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A ONE-FACTOR INTEREST RATE MODEL AND THE VALUATION OF LOANS WITH PREPAYMENT PROVISIONS

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ABSTRACT

This paper sets out, in algorithm form, a one-factor term structure model of interest rates and illustrates the application of such a model to the valuation of loans with prepayment provisions. It is mainly directed to actuaries who are not familiar with these models. The algorithms are presented in a form that can readily be implemented and adapted to other interest-sensitive assets and liabilities. A simple example illustrates the steps in the application of the term structure model. The paper outlines the features of standard prepayment models and the issues that arise when prepayments are incorporated in the implementation of a term structure model. A brief discussion of some problems in hedging a portfolio of loans with prepayment provisions completes the paper.

1. INTRODUCTION

Many financial products contain prepayment provisions. Loan contracts are often structured to provide the borrower with the option to prepay the loan at any time or on specific dates prior to the maturity date of the loan. Prepayment options are important aspects of these financial products. The most common investment product with this feature is the mortgage-backed security. Investment contracts issued by life insurance companies contain similar options in which the policyholder is allowed to surrender the policy with no surrender charge.

This paper considers the valuation of loan contracts with prepayment options. The loan contracts are fixed-interest-rate loans with no restrictions on the pattern of repayments. They can take the form of level-repayment contracts, interest-only and bullet-principal repayment contracts, or any other contractual repayment structure. In this paper the method used to price these contracts is a general algorithm-based approach that is not dependent on the structure of the loan cash flows. The algorithm allows for stochastic interest rates and incorporates a one-factor term structure of interest rates model. The algorithm is "arbitrage-free";

that is, the parameters of the one-factor term structure model are chosen to ensure that prices of traded zero-coupon bonds derived by using the algorithm are equal to the market prices of such bonds on the valuation date.

This paper does not develop new theoretical results. The approach used is the same as that of Jacob, Lord and Tilley [12]. A more detailed and comprehensive coverage of interest rate models is given in Tilley [29]. The paper aims to illustrate the practical implementation of some techniques of modern financial mathematics and economics as developed for the analysis of interest rate options. Prepared for actuaries who are unfamiliar with these ideas, the paper aims to assist in providing a basic understanding of these techniques. A glossary of terms at the end of the paper will assist those readers unfamiliar with the terminology.

The literature on modeling of interest rates and the application of these models to options has expanded over recent years. Examples include Black, Derman and Toy [4], Black and Karasinski [5], Bookstaber, Jacob and Langsam [6], Heath, Jarrow and Morton [9], Ho and Lee [10], Jamshidian [13], Miller ([17], [11], [19]), O'Brien [20], Pedersen and Shiu [21], Pedersen, Shiu and Thorlacius [22], and Ritchken and Sankarasubramanian [24]. A number of actuaries have actively contributed to the development of these techniques. Even so, the usefulness and application of these techniques are not well understood by actuaries not actively involved in investment issues; this paper aims to assist those actuaries.

Section 2 describes the prepayment option in these loan contracts. Section 3 sets out the algorithms that are the basis of the implementation of a one-factor arbitrage-free term structure model and shows how such a model can be used to value the prepayment option in such loan contracts. Section 4 outlines the main features of commonly used prepayment models. Section 5 defines the conventional risk statistics, including the delta, gamma, vega, duration, and convexity, that are used in the management of a portfolio of loan contracts with prepayment options. Section 6 briefly discusses the management of a portfolio of such loan contracts, including how the prepayment and interest rate risk of these loans might be hedged in financial markets.

2. PREPAYMENT RISK

This paper considers prepayment risk from the point of view of an issuer of loans with early repayment options. Prepayment risk arises in such a loan contract when the borrower is given the option to prepay a fixed-interest-rate loan prior to the maturity date of the loan without penalty. In this paper the loan contract analyzed is a fixed-interest-rate, fixed-term loan. It is assumed that under the terms of the loan agreement the borrower can repay the loan for the balance outstanding, regardless of current market interest rates, at any time during the term of the loan.

The value of early prepayment reflects the difference between the value of the outstanding loan repayments at the interest rate at the time of prepayment for the remaining term of the loan less the amount of the loan then outstanding (which is the value of the outstanding loan repayments at the contract interest rate). If interest rates have fallen, then the payoff from early prepayment would be positive. In options terminology the prepayment option would be "in the money." Similarly, if rates have risen, then the prepayment option would be "out of the money," because the payoff from prepayment would be negative.

This prepayment option is most commonly found in mortgage-backed securities. The prepayment option in these securities has been the subject of a number of papers, including Dunn and McConnell [7], Green and Shoven [8], Kang and Zenios [15], Kau, Keenan, Muller and Epperson [16], and Schwartz and Torous ([25], [26]). Bartlett [2] provides a comprehensive discussion of these securities.

3. VALUATION OF LOAN CONTRACTS WITH PREPAYMENT OPTIONS

The loan contract is the equivalent of a fixed-rate loan with an option to repay early. The prepayment option is a call option held by the customer on the loan contract with an exercise price equal to the loan's outstanding balance. The customer can call the loan contract by repaying the loan. Because the loan can be prepaid at any time, the option is equivalent to an American-style option to exchange the fixed-rate loan for a floating-rate loan for a term equal to the remaining term of the original loan. For interest-only loans, it is equivalent to an American-style option on an interest-paying bond with an original maturity equal to the original term of the loan.

American options, which allow for early exercise, are usually valued by assuming an optimal exercise policy. The optimal strategy for the customer would be to exercise the call option on the loan contract only when the difference between the value of the loan at the prevailing interest rates on any future date and the loan amount outstanding at that time exceeds the value of the prepayment option on that date, assuming an optimal exercise policy for the remaining term of the loan. Otherwise, the prepayment option should not be exercised because it is worth more "alive" than exercised.

In practice, there are many reasons why loan customers do not follow this optimal prepayment strategy. Some borrowers prepay when it does not appear to be optimal to do so, and not all borrowers prepay even when it would be optimal to do so. Such departures from this optimal exercise strategy arise for a host of reasons, including market frictions such as transaction costs and events such as divorce, death, change of job, and default. From the borrower's perspective, this suboptimal prepayment strategy is not necessarily suboptimal. A major difficulty in valuing loans with prepayment features is the allowance for this apparent suboptimal behavior.

The important point with this contract is that this apparent suboptimal exercise of the option provides positive value to the lender in all circumstances. This is also the case when the borrower does not prepay when it would be optimal to do so. If the loan can be issued for the cost of the prepayment option assuming an optimal exercise policy, then the lender need not consider an allowance for the suboptimal early prepayments in the pricing and could then allow profits from such prepayments to be recognized as they occur. The lender would also need to recognize profits from nonexercise of the prepayment option when it would be optimal to do so. Otherwise, an allowance for an expected pattern of prepayments must be included in the pricing.

A. Valuation of Prepayment Option—Optimal Exercise Policy

The algorithm used in this paper for the valuation and analysis of the prepayment option is based on a technique given by Jamshidian [13]. The algorithm is fast and efficient and allows the valuation of a range of interest-rate-related options. The basic approach is set out in this section, and a simple 12-month loan example is used to illustrate the implementation of the algorithms.

Interest rates in the model are stochastic and take discrete values based on a lattice. The simplest case is the binomial lattice, and because this has accepted usage, this is the basis adopted. Hull [11, Section 15.9] provides an introductory discussion of the binomial lattice for interest rates, and more details are found in Rendleman and Bartter [23] in an early application of the binomial lattice to interest rates.

Alternative lattice structures do have potential computational advantages, but this is unlikely to be an issue of concern with the algorithm recommended for the computations in this paper. Amin [1] and Kamrad and Ritchken [14] examine some of these alternatives and report their computational efficiency.

This lattice is constructed for the maximum time to be used in the valuation. For example, using a monthly time interval requires 60 time intervals for a five-year loan. A monthly interval should be accurate enough for many applications. In general, the number of intervals is an input variable. Hence, if M is the maximum time (in years) for construction of the interest-rate lattice and n is the number of time intervals into which this period is to be divided, then each time interval is of length $h=M/n$ years. For $M=5$ years and $n=60$ time intervals, the length of each is $5/60=1/12$ of a year, or one month.

Table 1 shows values of $s(i,t)$, the cumulative effect of up and down jumps in a binomial lattice, with an up jump counting for $+1$ and a down jump counting for -1, for a 12-month period, where t indicates time in months. The values for i are given by the row number, and the values for t are given by the column number. Nodes in the lattice are denoted by the pair (i,t). The probabilities of an up or down change in the lattice are taken to be ½. This probability is used for fast computation of values. Under this assumption, the expected value of $s(i,t)$ at time t is 0 and the variance of $s(i,t)$ at time t is t.

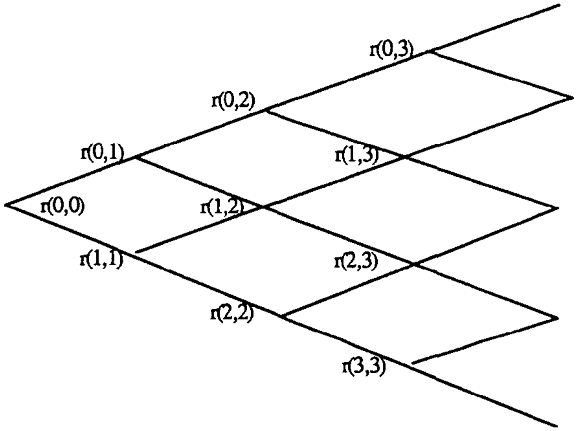
The one-period interest rates that apply for an investment from time t to $t+1$, denoted by $r(i,t)$, will be semiannual compounding per annum rates. This convention is used because market-based Treasury interest rates are quoted on this basis, and these rates provide the basis for pricing interest rate options. The $r(i,t)$ are functions of the one-period standard deviation, or volatility, of spot interest rates and the median future one-period interest rate. A diagram of a binomial lattice of the interest rates, $r(i,t)$, is illustrated in Figure 1.

TABLE 1

LATTICE OF $s(i,t)=t-2i$

i	t												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12
1		−1	0	1	2	3	4	5	6	7	8	9	10
2			−2	−1	0	1	2	3	4	5	6	7	8
3				−3	−2	−1	0	1	2	3	4	5	6
4					−4	−3	−2	−1	0	1	2	3	4
5						−5	−4	−3	−2	−1	0	1	2
6							−6	−5	−4	−3	−2	−1	0
7								−7	−6	−5	−4	−3	−2
8									−8	−7	−6	−5	−4
9										−9	−8	−7	−6
10											−10	−9	−8
11												−11	−10
12													−12

FIGURE 1
BINOMIAL LATTICE OF ONE-PERIOD INTEREST RATES



The choice of the function for $r(i,t)$ determines the limiting distribution of future one-period interest rates. Two common alternatives are the normal distribution generated by using Formula (1) and the lognormal distribution generated by using Formula (2).

The normal distribution specification, an additive model, is:

$$r(i,t) = f(t) + \left(\frac{\sigma_N(t)}{100}\right)s(i,t)\sqrt{h}, \tag{1}$$

and the lognormal specification, a multiplicative model, is:

$$r(i,t) = f(t)\exp\left[\frac{\sigma_L(t)}{100}\,s(i,t)\sqrt{h}\right], \tag{2}$$

where $f(t)$ is the median future one-period interest rate at time t; $\sigma_N(t)$ is the one-period interest rate volatility in absolute terms for time period t to $t+1$; $\sigma_L(t)$ is the one-period interest rate volatility in percentage terms; and h is the length of the time interval used. There are advantages and disadvantages to either model. There are also other models that can incorporate mean reversion and other distributions. For the illustrative example, the lognormal model is used.

Jamshidian [13] gives the following reasons for using the lognormal model:

- It does not allow negative interest rates, as is possible under the normal model.
- It allows the yield curve to move in a nonparallel fashion, unlike the normal model, which implies parallel moves, providing a more accurate basis for pricing interest rate options whose values depend on relative movements in yields for different maturities on the yield curve.
- The implied volatility curve for zero-coupon bonds derived from the resulting spot interest rates has higher volatilities for short-term bonds than for long-term bonds, unlike the normal model, which has approximately constant volatility for different term zero coupons. Higher volatility in short-term interest rates is an observed empirical fact for interest rates.
- The volatility parameter for the lognormal model is percentage yield volatility of the one-period forward interest rate and can be estimated from prices for options on forward interest rates.

Note also that yield volatility is used in Formulas (1) and (2), not price volatility. If options data give price volatility, then this must be converted into percentage yield volatility for the lognormal model. If the normal model is used, then price volatility must be converted into absolute dollar yield volatility. The Appendix details how to convert from one volatility to another. Volatility can be interpolated from a forward rate volatility curve or assumed constant for all periods for ease of computation.

When the formula for $r(i,t)$ is selected, it is then used to derive a lattice of present value factors to value cash flows. If $r(i,t)$ is a percent per annum semiannual compounding rate, then the discount factor for the lattice at node (i,t) is:

$$p(i,t) = \left[1 + \frac{r(i,t)}{200}\right]^{(-2h)}, \tag{3}$$

where h is the time interval. Formula (3) is readily adapted for other compounding frequencies for $r(i,t)$.

These present value factors are one-period discount factors that apply to the average value of the cash flows at the up and down jumps originating from the node (i,t). The average value is calculated with the probabilities of the up and down jumps, which are taken as ½ in the example.

Present values are determined by using backward recursion down the lattice starting at the last date of a cash flow.

The discount factors for each node of the lattice are determined by using forward recursion. This forward recursion produces "arbitrage-free" discount factors; this also implies that the $r(i,t)$ lattice is "arbitrage-free." This involves ensuring that the value (yield) of zero-coupon bonds maturing at the end of each time interval, determined using the interest rate lattice and the $p(i,t)$ discount factors, is equal to the current market price (yield) of those bonds.

