OPTION PRICING BY ESSCHER TRANSFORMS

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ABSTRACT

The Esscher transform is a time-honored tool in actuarial science. This paper shows that the Esscher transform is also an efficient technique for valuing derivative securities if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments. This family of processes includes the Wiener process, the Poisson process, the gamma process, and the inverse Gaussian process. An Esscher transform of such a stock-price process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primitive security is a martingale under the new probability measure. The price of any derivative security is simply calculated as the expectation, with respect to the equivalent martingale measure, of the discounted payoffs. Straightforward consequences of the method of Esscher transforms include, among others, the celebrated Black-Scholes option-pricing formula, the binomial option-pricing formula, and formulas for pricing options on the maximum and minimum of multiple risky assets. Tables of numerical values for the prices of certain European call options (calculated according to four different models for stock-price movements) are also provided.

1. INTRODUCTION

The Esscher transform [35] is a time-honored tool in actuarial science. Members of the Society of Actuaries were introduced to it by Kahn's survey paper [51] and Wooddy's Study Note [79]. In this paper we show that the Esscher transform is also an efficient technique for valuing derivative securities if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments. This family of processes includes the Wiener process, the Poisson process, the gamma process, and the inverse Gaussian process. Our modeling of stock-price movements by means of the gamma process and the inverse Gaussian process seems to be new. Straightforward consequences of the proposed method include, among

others, the celebrated Black-Scholes option-pricing formula, the bino-
mial option-pricing formula, and formulas for pricing options on the
maximum and minimum of multiple risky assets.

For a probability density function $f(x)$, let h be a real number such
that

$$M(h) = \int_{-\infty}^{\infty} e^{hx} f(x)\, dx$$

exists. As a function in x,

$$f(x; h) = \frac{e^{hx} f(x)}{M(h)}$$

is a probability density, and it is called the Esscher transform (parameter
h) of the original distribution. The Esscher transform was developed to
approximate the aggregate claim amount distribution around a point of
interest, x_0, by applying an analytic approximation (the Edgeworth se-
ries) to the transformed distribution with the parameter h chosen such
that the new mean is equal to x_0. When the Esscher transform is used
to calculate a stop-loss premium, the parameter h is usually determined
by specifying the mean of the transformed distribution as the retention
limit. Further discussions and details on the method of Esscher trans-
forms can be found in risk theory books such as [6], [7], [27], [38], and
[70]; see also Jensen's paper [49].

In this paper we show that the Esscher transform can be extended readily
to a certain class of stochastic processes, which includes some of those
commonly used to model stock-price movements. The parameter h is
determined so that the modified probability measure is an equivalent
martingale measure, with respect to which the prices of securities are
expected discounted payouts.

Our first application of the method of Esscher transforms is formula
(2.15), which is a general expression for the value of a European call
option on a non-dividend-paying stock and includes the Black-Scholes
option-pricing formula, the pure-jump option-pricing formula, and the
binomial option-pricing formula as special cases. We also introduce two
new models for stock-price movements; the first one is defined in terms
of the gamma process and the second in terms of the inverse Gaussian
process. Formulas for pricing European call options on stocks with such

price movements are also given, and numerical tables (calculated according to four different models) are provided.

In the second half of this paper, we extend the method of Esscher transforms to price derivative securities of multiple risky assets or asset pools. The main result is as follows: Assume that the risk-free force of interest is constant and denote it by δ. For $t \geq 0$, let $S_1(t)$, $S_2(t)$, ..., $S_n(t)$ denote the prices of n non-dividend-paying stocks or assets at time t. Assume that the vector

$$(\ln[S_1(t)/S_1(0)], \ln[S_2(t)/S_2(0)], ..., \ln[S_n(t)/S_n(0)])'$$

is governed by a stochastic process that has independent and stationary increments and that is continuous in probability. Let g be a real-valued measurable function of n variables. Then, for $\tau \geq 0$,

$$E^*[e^{-\delta\tau}S_j(\tau)g(S_1(\tau), S_2(\tau), ..., S_n(\tau))]$$
$$= S_j(0)E^{**}[g(S_1(\tau), S_2(\tau), ..., S_n(\tau))],$$

where the expectation on the left-hand side is taken with respect to the risk-neutral Esscher transform and the expectation on the right-hand side is taken with respect to another specified Esscher transform. It is shown that many classical option-pricing formulas are straightforward consequences of this result.

A useful introduction to the subject of options and other derivative securities can be found in Boyle's book [15], which was published recently by the Society of Actuaries. Kolb's book [52] is a collection of 44 articles on derivative securities by various authors; most of these articles are descriptive and not mathematical. For an intellectual history of option-pricing theory, see Chapter 11 of Bernstein's book [9].

In this paper the risk-free interest rate is assumed to be constant. We also assume that the market is frictionless and trading is continuous. There are no taxes, no transaction costs, and no restriction on borrowing or short sales. All securities are perfectly divisible. It is now understood that, in such a securities market model, the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure, with respect to which the price of a random payment is the expected discounted value. Some authors ([5], [34], [67]) call this result the "Fundamental Theorem of Asset Pricing." In a general setting, the equivalent martingale measure is not unique; the merit of the risk-neutral Esscher

transform is that it provides a general, transparent and unambiguous solution.

In the next section we use some basic ideas from the theory of stochastic processes. Two standard references are Breiman's book [18] and Feller's book [36].

2. RISK-NEUTRAL ESSCHER TRANSFORM

For $t \geq 0$, $S(t)$ denotes the price of a non-dividend-paying stock or security at time t. We assume that there is a stochastic process, $\{X(t)\}_{t \geq 0}$, with stationary and independent increments, $X(0)=0$, such that

$$S(t) = S(0)e^{X(t)}, \qquad t \geq 0. \tag{2.1}$$

For each t, the random variable $X(t)$, which may be interpreted as the continuously compounded rate of return over the t periods, has an infinitely divisible distribution [18, Proposition 14.16]. Let

$$F(x, t) = \Pr[X(t) \leq x] \tag{2.2}$$

be its cumulative distribution function, and

$$M(z, t) = E[e^{zX(t)}] \tag{2.3}$$

its moment-generating function. By assuming that $M(z, t)$ is continuous at $t=0$, it can be proved that

$$M(z, t) = [M(z, 1)]^t \tag{2.4}$$

([18, Section 14.4], [36, Section IX.5]). We assume that (2.4) holds.

For simplicity, let us assume that the random variable $X(t)$ has a density

$$f(x, t) = \frac{d}{dx} F(x, t), \quad t > 0;$$

then

$$M(z, t) = \int_{-\infty}^{\infty} e^{zx} f(x, t) \, dx.$$

Let h be a real number for which $M(h, t)$ exists. (It follows from (2.4) that, if $M(h, t)$ exists for one positive number t, it exists for all positive t.) We now introduce the Esscher transform (parameter h) of the process

$\{X(t)\}$. This is again a process with stationary and independent increments, whereby the new probability density function of $X(t)$, $t>0$, is

$$f(x, t; h) = \frac{e^{hx} f(x, t)}{\displaystyle\int_{-\infty}^{\infty} e^{hy} f(y, t)\, dy}$$

$$= \frac{e^{hx} f(x, t)}{M(h, t)}. \qquad (2.5)$$

That is, the modified distribution of $X(t)$ is the Esscher transform of the original distribution. The corresponding moment-generating function is

$$M(z, t; h) = \int_{-\infty}^{\infty} e^{zx} f(x, t; h)\, dx$$

$$= \frac{M(z + h, t)}{M(h, t)}. \qquad (2.6)$$

By (2.4),

$$M(z, t; h) = [M(z, 1; h)]^t. \qquad (2.7)$$

The Esscher transform of a single *random variable* is a well-established concept in the risk theory literature. Here, we consider the Esscher transform of a *stochastic process*. In other words, the probability measure of the process has been modified. Because the exponential function is positive, the modified probability measure is *equivalent* to the original probability measure; that is, both probability measures have the same null sets (sets of probability measure zero).

We want to ensure that the stock prices of the model are internally consistent. Thus we seek $h=h^*$, so that the discounted stock price process, $\{e^{-\delta t}S(t)\}_{t\geq 0}$, is a martingale with respect to the probability measure corresponding to h^*. In particular,

$$S(0) = E^*[e^{-\delta t} S(t)]$$

$$= e^{-\delta t} E^*[S(t)],$$

where δ denotes the constant risk-free force of interest. By (2.1), the parameter h^* is the solution of the equation

$$1 = e^{-\delta t} \, E^*[e^{X(t)}],$$

or

$$e^{\delta t} = M(1, t; h^*).$$ (2.8)

From (2.7) we see that the solution does not depend on t, and we may set $t = 1$:

$$e^{\delta} = M(1, 1; h^*),$$ (2.9)

or

$$\delta = \ln[M(1, 1; h^*)].$$ (2.10)

It can be shown that the parameter h^* is unique [40]. We call the Esscher transform of parameter h^* the *risk-neutral Esscher transform*, and the corresponding equivalent martingale measure the *risk-neutral Esscher measure*. Note that, although the risk-neutral Esscher measure is unique, there may be other equivalent martingale measures; see the paper by Delbaen and Haezendonck [30] for a study on equivalent martingale measures of compound Poisson processes.

To evaluate a derivative security (whose future payments depend on the evolution of the stock price), we calculate the expected discounted value of the implied payments; the expectation is with respect to the risk-neutral Esscher measure. Let us consider a European call option on the stock with exercise price K and exercise date τ, $\tau > 0$. The value of this option (at time 0) is

$$E^*[e^{-\delta\tau} (S(\tau) - K)_+],$$ (2.11)

where $x_+ = x$ if $x > 0$, and $x_+ = 0$ if $x \leq 0$. With the definition

$$\kappa = \ln[K/S(0)],$$ (2.12)

(2.11) becomes

$$e^{-\delta\tau} \int_\kappa^\infty [S(0)e^x - K] f(x, \tau; h^*) \, dx$$

$$= e^{-\delta\tau} S(0) \int_\kappa^\infty e^x f(x, \tau; h^*) \, dx - e^{-\delta\tau} K[1 - F(\kappa, \tau; h^*)].$$ (2.13)

It follows from (2.5), (2.6) and (2.8) that

$$e^x f(x, \tau; h^*) = \frac{e^{(h^* + 1)x} f(x, \tau)}{M(h^*, \tau)}$$

$$= \frac{M(h^* + 1, \tau)}{M(h^*, \tau)} f(x, \tau; h^* + 1)$$

$$= M(1, \tau; h^*) f(x, \tau; h^* + 1)$$

$$= e^{\delta\tau} f(x, \tau; h^* + 1). \tag{2.14}$$

Thus the value of the European call option with exercise price K and exercise date τ is

$$S(0)[1 - F(\kappa, \tau; h^* + 1)] - e^{-\delta\tau} K[1 - F(\kappa, \tau; h^*)]. \tag{2.15}$$

In Sections 3 and 4, this general formula is applied repeatedly. It is shown that (2.15) contains, among others, the celebrated Black-Scholes option-pricing formula as a special case.

