EDUCATION COMMITTEE

OF THE

SOCIETY OF ACTUARIES

FM STUDY NOTE

SUPPLEMENTARY NOTES FOR

FINANCIAL MATHEMATICS FOR ACTUARIES

by

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The Education and Examination Committee provides study notes to persons preparing for the examinations of the Society of Actuaries. They are intended to acquaint candidates with some of the theoretical and practical considerations involved in the various subjects. While varying opinions are presented where appropriate, limits on the length of the material and other considerations sometimes prevent the inclusion of all possible opinions. These study notes do not, however, represent any official opinion, interpretations or endorsement of the Society of Actuaries or its Education Committee. The Society is grateful to the authors for their contributions in preparing the study notes.

Purpose of this Supplementary Note

This note is provided as an accompaniment to *Financial Mathematics for Actuaries* by Wai-Sum Chan and Yiu-Kuen Tse, published by McGraw-Hill Education (Asia), 2011.

Financial Mathematics for Actuaries (FMA) is a textbook for students in Actuarial Science. It can be used for students preparing for the SOA Exam FM.

In this note we include some additional material required to meet the learning objectives of Exam FM. In particular, we provide an expanded version of Section 8.5 of FMA to cover the concepts of Redington and full immunization strategies.

Readers can replace Section 8.5 (pages 254-260) of FMA by this expanded version of Supplementary Note.

This note will be integrated with the FMA text formally in an "updated version" in late 2012 or early 2013.

This Supplementary Note can be downloaded freely from both authors' webpages:

- Wai-Sum Chan (http://ihome.cuhk.edu.hk/~b610769/FMASN.pdf)
- Yiu-Kuen Tse (http://www.mysmu.edu/faculty/yktse/FMASN.pdf)

Chapter 8

Bond Management

8.5 Immunization Strategies

Financial institutions are often faced with the problem of meeting a liability of a given amount some time in the future. For example, a corporation may be required to make a lump sum payment to its retirees one year later. To fix ideas, we consider a liability of amount V to be paid T periods later. A simple strategy to meet this obligation is to purchase a zero-coupon bond with face value V, which matures at time T. This strategy is called **cash-flow matching**. When cash-flow matching is adopted, the obligation is always met, even if there is fluctuation in the rate of interest.

However, zero-coupon bonds of the required maturity may not be available in the market. Institutions often fund their liabilities with coupon bonds, with a current value higher than or equal to the present value of their liabilities. However, when interest rate changes, the value of the bond will change, as well as the accumulation of the interest of the coupon payments. Consequently, the accumulated fund may not be able to meet the targeted liabilities.

Immunization is a strategy of managing a portfolio of assets such that the business is *immune* to interest-rate fluctuations. For the simple situation above, the **target-date immunization** strategy may be adopted. This involves holding a portfolio of bonds that will accumulate in value to V at time T at the current market rate of interest. The portfolio, however, should be constructed in such a way that its Macaulay duration D is equal to the targeted date of the liability T. Specifically, suppose the current yield rate is i, the current value of the portfolio of bonds, denoted by $P(i)$, must be

$$P(i) = \frac{V}{(1+i)^T}. \tag{8.20}$$

Now if the interest rate remains unchanged until time T, this bond portfolio will accumulate in value to V at the maturity date of the liability. If interest rate increases, the bond portfolio will drop in value. However, the coupon payments will generate higher interest and compensate for this. On the other hand, if interest rate drops, the bond portfolio value goes up, with subsequent slow-down in accumulation of interest. Under either situation, as we shall see, the bond portfolio value will finally accumulate to V at time T, provided the portfolio's Macaulay duration D is equal to T and there is only a one-time change in the rate of interest of a small amount.

We consider the bond value for a *one-time small change* in the rate of interest. If interest rate changes to $i + \Delta i$ immediately after the purchase of the bond, the bond price becomes $P(i+\Delta i)$ which, at time T, accumulates to $P(i+\Delta i)(1+i+\Delta i)^T$ if the rate of interest remains at $i+\Delta i$. We approximate $(1+i+\Delta i)^T$ to the first order in Δi to obtain (apply Taylor's expansion to $f(i) = (1+i)^T$)

$$(1+i+\Delta i)^T \approx (1+i)^T + T(1+i)^{T-1}\Delta i. \tag{8.21}$$

Using (8.10) and (8.21) we have

$$P(i+\Delta i)(1+i+\Delta i)^T \approx P(i)(1-D^*\Delta i)\left[(1+i)^T + T(1+i)^{T-1}\Delta i\right].$$

However, as $D^* = D/(1+i)$ and $T = D$, the above equation becomes

$$\begin{aligned} P(i+\Delta i)(1+i+\Delta i)^T &\approx P(i)\left[(1+i)^D - D^*\Delta i(1+i)^D + D(1+i)^{D-1}\Delta i\right] \\ &= P(i)(1+i)^D \\ &= V. \end{aligned} \tag{8.22}$$

Thus, for a small one-time change in interest rate, the bond accumulates to V at time T after the interest-rate change so that the business is immunized against interest-rate fluctuations.

Example 8.8: A company has to pay \$100 million 3.6761 years from now. The current market rate of interest is 5.5%. Demonstrate the funding strategy the company should adopt with the 6% annual coupon bond in Example 8.1. Consider the scenarios when there is an immediate one-time change in interest rate to (a) 5%, and (b) 6%.