To do this as efficiently as possible, it is necessary to determine current prices of single-dollar cash flows payable at the node (i,t) of the lattice with zero payable at every other point of the lattice. These prices are referred to as "state-contingent" prices. $G(i,t)$ is used to denote the price of a security that has a cash flow of \$1 at node (i,t) and 0 everywhere else. A zero-coupon bond maturing at time $t=T$ will have cash flows of \$1 for each node (i,T), $i=0$ to T, at time $t=T$ and 0 at every other point on the lattice. Therefore, the price of a zero-coupon bond maturing at time $t=T$ will be given by Formula (4):

$$P(T) = \sum_{i=0}^{i=T} G(i,T). \tag{4}$$

The zero-coupon yield curve, $y(t)$, for zero-coupon bonds maturing at the end of each of the time intervals in the lattice is used as input. This is for monthly time intervals for one year in the example in this paper. The price of a zero-coupon bond with face value of \$1 maturing at time $t=T$ with a per annum semiannual compounding yield of $y(T)\%$ is given by Formula (5):

$$P(T) = \left[1 + \frac{y(T)}{200}\right]^{(-2Th)}. \tag{5}$$

The valuation lattice is constructed by using forward recursion starting at node (0,0) in the lattice. The input is as follows:

- The $s(i,t)$ lattice (Table 1)
- The zero-coupon yield curve, $y(t)$, which is used to determine the price of zero-coupon bonds, $P(t)$ (see Table 2 for a numerical example)

- The annualized one-period spot rate percentage volatilities, $\sigma(t)$, for each period in the lattice (see Table 2 for a numerical example).

Table 3 gives the zero-coupon bond prices for each maturity by using Formula (5) for $P(t)$ and the zero-coupon bond yields in Table 2.

The output is as follows:

- The median forward interest rates, $f(t)$, for each value of t
- The "arbitrage-free" implied one-period spot interest rates at each node in the lattice corresponding to the $p(i,t)$ factors
- The one-period discount factors, $p(i,t)$, at each node in the lattice
- The current state-contingent prices, $G(i,t)$, for each node in the lattice.

The algorithm for determining these is based on a forward recursion algorithm for "state-contingent" prices, as given in Jamshidian [13]. State-contingent prices for single-node cash flows at the end of each time period are expressed in terms of the previously determined state-contingent prices for single-node cash flows at the start of each time period and a median forward interest rate. The median forward interest rate is determined so that the sum of the state-contingent prices for each node at the end of each time period equals the market price of the zero-coupon bond maturing at the end of the time interval.

The numerical values derived by applying the algorithm for a 12-month loan example are set out in Tables 4, 5, 6, and 7. The steps in the algorithm are given in Algorithm 1.

Algorithm 1. Forward Recursion: *Commence by initializing the time 0 values:*

$G(0,0)=1$
$p(0,0)=P(1)$
$r(0,0)=y(1)$
$f(0)=y(1)$

The algorithm then proceeds from $t=1$ to $n-1$ by using the following steps:

- *Begin with $i=0$ (so that $s(0,t)=t$).*
- *As a first guess, estimate the median interest rate by using the previous period's rate, $f(t)=f(t-1)$.*
- *Calculate the spot rate, $r(0,t)$ by using*

$$r(0,t) = f(t) \exp\left[\frac{\sigma_L(t)}{100} s(0,t)\sqrt{h}\right].$$

TABLE 2

ZERO-COUPON YIELD CURVE AND SPOT RATE VOLATILITIES

	t											
	1	2	3	4	5	6	7	8	9	10	11	12
$y(t)$	6.65	6.58	6.53	6.5	6.44	6.41	6.39	6.39	6.38	6.4	6.42	6.45
$\sigma(t)$	21	21	21	21	21	21	21	21	21	21	21	21

TABLE 3

ZERO-COUPON BOND PRICES

	t												
	0	1	2	3	4	5	6	7	8	9	10	11	12
$P(t)$	1	0.994563	0.989268	0.984064	0.978904	0.973935	0.968945	0.963973	0.958933	0.953989	0.948856	0.943721	0.938491

- *Calculate the discount factor corresponding to r(0,t) by using*

$$p(0,t) = \left[1 + \frac{r(0,t)}{200}\right]^{(-2h)}.$$

- *Calculate the state-contingent value of \$1 payable at time t by using the modified forward recursive relationship for i=0:*

$$G(0,t) = \tfrac{1}{2}p(0,t - 1)G(0,t - 1).$$

- *Continue for all nodes for fixed t by using, for i=1, ..., t−1:*

$$s(i,t) = t - 2i$$

$$r(i,t) = f(t)\exp\left[\frac{\sigma_L(t)}{100}\, s(i,t)\sqrt{h}\right]$$

$$p(i,t) = \left[1 + \frac{r(i,t)}{200}\right]^{(-2h)}$$

$$G(i,t) = \tfrac{1}{2}[p(i,t - 1)G(i,t - 1) + p(i - 1,t - 1)G(i - 1,t - 1)].$$

- *Finally, complete the process for i=t, so that s(t,t)=−t, and r(t,t) and p(t,t) are calculated as above, but G(t,t) is calculated by using the modified recursive relationship:*

$$G(t,t) = \tfrac{1}{2}[p(t - 1,t - 1)G(t - 1,t - 1)].$$

- *The values for G(i,t) and p(i,t) are then used to calculate the next period zero-coupon bond price using the relation:*

$$P^*(t + 1) = \sum_i G(i,t)p(i,t).$$

The first value derived for P(t+1) is based on an estimate for f(t) and is unlikely to be equal to the market price of the zero-coupon bond price derived from the input market yields and given in Table 3 for the example. If the value is not equal to P(t+1) in Table 3, at least within a reasonable margin of tolerance such as 10^{-5}, then an iterative method such as the secant method is used to determine the value of f(t) that equates P*(t+1) to the market price. Once P*(t+1) has converged to P(t+1), the zero-coupon bond price for next time point is then fitted.*

At the end of Algorithm 1, the values of $p(i,t)$ and $G(i,t)$ for all values of (i,t) in the lattice will have been derived so that they are "arbitrage-free" with respect to the zero-coupon yield curve. These values are all that are required to value the rational prepayment option.

For the 12-month loan example, the median future spot interest rates determined by using the above algorithm to fit the interest rate lattice to the market prices of the zero-coupon bonds are given in Table 4.

The fitted-interest-rate lattice is given in Table 5 as semiannual compounding rates. The corresponding discount factors for the interest rate lattice are given in Table 6; these are determined in conjunction with the values for $G(i,t)$ given in Table 7.

For the 12-month loan example, Table 7 gives the $G(i,t)$ values determined from Algorithm 1.

The values in the tables were derived by solving for the median forward rate in Formula (2) for $r(i,t)$ that produced values equal to the price of the zero-coupon bond for each maturity given in Table 3 using Algorithm 1.

Cash flows at future times can be valued by using the lattice of state-contingent prices simply by multiplying the cash flow at node (i,t) by $G(i,t)$ and summing over all values of i and t. This procedure works for standard loan cash flows and also for bond and other fixed-interest securities but not for loans and securities with prepayment options. These options require a backward recursion approach to determine the value of the security on future dates allowing for an optimal prepayment strategy.

The steps involved in the valuation of a loan with a prepayment option are, first, to determine the current values of the loan repayments at each node of the lattice. The contractual loan cash flows are set out in a lattice. The loan cash flow at node (i,t) in the lattice is denoted as $c(i,t)$. The value of the loan is derived by backward recursion through the lattice.

The example uses a 12-month interest-only fixed-rate loan with an interest rate of 6.449 percent per annum (semiannual compounding) for an amount of \$10,000. Monthly interest repayments for such a loan will be \$53.036 with repayment of the \$10,000 in a bullet payment in 12 months. The loan rate of 6.449 percent per annum (semiannual) was chosen because this is the "arbitrage-free" interest rate for such a loan based on the one-factor term structure model derived earlier, for which numerical values are found in Tables 5, 6, and 7. The lattice of loan cash flows is given in Table 8.

TABLE 4

MEDIAN SPOT INTEREST RATES $f(t)$ (% PER ANNUM SEMIANNUAL)

	t											
	0	1	2	3	4	5	6	7	8	9	10	11
$f(t)$	6.65	6.498	6.408	6.376	6.158	6.206	6.205	6.315	6.214	6.481	6.509	6.657

TABLE 5

ONE-PERIOD SPOT RATES $r(i,t)$ (% PER ANNUM SEMIANNUAL)

	t											
i	0	1	2	3	4	5	6	7	8	9	10	11
0	6.65	6.904105	7.23399	7.64771	7.847846	8.403308	8.927051	9.653111	10.09236	11.18385	11.93414	12.96831
1		6.115782	6.408	6.774481	6.951765	7.443803	7.907744	8.550901	8.939999	9.906858	10.57148	11.48756
2			5.676323	6.000958	6.158	6.593856	7.004823	7.574544	7.919214	8.775675	9.364412	10.17589
3				5.315758	5.454868	5.840958	6.205	6.709668	7.014984	7.773653	8.295166	9.01399
4					4.832022	5.174027	5.496502	5.943546	6.214	6.886043	7.348009	7.984756
5						4.583247	4.868901	5.264901	5.504474	6.099782	6.509	7.073042
6							4.312961	4.663745	4.875963	5.403298	5.765791	6.26543
7								4.131231	4.319217	4.78634	5.107442	5.550031
8									3.826041	4.239827	4.524265	4.916319
9										3.755716	4.007676	4.354965
10											3.550073	3.857707
11												3.417227

TABLE 6

DISCOUNT FACTORS $p(i,t)$

i	t = 0	1	2	3	4	5	6	7	8	9	10	11
0	0.994563	0.99436	0.994096	0.993765	0.993606	0.993164	0.992748	0.992175	0.991829	0.990972	0.990387	0.98958
1		0.994992	0.994758	0.994463	0.994321	0.993928	0.993558	0.993047	0.992738	0.991975	0.991452	0.99073
2			0.995346	0.995085	0.994959	0.994608	0.994279	0.993824	0.993549	0.992868	0.992403	0.99176
3				0.995638	0.995525	0.995214	0.994921	0.994515	0.994271	0.993665	0.99325	0.99268
4					0.996029	0.995752	0.995492	0.995131	0.994913	0.994374	0.994005	0.99349
5						0.996231	0.995999	0.995679	0.995485	0.995005	0.994676	0.99422
6							0.99645	0.996166	0.995993	0.995567	0.995274	0.99487
7								0.996598	0.996445	0.996066	0.995806	0.99544
8									0.996847	0.99651	0.996279	0.99596
9										0.996904	0.996699	0.99641
10											0.997072	0.99682
11												0.99718

TABLE 7

STATE-CONTINGENT PRICES $G(i,t)$

	t												
i	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0.497282	0.247238	0.122889	0.061062	0.030336	0.015064	0.007477	0.003709	0.00184	0.000911	0.000451	0.00022
1		0.497282	0.494634	0.36891	0.244495	0.151889	0.090547	0.052459	0.029757	0.01661	0.00915	0.004987	0.00269
2			0.247396	0.369143	0.367098	0.304177	0.226752	0.157709	0.104415	0.066641	0.041321	0.025039	0.01488
3				0.123122	0.244957	0.304554	0.302817	0.263367	0.209329	0.155935	0.110557	0.075409	0.04984
4					0.061293	0.152455	0.227452	0.263852	0.262245	0.23452	0.194074	0.151361	0.11261
5						0.030525	0.091108	0.158585	0.210234	0.235098	0.233562	0.212615	0.18088
6							0.015205	0.052947	0.105322	0.157092	0.19516	0.213278	0.21178
7								0.007575	0.030147	0.06747	0.1118	0.152784	0.18213
8									0.003775	0.016901	0.042023	0.076599	0.11418
9										0.001881	0.009359	0.025598	0.05089
10											0.000938	0.005132	0.01531
11												0.000468	0.00279
12													0.00023

TABLE 8

LOAN CASH-FLOW LATTICE $c(i,j)$

	t												
i	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.04
1		53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.04
2			53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.04
3				53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.04
4					53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.04
5						53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.04
6							53.036	53.036	53.036	53.036	53.036	53.036	10053.04
7								53.036	53.036	53.036	53.036	53.036	10053.04
8									53.036	53.036	53.036	53.036	10053.04
9										53.036	53.036	53.036	10053.04
10											53.036	53.036	10053.04
11												53.036	10053.04
12													10053.04

The value of the loan at node (i,t) is denoted as $v(i,t)$. The loan values are determined from Table 8 by using the following backward recursion algorithm.

Algorithm 2. Backward Recursion:

- *Initialize the value of the loan to zero at the end of the term of the loan, for $t=n$ and for $i=0$ to n ($n=12$ for the example):*

$$v(i,n) = 0.$$

- *Then calculate each value of $v(i,t)$ by using, for $t = n - 1$ to 0 and for $i = 0$ to t:*

$$v(i,t) = p(i,t)\{v(i,t + 1) + c(i,t + 1) + v(i + 1,t + 1) + c(i + 1,t + 1)\}\frac{1}{2}.$$

At the end of 12 months the value of the loan is zero because all repayments have been made. The values of the loan cash flows are derived by averaging the next period's loan values plus the loan cash flows for the two states that originate from the node and multiplying this by the one-period discount factor for that node. This gives the lattice of loan values in Table 9. An illustration of backward recursion is shown in Figure 2.

The next step is to generate a lattice of balances outstanding at the original loan yield rate. This is the amount due to be paid on early prepayment. These are the exercise prices of the prepayment option and are denoted by $b(i,t)$. They are given in Table 10 for the loan example. Note that these do not vary for different values of i, because the loan outstanding that is to be prepaid is the value of the outstanding repayments at the original loan interest rate.

The cash flow on optimal early prepayment is the difference between the value of the loan and the balance outstanding, provided that this is positive. These values are denoted by $o(i,t)$ and are determined as, for all t and i, $o(i,t)=\text{maximum}(v(i,t)-b(i,t),0)$. These values for the loan example are set out in Table 11.