2.1 Remarks

In the general case in which the distribution function $F(x, t)$ is not necessarily differentiable, we can define the Esscher transform in terms of Stieltjes integrals. That is, we replace (2.5) by

$$dF(x, t; h) = \frac{e^{hx} dF(x, t)}{\displaystyle\int_{-\infty}^{\infty} e^{hy} dF(y, t)}$$

$$= \frac{e^{hx} dF(x, t)}{M(h, t)}. \tag{2.1.1}$$

(In his paper [35] Esscher did not assume that the individual claim amount distribution function is differentiable.) Formula (2.15) remains valid.

That the condition of no arbitrage is intimately related to the existence of an equivalent martingale measure was first pointed out by Harrison and Kreps [42] and by Harrison and Pliska [43]. Their results are rooted in the idea of risk-neutral valuation of Cox and Ross [24]. For an insightful introduction to the subject, see Duffie's recent book [32]. In a finite discrete-time model, the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure ([28],

[67]). In a more general setting, the characterization is more delicate, and we have to replace the term "equivalent to" by "essentially equivalent to." Discussion of the details is beyond the scope of this paper; some recent papers are [4], [5], [23], [29], [44], [53], [59], [68], and [69].

The idea of changing the probability measure to obtain a consistent positive linear pricing rule had appeared in the actuarial literature in the context of equilibrium reinsurance markets ([12], [13], [19], [20], [39], and [73]); see also [77], [2], and [78].

Observe that the option-pricing formula (2.15) can be written as

$$S(0)\Pr[S(\tau) > K; h^* + 1] - e^{-\delta\tau} K\Pr[S(\tau) > K; h^*],$$

where the first probability is evaluated with respect to the Esscher transform with parameter $h^* + 1$, while the second probability is calculated with respect to the risk-neutral Esscher transform. Generalizations of this result are given in Section 6.

To construct a stochastic process $\{X(t)\}$ with stationary and independent increments, $X(0)=0$, and

$$M(z, t) = [M(z, 1)]^t,$$

we can apply the following theorem [18, Proposition 14.19]: Given the moment-generating function $\zeta(z)$ of an infinitely divisible distribution, there is a unique stochastic process $\{W(t)\}$ with stationary and independent increments, $W(0)=0$, such that

$$E[e^{zW(t)}] = [\zeta(z)]^t.$$

The normal distribution, the Poisson distribution, the gamma distribution, and the inverse Gaussian distribution are four examples of infinitely divisible distributions. In the following sections, we consider stock-price movements modeled with such processes.

3. THREE CLASSICAL OPTION FORMULAS

In this section we apply the results of Section 2 to derive European call option formulas in three classical models for stock-price movements. These three formulas can be found in textbooks on options, such as those by Cox and Rubinstein [26], Gibson [41] and Hull [47]. Note that Hull's book [47] is a textbook for the Society of Actuaries Course F-480 examination.

3.1 Logarithm of Stock Price as a Wiener Process

Here we make the classical assumption that the stock prices are log-normally distributed. Let the stochastic process $\{X(t)\}$ be a Wiener process with mean per unit time μ and with variance per unit time σ^2. Let $N(x; \mu, \sigma^2)$ denote the normal distribution function with mean μ and variance σ^2. Then

$$F(x, t) = N(x; \mu t, \sigma^2 t)$$

and

$$M(z, t) = \exp[(\mu z + \tfrac{1}{2} \sigma^2 z^2)t].$$

It follows from (2.6) that

$$M(z, t; h) = \exp\{[(\mu + h\sigma^2)z + \tfrac{1}{2} \sigma^2 z^2]t\}.$$

Hence the Esscher transform (parameter h) of the Wiener process is again a Wiener process, with modified mean per unit time

$$\mu + h\sigma^2$$

and unchanged variance per unit time σ^2. Thus

$$F(x, t; h) = N(x; (\mu + h\sigma^2)t, \sigma^2 t).$$

From (2.10) we obtain

$$\delta = (\mu + h^*\sigma^2) + \tfrac{1}{2} \sigma^2.$$

Consequently, the transformed process has mean per unit time

$$\mu^* = \mu + h^*\sigma^2$$

$$= \delta - (\sigma^2/2). \tag{3.1.1}$$

It now follows from (2.15) that the value of the European call option is

$$S(0)[1 - N(\kappa; (\mu^* + \sigma^2)\tau, \sigma^2\tau)] - e^{-\delta\tau} K[1 - N(\kappa; \mu^*\tau, \sigma^2\tau)]$$

$$= S(0)[1 - N(\kappa; (\delta + \tfrac{1}{2} \sigma^2)\tau, \sigma^2\tau)]$$

$$- e^{-\delta\tau} K[1 - N(\kappa; (\delta - \tfrac{1}{2} \sigma^2)\tau, \sigma^2\tau)]. \tag{3.1.2}$$

In terms of the standard normal distribution function Φ, this result can be expressed as

$$S(0)\Phi\left(\frac{-\kappa + (\delta + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - e^{-\delta\tau}K\Phi\left(\frac{-\kappa + (\delta - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right), \qquad (3.1.3)$$

which is the classical Black-Scholes option-pricing formula [11]. Note that μ does not appear in (3.1.3).

3.2 Logarithm of Stock Price as a Shifted Poisson Process

Next we consider the so-called pure jump model. The pricing of options on stocks with such stochastic movements was discussed by Cox and Ross [24]; however, they did not provide an option-pricing formula. The option-pricing formula for this model appeared several years later in the paper by Cox, Ross and Rubinstein [25, p. 255]; it was derived as a limiting case of the binomial option-pricing formula. (We deduce the binomial option-pricing formula by the Esscher transform method in Section 3.3.) A more thorough discussion of the derivation can be found in the paper by Page and Sanders [61].

Here the assumption is that

$$X(t) = kN(t) - ct, \qquad (3.2.1)$$

where $\{N(t)\}$ is a Poisson process with parameter λ, and k and c are positive constants. Let

$$\Lambda(x; \theta) = \sum_{0 \le j \le x} \frac{e^{-\theta}\theta^j}{j!}$$

be the cumulative Poisson distribution function with parameter θ. Then the cumulative distribution function of $X(t)$ is

$$F(x, t) = \Lambda\left(\frac{x + ct}{k}; \lambda t\right). \qquad (3.2.2)$$

Since

$$E[e^{zN(t)}] = \exp[\lambda t(e^z - 1)],$$

we have

$$\begin{aligned}
M(z, t) &= E(e^{z[kN(t)-ct]}) \\
&= e^{[\lambda(e^{zk}-1)-cz]t}, \qquad (3.2.3)
\end{aligned}$$

from which we obtain

$$M(z, t; h) = e^{[\lambda e^{hk}(e^{zk}-1)-cz]t}. \tag{3.2.4}$$

Hence the Esscher transform (parameter h) of the shifted Poisson process is again a shifted Poisson process, with modified Poisson parameter λe^{hk}. Formula (2.10) is the condition that

$$\delta = \lambda e^{h*k}(e^k - 1) - c. \tag{3.2.5}$$

Thus a derivative security is evaluated according to the modified Poisson parameter

$$\lambda* = \lambda e^{h*k}.$$
$$= (\delta + c)/(e^k - 1). \tag{3.2.6}$$

For example, the price of a European call option is, according to (2.15) and (3.2.2),

$$S(0)[1 - \Lambda((\kappa + c\tau)/k; \lambda*e^k\tau)]$$
$$- Ke^{-\delta\tau}[1 - \Lambda((\kappa + c\tau)/k; \lambda*\tau)]. \tag{3.2.7}$$

Formula (3.2.7) can be found in textbooks on options such as those by Cox and Rubinstein [26, p. 366], Gibson [41, p. 168] and Hull [47, p. 454]. Note that the Poisson parameter λ does not appear in (3.2.7).

3.3 Logarithm of Stock Price as a Random Walk

A very popular model for pricing options is the binomial model, which is a discrete-time model. Although this paper focuses on continuous-time models, we think that it is worthwhile to digress and derive the binomial option-pricing formula by the Esscher transform method, because of its importance in the literature. Indeed, the two papers in *TSA*, by Clancy [22] and Pedersen, Shiu, and Thorlacius [63], on the pricing of options on bonds, are based on models of the binomial type.

The binomial option-pricing formula was given in the papers by Cox, Ross and Rubinstein [25] and by Rendleman and Bartter [65]. In their paper [25], Cox, Ross and Rubinstein acknowledged their debt to Nobel laureate W.F. Sharpe for suggesting the idea.

Here, we assume that the stock price,

$$S(t) = S(0)e^{X(t)}, t = 0, 1, 2, ...,$$

is a discrete-time stochastic process. Let X_1, X_2, ... be a sequence of independent and identically distributed random variables. Define $X(0)=0$ and, for $t=1, 2, 3, ..., \tau$,

$$X(t) = X_1 + X_2 + \cdots + X_t. \tag{3.3.1}$$

Let Ω denote the set of points on which X_j has positive probability. Assume that Ω is finite and consists of more than one point; let a be its smallest element and b its largest. To avoid arbitrages, we suppose that

$$a < \delta < b.$$

Let us assume that $\{S(t)\}$ is a multiplicative binomial process; that is, Ω consists of exactly two points:

$$\Omega = \{a, b\}.$$

Suppose that

$$\Pr(X_j = b) = p$$

and

$$\Pr(X_j = a) = 1 - p.$$

Let

$$B(x; n, \theta) = \sum_{0 \leq j \leq x} \binom{n}{j} \theta^j (1 - \theta)^{n-j}$$

denote the cumulative binomial distribution function with parameters n and θ. Then the cumulative distribution function of $X(t)$ is

$$F(x, t) = \Pr\left(\sum_{j=1}^{t} X_j \leq x \right)$$

$$= B\left(\frac{x - at}{b - a}; t, p \right).$$

Since

$$M(z, t) = E[e^{zX(t)}]$$
$$= [(1 - p)e^{az} + pe^{bz}]^t, \qquad (3.3.2)$$

we have

$$M(z, t; h) = M(z + h, t)/M(h, t)$$
$$= \{[1 - \pi(h)]e^{az} + \pi(h)e^{bz}\}^t, \qquad (3.3.3)$$

where

$$\pi(h) = \frac{pe^{bh}}{(1 - p)e^{ah} + pe^{bh}}. \qquad (3.3.4)$$

Formula (2.9) is the condition that

$$e^{\delta} = [1 - \pi(h^*)]e^a + \pi(h^*)e^b, \qquad (3.3.5)$$

from which it follows that

$$\pi(h^*) = \frac{e^{\delta} - e^a}{e^b - e^a}. \qquad (3.3.6)$$

According to (2.15), the value of the European call option with exercise price K and exercise date τ is

$$S(0)\left[1 - B\left(\frac{\kappa - a\tau}{b - a}; \tau, \pi(h^* + 1)\right)\right]$$
$$- Ke^{-\delta\tau}\left[1 - B\left(\frac{\kappa - a\tau}{b - a}; \tau, \pi(h^*)\right)\right], \qquad (3.3.7)$$

where

$$\pi(h^* + 1) = \frac{\pi(h^*)e^b}{[1 - \pi(h^*)]e^a + \pi(h^*)e^b}$$
$$= \pi(h^*)e^{b-\delta}.$$

Note that it is not necessary to know the probability p to price the option, since it is replaced by $\pi(h^*)$.

