practical sense? Are distributions close to symmetrical or not? What are efficient computational tools for constructing scenarios? How does one make an effective choice of paths for mathematical techniques that use carefully selected sample paths rather than random scenarios?

The paper leaves unanswered our questions about the important and interrelated issues of auto correlation, long-term cycles, regression to the mean, and constancy over time of the underlying distributions. However, using better fitting stable Paretian distributions may speed up research progress in this area.

Spreadsheet Calculations of the Distributions

Computing the stable symmetric Paretian distribution on an electronic spreadsheet is reasonably straightforward. However, a few comments and tips may be helpful.

As expressed in the paper, the factorial and γ function terms may quickly overflow machine number limits. Many spreadsheets have built in functions that are the logarithms of factorials or gamma functions. Reexpressing terms using these log-factorial functions or calculating each term as a ratio to the preceding term are techniques around the overflow problem.

The second approximation, pertinent for large values of u, typically produces a series of progressively smaller terms, followed eventually by increasing terms. Satisfactory results, agreeing with the author's table, can be obtained by cutting off the series at the point the magnitude begins to increase. In my implementation of the approximation, I obtained slightly better results by using 4.03 instead of 4.0 as the constant in the criteria to choose between the first or second functions.

Fractals

Finally, the deep connection between the stable Paretian distributions and the study of fractal shapes should not go unnoticed. Self-similar patterns at various scales are the common theme.

It is likely, too, that fractals beget fractals. If natural and economic forces have fat-tailed stable or fractal distributions, they may impart this distribution onto phenomena that are affected by them.

The ubiquitousness of fractal shapes and phenomena in nature should give us pause before rejecting the same phenomena in economics.

THOMAS C. POWELL:

Cash-flow testing involves a daunting mix of scientific, regulatory, and professional challenges. Perhaps the Society should have an award for papers, such as this one, that display an exceptional measure of intestinal fortitude.

On the surface, Mr. Klein's paper seems to be directed at a technical issue, that is, the appropriateness of a particular interest rate generator (IRG), as measured by the effect that the choice of an IRG has on the results of a cash-flow test. In the course of his analysis, Mr. Klein works through the various trade-offs involved, particularly the trade-off between convenience and consistency with historical data.

We have been inundated with articles and papers about IRGs in the past few years, especially those based on the lognormal model. But there has not been a great deal of explanation on why IRGs should play a role in cash-flow testing. Much as I am impressed by Mr. Klein's thoroughness, I wish he had backed up a pace or two before starting his analysis.

The stated purpose of Mr. Klein's analysis is to arrest the actuarial profession's drift toward the effective adoption of the lognormal model as the benchmark IRG. This is important because, if a cash-flow test should require defending, it will be judged on its adherence to established doctrine.

General Acceptance in the Accounting and Actuarial Professions

In the opening sentence of the Introduction, Mr. Klein tells us that actuaries are "developing a consensus on the proper methodology for analyzing the cash flows of an insurance company."

The reference to consensus-building is the first of several references to some kind of general agreement among members of the profession. Later we see "virtually universally accepted," "currently common actuarial practice," and "currently accepted actuarial methodology."

In my opinion, the term "generally accepted," as in "generally accepted accounting principles" and "generally accepted actuarial standards," is a duplicitous phrase intended to impute general acceptance (in the vulgar sense) to a professional ukase. It is one thing to admit that a principle or practice has little basis other than the support of a number of presumably knowledgeable people; it is another thing to imply such support when it does not exist.

The accounting profession spells out the meaning of "general acceptance" in *Statement on Auditing Standards 69* (AICPA, June 1992). According to this document, established accounting principles stem first and foremost from statements of the Financial Accounting Standards Board (a creature of the AICPA) and last and least from the prevalence of industry practice. In other words, general acceptance, as commonly understood, has practically nothing to do with the AICPA's official definition of "generally accepted accounting principles." If generally accepted accounting principles were always in accord with general practice, then *Statement 69* must have been promulgated in order to use up paper and ink. But the fact is that situations often arise in which the plain meaning of a FASB pronouncement is ignored in favor of mob preference.

In the case of the actuarial profession, we have the Actuarial Standards Board. Like the Financial Accounting Standards Board, the Actuarial Standards Board's job is to announce, in some cases, that there is a consensus whether there is one or not.

Neither the AICPA nor the ASB is willfully rejecting science in favor of popularity; in fact, cause and effect is the ultimate criterion in allocating revenues and expenses to a particular accounting period. The concept of general acceptance reflects the professions' recognition that causes and effects are not always clear and that there is sometimes a need to rely on the judgment of experienced people. This need may arise from a public posing difficult questions or a profession volunteering to answer such questions.

The point is that Mr. Klein appears to be attacking the lognormal model on scientific grounds; his argument would be stronger without any reference to "consensus." One must admit, however, that current thinking about IRGs is an odd mixture of science and fashion.

My guess is that most actuaries believe that the lognormal model is reasonably accurate (based on the a priori reasoning that Mr. Klein sets forth in Section II-C) when applied to a "net" interest rate unsullied by reactions to inflation and fears of inflation. There is a problem in applying this hypothesis to historical data, however, in that the "net" interest rate is sometimes negative and the logarithm of the ratio successive rates does not exist. This does not bother me as much as it seems to bother some members of the profession; consistency with historical data is a nice thing to have, but if interest rates are continually buffeted by factors that are essentially political, we will never see a convincing stochastic model. If that is the case, then prevalence of practice is a legitimate criterion for selecting a model, assuming that some form of IRG is to be used at all.