Solution: From equation (8.20), the current value of the bond should be

$$\frac{100}{(1.055)^{3.6761}} = \$82.1338 \text{ million.}$$

From Example 8.1, the bond price is 101.7526% of the face value and the Macaulay duration is 3.6761 years, which is the target date for the payment. Hence, the bond purchased should have a face value of

$$\frac{82.13375}{1.017526} = \$80.7191 \text{ million.}$$

At the end of year 3, the accumulated value of the coupon payments is

$$80.7191 \times 0.06 s_{\overline{3}|0.055} = \$15.3432 \text{ million,}$$

and the bond price is (the bond will mature in 1 year with a 6% coupon payment and redemption payment of 80.7191)

$$\frac{80.7191 \times 0.06 + 80.7191}{1.055} = \$81.1017 \text{ million.}$$

Thus, the bond price plus the accumulated coupon values at time 3.6761 years is

$$(81.1017 + 15.3432)(1.055)^{0.6761} = \$100 \text{ million.}$$

Suppose interest rate drops to 5% immediately after the purchase of the bond, the accumulated coupon value 3 years later is

$$80.7191 \times 0.06 s_{\overline{3}|0.05} = \$15.2680 \text{ million,}$$

and the bond price at year 3 is

$$\frac{80.7191(1.06)}{1.05} = \$81.4879 \text{ million.}$$

The total of the bond value and the accumulated coupon payments at time 3.6761 years is

$$(81.4879 + 15.2680)(1.05)^{0.6761} = \$100 \text{ million.}$$

On the other hand, if the interest rate increases to 6% immediately after the purchase of the bond, the accumulated coupon value 3 years later is

$$80.7191 \times 0.06 s_{\overline{3}|0.06} = \$15.4186 \text{ million,}$$

and the bond price at year 3 is 80.7191 (this is a par bond with yield rate equal to coupon rate). Thus, the total of the bond value and the accumulated coupon payments at time 3.6761 years is

$$(80.7191 + 15.4186)(1.06)^{0.6761} = \$100 \text{ million.}$$

Thus, for an immediate one-time small change in interest rate, the bond accumulates to the targeted value of \$100 million at 3.6761 years, and the business is immunized. □

Example 8.9: A company has to pay \$100 million 4 years from now. The current market rate of interest is 5.5%. The company uses the 6% annual coupon bond in Example 8.1 to fund this liability. Is the bond sufficient to meet the liability when there is an immediate one-time change in interest rate to (a) 5%, and (b) 6%?

Solution: As the target date of the liability is 4 years and the Macaulay duration of the bond is 3.6761 years, there is a mismatch in the durations and the business is not immunized. To fund the liability in 4 years, the value of the bond purchased at time 0 is

$$\frac{100}{(1.055)^4} = \$80.7217 \text{ million,}$$

and the face value of the bond is

$$\frac{80.7217}{1.017526} = \$79.3313 \text{ million.}$$

If interest rate drops to 5%, the asset value at year 4 is

$$79.3313 \times 0.06 s_{\overline{4}|0.05} + 79.3313 = \$99.8470 \text{ million,}$$

so that the liability is under-funded. On the other hand, if the interest rate increases to 6%, the asset value at year 4 is

$$79.3313 \times 0.06 s_{\overline{4}\rceil 0.06} + 79.3313 = \$100.1539 \text{ million},$$

so that the liability is over-funded. □

Figure 8.4 describes the working of the target-date immunization strategy. The liability to be funded at time T is of amount V. At current rate of interest i, a bond of Macaulay duration $D = T$ and value of PV(V) is purchased. Suppose interest rate remains unchanged, the bond increases in value through Path I to V at time T. If interest rate drops, the value of the bond increases to above PV(V), but then increases in value at a slower rate and accumulates in value through Path II to V at time T. On the other hand, if interest rate increases, the value of the bond drops below PV(V), but will then increase in value at a faster rate through Path III, until it reaches V at time T.

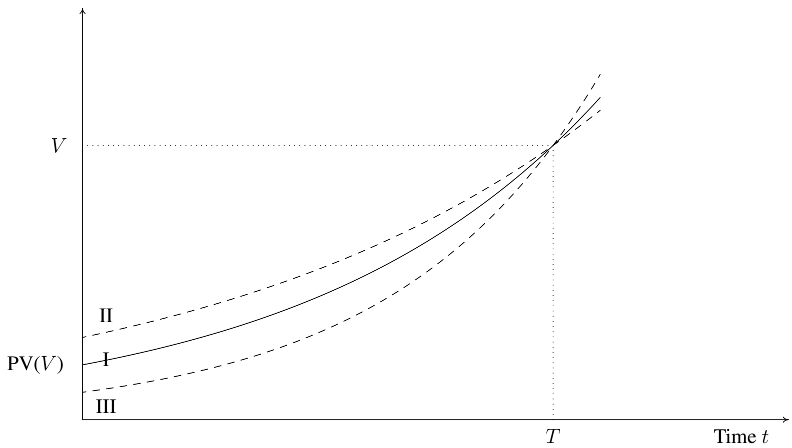

Figure 8.4: Illustration of target-date immunization

If a financial institution has multiple liability obligations to meet, the manager may adopt cash-flow matching to each obligation. This is a **dedication strategy** in which the manager selects a portfolio of bonds (zero-coupon or coupon bonds) to provide total cash flows in each period to match the required obligations. While this approach can eliminate interest-rate risk, it may be expensive to implement, or simply infeasible due to the constraints imposed on the selection of bonds. Alternatively, the manager may adopt target-date immunization for each liability obligation, matching the maturity date of the obligation with a bond (or portfolio of bonds) with the required duration and value.