4. TWO NEW MODELS

In this section we present two continuous-time models for stock-price movements. Similar to the pure jump model in Section 3.2, we assume here that

$$S(t) = S(0)e^{X(t)}$$
$$= S(0)e^{Y(t)-ct},$$

where c is a constant. The stochastic process $\{Y(t)\}$ in the first model is a gamma process and in the second model an inverse Gaussian process. These two stochastic processes have been used to model aggregate insurance claims [33]. Recall that, in the pure jump model, all jumps are of the same size. However, this is not the case in these two models.

4.1 Logarithm of Stock Price as a Shifted Gamma Process

We assume that

$$X(t) = Y(t) - ct, \qquad (4.1.1)$$

where $\{Y(t)\}$ is a gamma process with parameters α and β, and the positive constant c is a third parameter. Let $G(x; \alpha, \beta)$ denote the gamma distribution with shape parameter α and scale parameter β,

$$G(x; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-\beta y} \, dy, \qquad x \ge 0.$$

Then

$$F(x, t) = G(x + ct; \alpha t, \beta) \qquad (4.1.2)$$

and

$$M(z, t) = \left(\frac{\beta}{\beta - z} \right)^{\alpha t} e^{-ctz}, \qquad z < \beta. \qquad (4.1.3)$$

Hence

$$M(z, t; h) = \left(\frac{\beta - h}{\beta - h - z} \right)^{\alpha t} e^{-ctz}, \qquad z < \beta - h, \qquad (4.1.4)$$

which shows that the transformed process is of the same type, with β replaced by $\beta - h$. Formula (2.9) means that

$$e^{\delta} = \left(\frac{\beta - h^*}{\beta - h^* - 1}\right)^{\alpha} e^{-c}. \qquad (4.1.5)$$

Define

$$\beta^* = \beta - h^*.$$

It follows from (4.1.5) that

$$\beta^* = \frac{1}{1 - e^{-(c+\delta)/\alpha}}. \qquad (4.1.6)$$

According to (2.15) and (4.1.2), the value of the European call option is

$$S(0)[1 - G(\kappa + c\tau; \alpha\tau, \beta^* - 1)]$$
$$- Ke^{-\delta\tau}[1 - G(\kappa + c\tau; \alpha\tau, \beta^*)]. \qquad (4.1.7)$$

Note that the scale parameter β does not appear in (4.1.6) and (4.1.7).

4.2 Logarithm of Stock Price as a Shifted Inverse Gaussian Process

Here, we also assume that

$$X(t) = Y(t) - ct,$$

but $\{Y(t)\}$ is now an inverse Gaussian process with parameters a and b. Let $J(x; a, b)$ denote the inverse Gaussian distribution function,

$$J(x; a, b) = \Phi\left(\frac{-a}{\sqrt{2x}} + \sqrt{2bx}\right)$$
$$+ e^{2a\sqrt{b}} \Phi\left(\frac{-a}{\sqrt{2x}} - \sqrt{2bx}\right), \qquad x > 0, \quad (4.2.1)$$

where Φ is the standard normal distribution function. (Panjer and Willmot's book [62], which was published recently by the Society of Actuaries, has an extensive discussion on the inverse Gaussian distribution.) Then

$$F(x, t) = J(x + ct; at, b). \tag{4.2.2}$$

Since the moment-generating function of the inverse Gaussian distribution is

$$e^{a(\sqrt{b}-\sqrt{b-z})}, \qquad z < b,$$

we have

$$M(z, t) = e^{at(\sqrt{b}-\sqrt{b-z})-ctz}, \qquad z < b. \tag{4.2.3}$$

Consequently,

$$M(z, t; h) = e^{at(\sqrt{b-h}-\sqrt{b-h-z})-ctz}, \qquad z < b - h, \tag{4.2.4}$$

which shows that the transformed process is of the same type, with b replaced by $b-h$. Formula (2.10) leads to the condition

$$\delta = a\left(\sqrt{b - h^*} - \sqrt{b - h^* - 1}\right) - c. \tag{4.2.5}$$

Writing $b^*=b-h^*$, we have

$$\sqrt{b^*} - \sqrt{b^* - 1} = \frac{c + \delta}{a}, \tag{4.2.6}$$

which is an implicit equation for b^*. It follows from (2.15) that the value of the European call option with exercise price K and exercise date τ is

$$S(0)[1 - J(\kappa + c\tau; a\tau, b^* - 1)] - Ke^{-\delta\tau}[1 - J(\kappa + c\tau; a\tau, b^*)]. \tag{4.2.7}$$

Note that the parameter b does not appear in (4.2.6) and (4.2.7).

5. NUMERICAL EXAMPLES

In this section we present numerical values for various European call options for the four continuous-time models. These values illustrate quantitatively some of the verbal statements in Table 17.1 of Hull's book [47, p. 438]. We thank François Dufresne for his computer expertise.

If we assume that $\{X(t)\}$ is a Wiener process, only one parameter (σ^2, the variance per unit time) has to be estimated for applying Formula (3.1.3). This is a main reason for the popularity of the Black-Scholes formula. Suppose that, for a certain stock, $\sigma=0.2$ and $S(0)=100$. Consider a European call option with exercise price $K=90$ six months from now ($\tau=0.5$). With a constant risk-free force of interest $\delta=0.1$, the value of the European call option according to (3.1.3) is

$$100\Phi(1.1693) - 90e^{-0.05}\Phi(1.0279) = 15.29.$$

Table 1 gives the European call option values for various exercise prices K and times to maturity τ. For option values corresponding to different values of σ, see Table 14.1 of Ingersoll's book [48, p. 314].

TABLE 1

BLACK-SCHOLES OPTION PRICES
$[S(0)=100, \delta=0.1, \sigma=0.2]$

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.99	24.03	26.04	27.99
85	17.21	19.52	21.74	23.86
90	12.65	15.29	17.72	19.99
95	8.58	11.50	14.07	16.44
100	5.30	8.28	10.88	13.27
105	2.95	5.69	8.18	10.52
110	1.47	3.74	5.99	8.18
115	0.66	2.35	4.28	6.26
120	0.27	1.42	2.98	4.71

If the logarithm of the stock price does not follow a symmetric distribution, the assumption of a Wiener process is not appropriate. Suppose that the process $\{X(t)\}$ has mean per unit time μ, variance per unit time σ^2, and third central moment per unit time θ^3. Let $\gamma = \theta^3/\sigma^3$ denote the coefficient of skewness of $X(1)$. Then

$$\ln\{E[e^{zX(t)}]\} = \ln[M(z, t)]$$

$$= t\ln[M(z, 1)]$$

$$= t[\mu z + \sigma^2 z^2/2 + \theta^3 z^3/3! + \ldots]$$

$$= t[\mu z + \sigma^2 z^2/2 + \gamma\sigma^3 z^3/3! + \ldots]. \qquad (5.1)$$

In the following we assume, as in the Wiener process example, $\sigma=0.2$, $S(0)=100$ and $\delta=0.1$. Furthermore, we assume $\mu=0.1$ and $\gamma=1$.

5.1 Shifted Poisson Process Model

By (5.1) and (3.2.3), equating the first three central moments in the shifted Poisson process model yields the equations

$$\lambda k - c = \mu,$$

$$\lambda k^2 = \sigma^2$$

and

$$\lambda k^3 = \gamma \sigma^3,$$

from which we obtain

$$k = \gamma \sigma = 0.2,$$

$$\lambda = \gamma^{-2} = 1$$

and

$$c = (\sigma/\gamma) - \mu$$

$$= 0.1. \tag{5.1.1}$$

The resulting value for λ is not needed, since the calculations are done for λ^* in accordance with (3.2.6). Table 2 gives the European call option values computed with Formula (3.2.7) for various exercise prices K and times to maturity τ.

TABLE 2

POISSON PROCESS MODEL OPTION PRICES

$[S(0)=100,\ \delta=0.1,\ \mu=0.1,\ \sigma=0.2,\ \gamma=1]$

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.98	23.90	25.78	27.61
85	17.10	19.15	21.14	23.09
90	12.22	14.39	16.50	18.56
95	7.35	9.63	12.91	15.70
100	4.39	7.83	10.63	13.01
105	3.40	6.10	8.35	10.31
110	2.42	4.37	6.06	7.62
115	1.43	2.64	4.32	6.42
120	0.60	1.96	3.63	5.38

5.2 Shifted Gamma Process Model

By (5.1) and (4.1.3), matching the first three central moments in the shifted gamma process model yields the equations

$$(\alpha/\beta) - c = \mu,$$

$$\alpha/\beta^2 = \sigma^2$$

and

$$2\alpha/\beta^3 = \theta^3 = \gamma\sigma^3,$$

from which it follows that

$$\alpha = 4/\gamma^2 = 4,$$

$$\beta = 2/(\sigma\gamma) = 10$$

and

$$c = (2\sigma/\gamma) - \mu$$

$$= 0.3. \tag{5.2.1}$$

The resulting value for β is not needed, since the calculations are done for β^* in accordance with (4.1.6). Table 3 gives the European call option values computed with Formula (4.1.7) for various exercise prices K and times to maturity τ.

TABLE 3

GAMMA PROCESS MODEL OPTION PRICES

$[S(0)=100,\ \delta=0.1,\ \mu=0.1,\ \sigma=0.2,\ \gamma=1]$

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.98	23.90	25.78	27.62
85	17.10	19.15	21.18	23.24
90	12.22	14.50	16.89	19.17
95	7.60	10.59	13.20	15.59
100	4.66	7.61	10.18	12.55
105	2.93	5.45	7.80	10.03
110	1.88	3.91	5.96	7.99
115	1.23	2.82	4.55	6.35
120	0.82	2.05	3.48	5.05

5.3 Shifted Inverse Gaussian Process Model

By (5.1) and (4.2.3), matching the first three central moments in the shifted inverse Gaussian process model yields the equations

$$ab^{-1/2}/2 - c = \mu,$$

$$ab^{-3/2}/4 = \sigma^2$$

and

$$3ab^{-5/2}/8 = \theta^3 = \gamma\sigma^3,$$

from which it follows that

$$a = 3(6\sigma/\gamma^3)^{1/2} = 3(1.2)^{1/2},$$

$$b = 3/(2\sigma\gamma) = 7.5$$

and

$$c = (3\sigma/\gamma) - \mu$$

$$= 0.5.$$

The resulting value for b is not needed, since the calculations are done for b^* in accordance with (4.2.6):

$$\sqrt{b^*} - \sqrt{b^* - 1} = \frac{c + \delta}{a}$$

$$= \frac{0.2}{\sqrt{1.2}},$$

from which we obtain

$$b^* = 8^1/120.$$

(That b^* is a rational number is atypical.) Table 4 gives the European call option values computed with Formula (4.2.7) for various exercise prices K and times to maturity τ.