The Scientific Basis of Interest Rate Generators

I have often wondered what insight originally gave rise to the use of IRGs in actuarial applications. I suspect their use is rationalized by a line of reasoning such as the following:

(1) In performing an analysis of life insurance company operations, we do not generally use stochastic models for mortality, surrenders, expenses, and so forth. Variations in these parameters are commonly modeled by testing a range of values and summarizing the results. This approach is used because: (a) we think we have a basic understanding of the business decisions that affect these things and can therefore "choose" the experience we will have to some extent; (b) managers who use the report have a general understanding of these things (that is, what influences them) and therefore have a right to think they can exercise a measure of control over them; and (c) if we were to replace the deterministic model with a stochastic model, we might make the analysis incomprehensible to the user of our report, possibly at a great cost in time and effort.

(2) In the case of interest rates, we do not think that we are dealing with something we can understand and control. They are part of a more general environment, of baffling complexity, that is not materially affected by management decisions. This kind of uncertainty can be handled in various ways, including the use of Monte Carlo techniques.
(3) The actuarial profession has the requisite tools to model future interest rates stochastically and produce a credible analysis.
(4) Hence, the use of Monte Carlo techniques is a suitable response to the uncertainty associated with future interest rates.

In short, the universe of possible interest rate scenarios over the period of the test cannot be adequately modeled by a handful of preset scenarios. Hence, we use an IRG to select a random sample from this universe much as we pull colored balls from an urn.

The most vulnerable component of this argument is premise (3), especially if the touchstone of credibility is the fit of our IRG to historical data.

Consistency with Historical Data

In Section IV, Mr. Klein cites an experiment by Claire in which 99 scenarios were generated using a lognormal model. An additional test, based on actual rates from the 1980s, produced a test as unfavorable as any of the original 99. According to Mr. Klein: "This is evidence enough that the lognormal model is too far from reality to be useful."

Is it? *Actuarial Standard of Practice 7* appears to attribute its very existence to the "large increase in the level and volatility of investment rates of return that occurred in the 1970s and 1980s. . . ." Should we be surprised that actual rates from such a period produce an extreme result? In any case, Mr. Klein appears to dismiss the lognormal model because of its failure to fit the data from a particular historical era. This is a logical step from Mr. Klein's perspective, but the users of our reports may not agree that historical fit is an especially important criterion.

Suppose a manager who is paying for a cash-flow test tells us he or she is not interested in a Monte Carlo simulation; that he would prefer a range of preset scenarios so that he can apply his own judgment based on his perception of what the future holds for interest rates. Should the actuary responsible for the report take the position that he has professional expertise that dictates the use of a stochastic model?

The manager is treating the interest rate assumption the same way as other assumptions. Expected future expenses, for example, will be consistent with historical experience but may be modified by the manager's own assessment of the prevailing environment.

We may object that the manager has more control over the company's expenses than over market interest rates, but are we prepared to say that actuarial science has better answers? Presumably we want to contribute to the customers' understanding of the business and give them some idea of the problems they might anticipate under different interest rate scenarios. A stochastic model does not meet this goal. In fact, the implied actuarial soundness of such techniques may transform the cash-flow test from a projection into a prediction, at least in the customer's mind.

Finally, premise (3) is a testable hypothesis. Suppose our manager hires an actuarial consulting firm to set up a dummy company, perhaps something like Mr. Klein's example in Section III-B. Then suppose the manager employs several actuaries, working independently, to answer the question that Mr. Klein poses; that is, what reserve will reduce the probability of ruin in the next 10 years to 1 percent or less? Does the manager have a right to expect some uniformity of opinion among the actuarial subjects? I think so, if the expertise to address the question actually exists. The other predictive professions (psychics, fortune-tellers, etc.) submit to this sort of test—why not actuaries?

Principles of Actuarial Science

If actuaries claim to have expertise, I believe they have a duty to frame their argument in terms of established professional principles. Fortunately the Society of Actuaries has recently taken pains to systematize and set forth the fundamental principles that underlie actuarial science in "Principles of Actuarial Science" (*TSA* XLIV, 1992, 565–628). This paper cites two statistical principles:

> **Principle 1.1 (Statistical Regularity).** Phenomena exist such that, if a sequence of independent experiments is held under the same specified conditions, the proportion of occurrences of a given event stabilizes as the number of experiments becomes larger.

and

> **Principle 1.2 (Stochastic Modeling).** A phenomenon displaying statistical regularity can be described by a mathematical model that can estimate within any desired degree of uncertainty the proportion

of occurrences of a given event in a sufficiently long sequence of experiments.

In the context of a cash-flow test, the phenomenon being modeled is the interest rate scenario over some period of years or the change in the interest rate from period to period within a scenario.

Do periodic changes in interest rates display "statistical regularity"? I have the impression that Mr. Klein thinks they do, though he certainly does not equate regularity with simplicity, and in Section II-D he seems to favor the hypothesis that the parameters of a suitable IRG might drift over time.

If we *are* dealing with a statistically regular phenomenon, then Principle 1.2 seems to imply that we can accurately compute the probability of ruin over the next 10 years. Such a computation has only a priori validity, however, since there will be no way to establish, at the end of 10 years of observation, whether or not actual experience was consistent with our assumptions.

Relevance to the Appointed Actuary's Opinion

A statutory opinion requires various declarations, including that of conformity with the actuarial standards of practice promulgated by the American Academy of Actuaries. Perhaps it would help potential users of the opinion if members of the Society were to add a personal declaration on whether they think that the "expertise" on which the opinion is based exists.

The Actuarial Standards Board could assist the profession by identifying those standards whose principal goal is the protection of our turf from other professions. In some cases the presentation of the results of a stochastic interest rate model as a product of actuarial expertise may be a violation of the Academy's Code of Professional Conduct, that is, failure to "act honestly and in a manner to uphold the reputation of the actuarial profession and to fulfill the profession's responsibility to the public."