The manager may also consider the liability obligations as a whole and construct a portfolio to fund these obligations, with the objective of controlling for the interest-rate risk. A commonly adopted strategy is **duration matching**. To fix ideas, we assume a financial institution has a stream of liabilities $L_1, L_2, \cdots, L_N$ to be paid out at various times in the future. It will fund these liabilities with assets generating cash flows

$A_1, A_2, \cdots, A_M$ at various times in the future. For example, an insurance company may expect to pay claims or policy redemptions of amounts $L_1, L_2, \cdots, L_N$, and will receive policy premiums and bond incomes of amounts $A_1, A_2, \cdots, A_M$. Also, a pension fund may expect to pay retirees pensions of various amounts at various times, and will fund these with a portfolio of bonds providing coupon payments and redemption values at various times. We assume that the rate of interest i is flat for cash flows of all maturities and applies to both assets and liabilities.

We denote

$$\text{PV(assets)} = \sum_{j=1}^{M} \text{PV}(A_j) = V_A, \tag{8.23}$$

and

$$\text{PV(liabilities)} = \sum_{j=1}^{N} \text{PV}(L_j) = V_L. \tag{8.24}$$

Although the financial institution may initially have a portfolio of assets such that $V_A > V_L$, the values of the assets and liabilities may change differently when interest rate changes. Specifically, if interest rate increases and the drop in value in V_A is more than that in V_L, the liabilities may not be sufficiently funded. Duration matching is the technique of matching assets with liabilities to neutralize the interest-rate risk.

We denote the Macaulay durations of the assets and liabilities by D_A and D_L, respectively. The duration matching strategy involves constructing a portfolio of assets such that the following conditions hold:

1. $V_A \geq V_L$
2. $D_A = D_L$.

The first condition ensures that the liabilities are initially sufficiently funded. It is the second condition that gives this strategy its name. Note that the net worth of the financial institution is $V_A - V_L$, which will be denoted by S. Condition 2 ensures that, to the first-order approximation, the asset-liability ratio will not drop when interest rate changes. This result can be deduced as follows:

$$\begin{aligned}
\frac{d}{di}\left(\frac{V_A}{V_L}\right) &= \frac{V_L \dfrac{dV_A}{di} - V_A \dfrac{dV_L}{di}}{V_L^2} \\
&= \frac{V_A}{V_L}\left(\frac{1}{V_A}\frac{dV_A}{di} - \frac{1}{V_L}\frac{dV_L}{di}\right) \\
&= \frac{V_A}{V_L(1+i)}(D_L - D_A) \\
&= 0.
\end{aligned} \tag{8.25}$$

Example 8.10: A financial institution has to pay \$1,000 after 2 years and \$2,000 after 4 years. The current market interest rate is 10%, and the yield curve is assumed to be flat at any time. The institution wishes to immunize the interest rate risk by purchasing zero-coupon bonds which mature after 1, 3 and 5 years. One member in the risk management team of the institution, Alan, devised the following strategy:

- Purchase a 1-year zero-coupon bond with a face value of \$44.74,
- Purchase a 3-year zero-coupon bond with a face value of \$2,450.83,
- Purchase a 5-year zero-coupon bond with a face value of \$500.00.

(a) Find the present value of the liability. (b) Show that Alan's portfolio satisfies the conditions of the duration matching strategy. (c) Define surplus $S = V_A - V_L$, calculate S when there is an immediate one-time change of interest rate from 10% to (i) 9%, (ii) 11%, (iii) 15%, (iv) 30% and (v) 80%. (d) Find the convexity of the portfolio of assets and the portfolio of liabilities at $i = 10\%$.

Solution: (a) The present value of the liabilities is

$$V_L = 1{,}000\ (1.1)^{-2} + 2{,}000\ (1.1)^{-4} = \$2{,}192.47.$$

For (b), the present value of Alan's asset portfolio is

$$V_A = 44.74\ (1.1)^{-1} + 2{,}450.83\ (1.1)^{-3} + 500.00\ (1.1)^{-5} = \$2{,}192.47.$$

The Macaulay duration of the assets and liabilities can be calculated using equation (8.3) to give

$$\begin{aligned} D_A &= \left[1 \times 44.74\ (1.1)^{-1} + 3 \times 2{,}450.83\ (1.1)^{-3} + 5 \times 500.00\ (1.1)^{-5}\right]/2{,}192.47 \\ &= 3.2461 \text{ years}, \end{aligned}$$

$$\begin{aligned} D_L &= \left[2 \times 1{,}000\ (1.1)^{-2} + 4 \times 2{,}000\ (1.1)^{-4}\right]/2{,}192.47 \\ &= 3.2461 \text{ years}. \end{aligned}$$

Since $V_A = V_L$ and $D_A = D_L$, the conditions of the duration matching strategy are met for Alan's portfolio.