Notice that for lower future spot interest rates, the prepayment option is more "in the money." Because the prepayment option is assumed to be exercisable at any time, Table 11 does not give the state-contingent values of the prepayment option. At each node in the lattice it is necessary to check whether the prepayment option is worth more if left

TABLE 9

VALUE OF LOAN $v(i,t)$

i \ t	0	1	2	3	4	5	6	7	8	9	10	11	12
0	10000	9967.962	9939.114	9914.291	9894.439	9878.054	9868.342	9865.449	9871.501	9884.691	9910.943	9948.32	0
1		10035.3	10003.82	9975.93	9952.474	9932.102	9917.679	9909.329	9908.944	9914.896	9932.464	9959.895	0
2			10061.72	10031.08	10004.39	9980.45	9961.805	9948.565	9942.414	9941.886	9951.685	9970.228	0
3				10080.37	10050.79	10023.65	10001.23	9983.612	9972.301	9965.981	9968.837	9979.444	0
4					10092.22	10062.22	10036.42	10014.89	9998.967	9987.473	9984.131	9987.658	0
5						10096.62	10067.8	10042.78	10022.74	10006.63	9997.757	9994.973	0
6							10095.76	10067.63	10043.92	10023.69	10009.89	10001.49	0
7								10089.75	10062.77	10038.88	10020.69	10007.28	0
8									10079.55	10052.39	10030.29	10012.43	0
9										10064.4	10038.83	10017.01	0
10											10046.42	10021.08	0
11												10024.69	0
12													0

TABLE 10

LOAN OUTSTANDING LATTICE $b(i,t)$

	t												
i	0	1	2	3	4	5	6	7	8	9	10	11	12
0	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	0
1		10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	0
2			10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	0
3				10000	10000	10000	10000	10000	10000	10000	10000	10000	0
4					10000	10000	10000	10000	10000	10000	10000	10000	0
5						10000	10000	10000	10000	10000	10000	10000	0
6							10000	10000	10000	10000	10000	10000	0
7								10000	10000	10000	10000	10000	0
8									10000	10000	10000	10000	0
9										10000	10000	10000	0
10											10000	10000	0
11												10000	0
12													0

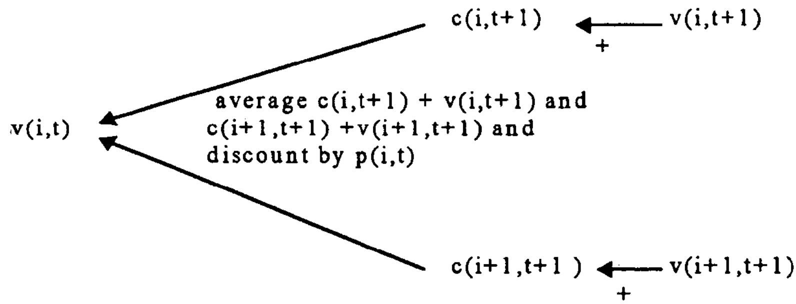
TABLE 11

EARLY-EXERCISE VALUES $o(i,t)$

i	t = 0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.004288	0	0	0	0	0	0	0	0	0	0	0	0
1		35.30308	3.821413	0	0	0	0	0	0	0	0	0	0
2			61.72284	31.08213	4.394019	0	0	0	0	0	0	0	0
3				80.37433	50.79331	23.65158	1.228116	0	0	0	0	0	0
4					92.21812	62.21785	36.41594	14.88889	0	0	0	0	0
5						96.61645	67.79727	42.77748	22.73893	6.628694	0	0	0
6							95.76283	67.62698	43.91642	23.68996	9.889743	1.485188	0
7								89.75412	62.77062	38.87682	20.68681	7.278335	0
8									79.54697	52.38788	30.29023	12.42969	0
9										64.40225	38.82807	17.00841	0
10											46.41547	21.07659	0
11												24.68994	0
12													0

FIGURE 2

BACKWARD RECURSION



unexercised and an optimal exercise policy followed for the remaining term of the loan.

The value of the optimal early-prepayment option is determined by stepping back through the optimal early-prepayment cash-flow lattice, allowing for the possibility that the value of the prepayment option at any node is worth more than the value that would be received by prepaying at that time. To do this, denote the optimal early-prepayment option value by $ov(i,t)$. The required value is $ov(0,0)$.

The following algorithm is then used to determine $ov(i,t)$.

Algorithm 3. Backward Recursion for Option Values:

- *Initialize the values at time $t=12$, for $i = 0$ to $n (n = 12)$:*

$$ov(i,n) = 0.$$

- *Calculate the option value, for $t = n - 1$ to 0 and for $i = 0$ to t:*

$$ov(i,t) = p(i,t)\{ov(i,t + 1) + ov(i + 1,t + 1)\}\frac{1}{2}.$$

- *If $o(i,t)$ is greater than $ov(i,t)$, then set $ov(i,t)=o(i,t)$.*

Note that whenever $o(i,t)$ exceeds $ov(i,t)$, then it is optimal to exercise the early-prepayment option in full and repay the whole loan. Otherwise, the option value is greater if kept "alive" and not exercised. The optimal exercise policy of this prepayment option therefore involves 100 percent prepayment of the loan whenever it is optimal to do so.

The values derived using Algorithm 3 are given in Table 12. For the example the prepayment option value is $ov(0,0)$=26.42259 per 10,000 face value for the interest-only loan.

B. Valuing the Prepayment Option Allowing for Realistic Prepayment Behavior

The actual prepayment behavior of loan customers does not conform to this ideal optimal prepayment behavior. Allowance for the actual pattern of exercise of the prepayment option is usually incorporated into the calculations.

A simple method for doing this is to assume a rate of prepayment as a proportion of the loan outstanding at each time period that is independent of the then-current interest rate. These rates are denoted by $q(t)$. These prepayment proportions can then be used to determine the altered principal repayment cash flows by adding the balance outstanding at time t multiplied by $q(t)$ to the existing principal repayment cash flows. The altered balance outstanding on the loan is then determined by deducting principal repayments including the early prepayments from the previous balance outstanding. This new balance outstanding is then used to establish the altered interest payments at the original loan interest rate. The sum of the principal repayments, allowing for previous early prepayments, and the interest payment is the altered loan cash flow $c(i,t)$.

The value of the loan at each node of the lattice is determined in the same way as before by using these altered loan cash flows. The balance outstanding of the loan is determined either by valuing the altered outstanding repayments at the original loan interest rate or by multiplying the original loan outstanding by a survivorship proportion of the principal outstanding by using the rates of prepayment $q(t)$.

The lattice approach is then used to value the prepayment option. The lattice approach as outlined so far allows only for prepayments that depend on the current interest rate and not on previous values of the interest rate.

The prepayment experience for mortgage-backed securities found in Becketti and Morris [3] and Bartlett [2] suggests that the prepayment rate should vary by:

- The original loan interest rate
- Time since the loan was issued

TABLE 12

PREPAYMENT OPTION VALUES $ov(i,t)$

i	t = 0	1	2	3	4	5	6	7	8	9	10	11	12
0	26.42259	12.40141	4.79104	1.387873	0.256931	0.02238	0	0	0	0	0	0	0
1		40.73266	20.15246	8.251119	2.536231	0.494789	0.045068	0	0	0	0	0	0
2			61.72284	32.26622	14.05788	4.606641	0.950555	0.090721	0	0	0	0	0
3				80.37433	50.79331	23.65158	8.312671	1.821328	0.182569	0	0	0	0
4					92.21812	62.21785	36.41594	14.88889	3.480176	0.367243	0	0	0
5						96.61645	67.79727	42.77748	22.73893	6.628694	0.738641	0	0
6							95.76283	67.62698	43.91642	23.68996	9.889743	1.485188	0
7								89.75412	62.77062	38.87682	20.68681	7.278335	0
8									79.54697	52.38788	30.29023	12.42969	0
9										64.40225	38.82807	17.00841	0
10											46.41547	21.07659	0
11												24.68994	0
12													0

- The interest rate several months previously
- The month of the year (seasonal)

and accurate estimation of such rates will be important.

To allow for prepayments on such a basis is more difficult than allowing for prepayments as a function only of time t. The value of the loan outstanding and the balance outstanding of the loan will be dependent on the history of interest rates during the life of the loan, because the prepayment proportions at each time will be dependent on the interest rate at that time, or at least several months previously, and the loan outstanding at any time will depend on all the previous prepayment proportions. The valuation lattice and the prepayment option valuation is then referred to as path-dependent because values at any time depend on the path of interest rates through the lattice.

The computation is more intensive than for the non-path-dependent case outlined so far. The valuation procedure is conceptually the same, but the interest rate lattice, and all the other lattices, require a separate node at each point for each distinct path through the lattice. In this case a down movement followed by an up movement in the lattice produces a node different than what is produced when an up movement is followed by a down movement. The binomial lattice used earlier assumed that an up move followed by a down move had the same financial effect on values as a down move followed by an up move.

The path-independent state lattice has $t+1$ nodes at the maturity date of the loan for a t period loan. In the 12-month-loan example with a monthly time interval, this produced 13 nodes at the maturity date ($i=0$ to 12). The path-dependent lattice will have 2^{n+1} nodes at maturity, because the nodes will double over each time interval in the lattice, resulting in $2^{13}=8,192$ for the 12-month example. This presents a computational problem where longer term loans are to be valued because the number of nodes explodes and cannot be handled even with powerful computers for reasons of both speed and memory. The method used to handle this is to value the option by sampling the possible paths. Simulation is the most common method of doing this.

Simulation is a powerful means of valuing interest-sensitive cash flows such as the loan-prepayment-option cash flows. Recent developments in the application of simulation to valuations allowing for optimal behavior are given in Tilley [29].

Applying simulation in a conventional manner to the prepayment feature problem requires some additional assumptions. The loan yield rate can be modeled as a normal or lognormal random variable. Mean reversion can also be incorporated. The valuation proceeds by generating standard normal random variables for each period for the number of simulations to be performed. The number of simulations required for accuracy could be determined by valuing options for which the value was known, such as traded bond options, and checking how close the simulated value was to the accurate value. Alternatively, an estimate of the standard error of the values calculated by using simulation can be determined and the number of simulations selected to reduce this standard error to an acceptably low level. The number of simulations required for any given accuracy will be reduced by using the variance reduction Monte-Carlo techniques referred to in Tilley [29]. Computational speed and accuracy inevitably require the use of such techniques in practical applications.

For this illustrative example, a flat yield curve has been assumed for the simulations. It would be possible to simulate values for one-period forward rates for every maturity of the yield curve at which cash flows occur. This would require a much higher number of simulated values and a longer computation time.

The interest rate level at the end of each month is generated by using the distributional assumption most suited for interest rate data. Interest rates are generated with a normal distribution using Formula (6):

$$r(t) = r(t-1) + \mu(t)h + \sigma_N(t)\sqrt{h}e(t), \tag{6}$$

where $e(t)$ is normal (0,1) random number; h is the time interval in years (for monthly values $\sqrt{h}=\sqrt{(1/12)}=0.288675$); and $\sigma_N(t)$ is the annual volatility of the one-period interest rate in absolute terms. Values for $\mu(t)$ and $\sigma_N(t)$ are determined so that values for conventional zero-coupon bonds calculated using simulation agree with the market values. The value for $r(0)$ is determined from the current yield curve.

The interest rate distribution is "arbitrage-free" in the sense that prices of cash flows calculated as expected present values using the parameters of this distribution will be consistent with the current zero-coupon bond yield curve.

The normal distribution allows negative values for one-period interest rates, and so it will usually be desirable to ensure positive values by generating returns from a lognormal distribution using Formula (7):

$$r(t) = r(t-1)(1 + \sigma_L(t)\sqrt{h}e(t)), \tag{7}$$

where $\sigma_L(t)$ is the percentage volatility of the interest rate.

Interest rates can be generated allowing for mean reversion by using Formula (8):

$$r(t) = r(t-1) + \kappa(\mu - r(t-1)) + \sigma_L(t)\sqrt{h}r(t-1)e(t), \tag{8}$$

where μ is the long-term mean interest rate and κ is the speed with which the current interest rate tends to move to the long-term interest rate.

Once a model has been selected, the market loan interest rate, $r(t)$, is generated. The proportion of the principal outstanding that is to be repaid is modeled as a function of the interest rate level or even the path of interest rates. Denote this by $q(t,r)$ to indicate that it is a function of the interest rate, r. It should consist of:

- A monthly proportion varying with the time since issue of the loan and the month of the year (to allow for seasonal effects)
- A proportion varying with the difference between the current rate, $r(t)$, and the original loan rate.

The contractual repayments, reduced by the proportion of the loan repaid, are valued at the market interest rate, $r(t)$, to obtain the value of the loan, $v(t)$. The value of the outstanding contractual repayments is determined at the original loan interest rate, $r(0)$, to get the balance outstanding, $b(t)$. The cash flow from the early-prepayment option, which could be a cost or benefit, is then determined by multiplying the proportion repaying by the difference between the value of the loan, $v(t)$, and the loan outstanding, $b(t)$. This gives $o(t)=q(t,r(t))\{v(t)-b(t)\}$.

The new principal outstanding is reduced by the amount of contractual repayments of principal and the proportion who repay early. The reduced contractual repayments are then determined based on the new principal outstanding.

This procedure continues for each month of the loan.

The early-prepayment-option cash flows, $o(t)$, are present-valued along the path of simulated interest rates by using the $r(t)$ interest rates generated for each month as discount rates. If $r(t)$ is a semiannual compounding rate, then the procedure obtains the value $ov(0)$ for this simulation as follows:

- Initialize $ov(n)=0$
- Then for $i=n-1$ to 0

$$ov(i) = o(i) + ov(i - 1) * \left[1 + \frac{r(i - 1)}{200}\right]^{(-2/12)}.$$

This procedure is repeated for the desired number of simulations. The prepayment-option cost is then estimated as the average of the simulated-option values, $ov(0)$, determined from each simulation.

Table 13 illustrates values derived for a 12-month loan for ten sample paths simulated by using the normal distribution model. They are provided only as an illustration of the procedure described above. The proportion prepaying has been modeled by using a prepayment function that generally corresponds to the Public Securities Association (PSA) standard prepayment assumptions as given in Bartlett [2] and most other mortgage-backed securities publications. The proportion prepaying has been determined by using Formula (10) given in Section 4 of the paper with g=0.008 and p=1.3.