General

Cash-flow tests are complex and controversial even in their simplest forms, and the target of Mr. Klein's analysis, the interest rate scenario, is important as well as complex. The heart of his analysis, addressing the issue of how close is close enough, requires an actual cash-flow test to measure the "distance" between the lognormal model and the stable Paretian model. This is a fairly esoteric metric, but the analysis would be incomplete without it.

In the back of my mind I have long held the suspicion that cash-flow testing is largely a waste of time. Unfortunately it is very difficult to analyze one's way through the forest of irrebuttable presumptions that passes for actuarial science these days. Is there not some point at which we admit that a problem is beyond our ken and settle for a dose of conservatism?

In any case, I am grateful for Mr. Klein's willingness to struggle with these messy issues. I hope he keeps it up.

ELIAS S. W. SHIU:

Mr. Klein is to be thanked for writing this interesting paper, which suggests that interest rate movements may be modeled with the stable Paretian distribution. There are four top-tier academic journals in finance: *Journal of Finance*, *Journal of Financial Economics*, *Journal of Financial and Quantitative Analysis*, and *Review of Financial Studies*. I do not think that, in the past 15 years, there is a single paper in any of these journals applying a non-normal stable distribution to model stock prices or interest rates. I checked the indexes of the recently published textbooks for doctoral students in finance ([4], [5], [6], [7], [8], [9]), and I could not find any reference to Dr. Benoit Mandelbrot or the stable Paretian distribution. Thus this is a most intriguing paper.

Some of the problems encountered in modeling the logarithm of the stock price as a non-normal stable distribution are discussed in the articles reprinted in the book edited by Cootner [3]. (References [3], [11] and [24] of the paper can be found in [3].) Although non-normal members of the stable family frequently fit the tails of the empirical distributions of stock prices better than the normal, there seems to be little empirical evidence to support adoption of the stable Paretian hypothesis over that of any leptokurtotic distribution. Indeed, with $\alpha \neq 2$, the first moment or expected value of the arithmetic price change does not exist; see Formula (7) below. A current approach to account for the "fat-tailed" phenomenon of the stock prices is to incorporate the feature of stochastic volatility in the modeling of the price dynamics.

To fix ideas, let me present a theoretical framework, as given by Nobel laureate Paul Samuelson [10], for suggesting that the logarithm of the stock price may have a stable Paretian distribution. For $t \geq 0$, let $S(t)$ denote the price of a stock a time t. We postulate that the stochastic process $\{S(t);\ t \geq 0\}$ is Markov. That is, the conditional distribution for future values of the stock, conditional on being at time t, depends only on its current value, and the inclusion of other information available as of time t will not alter this conditional probability. Hence, for $0 \leq t \leq T$, the conditional probability $Pr[S(T) \leq S | S(t) = s]$ is a well-defined quantity. Let us write

$$Pr[S(T) \leq S \mid S(t) = s] = P(t, s, T, S), \tag{1}$$

and call it the transition probability. By the Chapman-Kolmogorov equality, for $t < u < T$,

$$P(t, s, T, S) = \int_{y=0}^{y=\infty} P(u, y, T, S)\, d_y P(t, s, u, y). \tag{2}$$

Let us also assume that the transition probability depends only on the difference between t and T. That is, there exists a function $P(t, s, S)$ of three variables such that

$$P(t, s, T, S) = P(T - t, s, S). \tag{3}$$

In other words, we assume stationary transition probabilities. Motivated by the theory of compound interest, we further assume that there is a function in two variables, $P(t, S)$, such that, for $s > 0$,

$$P(t, s, S) = P(t, S/s). \tag{4}$$

In terms of this function, the Chapman-Kolmogorov equality (2) becomes, for $t < T$,

$$P(T, S/s) = \int_{y=0}^{y=\infty} P(T - t, S/y)\, d_y P(t, y/s), \tag{5}$$

which means that $\ln[S(t)]$ has an infinitely divisible distribution [1]. This is a motivation for using a stable distribution to model $\ln[S(t)]$.

Since

$$E[S(t)] = E[e^{\ln[S(t)]}], \tag{6}$$

$E[S(t)]$ is the value of the moment-generating function of the random variable $\ln[S(t)]$ evaluated at 1. If $\ln[S(t)]$ has a non-normal Paretian distribution, the moment-generating function of $\ln[S(t)]$ exists for no non-zero value, and hence

$$E[S(t)] = \infty, \tag{7}$$

which is one of the objections to using non-normal stable Paretian distributions for modeling stock prices.

Let us now turn to the problem of modeling interest rates. For a fixed positive number n, let $I(t)$ denote the n-year Treasury yield rate at time t. Let us see what problems occur, if in the paragraphs above we replace $S(t)$ by $I(t)$. It seems that the stochastic process $\{I(t)\}$ cannot be Markov, because the (conditional) distribution of $I(t+T)$ should depend not only on the value of $I(t)$ but also on the other values of the yield curve at time t. However, let us assume the process $\{I(t)\}$ is Markov and continue. The stationary transition probability assumption (3) is not unreasonable, unless we think that there should be features such as stochastic volatility in the model. (The paper [2] presents an empirical comparison of eight arbitrage-free term structure models and shows that those most successful in capturing the dynamics of the short-term interest rate are the models that allow the volatility of interest rate changes to be sensitive to the level of the riskless rate.) However, I do not see how assumption (4) makes sense at all in the context of interest rates. In the context of (stock) prices, $\ln[S(t+1)/S(t)]$ gives the force of return for the period $[t, t+1]$. On the other hand, I do not find the quantity

$$J(t) = \ln[I(t + 1)/I(t)]$$

meaningful. One can easily argue that one should consider some other expressions such as

$$\ln([1 + I(t + 1)]/[1 + I(t)]).$$

Finally, I also find Formula (7),

$$E[I(t)] = \infty,$$

disturbing.