For (c), when there is an immediate one-time shift in interest rate from 10% to 9%, using equation (8.2), we have

$$\begin{aligned} V_A &= 44.74\ (1.09)^{-1} + 2{,}450.83\ (1.09)^{-3} + 500.00\ (1.09)^{-5} = \$2{,}258.50, \\ V_L &= 1{,}000\ (1.09)^{-2} + 2{,}000\ (1.09)^{-4} = \$2{,}258.53, \\ S &= 2{,}258.50 - 2{,}258.53 = -\$0.03. \end{aligned}$$

We repeat the above calculations for interest rate of 11%, 15%, 30% and 80%. The results are summarized as follows:

Table 8.3 Results of Example 8.10

i	V_A	V_L	S
0.09	2,258.50	2,258.53	−0.03
0.10	2,192.47	2,192.47	0.00
0.11	2,129.05	2,129.08	−0.03
0.15	1,898.95	1,899.65	−0.70
0.30	1,284.61	1,291.97	−7.36
0.80	471.55	499.16	−27.61

For (d), using equation (8.15), the convexity of the assets is

$$\begin{aligned} C_A &= \frac{2 \times 1 \times 44.74\ (1.1)^{-1} + 4 \times 3 \times 2{,}450.83\ (1.1)^{-3} + 6 \times 5 \times 500.00\ (1.1)^{-5}}{(1.1)^2 \times 2{,}192.47} \\ &= 11.87, \end{aligned}$$

and the convexity of the liabilities is

$$\begin{aligned} C_L &= \left[3 \times 2 \times 1{,}000\ (1.1)^{-2} + 5 \times 4 \times 2{,}000\ (1.1)^{-4}\right] / [(1.1)^2 \times 2{,}192.47] \\ &= 12.17. \end{aligned}$$

□

Under duration matching, when there is an immediate one-time small shift in the interest rate, the surplus ($S = V_A - V_L$) should be preserved (i.e., non-negative). In Example 8.10, we notice that the values of assets and liabilities are matched at $i = 10\%$ with $V_A = V_L = \$2{,}192.47$. When there is an immediate one-time interest rate change, both asset and liability values move in the same direction. However, the rate of change of the value in the asset portfolio is slightly slower than that of the libability portfolio, always resulting in a deficit position. For example, when interest rate drops from 10% to 9%, the asset and liability values increase to \$2,258.50 and \$ 2,258.53, respectively. On the other hand, when interest rate shifts up to 15%, the asset and liability values decrease to \$1,898.95 and \$1,899.65, respectively, with a deficit position of $S = -\$0.70$.

It should be noted that the duration matching strategy is based on the first-order approximation. To improve the strategy, we may take into account the convexity of the asset and liability portfolios. Using second-order approximation, we re-consider the rate of change of the asset-liability ratio in equation (8.25) to obtain

$$\begin{aligned} \frac{d}{di}\left(\frac{V_A}{V_L}\right) &= \frac{V_A}{V_L}\left(\frac{1}{V_A}\frac{dV_A}{di} - \frac{1}{V_L}\frac{dV_L}{di}\right) \\ &= \frac{V_A}{V_L(1+i)}(D_L - D_A) + \frac{V_A}{V_L} \cdot \frac{1}{2}(C_A - C_L), \end{aligned} \tag{8.26}$$

where C_A and C_L are the convexity of the assets and liabilities, respectively. To protect the asset-liability ratio from dropping when interest rate changes, the **Redington immunization** strategy, named after the British actuary Frank Redington, imposes the following three conditions for constructing a portfolio of assets:

1. $V_A \geq V_L$
2. $D_A = D_L$
3. $C_A \geq C_L$.

These three conditions ensure that the right-hand side of equation (8.26) is always greater than or equal to zero under the second-order approximation. In Example 8.10, the convexity measures of the assets and the liabilities are $C_A = 11.87$ and $C_L = 12.17$, respectively. Since $C_A < C_L$, Alan's portfolio does not meet the conditions of the Redington immunization strategy.

Example 8.11: For the financial institution in Example 8.10, a risk consultant, Alfred, recommended the following strategy:

- Purchase a 1-year zero-coupon bond with a face value of \$154.16,
- Purchase a 3-year zero-coupon bond with a face value of \$2,186.04,
- Purchase a 5-year zero-coupon bond with a face value of \$660.18.

(a) Show that Alfred's portfolio satisfies the three conditions of the Redington immunization strategy. (b) Define surplus $S = V_A - V_L$, calculate S when there is an immediate one-time change of interest rate from 10% to (i) 9%, (ii) 11%, (iii) 15%, (iv) 30% and (v) 80%.

Solution: (a) The present value of Alfred's asset portfolio is

$$V_A = 154.16\ (1.1)^{-1} + 2{,}186.04\ (1.1)^{-3} + 660.18\ (1.1)^{-5} = \$2{,}192.47.$$

The Macaulay duration of the assets and liabilities can be calculated using equation (8.3) to give

$$\begin{aligned} D_A &= \left[1 \times 154.16\ (1.1)^{-1} + 3 \times 2{,}186.04\ (1.1)^{-3} + 5 \times 660.18\ (1.1)^{-5}\right] / 2{,}192.47 \\ &= 3.2461 \text{ years}, \end{aligned}$$

$$\begin{aligned} D_L &= \left[2 \times 1{,}000\ (1.1)^{-2} + 4 \times 2{,}000\ (1.1)^{-4}\right] / 2{,}192.47 \\ &= 3.2461 \text{ years}. \end{aligned}$$

Furthermore, using equation (8.15), we get

$$\begin{aligned} C_A &= \frac{2 \times 1 \times 154.16\ (1.1)^{-1} + 4 \times 3 \times 2{,}186.04\ (1.1)^{-3} + 6 \times 5 \times 660.18\ (1.1)^{-5}}{(1.1)^2 \times 2{,}192.47} \\ &= 12.17, \end{aligned}$$

$$\begin{aligned} C_L &= \left[3 \times 2 \times 1{,}000\ (1.1)^{-2} + 5 \times 4 \times 2{,}000\ (1.1)^{-4}\right] / [(1.1)^2 \times 2{,}192.47] \\ &= 12.17. \end{aligned}$$

Since $V_A = V_L$, $D_A = D_L$ and $C_A = C_L$, the conditions of the Redington immunization strategy are met for Alfred's strategy.