TABLE 4

INVERSE GAUSSIAN PROCESS MODEL OPTION PRICES

$[S(0)=100, \delta=0.1, \mu=0.1, \sigma=0.2, \gamma=1]$

Exercise Price (K)	Time to Maturity			
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$	$\tau = 1$
80	21.98	23.90	25.78	27.64
85	17.10	19.15	21.22	23.27
90	12.22	14.56	16.95	19.21
95	7.70	10.63	13.23	15.61
100	4.67	7.61	10.18	12.54
105	2.88	5.41	7.77	10.01
110	1.83	3.86	5.91	7.95
115	1.20	2.77	4.50	6.31
120	0.80	2.01	3.44	5.01

5.4 Remarks

The four continuous-time models have in common that, in each case, all but one parameter can be read off from the sample path of the process. The parameters that are not inherent in the sample paths are μ, λ, β, and b. In each case the probability measure is transformed by altering the respective parameter.

It can be shown that the limit for $\gamma \to 0$ of each of the models of Sections 3.2, 4.1 and 4.2 is the classical lognormal model of Section 3.1. In this sense these three models, in particular, Formulas (3.2.7), (4.1.7) and (4.2.7), are generalizations of the classical lognormal model and the Black-Scholes formula.

Stock-price models in the form of

$$S(t) = S(0)e^{ct - Y(t)},$$

as opposed to

$$S(t) = S(0)e^{Y(t) - ct},$$

are equally tractable. However, they are less realistic, since they imply a negative third central moment of the logarithm of stock prices.

Let us write down Equations (5.1.1), (5.2.1) and (5.3.1) in one place:

$$c = (\sigma/\gamma) - \mu, \tag{5.1.1}$$

$$c = (2\sigma/\gamma) - \mu, \tag{5.2.1}$$

$$c = (3\sigma/\gamma) - \mu. \tag{5.3.1}$$

It is interesting to observe how these three formulas for the downward drift coefficient c differ. It turns out that these processes are special cases of a general family, which has been studied by Dufresne, Gerber and Shiu [33] in the context of collective risk theory. For further elaboration, see Sections 5 and 6 of our paper [40].

Eight months after this paper was submitted for publication, Heston's paper [45] appeared. Heston [45] has also introduced the gamma process for modeling stock-price movements. His Formula (10a) can be shown to be the same as our Formula (4.1.7).

6. OPTIONS ON SEVERAL RISKY ASSETS

In this section we generalize the method of Esscher transforms to price derivative securities of multiple risky assets or asset pools. Some of the

related papers in the finance literature are [16], [17], [21], [37], [50], [56], [57], [58], [66], [75], and [76]. An obvious application of such results is portfolio insurance, or devising hedging strategies to protect portfolios of assets against losses ([3], [54], [55]). Other applications, such as the valuation of bonds involving one or more foreign currencies and pricing the quality option in Treasury bond futures, can be found in the cited references. In the actuarial literature, there are papers such as [3], [8], [14], [71] and [72]. The papers by Bell and Sherris [8] and by Sherris [72] study pension funds with benefit designs offering resignation, death and/or retirement benefits that are the greater of two alternative benefits. The two alternatives are typically a multiple of final (average) salary and the accumulation of contributions. Such a benefit design provides the plan participants an option on the maximum of two random benefit amounts.

For $t \geq 0$, let $S_1(t)$, $S_2(t)$, ..., $S_n(t)$ denote the prices of n non-dividend-paying stocks or assets at time t. Write

$$X_j(t) = \ln[S_j(t)/S_j(0)], \qquad j = 1, 2, ..., n, \qquad (6.1)$$

and

$$\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))'.$$

Let R^n denote the set of column vectors with n real entries. Let

$$F(\mathbf{x}, t) = \Pr[\mathbf{X}(t) \leq \mathbf{x}], \qquad \mathbf{x} \in R^n,$$

be the cumulative distribution function of the random vector $\mathbf{X}(t)$, and

$$M(\mathbf{z}, t) = E[e^{\mathbf{z}'\mathbf{X}(t)}], \qquad \mathbf{z} \in R^n,$$

its moment-generating function. In the rest of this paper we assume that $\{\mathbf{X}(t)\}_{t \geq 0}$ is a stochastic process with independent and stationary increments and that

$$M(\mathbf{z}, t) = [M(\mathbf{z}, 1)]^t, \qquad t \geq 0. \qquad (6.2)$$

For simplicity, we also assume that the random vector $\mathbf{X}(t)$ has density

$$f(\mathbf{x}, t) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F(\mathbf{x}, t), \qquad t > 0.$$

Then the modified density of $\mathbf{X}(t)$ under the Esscher transform with parameter vector $\mathbf{h}$ is

$$f(\mathbf{x}, t; \mathbf{h}) = \frac{e^{\mathbf{h}'\mathbf{x}}f(\mathbf{x}, t)}{M(\mathbf{h}, t)}$$

and the corresponding moment-generating function is

$$M(\mathbf{z}, t; \mathbf{h}) = M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t).$$

The Esscher transform (parameter vector $\mathbf{h}$) of the process $\{\mathbf{X}(t)\}$ is again a process with stationary and independent increments, and

$$M(\mathbf{z}, t; \mathbf{h}) = [M(\mathbf{z}, 1; \mathbf{h})]^t. \qquad (6.3)$$

In the general case where the density function $f(\mathbf{x}, t)$ may not exist, we define the Esscher transform in terms of Stieltjes integrals, as we did in (2.1.1).

The parameter vector $\mathbf{h} = \mathbf{h}^*$ is determined so that, for $j = 1, 2, \ldots, n$,

$$\{e^{-\delta t} S_j(t)\}_{t \geq 0}$$

is a martingale with respect to the modified probability measure. In particular,

$$S_j(0) = \mathrm{E}[e^{-\delta t} S_j(t); \mathbf{h}^*], \qquad t \geq 0, \quad j = 1, 2, \ldots, n. \qquad (6.4)$$

(Note that these conditions are independent of t.) The value of a derivative security is calculated as the expectation, with respect to the modified probability measure, of the discounted value of its payoffs.

Define

$$\mathbf{1}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)' \in R^n,$$

where the 1 in the column vector $\mathbf{1}_j$ is in the j-th position. Formulas (6.4) become

$$e^{\delta t} = M(\mathbf{1}_j, t; \mathbf{h}^*)$$

$$= [M(\mathbf{1}_j, 1; \mathbf{h}^*)]^t, \qquad t \geq 0, \quad j = 1, 2, \ldots, n, \qquad (6.5)$$

by (6.3). The following is the main result in this section.

Theorem

Let g be a real-valued measurable function of n variables. Then, for each positive t,

$$E[e^{-\delta t} S_j(t) g(S_1(t), S_2(t), \ldots, S_n(t)); \mathbf{h}^*]$$

$$= S_j(0) E[g(S_1(t), S_2(t), \ldots, S_n(t)); \mathbf{h}^* + \mathbf{1}_j]. \quad (6.6)$$

Proof

The proof follows the same line of argument that we used in deriving the European call option formula (2.15). The expectation on the left-hand side of (6.6) is obtained by integrating

$$e^{-\delta t} S_j(0) e^{x_j} g(S_1(0) e^{x_1}, \ldots, S_n(0) e^{x_n}) f(\mathbf{x}, t; \mathbf{h}^*)$$

with respect to $\mathbf{x} = (x_1, \ldots, x_n)'$ over R^n. Since

$$
\begin{aligned}
e^{x_j} f(\mathbf{x}, t; \mathbf{h}^*) &= \frac{e^{(\mathbf{h}^* + \mathbf{1}_j)' \mathbf{x}} f(\mathbf{x}, t)}{M(\mathbf{h}^*, t)} \\
&= \frac{M(\mathbf{h}^* + \mathbf{1}_j, t)}{M(\mathbf{h}^*, t)} f(\mathbf{x}, t; \mathbf{h}^* + \mathbf{1}_j) \\
&= M(\mathbf{1}_j, t; \mathbf{h}^*) f(\mathbf{x}, t; \mathbf{h}^* + \mathbf{1}_j) \\
&= e^{\delta t} f(\mathbf{x}, t; \mathbf{h}^* + \mathbf{1}_j),
\end{aligned}
$$

the result follows. $\square$

There is another way to derive the theorem. For $\mathbf{k} = (k_1, \ldots, k_n)'$, write

$$\mathbf{S}(t)^{\mathbf{k}} = S_1(t)^{k_1} \ldots S_n(t)^{k_n}.$$

Then

$$
\begin{aligned}
E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)); \mathbf{h}] &= \frac{E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)) e^{\mathbf{h}' \mathbf{X}(t)}]}{E[e^{\mathbf{h}' \mathbf{X}(t)}]} \\
&= \frac{E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)) \mathbf{S}(t)^{\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]} \\
&= \frac{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]} \frac{E[g(\mathbf{S}(t)) \mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]} \\
&= E[\mathbf{S}(t)^{\mathbf{k}}; \mathbf{h}] E[g(\mathbf{S}(t)); \mathbf{k} + \mathbf{h}].
\end{aligned}
$$

Now the theorem follows from this factorization formula (with $\mathbf{h} = \mathbf{h}^*$ and $\mathbf{k} = \mathbf{1}_j$) and (6.4).

One of the first papers generalizing the Black-Scholes formula to pricing derivative securities of more than one risky asset is by Margrabe [57]. Assuming that the asset prices are geometric Brownian motions, Margrabe [57] derived a closed-form formula for the value of an option to exchange one risky asset for another at the end of a stated period. In other words, he determined the value at time 0 of a contract whose only payoff is at time τ, the value of which is

$$[S_1(\tau) - S_2(\tau)]_+.$$

Corollary 1
The value at time 0 of an option to exchange $S_2(\tau)$ for $S_1(\tau)$ at time τ is

$$S_1(0)\mathrm{Pr}[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1] - S_2(0)\mathrm{Pr}[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2].$$

Proof
The option value at time 0 is

$$\mathrm{E}(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*).$$

Let $I(A)$ denote the indicator random variable of an event A. Then

$$[S_1(\tau) - S_2(\tau)]_+ = [S_1(\tau) - S_2(\tau)]I[S_1(\tau) > S_2(\tau)]$$

$$= S_1(\tau)I[S_1(\tau) > S_2(\tau)] - S_2(\tau)I[S_1(\tau) > S_2(\tau))].$$

Thus

$$\mathrm{E}(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*)$$

$$= \mathrm{E}(e^{-\delta\tau} S_1(\tau)I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^*) - \mathrm{E}(e^{-\delta\tau} S_2(\tau)I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^*)$$

$$= S_1(0)\mathrm{E}(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_1) - S_2(0)\mathrm{E}(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_2),$$

by the theorem. Since $\mathrm{E}[I(A)]=\mathrm{Pr}(A)$, the result follows. $\square$

In Section 7 we discuss the geometric Brownian motion assumption and show that Margrabe's formula is an immediate consequence of Corollary 1. Now we give another derivation for the European call option formula (2.15).