I would suggest that the model be refined so that there is no possibility for the interest rate to be very high or very close to zero. I also think that, in an interest rate evolution model, mean reversion is a desirable feature.

My final comment is motivated by the last-but-one paragraph in Section II-E of the paper. It reminded me of a remark by a judge [11]: "It seems paradoxical beyond endurance to rule that a manufacturer of shampoos may not endanger a student's scalp but a premier educational institution is free to stuff his skull with nonsense."

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STEPHEN J. STONE:

Mr. Klein is to be commended for his insightful and inciting article. He clearly demonstrates the validity of using the stable Paretian distribution as a model for interest rate changes. Also, his cash-flow analysis establishes the importance of the choice of a stochastic process for interest rate generation on cash-flow-testing results. However, I have three objections to his paper.

As an introduction to my first two objections, I briefly review the standard paradigm for the statistical modeling of time series. The first part of the paradigm is that there is a series of independent "shocks," which are referred to as white noise or an innovation process, typically denoted by ϵ_t. This innovation process is frequently assumed to be normally distributed, in which case it is referred to as a Gaussian innovation process. The next part of the paradigm is that there is a filter, denoted by $f(\cdot)$, which is a real-valued function that can be either linear or nonlinear. The last part of the paradigm is that the time series itself, which is denoted by J_t in Mr. Klein's paper, is the result of the filter applied to the innovation process. This can be expressed as $J_t = f(\epsilon_{t-i})$, where i denotes the set of the non-negative integers. Note that when one is choosing a random number generation algorithm for use in a Monte Carlo simulation for cash-flow testing, one is making an assumption about the distribution of ϵ_t, not the distribution of J_t.

My first objection to Mr. Klein's paper is the blurring of the distinction between J_t and ϵ_t. This blurring begins when he states "First, it is often asserted that interest rates change because of many small pieces of information moving through the markets. If it is assumed that these changes have distributions that are mutually independent, then the central limit theorem would encourage us to use the normal distribution for the change in interest rates, which is the sum of these many small changes." The "many small pieces of information moving through the market" mentioned in the first sentence of this quote are what drive the innovation process, that is, ϵ_t. In the first sentence he does not clearly state the relationship between ϵ_t and the interest rate changes, J_t. However, the truth of his assertion in the second sentence, that "the central limit theorem would encourage us to use the normal distribution for the changes in interest rates" is predicated on the choice of a filter to transform ϵ_t into J_t. This lack of clarity is likely to mislead many readers into assuming that the results of his paper have wider applicability than they in fact do.

Mr. Klein's intent was clearly to limit himself to a conceptual framework in which $J_t = \epsilon_t$, which is not a problem per se. However, he should have clarified that his corollary, which is that interest rate changes must have infinite variance, is predicated upon the assumption that $J_t = \epsilon_t$. This is an assumption that many practitioners prefer not to make and for which there is, a priori, no reason to make. Furthermore, there are many nonlinear filters that, when applied to Gaussian innovation processes, result in distributions of J_t that are leptokurtic; that is, they have excess kurtosis but finite variance. Indeed, it was the excess kurtosis of j_t, that is, the realization of the time series, that was responsible for the normal hypothesis being rejected. It is also possible that J_t has a conditional normal distribution, but j_t would fail a distributional test for normality.

Preferably, when testing the assumption of normality, one should do so on the realization of ϵ_t, which can be derived as the residuals of a time series model fitted to j_t. Whereas it is true that there might be situations in which one would want to reject the normal hypothesis and accept the stable Paretian hypothesis for ϵ_t, this will often not be the case. As I discuss below, it indeed may not be the case if one uses any of a variety of nonlinear models that are currently in vogue in the financial literature.

My objection on this point goes beyond Mr. Klein's lack of clarity on this important issue. In his section giving reasons why the stable Paretian hypothesis has been slow to gain acceptance, he leaves out a fifth, and very important, reason. It is that even some very simple nonlinear time series models not only explain the leptokurtic distribution of j_t, but also account for the positive serial correlation of the first moment of j_t and the conditional heteroscedasticity of the second moment of j_t, thereby explaining all the anomalies pointed out by Dr. Becker in his paper [1]. These models, in general, allow for insightful economic interpretations of these observed phenomena rather than having them become "meaningless," which is the rather existential implication of a stable Paretian assumption.

Admittedly, in Section II.D.4 of his paper Mr. Klein discusses the possibility that interest rates may be modeled as normal distributions with shifting parameters. However, the class of nonlinear Gaussian models that I have mentioned above may have stationary parameters, including constant μ and σ^2 for ϵ_t, and nevertheless exhibit nonstationary behavior in the realization of the time series. These models can also result in J_t having a conditional normal distribution, where the distribution is conditioned on the previous realization of the innovation process. This is an important distinction, since none of the four problems "that have not been adequately solved" apply. Furthermore, these models are currently an area of very active research.