For (b), when there is an immediate one-time shift in interest rate from 10% to 9%, using equation (8.2), we have

$$\begin{aligned} V_A &= 154.16\,(1.09)^{-1} + 2{,}186.04\,(1.09)^{-3} + 660.18\,(1.09)^{-5} = \$2{,}258.53, \\ V_L &= 1{,}000\,(1.09)^{-2} + 2{,}000\,(1.09)^{-4} = \$2{,}258.53, \\ S &= 2{,}258.53 - 2{,}258.53 = 0. \end{aligned}$$

We repeat the above calculations for interest rate of 11%, 15%, 30% and 80%. The results are summarized as follows:

Table 8.4 Results of Example 8.11

i	V_A	V_L	S
0.09	2,258.53	2,258.53	0.00
0.10	2,192.47	2,192.47	0.00
0.11	2,129.08	2,129.08	0.00
0.15	1,899.64	1,899.65	−0.02
0.30	1,291.40	1,291.97	−0.57
0.80	495.42	499.16	−3.74

□

The Redington immunization strategy works well for a one-time *small* shift in interest rate. It protects the net-worth position of the financial institution. However, for a *radical* change of interest rate, the net-worth position might not be immunized under the Redington strategy. In Example 8.11, the financial institution would experience a deficit of \$3.74 if the rate of interest had an extreme sudden jump from 10% to 80%.

Under certain conditions, it is possible to construct a portfolio of assets such that the net-worth position of the financial institution is guaranteed to be non-negative in any positive interest rate environment. A **full immunization** strategy is said to be achieved if under any one-time shift of interest rate from i_0 to i,

$$S(i) = V_A(i) - V_L(i) \geq 0, \qquad \text{for } i > 0.$$

We consider the example of a single liability of amount L to be paid T_L periods later. Full immunization strategy involves funding the liability by a portfolio of assets which will produce two cash inflows. The first inflow of amount A_1 is located at time T_1, which is Δ_1 periods *before* time T_L. The second inflow of amount A_2 is at time T_2, which is Δ_2 periods *after* time T_L. Figure 8.5 illustrates these three cashflows. It should be noted that all the values of $i_0, i, A_1, A_2, L, \Delta_1, \Delta_2, T_0, T_1$ and T_2 are positive, and Δ_1 is not necessarily equal to Δ_2.

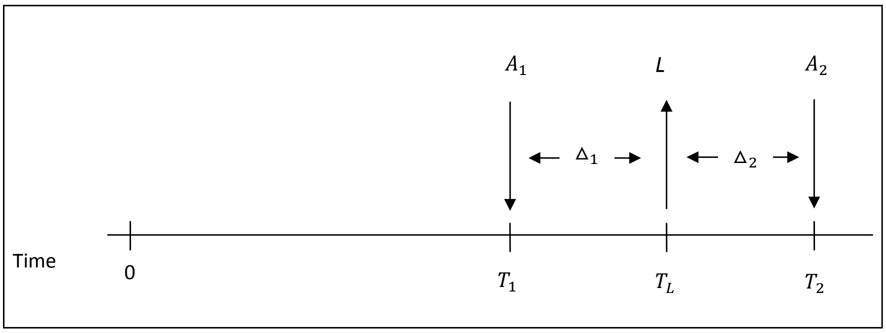

Figure 8.5 Illustration of cashflows in a full immunization strategy

In this particular example, the conditions for constructing a portfolio of assets under the full immunization strategy are:

1. $V_A = V_L$
2. $D_A = D_L$.

The above conditions can be rewritten as

1. $A_1(1+i_0)^{-T_1} + A_2(1+i_0)^{-T_2} = L(1+i_0)^{-T_L}$
2. $T_1A_1(1+i_0)^{-T_1} + T_2A_2(1+i_0)^{-T_2} = T_LL(1+i_0)^{-T_L}$.

The proof of full immunization (i.e., $S(i) \geq 0$, for $i > 0$) under the above two conditions are demonstrated in Exercise 8.41 at the end of this Chapter.

The full immunization strategy can be applied to financial institutions with multiple liabilities. We can simply deal with the liabilities one by one, and construct a pair of assets to immunize each of them.

Example 8.12: For the financial institution in Examples 8.10 and 8.11, an actuary, Albert, constructed the following strategy:

- Purchase a 1-year zero-coupon bond with a face value of \$454.55,
- Purchase a 3-year zero-coupon bond with a face value of \$1,459.09,
- Purchase a 5-year zero-coupon bond with a face value of \$1,100.00.

(a) Show that Albert's portfolio satisfies the conditions of the full immunization strategy.
(b) Define surplus $S = V_A - V_L$, calculate S when there is an immediate one-time change of interest rate from 10% to (i) 9%, (ii) 11%, (iii) 15%, (iv) 30% and (v) 80%.