The average discounted value of the $o(t)$ cash flows over all ten simulations is −0.07464. In practice, thousands of simulations are required to price these prepayment cash flows even when variance reduction techniques are used. The same random numbers are used for loans of different terms to produce consistent values and for computational efficiency.

4. PREPAYMENT MODELS

The prepayment experience in the mortgage-backed security market is discussed in Bartlett [2]. This reference indicates that prepayment rates vary with the original loan interest rate and that higher original interest rate loans have higher prepayment rates. An increase of 4 percent in the original interest rate can mean a four times higher prepayment rate. There is also a lag of about three months from when interest rates increase to when loans prepay early. Prepayments are seasonal, reflecting the timing of house sales, which are higher in summer and spring than in winter. Loans are not very sensitive to prepayment when rates rise during the first 2.5 years of loan issue.

The PSA standard that is used in the U.S. market has prepayment rates commencing at 0 percent in month 0 and increasing by 0.2 percent monthly to 6 percent in month 30 and a constant 6 percent thereafter.

Models that allow for prepayments for mortgage-backed securities have been set out in a number of studies. Examples of U.S. studies include

TABLE 13

SIMULATION OF PREPAYMENT OPTION VALUES

Month	$r(t)$	$q(t,r)$	Loan O/S	Value of Loan O/S	$o(t)$
			Simulation 1		
1	0.00553	0.00299	9191.42516	9188.03202	−0.01016
2	0.00484	0.00337	8378.43032	8406.98550	0.09625
3	0.00466	0.00366	7560.99131	7591.21424	0.11066
4	0.00407	0.00390	6739.08383	6781.22284	0.16435
5	0.00427	0.00410	5912.68347	5940.83988	0.11553
6	0.00362	0.00428	5081.76566	5114.36721	0.13951
7	0.00375	0.00443	4246.30571	4268.09752	0.09664
8	0.00373	0.00457	3406.27878	3421.01575	0.06740
9	0.00362	0.00470	2561.65992	2571.10000	0.04436
10	0.00396	0.00481	1712.42401	1716.26460	0.01848
11	0.00372	0.00511	858.54583	860.04073	0.00764
12	0.00000	0.00000	0.00000	0.00000	0.00000
			Simulation 2		
1	0.00509	0.00299	9191.42516	9212.11659	0.06195
2	0.00524	0.00337	8378.43032	8388.71854	0.03468
3	0.00526	0.00366	7560.99131	7568.53630	0.02763
4	0.00492	0.00390	6739.08383	6755.49711	0.06402
5	0.00508	0.00410	5912.68347	5921.83504	0.03755
6	0.00503	0.00428	5081.76566	5089.53646	0.03325
7	0.00550	0.00443	4246.30571	4245.90887	−0.00176
8	0.00520	0.00457	3406.27878	3408.56575	0.01046
9	0.00543	0.00470	2561.65992	2561.86055	0.00094
10	0.00568	0.00481	1712.42401	1711.88083	−0.00261
11	0.00587	0.00492	858.54583	858.20540	−0.00167
12	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 13—*Continued*

Month	$r(t)$	$q(t,r)$	Loan O/S	Value of Loan O/S	$\sigma(t)$
Simulation 3					
1	0.00561	0.00299	9191.42516	9183.42698	−0.02395
2	0.00543	0.00337	8378.43032	8380.15971	0.00583
3	0.00553	0.00366	7560.99131	7558.48468	−0.00918
4	0.00521	0.00390	6739.08383	6746.65675	0.02954
5	0.00532	0.00410	5912.68347	5916.13366	0.01416
6	0.00555	0.00428	5081.76566	5080.30070	−0.00627
7	0.00506	0.00443	4246.30571	4251.45674	0.02284
8	0.00494	0.00457	3406.27878	3410.73940	0.02040
9	0.00481	0.00470	2561.65992	2564.99967	0.01569
10	0.00451	0.00481	1712.42401	1714.86262	0.01173
11	0.00464	0.00492	858.54583	859.25299	0.00348
12	0.00000	0.00000	0.00000	0.00000	0.00000
Simulation 4					
1	0.00578	0.00299	9191.42516	9174.61617	−0.05033
2	0.00523	0.00337	8378.43032	8388.98600	0.03558
3	0.00523	0.00366	7560.99131	7570.00990	0.03302
4	0.00507	0.00390	6739.08383	6750.91862	0.04616
5	0.00524	0.00410	5912.68347	5918.05752	0.02205
6	0.00523	0.00428	5081.76566	5085.88863	0.01764
7	0.00556	0.00443	4246.30571	4245.11018	−0.00530
8	0.00514	0.00457	3406.27878	3409.01312	0.01251
9	0.00533	0.00470	2561.65992	2562.35766	0.00328
10	0.00539	0.00481	1712.42401	1712.61010	0.00090
11	0.00589	0.00492	858.54583	858.18624	−0.00177
12	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 13—*Continued*

Month	r(t)	q(t,r)	Loan O/S	Value of Loan O/S	o(t)
Simulation 5					
1	0.00585	0.00299	9191.42516	9170.66595	−0.06215
2	0.00602	0.00337	8378.43032	8353.33499	−0.08459
3	0.00634	0.00366	7560.99131	7528.40434	−0.11932
4	0.00682	0.00390	6739.08383	6698.72100	−0.15742
5	0.00619	0.00410	5912.68347	5895.88114	−0.06894
6	0.00637	0.00428	5081.76566	5065.84995	−0.06811
7	0.00652	0.00443	4246.30571	4232.98137	−0.05909
8	0.00609	0.00457	3406.27878	3400.97910	−0.02424
9	0.00599	0.00470	2561.65992	2559.01339	−0.01244
10	0.00588	0.00481	1712.42401	1711.37289	−0.00506
11	0.00599	0.00492	858.54583	858.09733	−0.00220
12	0.00000	0.00000	0.00000	0.00000	0.00000
Simulation 6					
1	0.00580	0.00299	9191.42516	9173.09867	−0.05487
2	0.00596	0.00337	8378.43032	8356.25944	−0.07473
3	0.00596	0.00366	7560.99131	7542.72463	−0.06688
4	0.00593	0.00390	6739.08383	6725.11188	−0.05449
5	0.00670	0.00410	5912.68347	5883.93745	−0.11795
6	0.00705	0.00428	5081.76566	5054.06065	−0.11856
7	0.00663	0.00443	4246.30571	4231.70009	−0.06477
8	0.00657	0.00457	3406.27878	3397.00558	−0.04241
9	0.00624	0.00470	2561.65992	2557.73194	−0.01846
10	0.00589	0.00481	1712.42401	1711.35273	−0.00516
11	0.00529	0.00492	858.54583	858.69488	0.00073
12	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 13—*Continued*

Month	$r(t)$	$q(t,r)$	Loan O/S	Value of Loan O/S	$\sigma(t)$
Simulation 7					
1	0.00549	0.00299	9191.42516	9190.16457	−0.00377
2	0.00549	0.00337	8378.43032	8377.25888	−0.00395
3	0.00577	0.00366	7560.99131	7549.64023	−0.04156
4	0.00598	0.00390	6739.08383	6723.64091	−0.06023
5	0.00602	0.00410	5912.68347	5899.68742	−0.05332
6	0.00628	0.00428	5081.76566	5067.48978	−0.06109
7	0.00661	0.00443	4246.30571	4231.90072	−0.06388
8	0.00698	0.00457	3406.27878	3393.52888	−0.05831
9	0.00646	0.00470	2561.65992	2556.60260	−0.02376
10	0.00630	0.00481	1712.42401	1710.29872	−0.01023
11	0.00664	0.00492	858.54583	857.54334	−0.00493
12	0.00000	0.00000	0.00000	0.00000	0.00000
Simulation 8					
1	0.00539	0.00299	9191.42516	9195.65644	0.01267
2	0.00555	0.00337	8378.43032	8374.64566	−0.01276
3	0.00555	0.00366	7560.99131	7557.98124	−0.01102
4	0.00542	0.00390	6739.08383	6740.34906	0.00493
5	0.00500	0.00410	5912.68347	5923.62068	0.04488
6	0.00529	0.00428	5081.76566	5084.95380	0.01364
7	0.00539	0.00443	4246.30571	4247.20793	0.00400
8	0.00576	0.00457	3406.27878	3403.81774	−0.01126
9	0.00605	0.00470	2561.65992	2558.68135	−0.01400
10	0.00627	0.00481	1712.42401	1710.37170	−0.00988
11	0.00630	0.00492	858.54583	857.83394	−0.00350
12	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 13—*Continued*

Month	$r(t)$	$q(t,r)$	Loan O/S	Value of Loan O/S	$o(t)$
Simulation 9					
1	0.00544	0.00299	9191.42516	9192.82046	0.00418
2	0.00536	0.00337	8378.43032	8383.43764	0.01688
3	0.00550	0.00366	7560.99131	7559.76330	−0.00450
4	0.00561	0.00390	6739.08383	6734.80844	−0.01667
5	0.00599	0.00410	5912.68347	5900.41709	−0.05033
6	0.00593	0.00428	5081.76566	5073.63464	−0.03479
7	0.00575	0.00443	4246.30571	4242.77710	−0.01565
8	0.00571	0.00457	3406.27878	3404.20513	−0.00948
9	0.00613	0.00470	2561.65992	2558.30997	−0.01574
10	0.00597	0.00481	1712.42401	1711.14988	−0.00613
11	0.00594	0.00492	858.54583	858.13743	−0.00201
12	0.00000	0.00000	0.00000	0.00000	0.00000
Simulation 10					
1	0.00575	0.00299	9191.42516	9176.14843	−0.04574
2	0.00577	0.00337	8378.43032	8364.58642	−0.04666
3	0.00551	0.00366	7560.99131	7559.43697	−0.00569
4	0.00574	0.00390	6739.08383	6731.03887	−0.03138
5	0.00580	0.00410	5912.68347	5904.94010	−0.03177
6	0.00616	0.00428	5081.76566	5069.64508	−0.05187
7	0.00623	0.00443	4246.30571	4236.71792	−0.04252
8	0.00608	0.00457	3406.27878	3401.10575	−0.02366
9	0.00627	0.00470	2561.65992	2557.56331	−0.01925
10	0.00644	0.00481	1712.42401	1709.93923	−0.01196
11	0.00635	0.00492	858.54583	857.79363	−0.00370
12	0.00000	0.00000	0.00000	0.00000	0.00000

Becketti and Morris [3], Green and Shoven [8], and Schwartz and Torous [25]. Prepayment models used in practice are often considered to be proprietary even though the basic form of such models is standard.

The basic form of these models allows for the following components:

(a) A proportion of loan amounts outstanding being prepaid, which varies through time and is assumed to be independent of interest rates. This proportion is assumed to increase to a maximum and then remain constant or decline slowly.

(b) An increase in this proportion whenever the difference between the original loan rate and an indicator of market interest rates increases above a threshold margin. This threshold margin reflects refinance costs.

(c) A reduction in the proportion of the time-dependent (non-interest-sensitive) prepayment proportions whenever the difference between the original loan rate and an indicator of market interest rates exceeds a "burnout" level beyond which it is assumed that all interest-sensitive loans will have prepaid.

The indicator of market interest rates is usually taken as the current fixed-interest refinance rate for a similar loan or a previous value of such a market rate, such as the minimum rate since issue of the loan or the rate three months previously. Another alternative is an average of several previous months market interest rates.

The usual form of these prepayment models can be written as follows:

$$q(t,r) = q(t)m(r) \tag{9}$$

where $q(t)$ is the interest-rate-independent prepayment proportion, which depends only on the time since issue of the loan, and $m(r)$ is a function of the market interest rates, which gives the proportionate increase in $q(t)$ resulting from the financial incentive to refinance for those borrowers who are considered to be interest-sensitive.

A multitude of formulas can be used for $q(t)$ and $m(r)$. Details can be found in Schwartz and Torous [25]. The aim should be to select a formula that can model a range of possibilities and that depends on as few parameters as possible.

All these prepayment models apply to a pool of loans, so that they are considered to be the average percentages of loans that prepay as a percentage of the balance outstanding for a large group of similar loans that were issued at the same time and for the same fixed interest rate. They do not apply directly to a single loan because in most cases a single loan will either fully prepay or will continue with the contractual payments. Estimation of prepayment rates will be dependent on a large enough volume of similar loans being issued, so that the prepayment functions can be considered as expected values. For smaller volumes of similar loans, the prepayment percentages vary from the model, and it is important to analyze the sensitivity of the prepayment model assumptions in the light of the expected statistical variation in such prepayment rates.

The form of the prepayment model should ideally be estimated from available data. The sensitivity of any pricing to the prepayment assumption must be examined before a particular model is adapted. As an example, based on Schwartz and Torous [25], the following formula captures a range of prepayment patterns. Assume $q(t,r)=q(t)m(r)$ with

$$q(t) = \frac{(gp)(gt)^{p-1}}{1 + (gt)^p}, \tag{10}$$

where g and p are parameters and t is the number of months since issue of the loan (t=0, 1, ..., 12 for a 12-month loan).

Parameters of g=0.008 and p=1.3 give $q(t)$ proportions similar to those of the PSA standard for U.S. mortgage-backed securities. Values of g=0.013 and p=1.9 give proportions approximately twice those of the PSA standard. The $q(t)$ proportions can be calculated as multiples of the PSA standard by following these two steps:

(i) Convert them to annual equivalents using the formula

$$cpr(t) = 1 - [1 - q(t)]^{12}$$

(ii) Multiply by $500/t$ for $t < 30$ or 16.67 for $t \geq 30$.

The prepayment proportions should be examined visually in a graph of $q(t)$ versus time to compare one set of assumptions against another.

The form of $m(r)$ can be specified in many ways. The following is one possible approach. Other specifications of $m(r)$ are possible, as given in Green and Shoven [8] and Schwartz and Torous [25], amongst others.