My second objection to Mr. Klein's paper is his lack of thoroughness in investigating the normal hypothesis. Mr. Klein establishes that the cost of rejecting the normal hypothesis, which would be the cost of the much higher reserve and surplus requirements, is potentially quite high. Therefore, given the large cost of a Type I error, that is, rejecting the null hypothesis when it is true, one should do so only after an exhaustive investigation. Given the data set in Mr. Klein's Table 2, I do not believe that the normal hypothesis should be rejected. If a time series has a stationary linear filter, then a Gaussian innovation process will result in a Gaussian distribution for J_t. As I show below, fitting a simple autoregressive model of order 2 to j_t results in residuals that cannot be rejected as normal at confidence levels above 81.9 percent. The acceptance of a normal hypothesis for ϵ_t in this case is equivalent to accepting a normal hypothesis for J_t, even though the j_t themselves fail a distributional test for normality. Due to the serial correlation present in j_t, a test of normality on the realization of ϵ_t is a more powerful test than a test on j_t and should therefore be preferred. Though this result may be data set dependent, it nevertheless highlights the importance of rigor in an analysis of this sort.

My third objection is to Mr. Klein's statement that "The reality of interest rate changes is that they are not lognormally distributed." This is a very unscientific statement. The α-level of the statistical hypothesis test performed by Mr. Klein was 0.05, which means he was willing to accept a 5 percent chance of rejecting the normal hypothesis if it were true. It is incorrect to call this a "reality." Furthermore, the p value of 0.037 for his χ^2 test of the normal hypothesis, which means the normal hypothesis would not be rejected at a confidence level of 96.3 percent or above, is hardly overwhelming evidence.

I would like to analyze the data set presented in Table 2 using both a linear and a nonlinear filter, as well as a combination of the two. The linear model I examine, as I mentioned above, is an autoregressive model of order 2 fit to j_t, that is, an AR(2) model. The nonlinear model I examine is an autoregressive conditional heteroscedastic model of order 1 fit to j_t, that is, an ARCH(1) model. In addition, I examine the results when an ARCH(1) model is fit to the residuals of the AR(2) model, that is, an AR(2)-ARCH(1) model. The class of ARCH models and its generalizations are widely used as filters for financial time series (see [3] and [4]).

I have chosen to use the Shapiro-Wilk test for normality since it is a more powerful test than the Pearson χ^2 test, as is explained in [2]. But first, I present the functional form for an AR(2)-ARCH(1) model. When J_t and ϵ_t are defined as above, it is given by:

$$J_t = \beta_0 + (\beta_1 \times J_{t-1}) + (\beta_2 \times \epsilon J_{t-2}) + (\epsilon_t \times \sqrt{V_t}), \qquad (1)$$

and

$$V_t = \alpha_0 + (\alpha_1 \times \epsilon J^2_{t-1}), \tag{2}$$

where the α's and β's are constants. When $\beta_0=\beta_1=\beta_2=0$, an ARCH(1) model results.

The following table presents p values for goodness-of-fit tests to a normal distribution for the residuals that are derived when the listed filter is fit to j_t, as derived from Table 2.

Filter	p value
None	0.014
AR(2)	0.181
ARCH(1)	0.059
AR(2)-ARCH(1)	0.469

It is difficult to draw conclusions of a general nature from these results because of their limited scope. However, it does seem highly likely that the use of a normal random number generator for cash-flow testing should not be a rejected proposition. Clearly, if one cares to assume that $J_t=\epsilon_t$, the stable Paretian distribution should be considered as a model for ϵ_t.

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SHAUN WANG:

Mr. Klein has presented an informative discussion that brings our attention to the significance of choosing a statistical model for interest rate changes in cash-flow analysis. He demonstrates two very different results of cash-flow analysis by choosing (1) the normal distribution and (2) the symmetric stable distribution for the log ratio of interest rates over successive periods.

Regarding the stable distribution, there has been an extensive literature on its use in modeling interest rate and stock price changes. Some authors are supportive (for example, Fama [2]), and others (for example, Officer [4]; Hsu et al. [3]) question the use of the stable distribution. My major concern about the use of the stable distribution is that only *symmetric* stable distributions are numerically and statistically tractable; on the other hand, many empirical data sets are found to be skewed. Becker [1, p. 53] showed that for the three-month maturity data he studied, 21 out of 26 data sets did *not* pass the skewness test (that is, 21 out of 26 data sets are skewed). In general, for severely skewed data, any symmetric distribution would cause inaccuracy in model fitting, and some skewed distribution should be used instead.

Mr. Klein studied two data sets of average yield to maturity on long-term Treasury bonds: (1) between years 1977–1990, and (2) between years 1953–1976. I got different numerical order statistics for both sets of data from the ones given by Mr. Klein. For the first data set (1977–1990), I got an order statistic that ranges from −0.11498740 to 0.13482772; the 50 percent truncated mean is 0.002146075; the empirical skewness index is −0.1538567. For the second data set (1953–1976), I got an order statistic that ranges from −0.07864313 to 0.06899287; the 50 percent truncated mean is 0.002673649; the empirical skewness index is −0.1617957. These numerical discrepancies do not qualitatively differ from Mr. Klein's major findings, though.

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(AUTHOR'S REVIEW OF DISCUSSIONS)

GORDON E. KLEIN:

I thank all the discussants for taking the time to contribute their comments. I think the discussions are a very valuable part of any *Transactions* paper, so I am delighted by the number and quality of discussions that this paper generated. It has been 14 years since a paper generated so many comments,* so obviously many actuaries have a lot of concerns in this area.

I address the discussions in alphabetical order.

Bill Bailey

Mr. Bailey uses a Kolmogorov-Smirnov (K-S) test to test the hypothesis that the data come from a symmetric distribution. He compares the data from Table 3 with a second sample based on the same data, but with each element replaced by one the same distance, but the opposite direction, from the sample mean, **x**. This procedure only makes sense if **x** is a good estimator of the mean of the distribution. For instance, a sample from a Cauchy distribution, for which the mean is infinite, would most likely fail this test, although the distribution is symmetric.