Solution: (a) The present value of Albert's asset portfolio is

$$V_A = 454.55\ (1.1)^{-1} + 1{,}459.09\ (1.1)^{-3} + 1{,}100.00\ (1.1)^{-5} = \$2{,}192.47.$$

Let A_1 and A_2 be the amount of 1-year and 3-year zero-coupon bonds that are needed to fully immunize the first liability of $L = 1{,}000$. Note that $T_1 = 1, T_L = 2$ and $T_2 = 3$. The two conditions for the full immunization strategy require

$$\begin{aligned} A_1(1.1)^{-1} + A_2(1.1)^{-3} &= 1{,}000(1.1)^{-2} \\ (1)A_1(1.1)^{-1} + (3)A_2(1.1)^{-3} &= (2)1{,}000(1.1)^{-2}. \end{aligned}$$

Solving the above system of equations, we get $A_1 = 454.55$ and $A_2 = 550.00$. Next, let A_1^* and A_2^* be the amounts of 3-year and 5-year zero-coupon bonds that would be needed to fully immunize the second liability $L = 2{,}000$. Note that now $T_1 = 3, T_L = 4$ and $T_2 = 5$. The two conditions for the full immunization strategy require

$$\begin{aligned} A_1^*(1.1)^{-3} + A_2^*(1.1)^{-5} &= 2{,}000(1.1)^{-4} \\ (3)A_1^*(1.1)^{-3} + (5)A_2^*(1.1)^{-5} &= (4)1{,}000(1.1)^{-4}. \end{aligned}$$

Solving the above system of equations, we get $A_1^* = 909.09$ and $A_2^* = 1,100.00$. The combined asset portfolio consists of a 1-year zero-coupon bond with a face value of \$454.55, a 3-year zero-coupon bond with a face value of (\$550.00 + \$909.09) = \$1,459.09 and a 5-year zero-coupon bond with a face value of \$1,100.00. This is indeed Albert's asset portfolio, which satisfies the full immunization conditions.

For (b), when there is an immediate one-time shift in interest rate from 10% to 9%, using equation (8.2), we have

$$\begin{aligned} V_A &= 454.55\ (1.09)^{-1} + 1{,}459.09\ (1.09)^{-3} + 1{,}100.00\ (1.09)^{-5} = \$2{,}258.62, \\ V_L &= 1{,}000\ (1.09)^{-2} + 2{,}000\ (1.09)^{-4} = \$2{,}258.53, \\ S &= 2{,}258.62 - 2{,}258.53 = 0.09. \end{aligned}$$

We repeat the above calculations for interest rate of 11%, 15%, 30% and 80%. The results are summarized as follows:

Table 8.5 Results of Example 8.12

i	V_A	V_L	S
0.09	2,258.62	2,258.53	0.09
0.10	2,192.47	2,192.47	0.00
0.11	2,129.17	2,129.08	0.09
0.15	1,901.53	1,899.65	1.88
0.30	1,310.04	1,291.97	18.07
0.80	560.93	499.16	61.76

□

To compare the duration matching, Redington and full immunization strategies, the results of Examples 8.10, 8.11 and 8.12 are plotted in Figure 8.6. The Redington strategy is an improved version of the duration matching method, and it ensures a relatively stable net-worth position when there is an immediate one-time small shift of the rate of interest. The full immunization strategy guarantees a non-negative surplus position regardless the size of the interest rate change. However, the surplus after the shift could be highly volatile. The cost of implementation of the immunization strategy, the availability of the required assets and the objectives of surplus management are the main factors for a financial institution to select an appropriate immunization strategy for its business. We also stress that the success of the immunization strategies discussed in this section depends on the assumption that the term structure is flat, and that the yield curve shift is parallel.

Finally, it should be noted that there might be numerous ways to design an asset portfolio to immunize a given liability portfolio against interest rate risk under each of the immunization strategies discussed in this subsection. For example, instead of employing 1-year and 3-year zero-coupon bonds to immunize the first liability in Example 8.12, one may use 1-year and 5-year bonds. It will lead to an alternative asset portfolio that also satisfies the full immunization conditions. In addition, while we have stated the duration matching, Redington immunization and full immunization conditions in terms of Macaulay durations, these conditions can be equivalently stated in terms of modified durations.