The following inputs are used:

(i) $a(t)$, the moving average of the market refinance fixed rates, $r(t)$, generated by using the simulation model for the previous t months. The value for t could be equal to six months to approximate the three-month lag typically found in studies of interest rate sensitivity of prepayments. For the first six months, $a(t)$ will be the average of all the monthly rates available until six months has passed.

(ii) r_l, the threshold point equal to the difference required between the original loan rate, $r(0)$, and the average of market rates at time t, $a(t)$, so that prepayments are influenced by falling interest rates.
(iii) r_h, the burnout point equal to the difference required between the original loan rate and the six-month average of market rates, at which point all interest-sensitive prepayments are assumed to have occurred.
(iv) $b=\ln(1+v/100)$, where v is the maximum percentage increase in the prepayment proportion assumed to occur at the burnout point.

The function $m(r)$ is then estimated as follows:

> If the original loan rate minus the moving average market rate at time t is greater than the threshold point and less than the burnout point (that is, $r_l<\{r(0)-a(t)\}<r_h$) and if the current moving average market rate is lower than the minimum of all its previous values (that is, $a(t)>\text{minimum } \{a(i),\ i<t\}$), then
>
> $$m(r) = \exp\left[b\,\frac{(r(0) - a(t))}{r_h} \right]; \qquad (11)$$
>
> otherwise, $m(r) = 1$.

Suitable values for the input parameters are based on any repayment experience available. A slowing down of the rate of prepayments also can be modeled whenever prepayments over the life of the loans issued at a particular time have been higher than expected. To do this, the prepayment proportion, $q(r,t)$, would be multiplied by the factor $\exp(-c\{\ln(ob(t)/sb(t)\})$, where $ob(t)$ is the actual balance outstanding on the loans issued and $sb(t)$ is the originally scheduled balance outstanding. This factor is included in some of the U.S. studies of mortgage-backed securities, such as Schwartz and Torous [25].

The simplest procedure for determining the parameters is to select the values of g, p, b, r_l, and r_h that best fit the available or expected loan experience data. This loan experience could be based on forecast experience or on historical data. The historical data required would be

- The month of issue
- The fixed interest rate
- The actual balances outstanding for each month since issue
- The contractual balances outstanding had the loan followed the contractual repayment pattern for each month since issue
- The market interest rate for each month since issue.

The best fit can be determined by using a number of techniques of which least squares would be the most straightforward. The U.S. studies referred to in this paper have also used other statistical techniques such as maximum likelihood.

Figures 3 to 7 give simulations over ten years of sample prepayment rates by using Formulas (9), (10), and (11) for $q(r,t)$. The prepayment rates have been designed to be similar to the PSA standard with an allowance for interest-sensitive payments. They are given only as a graphical illustration and are not designed to be based on any particular historical loan prepayment experience.

5. RISK STATISTICS

As with any portfolio of interest-sensitive financial contracts, it is essential in the management of the portfolio to determine the sensitivities of the portfolio value to the underlying factors that affect the value. For options, these sensitivities, or risk statistics, are the delta, gamma, theta, and vega. The delta measures the change in the option value for a small

FIGURE 3

SIMULATION 1

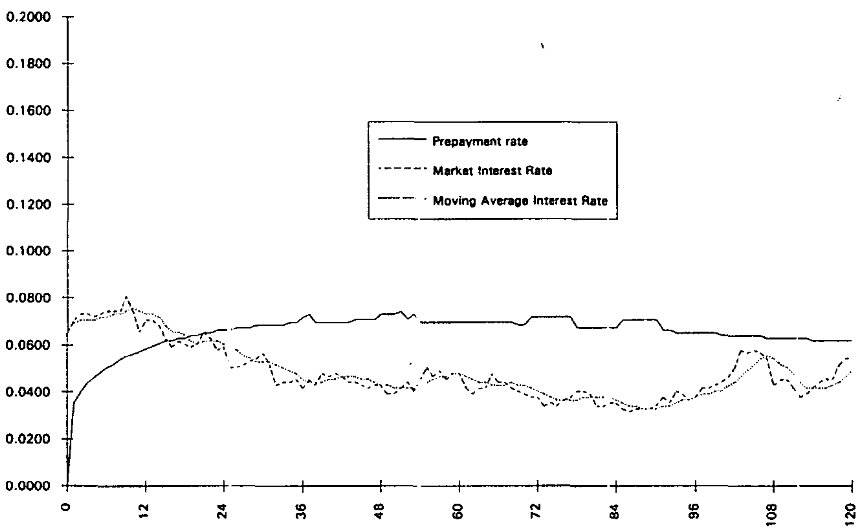


FIGURE 4
SIMULATION 2

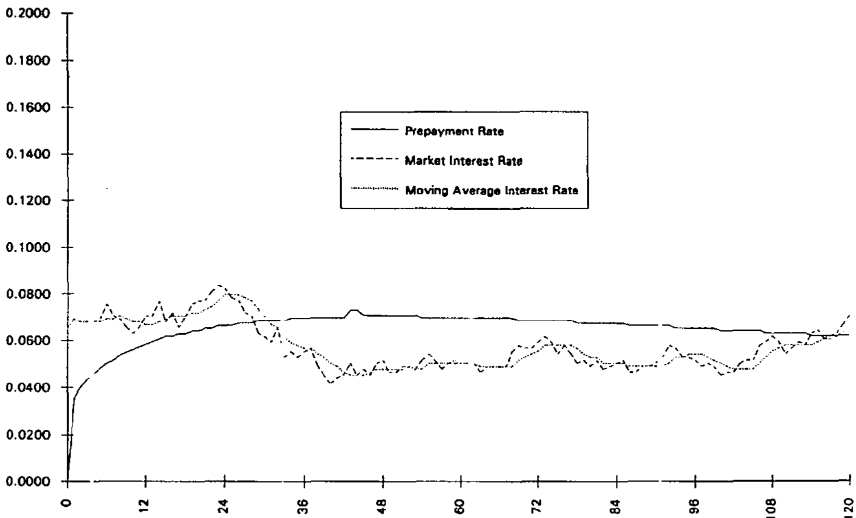


FIGURE 5
SIMULATION 3

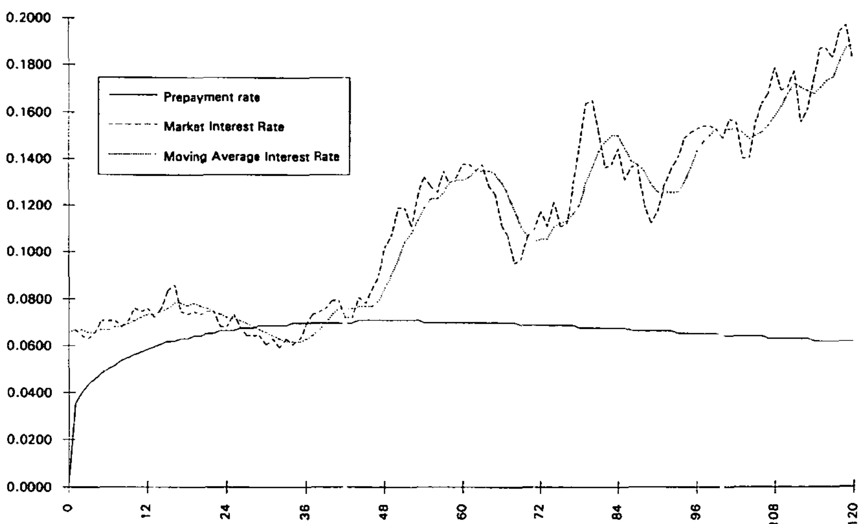


FIGURE 6
SIMULATION 4

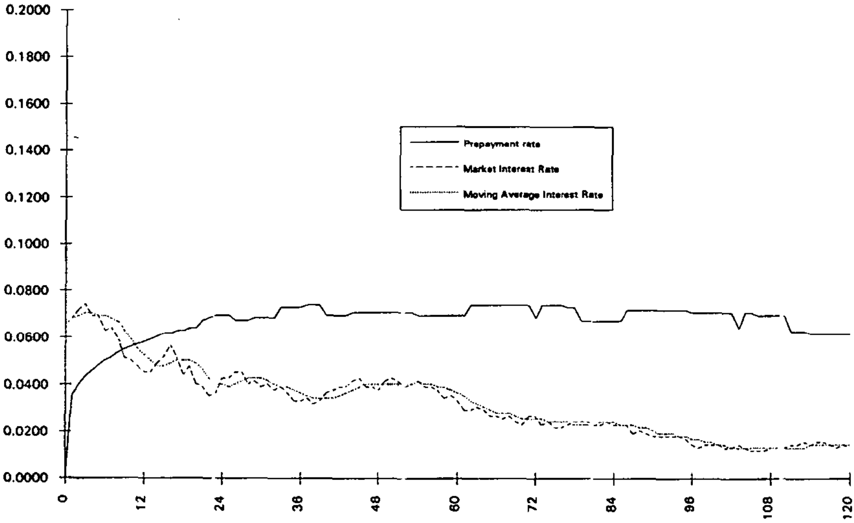
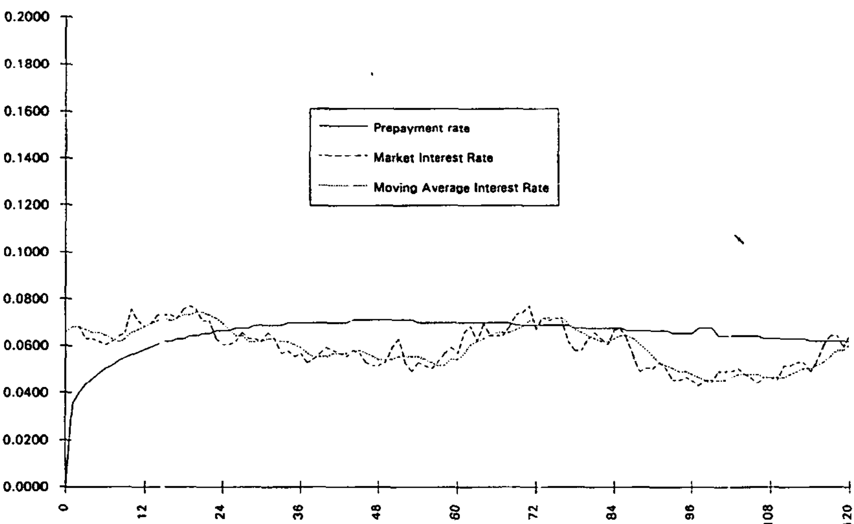


FIGURE 7
SIMULATION 5



change in the value of the underlying contract. The gamma measures the sensitivity of the delta to changes in the value of the underlying contract. The theta measures the change in the value of the option as time changes. The vega measures the sensitivity of the option value to changes in the volatility. Hull [11, chapter 13] provides a comprehensive description of the risk statistics used in the management and hedging of options positions.

Since this loan contract is an interest-rate-related instrument, the interest-rate-related risk statistics such as duration and convexity also must be evaluated. Duration measures the sensitivity of the option value to changes in interest rates, and convexity measures the sensitivity of the duration to changes in interest rates.

The prepayment option is an American-style option, and to evaluate risk statistics, a numerical technique must be used. This numerical technique is equivalent to a discrete approximation to the partial differential of the option value. For the lattice model, the formulas for the delta and gamma are found in Hull [11, page 341]. The underlying asset is taken to be the loan.

A. *Option Delta*

For the optimal prepayment option valuation, the delta is calculated by using the following lattice values:

$$\frac{ov(0,2) - ov(2,2)}{v(0,2) - v(2,2)}.$$

For the 12-month-loan example, the value of the loan and the value of the option are found in Tables 9 and 12, respectively. The option delta, measuring sensitivity to changes in the underlying loan value, is:

$$\frac{4.79104 - 61.72284}{9939.114 - 10061.72} = 0.464348.$$

For the simulation option value, the delta is calculated by recalculating the option value for a small change, usually 0.01 percent, in the loan value, so that the formula will be:

$$\frac{ov^*(0) - ov(0)}{v^* - v},$$

where $v^* = v(1.0001)$ is the current value of the loan, v, increased by 0.01% and ov^* is the option value using the identical random numbers, $e(t)$, for ov but with a starting market interest rate, $r^*(0)$, determined by equating the value of the outstanding contractual loan prepayments by v^*.

Note that a much larger number of simulations are usually required than for the estimation of the option value to ensure an accurate calculation of the option delta, even when variance reduction techniques are used.

B. Option Gamma

For the optimal prepayment option, the gamma is calculated by using the following lattice values:

$$\frac{\left[\dfrac{ov(0,2) - ov(1,2)}{v(0,2) - v(1,2)} - \dfrac{ov(1,2) - ov(2,2)}{v(1,2) - v(2,2)}\right]}{\dfrac{v(0,2) - v(2,2)}{2}}.$$

The values from Tables 9 and 12 give:

$$\frac{\left[\dfrac{4.79104 - 20.15246}{9939.114 - 10003.82} - \dfrac{20.15246 - 61.72284}{10003.82 - 10061.72}\right]}{\dfrac{9939.114 - 10061.72}{2}} = 0.007839.$$

The gamma for the simulation option value uses an approximation to the second derivative of the option value with respect to the loan value given by:

$$\frac{ov^*(0) - 2ov(0) + ov^{**}(0)}{[v^* - v]^2},$$

where ov^* and v^* are as for the option delta and $v^{**} = 0.9999v$, so that ov^{**} is the option value corresponding to a starting market interest rate, which equates the outstanding contractual loan repayments, adjusted for any previous early proportion of prepayments, to v^{**} using the same set of random numbers used to calculate the option value. Once again, care has to be exercised to ensure that a large enough number of simulations are used to guarantee a reasonable accuracy in calculating these numerical approximations to the option derivative.

C. Option Theta

For the optimal prepayment option, the lattice values used for the option theta are given by:

$$\frac{ov(1,2) - ov(0,0)}{2h},$$

where $h=1/12$.