Corollary 2
Formula (2.15) holds.

Proof

Consider $n=2$ with $S_1(t)=S(t)$ and $S_2(t)=Ke^{\delta(t-\tau)}$. Then

$$\mathbf{X}(t) = (X_1(t), X_2(t))' = (X(t), \delta t)',$$

$$M(\mathbf{z}, t) = M(z_1, t)e^{z_2\delta t},$$

and

$$M(\mathbf{z}, t; \mathbf{h}) = M(z_1, t; h_1)e^{z_2\delta t}. \tag{6.7}$$

Since the parameter h_2 does not appear in the right-hand side of (6.7), the parameter h_2^* is arbitrary, and $h_1^*=h^*$. Thus the value of the European call option is

$E^*(e^{-\delta\tau}[S(\tau) - K]_+)$

$\quad = E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*)$

$\quad = S_1(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1] - S_2(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2]$

$\quad = S(0)\Pr[S(\tau) > K; h^* + 1] - e^{-\delta\tau} K\Pr[S(\tau) > K; h^*]$

$\quad = S(0)\{1 - \Pr[S(\tau) \le K; h^* + 1]\} - e^{-\delta\tau} K\{1 - \Pr[S(\tau) \le K; h^*]\},$

which is formula (2.15). □

Margrabe's work [57] was extended by Stulz [75], who also assumed that the asset prices are geometric Brownian motions. By laborious calculation, Stulz derived formulas for valuing options on the maximum and the minimum of two risky assets; that is, he found the value at time 0 of a contract with payoff at time τ

$$(\text{Max}[S_1(\tau), S_2(\tau)] - K)_+$$

and the value at time 0 of a contract with payoff at time τ

$$(\text{Min}[S_1(\tau), S_2(\tau)] - K)_+.$$

These two option formulas of Stulz were generalized to the case of n risky assets by Johnson [50]. Indeed, one may further ask the following questions: How much should one pay at time 0 to obtain (the value of) the second-highest value asset at time τ? The third-highest value asset? The k-th highest value asset? More generally, what is the value of the

European call option on the k-th highest value asset at time τ with exercise price K? Note again that, in the papers quoted in this paragraph, the asset prices are assumed to be geometric Brownian motions.

For a fixed time τ, $\tau > 0$, let $\mathbb{S}$ denote the set consisting of the random variables $\{S_j(\tau); j = 1, 2, \ldots, n\}$. Let $S_{[k]}$ denote the random variable defined by the k-th highest value of $\mathbb{S}$. Thus, $S_{[1]}$ and $S_{[n]}$ denote the maximum and minimum of $\mathbb{S}$, respectively.

Corollary 3

Assume that $X(t)$ has a continuous distribution. Then the option to obtain the k-th highest value asset at time τ is worth

$$\sum_{j=1}^{n} S_j(0) \Pr(S_j(\tau) \text{ ranks } k\text{-th among } \mathbb{S}; \mathbf{h}^* + \mathbf{1}_j) \qquad (6.8)$$

at time 0.

Proof

The option value at time 0 is

$$E(e^{-\delta\tau} S_{[k]}; \mathbf{h}^*).$$

Since $X(\tau)$ has a continuous distribution, we have the identity

$$S_{[k]} = \sum_{j=1}^{n} S_j(\tau) I[S_j(\tau) \text{ ranks } k\text{-th among } \mathbb{S}].$$

Formula (6.8) now follows from the theorem. $\square$

Corollary 4

Assume that $X(t)$ has a continuous distribution. Then the European call option on the k-th highest value asset at time τ with exercise price K is worth

$$\sum_{j=1}^{n} S_j(0) \Pr(S_j(\tau) > K \text{ and } S_j(\tau) \text{ ranks } k\text{-th among } \mathbb{S}; \mathbf{h}^* + \mathbf{1}_j)$$
$$- e^{-\delta\tau} K \Pr(S_{[k]} > K; \mathbf{h}^*) \qquad (6.9)$$

at time 0.

The proof for Corollary 4 is essentially a combination of the proofs for Corollary 2 and Corollary 3. Note that, when $K=0$, Corollary 4 becomes Corollary 3.

There are obviously many other applications of the theorem. For example, in a paper recently published in the *Journal of the Institute of Actuaries*, Sherris [71] analyzed the "capital gains tax option," whose payoff at time τ is

$$(S(\tau) - \text{Max}[C(\tau), K])_+,$$

where $S(t)$ denotes the price of a risky asset at time t and $C(t)$ denotes the value of an index at time t. Sherris's result follows from the formula

$$(S - \text{Max}(C, K))_+ = S\,I(S > C \text{ and } S > K)$$

$$- [C\,I(S > C > K) + K\,I(S > K > C)].$$

Let us end this section by showing that an American call option on the maximum of n non-dividend-paying stocks is never optimally exercised before its maturity date. Consequently, the value of the American option is given by Corollary 4 (with $k=1$). The proof is by two applications of Jensen's inequality:

$$E[e^{-\delta t}\,(\text{Max}\{S_j(t)\} - K)_+; \mathbf{h}^*] \geq (E[e^{-\delta t}\,\text{Max}\{S_j(t)\}; \mathbf{h}^*] - e^{-\delta t}\,K)_+$$

$$\geq (\text{Max}\{E[e^{-\delta t}\,S_j(t); \mathbf{h}^*]\} - e^{-\delta t}\,K)_+$$

$$= (\text{Max}\{S_j(0)\} - e^{-\delta t}\,K)_+$$

$$\geq (\text{Max}\{S_j(0)\} - K)_+$$

For $t>0$ and $\delta>0$, the last inequality is strict if the option is currently in the money, that is, if

$$\text{Max}\{S_j(0)\} > K.$$

7. LOGARITHMS OF STOCK PRICES
AS A MULTIDIMENSIONAL WIENER PROCESS

In the finance literature, the usual distribution assumption on the prices of the primitive securities is that they are geometric Brownian motions. In other words, $\{\mathbf{X}(t)\}$ is assumed to be an n-dimensional Wiener process. We now show that many results on options and derivative securities in

the literature are relatively straightforward consequences of the theorem and its corollaries.

Following the notation in Chapter 12 of Hogg and Craig's textbook [46] for the Course 110 examination, we let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)'$ and $V = (\sigma_{ij})$ denote the mean vector and the covariance matrix of $X(1)$, respectively. It is assumed that V is nonsingular. For $t > 0$, the density function of $X(t)$ is

$$f(\mathbf{x}, t) = \frac{1}{(2\pi)^{n/2} |tV|^{1/2}} e^{-(\mathbf{x} - t\mu)'(2tV)^{-1}(\mathbf{x} - t\mu)}, \qquad \mathbf{x} \in R^n.$$

It can be shown [46, Section 12.1] that

$$M(\mathbf{z}, t) = \exp[t(\mathbf{z}'\mu + \tfrac{1}{2}\mathbf{z}'V\mathbf{z})], \qquad \mathbf{z} \in R^n.$$

Thus, for $\mathbf{h} \in R^n$,

$$M(\mathbf{z}, t; \mathbf{h}) = M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t)$$

$$= \exp\{t[\mathbf{z}'(\mu + V\mathbf{h}) + \tfrac{1}{2}\mathbf{z}'V\mathbf{z}]\}, \qquad \mathbf{z} \in R^n,$$

which shows that the Esscher transform (parameter vector $\mathbf{h}$) of the n-dimensional Wiener process is again an n-dimensional Wiener process, with modified mean vector per unit time

$$\mu + V\mathbf{h}$$

and unchanged covariance matrix per unit time V. Equations (6.5) mean that, for $j = 1, 2, \ldots, n$,

$$\delta = \mathbf{1}_j'(\mu + V\mathbf{h}^*) + \tfrac{1}{2}\mathbf{1}_j'V\mathbf{1}_j,$$

from which we obtain

$$\mu + V\mathbf{h}^* = (\delta - \tfrac{1}{2}\sigma_{11}, \delta - \tfrac{1}{2}\sigma_{22}, \ldots, \delta - \tfrac{1}{2}\sigma_{nn})'. \qquad (7.1)$$

Consequently, the mean vector per unit time of the modified process with parameter vector $\mathbf{h}^* + \mathbf{1}_j$ is

$$\mu + V(\mathbf{h}^* + \mathbf{1}_j) = (\delta + \sigma_{1j} - \tfrac{1}{2}\sigma_{11}, \delta + \sigma_{2j} - \tfrac{1}{2}\sigma_{22}, \ldots,$$

$$\delta + \sigma_{nj} - \tfrac{1}{2}\sigma_{nn})'. \qquad (7.2)$$

Note that the right-hand sides of (7.1) and (7.2) do not contain any elements of μ.

To derive Margrabe's [57] main result, we evaluate the expectation

$$E(e^{-\delta\tau} [S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*),$$

which, by Corollary 1, is

$$S_1(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1] - S_2(0)\Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2]$$

$$= S_1(0)\Pr[Y < \xi; \mathbf{h}^* + \mathbf{1}_1] - S_2(0)\Pr[Y < \xi; \mathbf{h}^* + \mathbf{1}_2],$$

where

$$Y = X_2(\tau) - X_1(\tau) \qquad (7.3)$$

and

$$\xi = \ln[S_1(0)/S_2(0)]. \qquad (7.4)$$

Now, Y is a normal random variable with respect to any Esscher transform,

$$E(Y; \mathbf{h}^* + \mathbf{1}_1) = [(\delta + \sigma_{21} - \tfrac{1}{2}\sigma_{22}) - (\delta + \sigma_{11} - \tfrac{1}{2}\sigma_{11})]\tau$$

$$= (-\tfrac{1}{2}\sigma_{11} + \sigma_{21} - \tfrac{1}{2}\sigma_{22})\tau$$

and

$$E(Y; \mathbf{h}^* + \mathbf{1}_2) = [(\delta + \sigma_{22} - \tfrac{1}{2}\sigma_{22}) - (\delta + \sigma_{12} - \tfrac{1}{2}\sigma_{11})]\tau$$

$$= (\tfrac{1}{2}\sigma_{11} - \sigma_{12} + \tfrac{1}{2}\sigma_{22})\tau.$$

The variance of Y does not depend on the parameter vector; it is

$$(\sigma_{11} - 2\sigma_{12} + \sigma_{22})\tau.$$

With the definition

$$\upsilon^2 = \sigma_{11} - 2\sigma_{12} + \sigma_{22} \qquad (7.5)$$

(the variance per unit time of the process $\{X_1(t) - X_2(t)\}$), we have

$$E(Y; \mathbf{h}^* + \mathbf{1}_1) = -\upsilon^2\tau/2,$$

$$E(Y; \mathbf{h}^* + \mathbf{1}_2) = \upsilon^2\tau/2$$

and

$$\text{Var}(Y) = \upsilon^2\tau.$$

Thus the value (at time 0) of the option to exchange $S_2(\tau)$ for $S_1(\tau)$ at time τ is

$$S_1(0)\Phi\left(\frac{\xi + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right) - S_2(0)\Phi\left(\frac{\xi - \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right), \qquad (7.6)$$

which is the formula on p. 179 of Margrabe's paper [57].