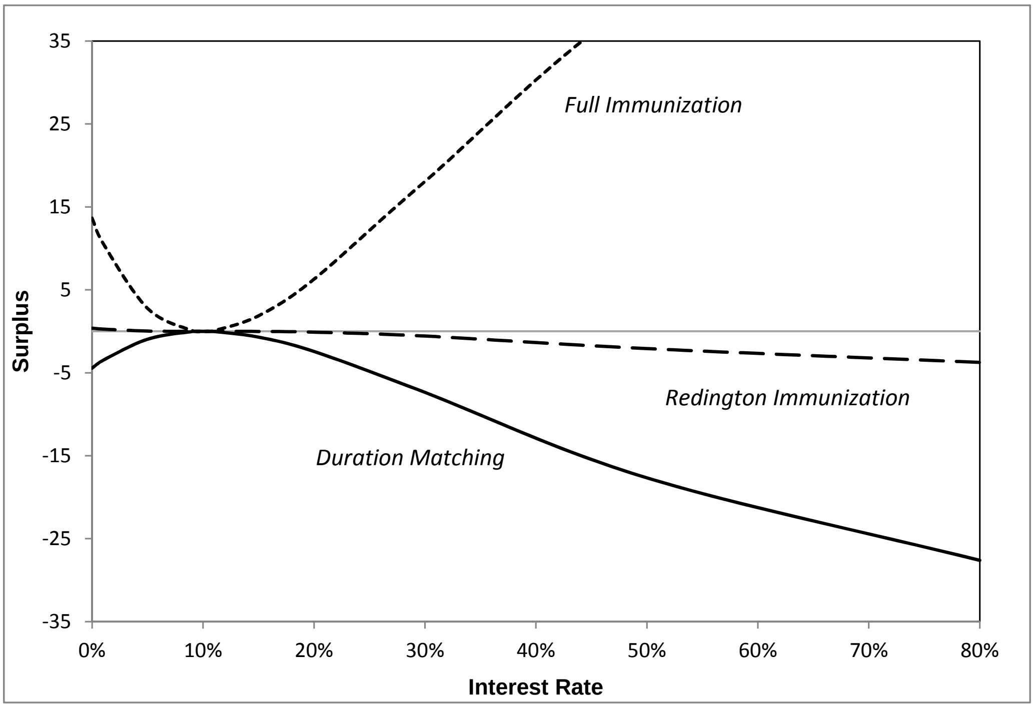

Figure 8.6 Surplus positions under various immunization strategies

Additional Exercise

8.41 Consider the full immunization conditions and the corresponding cash-flow diagram in Figure 8.5.

(a) Show that the two full immunization conditions can be expressed as:

$$\begin{aligned} A_1(1+i_0)^{-T_1} + A_2(1+i_0)^{-T_2} &= L(1+i_0)^{-T_L} \\ T_1A_1(1+i_0)^{-T_1} + T_2A_2(1+i_0)^{-T_2} &= T_LL(1+i_0)^{-T_L} \end{aligned}$$

(b) Based on the system of equations in (a), show that

$$A_1 = \left(\frac{\Delta_2}{\Delta_1}\right) A_2\,(1+i_0)^{-(\Delta_1+\Delta_2)}$$

(c) Substitute the result in (b) into the first condition in (a), show that

$$L = A_2(1+i_0)^{-\Delta_2}\left(\frac{\Delta_2}{\Delta_1}+1\right)$$

(d) Let $S(i) = V_A - V_L$, at interest rate i, show that

$$S(i) = C(i) \times \eta(i)$$

where $C(i)$ is a positive function of i, and

$$\eta(i) = \left[\left(\frac{\Delta_2}{\Delta_1}\right)\left(\frac{1+i}{1+i_0}\right)^{\Delta_1} + \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2} - \left(\frac{\Delta_2}{\Delta_1}+1\right)\right]$$

(e) Show that

$$\eta'(i) = \frac{d\eta(i)}{di} = \frac{\Delta_2}{1+i}\left[\left(\frac{1+i}{1+i_0}\right)^{\Delta_1} - \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2}\right]$$

(f) When $0 < i < i_0$, show that $\eta(i)$ decreases when i increases. When $i > i_0 > 0$, show that $\eta(i)$ increases when i increases.

(g) Show that $\eta(i)$ achieves an absolute minimum at $i = i_0$ and $\eta(i_0) = 0$.

(h) Given the conditions in (a) and by combining the results from (b) to (g), show that $S(i) = V_A - V_L \geq 0$, for all $i > 0$.

Solution to Additional Exercise

8.41 (a) The first condition of the full immunization strategy is $V_A = V_L$, and as

$$\begin{aligned} V_A &= A_1(1+i_0)^{-T_1} + A_2(1+i_0)^{-T_2} \\ V_L &= L(1+i_0)^{-T_L}, \end{aligned}$$

the two conditions in (a) are exactly the same as the two conditions of the full immunization strategy.

(b) Multipling T_L to both the left-hand side and right-hand side of the first condition in (a), we have

$$T_L A_1(1+i_0)^{-T_1} + T_L A_2(1+i_0)^{-T_2} = T_L L(1+i_0)^{-T_L}.$$

Substituting the above result into the second condition in (a), we get

$$T_1 A_1(1+i_0)^{-T_1} + T_2 A_2(1+i_0)^{-T_2} = T_L A_1(1+i_0)^{-T_1} + T_L A_2(1+i_0)^{-T_2},$$

so that

$$A_1(T_L - T_1)(1+i_0)^{-T_1} = A_2(T_2 - T_L)(1+i_0)^{-T_2}.$$

Note that $\Delta_1 = (T_L - T_1), \Delta_2 = (T_2 - T_L), \Delta_1 + \Delta_2 = (T_2 - T_1)$. Hence,

$$A_1 = \left(\frac{\Delta_2}{\Delta_1}\right) A_2 \,(1+i_0)^{-(\Delta_1+\Delta_2)}.$$

(c) Substituting the result in (b) into the first equation in (a), we get

$$\begin{aligned} L(1+i_0)^{-T_L} &= A_1(1+i_0)^{-T_1} + A_2(1+i_0)^{-T_2} \\ &= \left(\frac{\Delta_2}{\Delta_1}\right) A_2 \,(1+i_0)^{-(\Delta_1+\Delta_2)-T_1} + A_2(1+i_0)^{-T_2}, \end{aligned}$$

so that

$$\begin{aligned} L &= \left(\frac{\Delta_2}{\Delta_1}\right) A_2 \,(1+i_0)^{-(\Delta_1+\Delta_2)-T_1+T_L} + A_2(1+i_0)^{-T_2+T_L} \\ &= A_2 \left[\left(\frac{\Delta_2}{\Delta_1}\right)(1+i_0)^{-(\Delta_1+\Delta_2)+\Delta_1} + (1+i_0)^{-\Delta_2}\right] \\ &= A_2(1+i_0)^{-\Delta_2}\left(\frac{\Delta_2}{\Delta_1} + 1\right). \end{aligned}$$