Using the values from Table 12 for the 12-month loan example gives:

$$\frac{20.15246 - 26.42259}{0.166667} = -37.62078.$$

The theta for the simulation option valuation is determined by using:

$$\frac{ov^*(1) - ov(0)}{h},$$

where $ov^*(1)$ is the value of the option at the end of the following month when the loan term will be reduced by one month and allowance is made for the contractual repayments due over the next month. The same random numbers are used in the calculations as when $ov(0)$ is calculated.

D. Option Vega

For the optimal exercise option value, the option vega cannot be derived from the lattice values; it must be approximated by recalculating the option value for a small change in the input volatilities. The same approximation can be used in the simulation calculations. The formula is:

$$\frac{ov(\sigma 1.01) - ov(0)}{\sigma .01},$$

where $ov(\sigma 1.01)$ is the option value calculated by using a volatility of 1.01 times the volatility used to derive the option value $ov(0)$.

E. Option Duration

Option deltas given above measure the sensitivity of the option value to changes in the value of the underlying loan. They can be interpreted as the usual option hedge ratios if the option is to be replicated by using loan instruments with the same cash flow and value characteristics.

Hedging could also be considered by using financial instruments with different cash-flow characteristics to the loan but with similar sensitivity to general levels of interest rates. The option duration is a measure of the sensitivity of the change in the value of the option to changes in interest rates used to determine the value of the loan.

The loan option duration can be estimated by using the formula:

$$D[\text{option}\delta]\,\frac{v(0,0)}{ov(0,0)},$$

where D is the modified duration of the outstanding loan repayments. The modified duration is the proportionate sensitivity of the loan to changes in the loan interest rate. Because the option sensitivity to interest rates is given by the partial differential of the option value with respect to interest rates, the above formula can be derived by recognizing that the option value is a function of the loan value.

Option duration can also be approximated by using the formula:

$$\frac{ov(r^+) - ov}{0.0001ov},$$

where $ov(r^+)$ is the option value calculated for a 1-basis-point increase in the interest rates used to calculate the option value.

F. Option Convexity

The prepayment option convexity can be approximated numerically by recalculating the option value for a 1-basis-point decrease in interest rates as well as a 1-basis-point increase and by using the approximate formula:

$$\frac{ov(r^+) - 2ov + ov(r^-)}{(0.0001)^2\,ov}.$$

The option duration and option convexity are closely related to the option delta and gamma for interest rate options, because the underlying value of the loan is a function of the interest rate as well as the option value.

Duration and convexity measures would be the common measures of interest sensitivity, because these are computed for most interest-sensitive assets, including interest-rate-related options such as bond options.

Accurate calculation of risk statistics, and also of values, of these interest-sensitive cash flows requires significant computation time. Developments in computational techniques are constantly improving the methods and accuracy of calculations for these statistics. One such technique that improves the speed of computation of risk statistics attributed to Eric Reiner is given in the Tompkins and Field book [28, pages 44–45].

6. HEDGING

The ideal hedge instrument for the prepayment option is an American-style call option on an amortizing reducing-term interest rate swap. Such a swap pays a fixed interest rate on a reducing balance outstanding in exchange for a floating interest rate on the same reducing balance. Any alternative hedging strategies involve either overhedging or underhedging. The amortization of the interest rate swap needs to correspond to the prepayment pattern of the loan. In the case of interest-only loans, the ideal hedge for the prepayment option would be an American-style physical bond option for the term of the loan, because a physical bond pays interest only in the same way as an interest-only loan.

The ideal hedge instrument often does not exist, and alternative ways of hedging the option must be considered. In theory the actual hedging instruments can be used to value the option, because the market price of the hedge instruments that perfectly hedge the option will be the cost of the prepayment option.

To evaluate alternative hedging methods, the pricing basis must be considered as well. If the prepayment option is to be priced on a optimal exercise basis and this is the price charged to the borrower, then the option cash flows determined by using the lattice approach are appropriate for determining the required hedging. If realistic allowance for prepayments is made in the value of the option, then it will be necessary to attempt to hedge the expected cash flows based on estimated prepayment proportions, and the accuracy of the hedge will depend on the accuracy of the estimate of the proportions prepaying. In this case, assessment of the risk of the hedging strategy will be needed.

A simple approach to hedging the prepayment option is to use an immunization or dynamic hedging approach. Hedge instruments are chosen to match the prepayment option duration and convexity. Each month the portfolio of hedge instruments is rebalanced to match the altered duration and convexity of the prepayment option. This strategy is likely to be relatively risky, because it relies on the accuracy of the estimates of interest rate sensitivity (in the form of duration and convexity) of the hedge instruments and the prepayment option. In theory, any interest-sensitive hedge instrument for which an estimate of the duration and convexity was available could be used. This would include instruments such as bond options, swaptions (options on interest rate swap agreements), cap agreements (agreements that place a cap on interest rates payable over fixed time intervals), and floor agreements (agreements that place a floor on interest rates payable over fixed time intervals). The selection of hedge instrument should take into account the liquidity and depth of the market for the instrument, transactions costs, and other market-related factors.

A better approach would be to attempt to match the cash flows of the prepayment option more exactly. Although the prepayment option is American-style, most over-the-counter interest rate options are European-style. A portfolio of European-style options will not match the prepayment option cash flows well unless it is possible to sell the European options when it is optimal to prepay the loan agreement. Even so, the sale of the European-style options will be for less than the payoff on early exercise of the prepayment option, because American-style options usually have higher values than European options.

Even when American-style options are available, these are not usually on an underlying instrument that is equivalent to the loan. Using American-style options will involve constructing a portfolio of options on different underlying instruments. Such a portfolio of options is unlikely to exactly replicate the prepayment option, because it will not recognize the interdependencies of the underlying instruments that are needed when they are put together to form the loan cash flows. Purchasing a portfolio of options will involve paying too much for the hedge.

The hedging of loan contracts with prepayment features requires skill as well as a knowledge of financial market instruments and the interest sensitivity of these different instruments.

7. CONCLUSIONS

This paper has described computational algorithms that are used to value loan contracts with prepayment features. It uses a simple example to illustrate the techniques. The computational algorithms have been presented in a form that can be implemented rather than in the mathematical form that these term structure models are usually presented.

Prepayments introduce the computational problem of path dependence because the prepayment rates depend not only on the level of interest rates but also on the future path taken by interest rates. Conceptually the valuation is no different from when there is no path dependence, but computationally it is more difficult.

Prepayment models usually include an underlying rate of prepayment as well as the dependence of these prepayment rates on interest rates, the time since issue of the loan, and other factors such as seasonal variation.

Hedging the interest rate in loan contracts with prepayments is more difficult than for most interest-rate-dependent securities because prepayment behavior is difficult to predict. The computation of risk statistics is also more demanding because of the path dependence. Computational techniques such as variance reduction are essential.

The simple model covered in this paper is relatively easy to implement and can be used to assess interest-rate-related options in assets and liabilities.

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APPENDIX

PRICE AND YIELD VOLATILITY

Let $P(r)$ be the price of security at yield r. The current values of P and r are known. The absolute volatility of P and r is denoted by $\sigma(P)$ and $\sigma(r)$, respectively. The percentage volatility of P and r are simply $\sigma(P)/P$ and $\sigma(r)/r$, respectively. The relationship between price and yield volatility can be approximated by using Ito's lemma and assuming a diffusion process for both P and r. In this case:

$$dP = u(P)dt + \sigma(P)dZ$$

and

$$dr = u(r)dt + \sigma(r)dZ,$$

and since $P=P(r)$, then:

$$dP = \{P_r u(r) + \tfrac{1}{2}P_{rr}\sigma^2(r)\}dt + P_r\sigma(r)dZ,$$

where P_r denotes the partial derivative of the price with respect to the interest rate.

It then follows that:

$$\sigma(P) = P_r \sigma(r),$$

and since the modified duration of a security is defined as $-P_r/P$, this gives:

$$\sigma(P) = PD\sigma(r),$$

where D is the modified duration of the security. Hence:

absolute price volatility = price × modified duration
× absolute yield volatility.

We also have

$$\sigma(P)/P = D\,\{\sigma(r)/r\}r$$

or

percentage price volatility = modified duration
× percentage yield volatility × yield.

GLOSSARY OF KEY TERMS

Arbitrage-free. An interest rate or price model is arbitrage-free, with respect to the securities being priced, if it is not possible to construct portfolios of these securities with no net future cash flows that require a non-zero net initial investment at the market prices of these securities.

Backward recursion. A calculation procedure in which values at an earlier time point are calculated from those at the next time point by using a prespecified algorithm. The recursion commences with the final time point values and produces intermediate time point values one step at a time backwards through time to ultimately provide the initial values.

Convexity. A measure of the curvature of the price to yield relationship for a set of cash flows. It is usually determined as the second derivative of the price with respect to the yield as a proportion of the price.

Delta. A measure of the sensitivity of the value of an option on an underlying security or commodity to changes in the value of the underlying security or commodity. It is usually determined as the partial differential of the option value with respect to the underlying security or commodity value.

Duration. A measure of the weighted average term to receipt of a set of cash flows that is closely related to the sensitivity of the value of a set of cash flows to changes in the yield used to value the cash flows. This sensitivity is referred to as modified duration.

Forward interest rate. An interest rate as of a specified date that applies for a fixed time period commencing on a future date. These future dates and time periods are also referred to as forward dates and time periods. Forward interest rates on or before the current date can in theory be determined from market yields and are not random variables, whereas forward interest rates that apply as at future dates will not be known at the current date and are therefore random variables.

Forward rate volatility curve. The curve of forward rate volatility as a function of the forward date to which the forward rate volatility applies.

Forward recursion. A calculation procedure in which values at a later time point are calculated from those at the previous time point. The recursion commences with the initial time point values and produces intermediate time point values one step at a time forwards through time to ultimately provide the final time point values.

Gamma. A measure of the curvature of the option value as a function of the underlying asset value. Defined as the second partial differential of the option value with respect to the underlying asset value, it is the same as the partial differential of the option delta with respect to the underlying asset value.

Interest rate volatility. A measure of the variability, or volatility, of interest rates. It usually refers to the diffusion parameter, σ, in the continuous time stochastic differential equation for the interest rate, $di=\mu(i)dt+\sigma(i)dZ$, where dZ is a standardized Brownian motion. The volatility parameter is often estimated by the historical interest rate sample standard deviation. In this sense volatility is another term for standard deviation. If $\sigma(i)=\sigma$, a constant, then volatility is estimated in absolute terms, whereas if $\sigma(i)=\sigma i$ so that $\sigma(i)/i=\sigma$, a constant, then volatility is estimated in percentage terms.

Mean reversion. The tendency of a future interest rate to revert back to a long-run equilibrium interest rate. In a stochastic interest rate model, the expected next period interest rate, conditional on all past values of the interest rates, reverts towards a long-run unconditional mean interest rate.

One-period discount factor. A discount, or present value, factor that is used to present value expected cash flows back in time for one future period. For a discrete binomial term structure model based on monthly time intervals, each future monthly time interval will have an associated one-period discount factor for present valuing expected cash flows from the end of the time interval to each node at the beginning of the time interval.

One-factor interest rate model. A stochastic interest rate model that incorporates one factor as the only source of randomness in the model. The factor is usually taken as the one-period spot interest rate.

Spot interest rates. These are the yields to maturity on zero-coupon (single-payment) bonds for different maturities. The one-period spot interest rate is the yield to maturity on a zero-coupon bond maturing in one period's time. For a model with a monthly time interval, the spot interest rates will be for zero-coupon bonds maturing in 1 month, 2 months, 3 months, and so on to the longest maturity date in the model.

Theta. A measure of the change in value of an option on an underlying security or commodity with respect to time. It is usually determined as the partial differential of the option value with respect to time.

Vega. A measure of the sensitivity of the value of an option on an underlying security or commodity to changes in the volatility of the value of the underlying security or commodity. It is usually determined as the partial differential of the option value with respect to the volatility parameter.

Zero-coupon yield curve. The curve of yields to maturity for zero-coupon bonds as a function of time to maturity. These yields are also referred to as spot interest rates.

DISCUSSION OF PRECEDING PAPER

PHILIPPE ARTZNER*:

I suggest adding to this interesting paper some precisions on prepayment risk and on default risk and some remarks about the links with traditional educational material.

Prepayment Risk

In Section 2 the author defines prepayment risk as arising from the possibility that the lender will repay the loan "regardless of current market interest rates." At first sight, this refers only to *interest rate risk*.

Section 3 indeed mentions that "some borrowers prepay when it does not appear to be optimal to do so," and Section 7 warns that this prepayment behavior "is difficult to predict." The expression "prepayment risk" should be reserved for this very problem of discrepancy between actual prepayment rates and forecasted ones. The prepayment patterns presented in Formula (10) of Section 4 introduce randomness risk, only through the randomness of interest rates and not through the nonoptimal behavior of borrowers. Griffin's paper in the *Transactions* and the discussions [2] raised the question of possible diversification of this discrepancy risk or of its possible reward.

Default Risk

The developments of the paper aim at the mortgage markets for individuals, as shown in the third paragraph of Section 3, which alludes to the risk of default. Although the first events mentioned are ingredients to the prepayment risk, the default risk would deserve a treatment on its own [1].

Additional Remarks

It would be interesting, at least from an educational point of view, to mention the relation between prepayments (prepayment options) and swaps (swaptions) before the section on hedging. In the same spirit, note that Algorithm 3 pertains to optimal stopping problems.

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Finally, let me mention the challenging problem of determining the "level premium" that should be asked from the borrower for the right of prepaying the loan.

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ELIAS S.W. SHIU:

Mr. Sherris is to be congratulated for this paper, explaining the valuation of interest-sensitive cash flows. The numerical examples are illuminating. The purpose of this discussion is to supplement this fine exposition with some technical details about binomial lattices. In particular, I present the development of an additive model related to Formula (1).