It is somewhat surprising that (7.6) does not depend on the risk-free force of interest, δ. Note also that, if $S_2(t) = Ke^{-\delta(\tau - t)}$, (7.6) becomes the Black-Scholes formula (3.1.3).

Next we calculate the value (at time 0) of the option to receive the greater of $S_1(\tau)$ and $S_2(\tau)$ at time τ. Because of the identity

$$\text{Max}[S_1(\tau), S_2(\tau)] = S_2(\tau) + [S_1(\tau) - S_2(\tau)]_+,$$

the option value is

$$S_2(0) + e^{-\delta\tau}\, \text{E}([S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*),$$

which, by (7.6), is

$$S_1(0)\Phi\left(\frac{\xi + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right) + S_2(0)\left[1 - \Phi\left(\frac{\xi - \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right)\right]$$

$$= S_1(0)\Phi\left(\frac{\xi + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right) + S_2(0)\Phi\left(\frac{-\xi + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right)$$

$$= S_1(0)\Phi\left(\frac{\ln[S_1(0)/S_2(0)] + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right)$$

$$+ S_2(0)\Phi\left(\frac{\ln[S_2(0)/S_1(0)] + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}\right). \qquad (7.7)$$

This result can also be obtained by applying Corollary 3 (with $n=2$). Again, it is noteworthy that (7.7) does not depend on δ.

Let us also derive the results in Stulz's paper [75] and in Johnson's paper [50]. By Corollary 4 (with $n=2$),

$$E(e^{-\delta\tau}\{Max[S_1(\tau), S_2(\tau)] - K\}_+; \mathbf{h}^*)$$

$$= E[e^{-\delta\tau}(S_{[1]} - K)_+; \mathbf{h}^*],$$

$$= S_1(0)Pr[S_1(\tau) > K \text{ and } S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1]$$

$$+ S_2(0)Pr[S_2(\tau) > K \text{ and } S_2(\tau) > S_1(\tau); \mathbf{h}^* + \mathbf{1}_2]$$

$$- Ke^{-\delta\tau} Pr[S_1(\tau) > K \text{ or } S_2(\tau) > K; \mathbf{h}^*]. \qquad (7.8)$$

First, we evaluate the last probability term,

$$Pr[S_1(\tau) > K \text{ or } S_2(\tau) > K; \mathbf{h}^*] = 1 - Pr[S_1(\tau) \le K \text{ and } S_2(\tau) \le K; \mathbf{h}^*].$$

Similar to (2.12), define

$$\kappa_1 = \ln[K/S_1(0)] \qquad (7.9)$$

and

$$\kappa_2 = \ln[K/S_2(0)]. \qquad (7.10)$$

Then

$$Pr[S_1(\tau) \le K \text{ and } S_2(\tau) \le K; \mathbf{h}^*]$$

$$= Pr[X_1(\tau) \le \kappa_1 \text{ and } X_2(\tau) \le \kappa_2; \mathbf{h}^*]. \qquad (7.11)$$

By (7.1)

$$E[X_1(\tau); \mathbf{h}^*] = (\delta - \tfrac{1}{2} \sigma_{11})\tau$$

and

$$E[X_2(\tau); \mathbf{h}^*] = (\delta - \tfrac{1}{2} \sigma_{22})\tau.$$

Let $\Phi_2(a, b; \rho)$ denote the bivariate cumulative standard normal distribution with upper limits of integration a and b and coefficient of correlation ρ. (For various properties of Φ_2, see Section 26.3 in the book by Abramowitz and Stegun [1]). Write

$$\sigma_i = \sqrt{\sigma_{ii}} \qquad (7.12)$$

and

$$\rho_{ij} = \sigma_{ij}/(\sigma_i\sigma_j). \qquad (7.13)$$

Then the probability defined by (7.11) is

$$\Phi_2\left(\frac{\kappa_1 - (\delta - \sigma_1^2/2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\kappa_2 - (\delta - \sigma_2^2/2)\tau}{\sigma_2\sqrt{\tau}}, \rho_{12}\right). \quad (7.14)$$

To obtain approximate numerical values for (7.14), we can use formulas (26.3.11) and (26.3.20) together with Figures 26.2, 26.3 and 26.4 in the book by Abramowitz and Stegun [1]. An algorithm to calculate the bivariate cumulative standard normal distribution to four-decimal-place accuracy can be found in the paper by Drezner [31]. The Drezner algorithm (with a typo corrected) can be found in Appendix 10B in Hull's book [47, p. 245] and in Appendix 13.1 in the book by Stoll and Whaley [74, p. 338].

Next, we evaluate the first probability term in (7.8),

$$\Pr[S_1(\tau) > K \text{ and } S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1]$$

$$= \Pr[-X_1(\tau) < -\kappa_1 \text{ and } X_2(\tau) - X_1(\tau) < \xi; \mathbf{h}^* + \mathbf{1}_1], \quad (7.15)$$

where the constants κ_1 and ξ are defined by (7.9) and (7.4), respectively. Now,

$$E[-X_1(\tau); \mathbf{h}^* + \mathbf{1}_1] = -(\delta + \sigma_{11} - \tfrac{1}{2}\sigma_{11})\tau$$

$$= -(\delta + \tfrac{1}{2}\sigma_{11})\tau,$$

$$E[X_2(\tau) - X_1(\tau); \mathbf{h}^* + \mathbf{1}_1] = (-\tfrac{1}{2}\sigma_{11} + \sigma_{12} - \tfrac{1}{2}\sigma_{22})\tau$$

$$= -\upsilon^2\tau/2,$$

$$\text{Var}[X_2(\tau) - X_1(\tau); \mathbf{h}] = (\sigma_{11} - 2\sigma_{12} + \sigma_{22})\tau$$

$$= \upsilon^2\tau$$

and

$$\text{Cov}(-X_1(\tau), X_2(\tau) - X_1(\tau); \mathbf{h}) = \text{Cov}(X_1(\tau), X_1(\tau) - X_2(\tau); \mathbf{h})$$

$$= (\sigma_{11} - \sigma_{12})\tau$$

$$= [\sigma_1(\sigma_1 - \rho_{12}\sigma_2)]\tau,$$

where υ is defined by (7.5). Thus (7.15) can be expressed as

$$\Phi_2\left(\frac{-\kappa_1 + (\delta + \tfrac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\xi + (\upsilon^2\tau/2)}{\upsilon\sqrt{\tau}}; \frac{\sigma_1 - \rho_{12}\sigma_2}{\upsilon}\right). \quad (7.16)$$

By symmetry, we can write down the expression, in terms of the distribution Φ_2, for the second probability term in (7.8). Hence the value at time 0 of the European call option on the maximum of two risky assets with exercise price K and exercise date τ is

$$
S_1(0)\Phi_2\left(\frac{-\kappa_1 + (\delta + \tfrac{1}{2}\,\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln[S_1(0)/S_2(0)] + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}; \frac{\sigma_1 - \rho_{12}\sigma_2}{\upsilon}\right)
$$

$$
+ S_2(0)\Phi_2\left(\frac{-\kappa_2 + (\delta + \tfrac{1}{2}\,\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln[S_2(0)/S_1(0)] + \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}; \frac{\sigma_2 - \rho_{12}\sigma_1}{\upsilon}\right)
$$

$$
- Ke^{-\delta\tau}\left[1 - \Phi_2\left(\frac{\kappa_1 - (\delta - \tfrac{1}{2}\,\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\kappa_2 - (\delta - \tfrac{1}{2}\,\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}; \rho_{12}\right)\right],
$$

$$(7.17)$$

which is the same as equation (6) in Johnson's paper [50, p. 281].

Let us also consider the expectation

$$
E[e^{-\delta\tau}(\mathrm{Min}[S_1(\tau), S_2(\tau)] - K)_+; \mathbf{h}^*],
$$

which by Corollary 4 (with $n=2$) is

$$
S_1(0)\mathrm{Pr}[K < S_1(\tau) < S_2(\tau); \mathbf{h}^* + \mathbf{1}_1]
$$

$$
+ S_2(0)\mathrm{Pr}[K < S_2(\tau) < S_1(\tau); \mathbf{h}^* + \mathbf{1}_2]
$$

$$
- Ke^{-\delta\tau}\,\mathrm{Pr}[K < S_1(\tau) \text{ and } K < S_2(\tau); \mathbf{h}^*]
$$

$$
= S_1(0)\mathrm{Pr}[-X_1(\tau) < -\kappa_1 \text{ and } X_1(\tau) - X_2(\tau) < \ln[S_2(0)/S_1(0)]; \mathbf{h}^* + \mathbf{1}_1]
$$

$$
+ S_2(0)\mathrm{Pr}[-X_2(\tau) < -\kappa_2 \text{ and } X_2(\tau) - X_1(\tau) < \ln[S_1(0)/S_2(0)]; \mathbf{h}^* + \mathbf{1}_2]
$$

$$
- Ke^{-\delta\tau}\,\mathrm{Pr}[-X_1(\tau) < -\kappa_1 \text{ and } -X_2(\tau) < -\kappa_2; \mathbf{h}^*].
$$

By a calculation similar to the above, we obtain the value at time 0 of the European call option on the minimum of two risky assets with exercise price K and exercise date τ:

$$S_1(0)\Phi_2\left(\frac{-\kappa_1 + (\delta + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln[S_2(0)/S_1(0)] - \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}; \frac{\rho_{12}\sigma_2 - \sigma_1}{\upsilon}\right)$$

$$+ S_2(0)\Phi_2\left(\frac{-\kappa_2 + (\delta + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln[S_1(0)/S_2(0)] - \upsilon^2\tau/2}{\upsilon\sqrt{\tau}}; \frac{\rho_{12}\sigma_1 - \sigma_2}{\upsilon}\right)$$

$$- Ke^{-\delta\tau}\Phi_2\left(\frac{-\kappa_1 + (\delta - \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \frac{-\kappa_2 + (\delta - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}; \rho_{12}\right). \quad (7.18)$$

This is the same as formula (11) in Stulz's paper [75, p. 165] (both $\sigma^2\sqrt{\tau}$ should be $\sigma_2\tau$) and formula (8) in Johnson's paper [50, p. 281].