As discussed in Section II-D.1.c of the paper, **x** is not a good estimator of the mean of a stable Paretian distribution. A better estimator is the 50 percent truncated mean. I reran the K-S test using this estimator of the mean. The statistic $D_{167,167}$ turned out to be 0.0958. It follows that the null hypothesis of symmetry is not rejected by this test.

Next Mr. Bailey uses another form of the K-S test to test the goodness-of-fit of the fitted stable Paretian distribution. His results do not reject the null hypothesis that the data come from the distribution with the parameters as estimated. He then warns that the assumption of $\delta=0$ may be questionable, since the goodness-of-fit test with $\delta=0$ and the other parameters unchanged is failed. My reason for using this assumption was that to do otherwise introduces a "drift" into the process over time. The estimator of the mean change in interest rates is largely a function of where interest rates are now relative to where they have been. If one does not wish to assume that they will continue in that same direction, then the assumption that $\delta=0$ is appropriate.

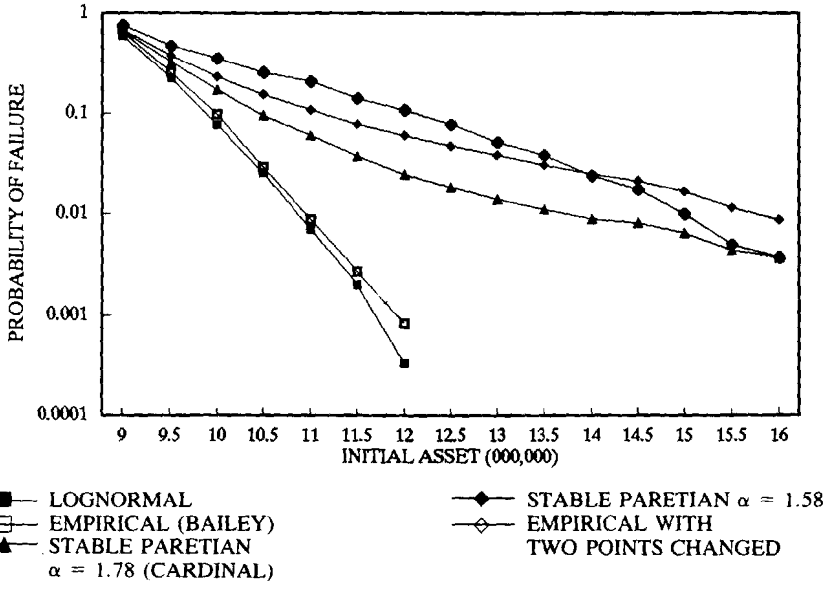
*See LECKIE, ROBIN B. "Some Actuarial Considerations for Mutual Companies," *TSA* XXXI (1979): 187–259.

Mr. Bailey states, "We have no evidence that the tail of the resulting stable Paretian distribution (that is, beyond the largest value in the observed data) is, or is not, representative of what can be expected there." I agree with this much of the statement. But he goes on to state, "I happen to believe that the appending of tails is unnecessary and perhaps even misleading." I do not agree with this. His position is equivalent to the belief that the probability of an event more extreme than has been observed is 0. This is clearly an understatement of the actual probability. I have shown that what is in those tails is very important. To chop them off at the point of the most extreme observation to date is certainly to understate the probability of ruin. In order to demonstrate this, I ran 6,000 scenarios using the empirical interest-rate generator suggested by Mr. Bailey. The empirical interest-rate generator assigns equal probabilities (1/167) to each of the 167 outcomes shown in Table 3. The annual change is based on 12 draws from this distribution.

Figure 1 is an expansion of Figure 6 from the paper. It includes the graph of the probability of negative terminal surplus as a function of initial assets for the two interest-rate generators from the paper. Unlike Figure 6, the left scale is logarithmic. In addition, it includes the same function for the empirical interest-rate generator. It can be seen that the empirical distribution, by eliminating any positive density in the tails, understates the probability of failure relative to the other interest-rate generators. This shows again that the crucial question is what is in the tails.

One additional comment is in order. The K-S statistic is one measure of the distance between two distribution functions. It is, however, not the relevant measure for determining how different two interest-rate generators are for cash-flow testing. The relevant measure is the absolute value of the difference between the expected value of the utility of the final surplus under one and the expected value of the utility of the final surplus under the other. This measure can vary greatly due to a change in the interest-rate generator that causes only a small change in the K-S measure. For instance, the K-S distance between the empirical distribution advocated by Bailey and the stable Paretian distribution is small, but the expected-value-of-the-utility-of-the-final-surplus distance is large. It does not make sense to say that these two interest-rate generators are close to each other, yet this is exactly what the K-S measure says. Likewise, if two interest-rate generators produced a K-S statistic that showed they were significantly different, but they produced final surpluses that

FIGURE 1
PROBABILITY OF FAILURE
AS A FUNCTION OF INITIAL ASSETS



were quite close together, then we would not quibble over which one was appropriate for cash-flow testing.

As an illustration of this point, I devised another interest-rate generator. It is just like the empirical one at 165 of the 167 points. However, the largest and smallest values from Table 3 were replaced by 2.0 and 0.5, respectively. The results of the cash-flow test using this distribution are shown in Figure 1. The K-S measure between this modified empirical distribution and the original distribution is 0.006, so the K-S test does not distinguish between the two. Using the correct measure, however, they are as different as the normal and the stable Paretian. This is true despite the fact that neither has positive density in the tails and both have finite moments of all orders.

The point that must be recognized is this: Two distributions may be close together by one measure and far apart by another. In more mathematical terms, we have two metric spaces that are not homeomorphic (that is, there is not a bicontinuous bijective function between them).

Tim Cardinal

Mr. Cardinal refines many of the results of this paper and reinforces its conclusions.