(d) Using the results in (b) and (c) and substituting them into the definition of

$S(i)$, we obtain

$$\begin{aligned}
S(i) &= V_A - V_L \\
&= A_1(1+i)^{-T_1} + A_2(1+i)^{-T_2} - L(1+i)^{-T_L} \\
&= \left(\frac{\Delta_2}{\Delta_1}\right) A_2\,(1+i_0)^{-(\Delta_1+\Delta_2)}\,(1+i)^{-T_1} + A_2(1+i)^{-T_2} \\
&\quad - A_2(1+i_0)^{-\Delta_2}\left(\frac{\Delta_2}{\Delta_1}+1\right)(1+i)^{-T_L} \\
&= A_2(1+i)^{-T_L}\left[\left(\frac{\Delta_2}{\Delta_1}\right)(1+i_0)^{-(\Delta_1+\Delta_2)}\,(1+i)^{-T_1+T_L} + (1+i)^{-T_2+T_L}\right. \\
&\quad \left. - (1+i_0)^{-\Delta_2}\left(\frac{\Delta_2}{\Delta_1}+1\right)\right] \\
&= A_2(1+i)^{-T_L}\left[\left(\frac{\Delta_2}{\Delta_1}\right)(1+i_0)^{-(\Delta_1+\Delta_2)}\,(1+i)^{\Delta_1} + (1+i)^{-\Delta_2}\right. \\
&\quad \left. - (1+i_0)^{-\Delta_2}\left(\frac{\Delta_2}{\Delta_1}+1\right)\right] \\
&= A_2(1+i)^{-T_L}(1+i_0)^{-\Delta_2}\left[\left(\frac{\Delta_2}{\Delta_1}\right)\left(\frac{1+i}{1+i_0}\right)^{\Delta_1}\right. \\
&\qquad \left. + \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2} - \left(\frac{\Delta_2}{\Delta_1}+1\right)\right] \\
&= C(i)\times\eta(i),
\end{aligned}$$

where

$$C(i) = A_2(1+i)^{-T_L}(1+i_0)^{-\Delta_2}$$

is a positive function of i.

(e) Note that

$$\begin{aligned}
\eta'(i) &= \frac{d}{di}\left[\left(\frac{\Delta_2}{\Delta_1}\right)\left(\frac{1+i}{1+i_0}\right)^{\Delta_1} + \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2} - \left(\frac{\Delta_2}{\Delta_1}+1\right)\right] \\
&= \left(\frac{\Delta_2}{\Delta_1}\right)\frac{\Delta_1}{(1+i_0)^{\Delta_1}}(1+i)^{\Delta_1-1} + \frac{-\Delta_2}{(1+i_0)^{-\Delta_2}}(1+i)^{-\Delta_2-1} \\
&= \frac{\Delta_2}{1+i}\left[\left(\frac{1+i}{1+i_0}\right)^{\Delta_1} - \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2}\right].
\end{aligned}$$

(f) When $0 < i < i_0$,

$$\left(\frac{1+i}{1+i_0}\right) < 1 \quad \text{implies} \quad \left(\frac{1+i}{1+i_0}\right)^{\Delta_1} < 1 \quad \text{for all positive } \Delta_1.$$

$$\left(\frac{1+i}{1+i_0}\right) < 1 \text{ implies } \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2} > 1 \quad \text{for all positive } \Delta_2.$$

Therefore, $\eta'(i) < 0$ when $0 < i < i_0$, and $\eta(i)$ is a decreasing function for $0 < i < i_0$. Similarly, we can show that $\eta(i)$ is an increasing function for $i > i_0 > 0$.

(g) Setting $\eta'(i) = 0$, for $\Delta_1 > 0, \Delta_2 > 0, i_0 > 0$, and $i > 0$, we have

$$\left(\frac{1+i}{1+i_0}\right)^{\Delta_1} = \left(\frac{1+i}{1+i_0}\right)^{-\Delta_2},$$

so that

$$(1+i)^{\Delta_1+\Delta_2} = (1+i_0)^{\Delta_1+\Delta_2}.$$

Hence, $\eta(i)$ achieves a minimum at $i = i_0$. When $i = i_0$, $\eta(i)$ in (d) is equal to

$$\begin{aligned}\eta(i_0) &= \left[\left(\frac{\Delta_2}{\Delta_1}\right)\left(\frac{1+i_0}{1+i_0}\right)^{\Delta_1} + \left(\frac{1+i_0}{1+i_0}\right)^{-\Delta_2} - \left(\frac{\Delta_2}{\Delta_1}+1\right)\right] \\ &= 0.\end{aligned}$$

(h) Given the conditions in (a) and combining the results from (b) to (g), we have $S(i) = C(i) \times \eta(i)$ and $C(i)$ is a positive function. The $\eta(i)$ function is decreasing for $0 < i < i_0$ and increasing for $i > i_0 > 0$ with an absolute minimum at $\eta(i_0) = 0$. Therefore, $S(i) \geq 0$, for all $i > 0$.