Because of the identity

$$(\text{Max}[S_1(\tau), S_2(\tau)] - K)_+ + (\text{Min}[S_1(\tau), S_2(\tau)] - K)_+$$
$$= [S_1(\tau) - K]_+ + [S_2(\tau) - K]_+,$$

the sum of (7.17) and (7.18) should be

$$S_1(0)\Phi\left(\frac{-\kappa_1 + (\delta + \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}\right) - Ke^{-\delta\tau}\Phi\left(\frac{-\kappa_1 + (\delta - \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}\right)$$

$$+ S_2(0)\Phi\left(\frac{-\kappa_2 + (\delta + \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}\right) - Ke^{-\delta\tau}\Phi\left(\frac{-\kappa_2 + (\delta - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}\right).$$

We can verify this algebraically by applying the formulas

$$\Phi_2(a, b; \rho) + \Phi_2(a, -b; -\rho) = \Phi(a)$$

and

$$\Phi_2(a, b; \rho) - \Phi_2(-a, -b; \rho) = \Phi(a) - \Phi(-b)$$
$$= \Phi(a) + \Phi(b) - 1.$$

Johnson [50] also gave formulas for European options on the maximum and the minimum of n risky assets with exercise price K. These formulas are of course special cases of Corollary 4. Let us end this section by showing how to evaluate the first probability term in (6.9) (with $k=1$),

$Pr[S_1(\tau) > K$ and $S_1(\tau)$ ranks first among $\mathbb{S}$; $\mathbf{h}^* + \mathbf{1}_1]$

$$= Pr[S_1(\tau) > K, S_1(\tau) > S_2(\tau), \ldots, S_1(\tau) > S_n(\tau); \mathbf{h}^* + \mathbf{1}_1]. \quad (7.19)$$

Write

$$\mathbf{W} = (0, X_2(\tau), X_3(\tau), \ldots, X_n(\tau))',$$

$$\mathbf{1} = (1, 1, 1, \ldots, 1)'$$

and

$$\mathbf{s} = (\ln[S_1(0)/K], \ln[S_1(0)/S_2(0)], \ln[S_1(0)/S_3(0)], \ldots, \ln[S_1(0)/S_n(0)])'.$$

Let $N_n(\mathbf{x}; \boldsymbol{\mu}, \mathbf{V})$ denote the n-dimensional normal distribution function with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$. Then the probability expressed by (7.19) is the same as

$Pr[\mathbf{W} - X_1(\tau)\mathbf{1} < \mathbf{s}; \mathbf{h}^* + \mathbf{1}_1]$

$$= N_n(\mathbf{s}; E[\mathbf{W} - X_1(\tau)\mathbf{1}; \mathbf{h}^* + \mathbf{1}_1], \tau\mathbf{Y}), \quad (7.20)$$

where $\tau\mathbf{Y} = (\tau y_{ij})$ denotes the covariance matrix of the random vector $\mathbf{W} - X_1(\tau)\mathbf{1}$. By (7.2),

$E[\mathbf{W} - X_1(\tau)\mathbf{1}; \mathbf{h}^* + \mathbf{1}_1] = E(\mathbf{W}; \mathbf{h}^* + \mathbf{1}_1) - E[X_1(\tau)\mathbf{1}; \mathbf{h}^* + \mathbf{1}_1]$

$$= (0, \delta + \sigma_{21} - \tfrac{1}{2}\sigma_{22}, \ldots, \delta + \sigma_{n1} - \tfrac{1}{2}\sigma_{nn})'$$

$$- (\delta + \sigma_{11} - \tfrac{1}{2}\sigma_{11})\mathbf{1}$$

$$= (-\delta, \sigma_{21} - \tfrac{1}{2}\sigma_{22}, \ldots, \sigma_{n1} - \tfrac{1}{2}\sigma_{nn})'$$

$$- \tfrac{1}{2}\sigma_{11}\mathbf{1}.$$

To find the matrix $\mathbf{Y}$, which is independent of $\mathbf{h}^*$, observe that

$$[\mathbf{W} - X_1(\tau)\mathbf{1}][\mathbf{W} - X_1(\tau)\mathbf{1}]' = \mathbf{W}\mathbf{W}' - X_1(\tau)[\mathbf{W}\mathbf{1}' + \mathbf{1}\mathbf{W}'] + [X_1(\tau)]^2\mathbf{1}\mathbf{1}'.$$

Thus

$$y_{11} = \sigma_{11};$$

for $i > 1$,

$$y_{i1} = y_{1i} = -\sigma_{i1} + \sigma_{11};$$

and, for $i \neq 1$ and $j \neq 1$,

$$y_{ij} = y_{ji} = \sigma_{ij} - (\sigma_{i1} + \sigma_{j1}) + \sigma_{11}.$$

As a test of understanding of the method presented in this paper, the interested reader is encouraged to work out all the probability terms for evaluating the options with payoffs $(S_{[1]} - K)_+$ and $(S_{[n]} - K)_+$. Answers can be checked against the published formulas in Johnson's paper [50].

8. CONCLUSION

The option-pricing theory of Black and Scholes [11] is perhaps the most important advance in the theory of financial economics in the past two decades. Their theory has been extended in many directions, usually by applying sophisticated mathematical tools such as stochastic calculus and partial differential equations. A fundamental insight in the development of the theory was provided by Cox and Ross [24] when they pointed out the concept of risk-neutral valuation. This idea was further elaborated on by Harrison and Kreps [42] and by Harrison and Pliska [43] under the terminology of equivalent martingale measure.

Under the assumption of a constant risk-free interest rate, this paper shows how such equivalent martingale measures can be determined for a large class of stochastic models of asset price movements. Any Esscher transform of the stochastic process $\{X(t)\}$ provides an equivalent probability measure for the process; the parameter vector $\mathbf{h}^*$ is chosen such that the equivalent probability measure is also a martingale measure for the discounted value of each primitive security. The price of a derivative security is calculated as the expectation, with respect to the equivalent martingale measure, of the discounted payoffs. In other words, after an appropriate change of probability measure, the price of each security is simply an actuarial present value.

We hope that this paper helps demystify the procedure for valuing European options and other derivative securities. If actuaries can project the cash flow of a derivative security, they can value it by using what they learned as actuarial students—by discounting and averaging. The one difference is that averaging is done with respect to the risk-neutral Esscher measure, which this paper shows how to determine.

REFERENCES

1. ABRAMOWITZ, M., AND STEGUN, I.A. *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards, 1964. Reprint. New York: Dover Publications.

2. ALBRECHT, P. "Premium Calculation without Arbitrage? A Note on a Contribution by G. Venter," *ASTIN Bulletin* 22 (1992): 247–54.

3. ALBRECHT, P., AND MAURER, P. "Portfolio Insurance, Strategien zur Wertsicherung von Aktien-Portefeuilles," *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* 20 (1992): 337–62.

4. BACK, K. "Asset Pricing for General Processes," *Journal of Mathematical Economics* 20 (1991): 371–95.

5. BACK, K., AND PLISKA, S.R. "On the Fundamental Theorem of Asset Pricing with an Infinite State Space," *Journal of Mathematical Economics* 20 (1991): 1–18.

6. BEARD, R.E., PENTIKÄINEN, T., AND PESONEN, E. *Risk Theory: The Stochastic Basis of Insurance.* 3rd ed. London: Chapman and Hall, 1984.

7. BEEKMAN, J.A. *Two Stochastic Processes.* Stockholm: Almqvist & Wiksell, 1974.

8. BELL, I., AND SHERRIS, M. "Greater of Benefits in Superannuation Funds," *Quarterly Journal of the Institute of Actuaries of Australia* (June 1991): 47–63.

9. BERNSTEIN, P.L. *Capital Ideas: The Improbable Origins of Modern Wall Street.* New York: Free Press, 1992.

10. BHATTACHARYA, S., AND CONSTANTINIDES, G., ED. *Theory of Valuation: Frontier of Modern Financial Theory*, Vol. 1. Totowa, N.J.: Rowman & Littlefield, 1989.

11. BLACK, F., AND SCHOLES, M. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (1973): 637–59. Reprinted in [52], 157–74, and in [55], 226–43.

12. BORCH, K. "The Safety Loading of Reinsurance Premiums," *Skandinavisk Aktuarietidskrift* (1960): 163–184. Reprinted in [13], 61–81.

13. BORCH, K. *Economics of Insurance.* Amsterdam: Elsevier, 1990.

14. BOYLE, P.P. "Valuation of Derivative Securities Involving Several Assets Using Discrete Time Methods," *Insurance: Mathematics and Economics* 9 (1990): 131–39.

15. BOYLE, P.P. *Options and the Management of Financial Risk.* Schaumburg, Ill.: Society of Actuaries, 1992.

16. BOYLE, P.P., EVNINE, J., AND GIBBS, S. "Numerical Evaluation of Multivariate Contingent Claims," *Review of Financial Studies* 2 (1989): 241–50.

17. BOYLE, P.P., AND TSE, Y.K. "An Algorithm for Computing Values of Options on the Maximum or Minimum of Several Assets," *Journal of Financial and Quantitative Analysis* 25 (1990): 215–27.

18. BREIMAN, L. *Probability*. Menlo Park, Calif.: Addison-Wesley, 1968. Reprint. Philadelphia: Society of Industrial and Applied Mathematics, 1992.

19. BÜHLMANN, H. "An Economic Premium Principle," *ASTIN Bulletin* 11 (1980): 52–60.

20. BÜHLMANN, H. "The General Economic Premium Principle," *ASTIN Bulletin* 14 (1984): 13–21.

21. CHEYETTE, O. "Pricing Options on Multiple Assets," *Advances in Futures and Options Research* 4 (1990): 69–81.

22. CLANCY, R.P. "Options on Bonds and Applications to Product Pricing," *TSA* XXXVII (1985): 97–130; Discussion 131–51.

23. COX, J.C., AND HUANG, C.F. "Option Pricing Theory and Its Applications." In *Theory of Valuation: Frontier of Modern Financial Theory*, Vol. 1, 272–88, edited by S. Bhattacharya and G. Constantinides. Totowa, N.J.: Rowman & Littlefield, 1989.

24. COX, J.C., AND ROSS, S.A. "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics* 3 (1976): 145–66.

25. COX, J.C., ROSS, S.A., AND RUBINSTEIN, M. "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (1979): 229–63. Reprinted in [55], 244–77.

26. COX, J.C., AND RUBINSTEIN, M. *Options Markets*. Englewood Cliffs, N.J.: Prentice-Hall, 1985.

27. CRAMÉR, H. "Mathematical Risk Theory," *Skandia Jubilee Volume*. Stockholm: Nordiska Bokhandeln, 1955.

28. DALANG, R.C., MORTON, A., AND WILLINGER, W. "Equivalent Martingale Measures and No-Arbitrage in Stochastic Securities Market Models," *Stochastics and Stochastic Reports* 29 (1990): 185–201.

29. DELBAEN, F. "Representing Martingale Measures When Asset Prices Are Continuous and Bounded," *Mathematical Finance* 2 (1992): 107–30.

30. DELBAEN, F., AND HAEZENDONCK, J. "A Martingale Approach to Premium Calculation Principles in an Arbitrage Free Market," *Insurance: Mathematics and Economics* 8 (1989): 269–77.