He first questions the particular definition of the stable Paretian distributions used in the paper. Zolotarev [3] states that, "There are many different criteria for a distribution function to belong to the family [of stable laws], and, if desired, any one of them can be taken as the original definition of stable laws." I have tried to present just enough of the theory of stable laws that my main argument can be followed. I think my definition is much easier to grasp than the alternative. The reader whose interest goes beyond the arguments presented here can turn to a source such as Zolotarev for a much more complete coverage of the theory.

I thank Mr. Cardinal for bringing additional references to my attention, such as the papers by Koutrouvelis (his ref. [5]), by McCulloch (his ref. [6]), and by Paulson, Holcomb, and Leitch (his ref. [7]). These papers present improved methods for estimating the parameters of the stable Paretian distribution. I reran the cash-flow test using the following parameters for the distribution of monthly changes: $\alpha=1.780$, $\beta=0$, $c=0.002172$, $\delta=0$. The results are plotted along with the results from Table 6 from the paper in Figure 1. As one might expect, the results fall between those of the normal case and those of the $\alpha=1.580$ case. Using Mr. Cardinal's parameters, it takes \$13,750,000 of initial assets (versus \$10,850,000 in the normal case and \$15,850,000 in the $\alpha=1.580$ case) to reduce the probability of failure to 1 percent.

In my review of Mr. Bailey's discussion, I discussed the need to look at the correct measure of the difference between two distributions. A similar problem arises in Mr. Cardinal's discussion. The regression technique of Koutrouvelis is based on the premise that the important measures are the differences between the actual parameters and their estimates. The procedure of Miller, Wichern, and Hsu is based on the premise that the χ^2 statistic is the important measure. In both cases, it is possible that the results of the cash-flow test are less accurate due to procedures that focus on the wrong measure.

Mr. Cardinal provides statistical bases for several observations made in the paper, including that the data are close to symmetric, that the mean is close to 0, that α is consistent between the two sets of data, whereas c is not, and that the results of Test 1 and Test 2 cast doubt on the assumption of independence. In each case his analysis reinforces the statements in this paper.

Beda Chan

Dr. Chan's discussion provides additional references and points out that the data fail the runs test for independence. The question of dependence needs to be addressed. It is my hope that research in this area will not exclude the stable Paretian distribution.

Steve Craighead

Mr. Craighead points out that an estimate of the parameters for daily data is not consistent with the estimates for monthly data. If the data were realizations of independent, identically distributed stable Paretian random variables, then the monthly γ should be the sum of the daily γ's. The effect that Mr. Craighead observes is consistent with other results that have questioned the assumption of independence.

Martin Den Heyer

Mr. Den Heyer tackles the issue of dependence head-on. I think he has outlined a direction for future research that should prove fruitful. There is little else that I could add to this discussion, which I think adds a great deal to the paper.

John Dutemple

Mr. Dutemple makes an important point. We actuaries need to become better at realizing the limitations in our models. We need to be able to quantify the effects of the simplifications in our models.

I think this paper is too technical to add to the syllabus, but a paper such as I presented at the 1991 Valuation Actuary Symposium [1] would be a good supplement to the current material on cash-flow testing.

Paul Huber

Mr. Huber "questions the validity of the paper's results and suggests that the paper presents a biased perspective in favour of the stable Paretian hypothesis." I fail to see which results he questions, and I think he goes farther out of his way to support the lognormal hypothesis than I went to support the stable Paretian hypothesis.

The main result of my paper is that the results of cash-flow testing are extremely sensitive to the model chosen for future interest-rate changes. It does not advocate the assumption that interest-rate changes are independent and identically distributed, whether lognormal or stable Paretian. It starts with this assumption and finds it questionable. Since many actuaries are using this questionable assumption, it is not unreasonable to demonstrate that the results based on this and the assumption of lognormality vary greatly from those in the stable Paretian case, especially since the arguments typically given for the lognormal assumption also support the stable Paretian assumption.

Mr. Huber cites a number of studies on stock price changes. These studies reinforce my point that independent, identically distributed stable Paretian random variables are not a good model. He suggests other leptokurtotic distributions. This reinforces my point that the extremes of the distribution are what matter.

Mr. Huber and Dr. Shiu both question the stable Paretian hypothesis on the grounds that it leads to infinite expected values. McCulloch [2, p. 618] discusses this and concludes that "[t]here is therefore no theoretical reason to reject log-stable price movements on the grounds that they give infinite expected future prices and expected rates of return." The solution to the problem is provided by utility theory. "[A] risky asset can have an infinite expected value and still have a finite market price." The distribution used in the paper was truncated, so that its mean, variance, and all moments exist.

Mr. Huber attempts to fit the lognormal distribution to three subperiods of the data. He performs a number of statistical tests, always with lognormality as the null hypothesis. He uses whatever significance level is necessary in order for each statistic to be not significant. In one subperiod, he has only 18 data points. In the final subperiod, his estimate of α is 1.50, but he cannot bring himself to question normality even here.

Of course it is always possible to divide the data into periods where statistical tests for normality will not fail. The questions then become (1) how to model the shifts from one period to another and (2) how to draw the borders between the past subperiods. Mr. Huber states that, "[i]f these changes can be explained in economic terms or in terms of a change of policy and are found to be more or less permanent, then it is not appropriate to model future interest rates on the basis of the data that occurred before the changes." How does one decide whether a change in policy is permanent? One cannot simply ignore the possibility that changes of policy will happen in the future. Mr. Huber admits as much in his final paragraph. The logical conclusion to be drawn from his position is that we must estimate parameters using only data since the time of the last policy change and that we can use these parameters to model interest rates only until the next policy change. If the last change in policy was fairly recent, this leaves us in a position of being unable to model interest-rate changes at all. And if we model a change of policy in the near future, then we can only model interest rates for a short time.