The binomial lattice model in Section 3 is based on the multiplicative model as prescribed by Formula (2), which may be rewritten as

$$r(i, t) = r(0, t)\, a(t)^i, \quad 0 \le i \le t, \tag{D.1}$$

where

$$a(t) = \exp[-\sigma_L(t)\sqrt{h}/50]. \tag{D.2}$$

The "base" rate at time t, $r(0, t)$, is determined so that the current price of the $t+1$ period zero-coupon bond, $P(t+1)$, is replicated by the model,

$$P(t+1) = \sum_{i=0}^{t} \frac{G(i, t)}{\left[1 + \dfrac{r(0, t)}{200}\, a(t)^i\right]^{2h}}. \tag{D.3}$$

To solve for $r(0, t)$, we need to use an iterative method such as the Newton-Raphson method or the secant method.

Let us now consider the additive model as prescribed by Formula (1), which may be rewritten as

$$r(i, t) = r(0, t) - i\, b(t), \quad 0 \le i \le t, \tag{D.4}$$

where

$$b(t) = \sigma_N(t)\sqrt{h}/50. \tag{D.5}$$

Because, for small x (in absolute value),

$$\frac{1}{1+x} \approx e^{-x},$$

we are motivated to model the forces of interest. In other words, we replace Condition (D.4) by

$$\delta(i, t) = \delta(0, t) - i\,c(t), \quad 0 \le i \le t, \tag{D.6}$$

where $c(t)$ is a deterministic function of t. In terms of the notation in the paper,

$$p(i, t) = e^{-h\delta(i,t)}. \tag{D.7}$$

Here the formula analogous to (D.3) is

$$P(t+1) = \sum_{i=0}^{t} G(i, t)\, e^{-h\delta(i,t)}$$

$$= e^{-h\delta(0,t)} \sum_{i=0}^{t} G(i, t)\, e^{hic(t)}. \tag{D.8}$$

An advantage of Condition (D.6) is now clear. There is no need to use an iterative method to solve for the "base" rate $\delta(0, t)$ for each t.

Let us further examine this additive model. For simplicity we assume $h=1$. For $m \le t$, let $p(m, t; i)$ denote the price at node (i, m) of the zero-coupon bond paying 1 at time t. In terms of the notation in the paper,

$$p(t, t+1; i) = p(i, t)$$

and

$$p(0, t; 0) = P(t).$$

It immediately follows from (D.6) that, for $0 \le i, j \le t$,

$$p(t, t+1; j) = p(t, t+1; i)e^{(j-i)c(t)}. \tag{D.9}$$

For $0 \le i \le m$, we assume that the risk-neutral probability of moving from node (i, m) to $(i+1, m+1)$ is $\pi(m)$ (which is independent of i), and therefore the risk-neutral probability of moving from (i, m) to $(i, m+1)$ is $1-\pi(m)$. In the paper, $\pi(m)=1/2$ for all m. Then, for $0 \le i \le m < t$,

$$p(m, t; i) = p(m, m+1; i)\{\pi(m)\, p(m+1, t; i+1) + [1-\pi(m)]p(m+1, t; i)\}. \tag{D.10}$$

With $m=t-2$, (D.10) becomes

$$\begin{aligned} p(t-2, t; i) &= p(t-2, t-1; i)\{\pi(t-2)p(t-1, t; i+1) \\ &\quad + [1-\pi(t-2)]p(t-1, t; i)\} \\ &= p(t-2, t-1; i)\{\pi(t-2)e^{c(t-1)} \\ &\quad + [1-\pi(t-2)]\}p(t-1, t; i), \end{aligned} \tag{D.11}$$

because of (D.9). It follows from (D.11) and (D.9) that, for $0 \le i, j \le t-2$,

$$p(t-2, t; j) = p(t-2, t; i)e^{(j-i)[c(t-2)+c(t-1)]}. \tag{D.12}$$

With $m=t-3$, (D.10) becomes

$$\begin{aligned} p(t-3, t; i) &= p(t-3, t-2; i)\{\pi(t-3)p(t-2, t; i+1) \\ &\quad + [1-\pi(t-3)]p(t-2, t; i)\} \\ &= p(t-3, t-2; i)\{\pi(t-3)e^{c(t-2)+c(t-1)} \\ &\quad + [1-\pi(t-3)]\}p(t-2, t; i), \end{aligned}$$

because of (D.12). Applying (D.11) yields

$$\begin{aligned} p(t-3, t; i) &= p(t-3, t-2; i)\{\pi(t-3)e^{c(t-2)+c(t-1)} \\ &\quad + [1-\pi(t-3)]\}p(t-2, t-1; i)\{\pi(t-2)e^{c(t-1)} \\ &\quad + [1-\pi(t-2)]\}p(t-1, t; i). \end{aligned} \tag{D.13}$$

It follows from (D.13) and (D.9) that, for $0 \le i$, $j \le t-2$,

$$p(t-3, t; j) = p(t-3, t; i)e^{(j-1)[c(t-3)+c(t-2)+c(t-1)]}. \tag{D.14}$$

The pattern is now clear. Define

$$g(j, s) = [1-\pi(j)] + \pi(j)\exp[c(j+1) + c(j+2) + \cdots + c(s)], \quad j < s,$$

and

$$g(s, s) = 1.$$

Then, for $0 \leq i \leq m < t$,

$$p(m, t; i) = \prod_{j=m}^{t-1} [p(j, j + 1; i) g(j, t - 1)]. \tag{D.15}$$

Applying (D.15) twice (with $m=0$, $i=0$, $t=t+1$ and $t=t$) and dividing yields

$$\frac{p(0, t + 1; 0)}{p(0, t; 0)} = p(t, t + 1; 0) \frac{\prod_{j=0}^{t} g(j, t)}{\prod_{j=0}^{t-1} g(j, t - 1)}$$

$$= p(t, t + 1; 0) \frac{\prod_{j=0}^{t-1} g(j, t)}{\prod_{j=0}^{t-2} g(j, t - 1)}. \tag{D.16}$$

It follows from (D.16) that, for $m=1, 2, 3, \ldots, 0 \leq i \leq m$,

$$p(m, m + 1; i) = p(m, m + 1; 0)\, e^{ic(m)}$$

$$= \frac{p(0, m + 1; 0)}{p(0, m; 0)} \frac{\prod_{j=0}^{m-2} g(j, m - 1)}{\prod_{j=0}^{m-1} g(j, m)} e^{ic(m)}. \tag{D.17}$$

Thus the binomial lattice model is completely specified as soon as the risk-neutral probabilities, $\{\pi(m)\}$, and volatility numbers, $\{c(m)\}$, are given. The current bond prices, $\{p(0, m; 0)\}$, are replicated in the model because they are incorporated through (D.17). The path-breaking Ho and Lee model [1] is the special case where

$$\pi(0) = \pi(1) = \pi(2) = \cdots$$

and

$$c(1) = c(2) = c(3) = \cdots.$$

Because

$$\prod_{j=m}^{t-1}\left[\frac{\prod_{k=0}^{j-2} g(k, j-1)}{\prod_{k=0}^{j-1} g(k, j)}\, g(j, t-1)\right] = \left[\frac{\prod_{k=0}^{m-2} g(k, m-1)}{\prod_{k=0}^{t-2} g(k, t-1)}\right]\prod_{j=m}^{t-1} g(j, t-1)$$

$$= \frac{\prod_{k=0}^{m-2} g(k, m-1)}{\prod_{k=0}^{m-1} g(k, t-1)},$$

it follows from (D.15) and (D.17) that, for $0 \le i \le m < t$,

$$p(m, t; i) = \frac{p(0, t; 0)}{p(0, m; 0)}\left[\frac{\prod_{j=0}^{m-2} g(j, m-1)}{\prod_{j=0}^{m-1} g(j, t-1)}\right]$$

$$\times \exp(i[c(m) + c(m+1) + \cdots + c(t-1)]). \quad \text{(D.18)}$$

It is surprising that the bond price $p(m, t; i)$ does not depend on the risk-neutral probabilities $\{\pi(m), \pi(m+1), \pi(m+2), \ldots, \pi(t-2)\}$ because the formula for $p(m, t; i)$ was derived by backward induction starting with the boundary condition

$$p(t, t; i) = p(t, t; i+1) = \cdots = p(t, t; i+t-m) = 1;$$

see (D.10) and (D.15).

There is a "deficiency" in a binomial lattice model satisfying

$$r(0, t) > r(1, t) > r(2, t) > \cdots > r(t, t)$$

or

$$r(0, t) < r(1, t) < r(2, t) < \cdots < r(t, t).$$

Unless we diminish the volatilities of the one-period interest rates as t becomes large, the model will have high interest rates or negative interest rates or both.

The introduction of the state-contingent prices or Arrow-Debreu prices, $G(i, t)$, has greatly simplified the construction of the multiplicative binomial lattice. Jamshidian [2, p. 62, p. 74] wrote:

> Ironically, forward induction is virtually unknown, if less so among practitioners than academicians. To the best of my knowledge, there is no written public account of it, nor of the even more fundamental binomial forward equation, save for my appendix in Merrill Lynch & Co., Inc. [1990]. . . . While I came upon forward induction independently, and by mid-1989 had internally released it in software and lectured and written about it, I claim to be neither the sole nor the first discoverer of this method. Yet the method appears to be known to few others in Wall Street.

I would like to take this opportunity to give credit to Kevin Buhr, currently a doctoral student at the University of Wisconsin, Madison. In 1990, Kevin was a sophomore summer intern at the Great-West Life Assurance Company in Winnipeg. That summer we were interested in building binomial lattices to study the price behavior of mortgage-backed securities. Kevin came up with the idea of forward induction independently. Of course, he never studied "Arrow-Debreu securities."

I would like to suggest an exercise for the interested reader. The prepayment option value in the example in Section 3.A of the paper is 26.42259 for 10,000 of loan. This means that the initial market value of the loan is 9973.57741. My suggestion is to redo Table 9 to obtain this value directly. The one change is that, in the backward recursion, whenever a value is greater than 10,000, it is replaced by 10,000.

My final comment is to recommend some references on pricing mortgage-backed securities. Two instructive papers are Richard and Roll [5] and Richard [4]. Number 3 in Volume 4 of the *Journal of Fixed Income* is a special issue focusing on recent changes in prepayment modeling and hedging for mortgage-backed securities. Again, I thank Mr. Sherris for writing this useful exposition.

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YONG YAO*:

It is a pleasure to read Mr. Sherris's informative paper. He discusses the valuation of interest-sensitive assets and liabilities based on a model of short-term interest rates. The computational algorithms described in the paper are easy to implement. As pointed out by Mr. Sherris, we need to "ensure that prices of zero-coupon bonds derived by using the algorithm are equal to the market prices of such bonds on the valuation date."

Recent research has led to the construction of models that can perfectly replicate the current term structure. An excellent survey of these models, as well as other continuous time models of term structure of interest rates, is given by Vetzal [7]. One such model is the Heath-Jarrow-Morton (HJM) model [4], which has found increasing favor with traders of fixed-income derivative securities. The HJM approach begins by taking the state vector to be the entire forward rate curve and assuming that its movement is driven by a finite number of standard Brownian motions (see Equation (4) below). Restrictions are then imposed on the drift of the process to guarantee that no riskless arbitrage opportunities arise (see Equation (5) below). Like the Black-Scholes model of stock options, the HJM model requires no assumptions about investor preferences. The contingent claim prices do not explicitly depend on the market prices of risk and are determined by the volatility structure of interest rates. In particular, estimates of drift or expected rate changes are not needed. In general, in this framework, the spot rate process underlying the bond prices is path-dependent, and the movements of the spot rate may depend

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on its entire history. This makes the model difficult to implement from a computational point of view. Fortunately, a special case of the HJM model has been found ([2], [5], and [6]), where the path-dependence nature of the short rates is captured by a single sufficient statistic (see $\phi(t)$ in Equation (9) below), and the bond prices are easy to calculate (see Formula (10) below).

Term Structure of Interest Rates

A default-free, pure-discount bond maturing at time $T \geq 0$ is a security that will pay one dollar at T and nothing at any other time. Zero-coupon Treasury securities or stripped Treasuries can be viewed as examples of default-free, pure-discount bonds. Any noncallable default-free bond can be considered as a package of these bonds.

We denote the price of this default-free, pure-discount bond prevailing at time t, $0 \leq t \leq T$, as $P(t, T)$. At maturity, we require that

$$P(T, T) = 1.$$

The instantaneous forward rate prevailing at time $t<T$, $f(t, T)$, is given by

$$f(t, T) = -\frac{\partial(\ln P(t, T))}{\partial T}. \tag{1}$$

It follows from (1) that, for $t<T$,

$$P(t, T) = \exp\left[-\int_{s=t}^{s=T} f(t, s)ds\right]. \tag{2}$$

The (instantaneous) spot rate prevailing at time t is given by

$$r(t) = f(t, T)|_{T=t} = \lim_{\Delta \to 0+} f(t, t + \Delta). \tag{3}$$

For a better understanding of the HJM model, we first explain the term structure dynamics in a simple setting, in which all the future movements of the interest rates are known with certainty. In this case, we know from the theory of compound interest that

$$P(t, T) = \exp\left[-\int_t^T r(s)ds\right] = P(0, T)/P(0, t).$$

From Formula (2), we have

$$\exp\left[-\int_t^T r(s)ds\right] = \exp\left[-\int_{s=t}^{s=T} f(t, s)ds\right],$$

and hence

$$f(t, T) \equiv r(T) \quad \text{for all } T \geq t \geq 0.$$

So we have $f(t, T) \equiv f(0, T) \equiv r(T)$ for all $T \geq t \geq 0$.

The HJM Model

Let τ be a fixed positive number. For each $T \in [0, \tau]$, the instantaneous forward rate $f(t, T)$ is assumed to satisfy the following equation:

$$f(t, T) = f(0, T) + \int_0^t \mu(s, T)ds + \sum_{j=1}^{n} \int_0^t \sigma_i(s, T)dZ_i(s) \quad \text{for all } t < T, \tag{4}$$

where $\{f(0, T): T \in [0, \tau]\}$ is the initial forward rate curve calculated from the current market prices $\{P(S): S \in [0, \tau]\}$, and $\{Z_i(t); t \geq 0\}$ are n independent one-dimensional standard Brownian motions. In the absence of riskless arbitrages, it can be shown that

$$\mu(t, T) = \sum_{i=1}^{n} \sigma_i(t, T)\left[\int_t^T \sigma_i(t, u)du - \lambda_i(t)\right], \tag{5}$$

where $\{\lambda_1(s), \lambda_2(s), \ldots, \lambda_n(s)\}$ are the *market prices of risk* satisfying some technical conditions (see the Heath-Jarrow-Morton paper [4]).