I agree with Mr. Huber that independent, identically distributed random variables are not adequate to model interest-rate changes. However, I think that the assumption of independence should be abandoned first. I agree with Mr. Den Heyer that the stable Paretian distribution should be explored in the absence of the assumption of independence. I see no reason to dismiss this distribution from further study.

Merlin Jetton

Mr. Jetton raises the important point that a model should be chosen based on the aspects of reality that one wants to capture. In particular, if the probability of failure is the aspect of reality that is deemed relevant, then the tails of the distribution of interest-rate changes are very important.

He is correct that the data do not show the scaling properties that they would if they were realizations of independent, identically distributed stable Paretian random variables. However, they also do not scale as if they were lognormal. Again, I think the assumption of independence must be abandoned.

Tom Mitchell

Mr. Mitchell makes a number of useful observations. He points out that, while the sum of n random variables whose values are limited to a bounded set converges to the normal distribution as n increases, the convergence may be very slow.

I comment on only one of the areas Mr. Mitchell proposes for further research. He questions the effect on duration and convexity of assuming the stable Paretian distribution. The answer is that there is no effect whatsoever. Duration and convexity merely measure sensitivity of price to changes in interest rates. The likelihood of these changes has no effect on either measure. This is a major limitation of duration and convexity as measures of risk.

Tom Powell

I appreciate Mr. Powell's comments about the notion of "general acceptance." I certainly did not mean to imply that all actuaries agree with what I see as a consensus on how to do cash-flow testing. I think that there are many aspects of cash-flow testing that have appeared in the actuarial literature a number of times without being seriously questioned in the literature. My purpose is to question one such aspect.

Mr. Powell questions my statement that Ms. Claire's cash-flow test with actual historical interest rates showed that the lognormal distribution was too far from reality. Her cash-flow test was performed in 1991. The first 100 scenarios were based on a lognormal model. The parameters were presumably based on historical experience. The volatile interest rates of the 1980s should have been part of that experience. Nevertheless, the scenario based on actual interest rates was as bad as any of the ones generated by the lognormal distribution. It seems to me that this shows that the model was not reflecting the reality on which it was based.

I would be as interested as Mr. Powell to see the results of several actuaries setting reserves for the same company using cash-flow testing. I suspect that there would not be a great deal of uniformity in the results.

Mr. Powell states that, if interest rates exhibit statistical regularity, then an actuary should be able to give the probability of ruin to any degree of accuracy (based on the Principles of Actuarial Science). I find interesting the degree to which actuaries are stumped by the question: "What percentage of the scenarios can be failed in a cash-flow test." This question was raised at the Postmortem on Cash-Flow Testing seminar sponsored by the Society in the spring of 1992. The lack of uniformity in answers to this question must be as great as the lack of uniformity in the actual percentage that would be failed if several actuaries tested the same company. While there is certainly no consensus on this question, there appears to be more agreement that cash-flow testing should not be thought of as any sort of statistical test. There seems to be recognition that the outcomes cannot simply be assigned equal probabilities. It is not clear to me what the value of cash-flow testing is if it is not a statistical procedure. What meaning can be assigned to failure of 10 scenarios out of 1,000 if not that the probability of the reserves being adequate is 99 percent?

I appreciate Mr. Powell's willingness to admit that he thinks cash-flow testing is a waste of time. Several actuaries have expressed this opinion to me over the last few years. I also appreciate his insight into the use of cash-flow testing as a metric. This served as the inspiration for part of my reviews of Mr. Bailey's and Mr. Cardinal's papers.

Elias Shiu

Some of Dr. Shiu's comments are similar to those of Mr. Huber. My replies to Dr. Shiu are the same.

In the first draft of my paper, I considered the random variable

$$\frac{\log_e(1 + I_{t+1})}{\log_e(1 + I_t)}.$$

The reviewers questioned this formulation on the grounds that the precedent in the actuarial literature is to consider

$$\frac{\log_e(I_{t+1})}{\log_e(I_t)}.$$

The advantage of the latter formulation is that the interest rate cannot be negative. I followed the precedent in the revision of the paper. The results differed very little between the two formulations. This is not to imply that this would always be the case.

Steve Stone

Mr. Stone expands on the idea of going beyond independent, identically distributed random variables. It would be interesting to see a comparison of a cash-flow test based on the AR(2)-ARCH(1) model he describes with the ones in the paper. Unfortunately, I was unable to make such a comparison.

I find it interesting that Mr. Stone considers only the cost of rejecting the lognormal distribution if it is in fact appropriate. By this line of thinking, Executive Life would simply have been wasting money if it had set up reserves for the possibility of default on its junk bonds. There is also a cost to accepting the lognormal hypothesis if it turns out to be wrong.

Mr. Stone accuses me of being "very unscientific" for stating that "the reality of interest rate changes is that they are not lognormally distributed." I stand by my statement. While the evidence for one particular time series was the "hardly overwhelming" p value of 3.7 percent, one could produce many other time series of interest rates for various countries, time periods, and points on the yield curve. The product of the p values for the tests of each of these time series is the probability that interest rates are produced by a stationary lognormal stochastic process. At what value would Mr. Stone find the evidence overwhelming? Does he find the rejection of astrology unscientific on the same grounds?

Shaun Wang

Mr. Wang raises the issue of asymmetry. Mr. Cardinal addressed this in his discussion.

I have clarified the calculation of the order statistics in the final version of the paper.

Conclusion

Again, I thank each of the discussants for his contribution to this paper.

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