Substituting Equation (5) into (4), we obtain an arbitrage-free characterization of the term structure in terms of forward rates:

$$f(t, T) - f(0, T) = \sum_{i=1}^{n} \int_0^t \sigma_i(s, T)\left[\int_s^T \sigma_i(s, u)du - \lambda_i(s)\right]ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s, T)dZ_i(s), \tag{6}$$

or equivalently,

$$df(t,T) = \sum_{i=1}^{n} \sigma_i(t,T)\left[\int_t^T \sigma_i(t,u)du - \lambda_i(t)\right]dt + \sum_{i=1}^{n} \sigma_i(t,T)dZ_i(t). \quad (7)$$

A Special Case of the HJM Model

Let $k(x)$ be a deterministic function, and denote

$$K(t, T) = \exp\left[-\int_t^T k(x)dx\right].$$

In the HJM model, if we choose

$$\sigma_i(t, T) = \sigma_i(t, t)K(t, T) \quad \text{for } t \le T, \text{ and } i = 1, 2, \ldots, n, \quad (8)$$

then the movement of the spot rate will be determined by

$$dr(t) = \left\{k(t)[f(0, t) - r(t)] + \phi(t) - \sum_{j=1}^{n} \sigma_i(t, t)\,\lambda_i(t) + \frac{d}{dt}f(0, t)\right\}dt + \sum_{i=1}^{n} \sigma_i(t, t)dZ_i(t), \quad (9)$$

and the prices of default-free, pure-discount bonds are given by

$$P(t, T) = \frac{P(T)}{P(t)} \exp\{-\beta(t, T)[r(t) - f(0, t)] - \frac{1}{2}[\beta(t, T)]^2\phi(t)\}, \quad (10)$$

where

$$\phi(t) = \sum_{j=1}^{n} \int_0^t [\sigma_i(s, t)]^2\, ds = \sum_{i=1}^{n} \int_0^t [\sigma_i(s, s)K(s, t)]^2\, ds,$$

$$\beta(t, T) = \int_t^T K(t, u)du,$$

and $\{P(t): t \in [0, \tau]\}$ are the current market prices.

Remarks

1. It follows from Equation (10) that bond prices at time t are expressed in terms of the current prices, the spot rate at time t, and the accumulated variance of the forward rates up to time t.

2. Bond prices at time 0, $\{P(0, T)\}$, are consistent with the current market prices $\{P(T)\}$ automatically.

Proof of Equations (9) and (10)

For simplicity, we prove the case $n=1$, and omit all the subscripts. Set

$$\Delta(t; T) = f(t, T) - f(0, T).$$

It follows from (8) that

$$\begin{aligned} \sigma(s, T) &= \sigma(s, s)\, K(s, T) \\ &= \sigma(s, s)\, K(s, t)\, K(t, T) \\ &= \sigma(s, t)\, K(t, T). \end{aligned}$$

Applying this formula to Equation (6), we have

$$\begin{aligned} \Delta(t; T) = &\int_0^t \sigma(s, t) K(t, T) \left[\int_s^T \sigma(s, u) du - \lambda(s) \right] ds \\ &+ \int_0^t \sigma(s, t) K(t, T) dZ(s). \end{aligned}$$

Hence

$$\begin{aligned} &\Delta(t; T) - K(t, T) \int_0^t \sigma(s, t) \left[\int_s^T \sigma(s, u) du \right] ds \\ &\quad = K(t, T) \left[- \int_0^t \sigma(s, t) \lambda(s) ds + \int_0^t \sigma(s, t) dZ(s) \right], \end{aligned}$$

or

$$\frac{\Delta(t;T)}{K(t,T)} - \int_0^t \sigma(s, t) \left[\int_s^T \sigma(s, u) du \right] ds = - \int_0^t \sigma(s, t) \lambda(s) ds + \int_0^t \sigma(s, t) dZ(s).$$

Since the right-hand side of this equation is independent of T, the left-hand side is also independent of T. So we have, for each $S \geq t$,

$$\frac{\Delta(t;T)}{K(t,T)} - \int_0^t \sigma(s,t)\left[\int_s^T \sigma(s,u)du\right]ds \equiv \frac{\Delta(t;S)}{K(t,S)} - \int_0^t \sigma(s,t)\left[\int_s^S \sigma(s,u)du\right]ds.$$

With $S=t$ and noting that $K(t, t)=1$, we obtain

$$\frac{\Delta(t;T)}{K(t,T)} - \int_0^t \sigma(s,t)\left[\int_s^T \sigma(s,u)du\right]ds = \Delta(t;t) - \int_0^t \sigma(s,t)\left[\int_s^t \sigma(s,u)du\right]ds,$$

which simplifies to

$$\begin{aligned}
\Delta(t;T) &= K(t,T)\left\{\Delta(t;t) - \int_0^t \sigma(s,t)\left[\int_s^t \sigma(s,u)du\right]ds \right.\\
&\qquad \left. + \int_0^t \sigma(s,t)\left[\int_s^T \sigma(s,u)du\right]ds\right\}\\
&= K(t,T)\left\{\Delta(t;t) + \int_0^t \sigma(s,t)\left[\int_t^T \sigma(s,u)du\right]ds\right\}\\
&= K(t,T)\left\{\Delta(t;t) + \left[\int_0^t [\sigma(s,t)]^2 ds\right]\left[\int_t^T K(t,u)du\right]\right\}\\
&= K(t,T)\left\{\Delta(t;t) + \phi(t)\left[\int_t^T K(t,u)du\right]\right\} \qquad (11)\\
&= K(t,T)[\Delta(t;t) + \phi(t)\beta(t,T)].
\end{aligned}$$

Because $\frac{\partial}{\partial T}\beta(t,T) = K(t,T)$, we have

$$\Delta(t;T) = \frac{d}{dT}\left[\Delta(t;t)\beta(t,T) + \frac{1}{2}[\beta(t,T)]^2\phi(t)\right]. \qquad (12)$$

We are now ready to prove (9). By (3), we have

$$dr(t) = [df(t,T)]|_{T=t} + \left\{\left[\frac{\partial}{\partial T} f(t,T)\right]\bigg|_{T=t}\right\}dt.$$

From Equation (7),

$$df(t, T)|_{T=t} = -\sigma(t, t)\lambda(t)dt + \sigma(t, t)dZ(t).$$

From Formula (11),

$$f(t, T) = f(0, T) + K(t, T)[\Delta(t; t) + \phi(t)\beta(t, T)],$$

and hence

$$\left[\frac{\partial}{\partial T} f(t, T)\right]\Bigg|_{T=t} = \frac{d}{dt} f(0, t) - k(t)[\Delta(t; t)] + \phi(t)$$

$$= \frac{d}{dt} f(0, t) + k(t)[f(0, t) - r(t)] + \phi(t).$$

So we can get

$$dr(t) = -\sigma(t, t)\lambda(t)dt + \sigma(t, t)dZ(t) + \left\{k(t)[f(0, t) - r(t)] + \phi(t) + \frac{d}{dt} f(0, t)\right\}dt,$$

which is the one-dimensional case of (9).

Next we prove (10). Recall that $\Delta(t; T) = f(t, T) - f(0, T)$. From (2), we obtain

$$P(t, T) = \exp\left[-\int_{s=t}^{s=T} f(t, s)ds\right]$$

$$= \exp\left\{-\int_{s=t}^{s=T} [f(0, s) + \Delta(t; s)]ds\right\}$$

$$= \exp\left[-\int_{s=t}^{s=T} f(0, s)ds\right] \exp\left[-\int_{s=t}^{s=T} \Delta(t; s)ds\right]$$

Because of $\exp\left[-\int_{s=t}^{s=T} f(0, s)ds\right] = \frac{P(T)}{P(t)}$,

$$P(t, T) = \frac{P(T)}{P(t)} \exp\left[-\int_{s=t}^{s=T} \Delta(t; s)ds\right]$$

$$= \frac{P(T)}{P(t)} \exp\left\{ -\beta(t, T)[r(t) - f(0, t)] - \frac{1}{2}[\beta(t, T)]^2\phi(t) \right\},$$

by (12).

Two Examples

We present two examples to explain how to use this result.

Example 1: Set $n=1$, $\sigma(t, t)=\sigma|r(t)|^{1/2}$, $k(x)=k$, where σ and k are two positive constants. We choose $\lambda(t)=0$ and use the following approximations:

$$\phi(t) = \int_0^t [\sigma(s, s)K(s, t)]^2\, ds \approx \sigma^2 \sum_{i=0}^{t-1} |r(i)|\, e^{-2k(t-i)}$$

$$r(t) = r(t-1) + \mu(t)h + \sigma[h|r(t-1)|]^{1/2}e(t),$$

where

$$\mu(t) = k[f(0, (t-1)h) - r(t-1)] + \phi(t) + \frac{[f(0, th) - f(0, (t-1)h)]}{h},$$

$e(t)$ is a normal (0, 1) random number; and h is the length of the time interval in years. From (10), we can calculate the prices of pure-discount bonds. Then we use simulation to value loans with prepayment provisions.

Example 2: If $\sigma(t, t)\neq 0$ for all $t\geq 0$, we can simplify the model described in this discussion by choosing $\lambda(t)$ such that

$$\phi(t) - \sigma(t, t)\lambda(t) = 0.$$

For simplicity, we set $\sigma(t, t)=\sigma$ and $k(x)=k$, where σ and k are two positive constants. Then

$$\beta(t, T) = \frac{1}{k}(1 - e^{-k(T-t)}),$$

$$\phi(t) = \frac{\sigma^2}{2k}(1 - e^{-2kt}),$$

and

$$\lambda(t) = \frac{\phi(t)}{\sigma(t, t)} = \sigma \frac{(1 - e^{-2kt})}{2k},$$

which is a increasing function of t.

In this case, the movements of the spot rate are determined by

$$dr(t) = \left\{ k[f(0, t) - r(t)] + \frac{d}{dt} f(0, t) \right\} dt + \sigma dZ(t).$$

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(AUTHOR'S REVIEW OF DISCUSSIONS)

MICHAEL SHERRIS:

I thank Dr. Artzner, Dr. Shiu and Mr. Yao for their informative discussions of the paper.

Dr. Artzner raises important issues in prepayment and default risk. The pricing model in the paper does not price prepayment risk as defined by Dr. Artzner. It is concerned only with interest rate risk and the extent to which the prepayments are a function of interest rates. This is an important issue because it has parallels with insurance products. Lapse rates in insurance products can be sensitive to interest rates and other factors in a similar way to prepayments in mortgage-backed securities. The pricing of lapse risk is an issue separate from the pricing of interest rate risk in these products. The difficulty in insurance products is determining a price for this lapse risk. The same problem applies to mortgage-backed securities. In the case of mortality, it is possible to assume that the probabilities of death for different policies are relatively independent and the mortality risk can be diversified to a significant extent. To the extent that lapse risks can be diversified, then there is a similarity to mortality. However, prepayments on mortgage-backed securities tend to be correlated, and a significant component of the correlation is often the common effect of interest rate moves.

In Section 3 of the paper, in the first paragraph, I state that "the option is equivalent to an American-style option to exchange the fixed-rate loan for a floating-rate loan for a term equal to the remaining term of the original loan." Since a swap is a contract to exchange a fixed-rate loan for a floating-rate loan, I have introduced this idea prior to the section on hedging, but as noted by Dr. Artzner, I did not spend any time discussing swaps or swaptions.

Determining the "level premium," which would normally be incorporated in the interest rate or yield on the mortgage-backed security, is a difficult theoretical problem. In practice, this is how these securities price the prepayment option. This problem is similar to that of pricing a level-premium insurance contract that provides a guaranteed surrender value.

I also thank Dr. Artzner for pointing out his paper with Delbaen [1], the paper by Griffin [2], and, in particular, the discussion of Griffin's paper. On reading the Artzner and Delbaen paper, I realized that I had attended a presentation of that paper to the Erasmus University Conference on Insurance, Solvency and Finance in 1991. I recall enjoying Dr. Artzner's presentation very much. However, it also became clear that I had not appreciated the significance of the issues he discussed at the time.

Dr. Shiu has provided a general specification of the mathematics for the additive model. This allows the fitting of the model parameters to the current zero-coupon bond yield curve directly and also allows the analytical specification of the possible yield curves on any future date. The original intention of my paper was to set out the steps in developing a simple term structure model in a manner that an actuary or student, with limited knowledge of term structure models, could follow and use to develop simple programs for valuing interest rate instruments including bonds and interest rate options. For this reason, the paper does not contain any mathematical development of the formulas. Dr. Shiu's discussion indicates the usefulness of a mathematical approach to term structure models, and I highly recommend it to readers of my paper.

I was also interested in the result in Equation (D.18), where the bond price does not depend on the risk-neutral probabilities. This result appears to be related to the results in the paper by Heath, Jarrow and Morton [3, p. 420], which states that

> We show, however, in the limit that contingent claim values only depend on the volatility parameters, and not the pseudo probability. This is analogous to the situation that occurs with the binomial approximation to the Black-Scholes model.

Mr. Yao's discussion develops continuous-time results similar to the discrete-time results of Dr. Shiu. Mr. Yao develops a continuous-time bond price formula in terms of the current market bond prices. The Heath-Jarrow-Morton approach to modeling interest rates is quite general. I found the derivation of the bond price formula in this discussion informative. The examples presented are very instructive and indicate how the results derived in the discussion can be used in the problems discussed in my paper.

I conclude by noting that the discussions of papers in the *Transactions* are a very valuable component of the published papers. More often than not, the discussions include original ideas and add much to the published papers. The discussions of my paper have demonstrated this, and I thank the discussants for their contributions.

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