31. DREZNER, Z. "Computation of the Bivariate Normal Integral," *Mathematics of Computation* 32 (1978): 277–79.

32. DUFFIE, D. *Dynamic Asset Pricing Theory*. Princeton: Princeton University Press, 1992.

33. DUFRESNE, F., GERBER, H.U., AND SHIU, E.S.W. "Risk Theory with the Gamma Process," *ASTIN Bulletin* 21 (1991): 177–92.

34. DYBVIG, P.H., AND ROSS, S.A. "Arbitrage." In *The New Palgrave: A Dictionary of Economics*, Vol. 1, edited by J. Eatwell, M. Milgate and P. Newman, 100–106. London: Macmillan, 1987. Reprinted in Volume 1 of [60], 43–50.

35. ESSCHER, F. "On the Probability Function in the Collective Theory of Risk," *Skandinavisk Aktuarietidskrift* 15 (1932): 175–95.

36. FELLER, W. *An Introduction to Probability Theory and Its Applications*, Vol. 2. 2nd ed. New York: Wiley, 1971.

37. GASTINEAU, G. "An Introduction to Special-Purpose Derivatives: Options with a Payout Depending on More Than One Variable," *Journal of Derivatives* 1, no. 1 (Fall 1993): 98–104.

38. GERBER, H.U. *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation Monograph Series No. 8. Homewood, Ill.: Irwin, 1979.

39. GERBER, H.U. "Actuarial Applications of Utility Functions." In *Advances in the Statistical Sciences*. Vol. VI of *Actuarial Science*, edited by I.B. MacNeill and G.J. Umphrey, 53–61. Dordrecht, Holland: Reidel, 1987.

40. GERBER, H.U., AND SHIU, E.S.W. "Martingale Approach to Pricing Perpetual American Options." In *Proceedings of the 4th AFIR International Colloquium, Orlando*, April 20–22, 1994, 659–89.

41. GIBSON, R. *Option Valuation: Analyzing and Pricing Standardized Option Contracts*. New York: MacGraw-Hill, 1991.

42. HARRISON, J.M., AND KREPS, D.M. "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* 20 (1979): 381–408.

43. HARRISON, J.M., AND PLISKA, S. "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications* 11 (1981): 215–60.

44. HEATH, D.C., AND JARROW, R.A. "Arbitrage, Continuous Trading, and Margin Requirements," *Journal of Finance* 42 (1987): 1129–42.

45. HESTON, S.L. "Invisible Parameters in Option Prices," *Journal of Finance* 48 (1993): 933–47.

46. HOGG, R.V., AND CRAIG, A.T. *Introduction to Mathematical Statistics*. 4th ed. New York: Macmillan, 1978.

47. HULL, J. *Options, Futures, and Other Derivative Securities*, 2nd ed. Englewood Cliffs, N.J.: Prentice-Hall, 1993.

48. INGERSOLL, J.E., JR. *Theory of Financial Decision Making*. Totowa. N.J.: Rowan and Littlefield, 1987.

49. JENSEN, J.L. "Saddlepoint Approximations to the Distribution of the Total Claim Amount in Some Recent Risk Models," *Scandinavian Actuarial Journal* (1991): 154–68.

50. JOHNSON, H. "Options on the Maximum or the Minimum of Several Assets," *Journal of Financial and Quantitative Analysis* 22 (1987): 277–83.

51. KAHN, P.M. "An Introduction to Collective Risk Theory and Its Application to Stop-Loss Reinsurance," *TSA* XIV (1962): 400–25; Discussion, 426–49.

52. KOLB, R.W. ED. *The Financial Derivatives Reader*. Miami: Kolb Publishing, 1992.

53. LAKNER, P. "Martingale Measures for a Class of Right-Continuous Processes," *Mathematical Finance* 3 (1993): 43–53.

54. LELAND, H.E. "Portfolio Insurance." In *The New Palgrave Dictionary of Money and Finance*, Vol. 3, edited by P.K. Newman, M. Milgate, and J. Eatwell, 154–56. London: Macmillan, 1992.

55. LUSKIN, D.L. ED. *Portfolio Insurance: A Guide to Dynamic Hedging*. New York: Wiley, 1988.

56. MADAN, D.B., MILNE, F., AND SHEFRIN, H. "The Multinomial Option Pricing Model and Its Brownian and Poisson Limits," *Review of Financial Studies* 2 (1989): 251–65.

57. MARGRABE, W. "The Value of an Option to Exchange One Asset for Another," *Journal of Finance* 33 (1978): 177–86.

58. MARGRABE, W. "Triangular Equilibrium and Arbitrage in the Market for Options to Exchange Two Assets," *Journal of Derivatives* 1, no. 1 (Fall 1993): 60–69.

59. MÜLLER, S.M. "On Complete Securities Markets and the Martingale Property of Securities Prices," *Economic Letters* 31 (1989): 37–41.

60. NEWMAN, P.K., MILGATE, M., AND EATWELL, J., EDS. *The New Palgrave Dictionary of Money and Finance*, 3 vols. London: Macmillan, 1992.

61. PAGE, F.H., JR., AND SANDERS, A.B. "A General Derivation of the Jump Process Option Pricing Formula," *Journal of Financial and Quantitative Analysis* 21 (1986): 437–46.

62. PANJER, H.H., AND WILLMOT, G.E. *Insurance Risk Models*. Schaumburg, Ill.: Society of Actuaries, 1992.

63. PEDERSEN, H.W., SHIU, E.S.W., AND THORLACIUS, A.E. "Arbitrage-Free Pricing of Interest-Rate Contingent Claims," *TSA* XLI (1989): 231–65; Discussion 267–79.

64. PLATT, R.B., ED. *Controlling Interest Rate Risk: New Techniques and Applications for Money Management*. New York: Wiley, 1986.

65. RENDLEMAN, R.J., JR., AND BARTTER, B.J. "Two State Option Pricing," *Journal of Finance* 34 (1979): 1093–1110. Reprinted in [52], 139–56.

66. RITCHEN, P., AND SANKARASUBRAHMANIAN, L. "Pricing the Quality Option in Treasury Bond Futures," *Mathematical Finance* 2 (1992): 197–214.

67. SCHACHERMAYER, W. "A Hilbert Space Proof of the Fundamental Theorem of Asset Pricing in Finite Discrete Time," *Insurance: Mathematics and Economics* 11 (1992): 249–57.

68. SCHACHERMAYER, W. "A Counterexample to Several Problems in the Theory of Asset Pricing," *Mathematical Finance* 3 (1992): 217–29.

69. SCHWEIZER, M. "Martingale Densities for General Asset Prices," *Journal of Mathematical Economics* 21 (1992): 363–78.

70. SEAL, H.L. *The Stochastic Theory of a Risk Business*. New York: Wiley, 1969.

71. SHERRIS, M. "Reserving for Deferred Capital Gains Tax (An Application of Option Pricing Theory)," *Journal of the Institute of Actuaries* 119 (1992): 45–67.

72. SHERRIS, M. "Contingent Claims Valuation of 'Greater of' Benefits," *ARCH* 1993.1, 291–309.

73. SONDERMAN, D. "Reinsurance in Arbitrage-Free Markets," *Insurance: Mathematics and Economics* 10 (1991): 191–202.

74. STOLL, H.R., AND WHALEY, R.E. *Futures and Options: Theory and Applications*. Cincinnati: South-Western, 1993.

75. STULZ, R. "Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications," *Journal of Financial Economics* 10 (1982): 161–85.

76. TILLEY, J.A., AND LATAINER, G.D. "A Synthetic Option Framework for Asset Allocation," *Financial Analysts Journal* 41 (May/June 1985): 32–43. Reprinted as Chapter 12 of [64], 384–400.

77. VENTER, G.G. "Premium Calculation Implications of Reinsurance without Arbitrage," *ASTIN Bulletin* 21 (1991): 223–30.

78. VENTER, G.G. "Premium Calculation without Arbitrage: Author's Reply on the Note by P. Albrecht," *ASTIN Bulletin* 22 (1992): 255–56.

79. WOODDY, J.C. "Risk Theory," *Part 5 Study Note 55.1.71*. Chicago: Society of Actuaries, 1971.

DISCUSSION OF PRECEDING PAPER

JACQUES F. CARRIERE:

First, I greatly enjoyed this paper. Second, I want to embellish it by showing that the valuation formula (2.15) for a European call option is the same as the formula for the American call option because the early-exercise privilege has no value.

Let us give some definitions and facts about martingales, submartingales and stopping times. Let $\{e^{-\delta t}S_t, \mathcal{H}_t\}_{t\geq 0}$ denote a discounted price process adapted to the history (sub-σ-field) $\mathcal{H}_t$. In this discussion $\mathcal{H}_t = \sigma\{S_s: s\leq t\}$. Let us assume that this process is a martingale with a measure P^* and an expectation operator E^*. From the definition of a martingale, we know that $E^*(e^{-\delta t}S_t|\mathcal{H}_s) = e^{-\delta s}S_s$ for all $0\leq s\leq t$. The intrinsic value of the American option at time t is $I_t = \max\{0, S_t - K\}$. Let us prove that the process $\{e^{-\delta t}I_t, \mathcal{H}_t\}_{t\geq 0}$ is a submartingale. This is true because

$$E^*(e^{-\delta t}I_t|\mathcal{H}_s) \geq \max\{0, E^*(e^{-\delta t}S_t|\mathcal{H}_s) - e^{-\delta t}K\}$$

$$= \max\{0, e^{-\delta s}S_s - e^{-\delta t}K\} \geq \max\{0, e^{-\delta s}S_s - e^{-\delta s}K\}$$

$$= e^{-\delta s}I_s.$$

Let $\tau \in \mathcal{R}$ be the latest time that the option can be exercised and let ξ denote any stopping time in the class $\mathcal{C}$, where $\mathcal{C}$ is the collection of all stopping times with the property that $P^*(0\leq\xi\leq\tau) = 1$. Remember that ξ is a stopping time whenever $[\xi\leq t]\in\mathcal{H}_t$, $\forall t\geq 0$. Intuitively, the class $\mathcal{C}$ represents all the trading strategies available to the investor. Using Doob's optional sampling theorem (Wong and Hajek[*]), we find that

$$E^*(e^{-\delta\xi}I_\xi|\mathcal{H}_0) \leq E^*(e^{-\delta\tau}I_\tau|\mathcal{H}_0).$$

This means that the optimal stopping time is simply τ because

$$\sup_{\xi\in\mathcal{C}} E^*(e^{-\delta\xi}I_\xi|\mathcal{H}_0) = E^*(e^{-\delta\tau}I_\tau|\mathcal{H}_0).$$

In other words, the optimal trading strategy is to hold the call option until the latest exercise date τ. So the value of the American option, denoted as

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[*]WONG, E. AND HAJEK, B. *Stochastic Processes in Engineering Systems*. New York: Springer-Verlag, 1985, p. 213.

$$\sup_{\xi \in \mathscr{C}} E^*(e^{-\delta\xi}I_\xi|\mathscr{H}_0),$$

is equal to the value of the European option, denoted as $E^*(e^{-\delta\tau}I_\tau|\mathscr{H}_0)$. The preceding result in continuous time also holds in discrete time, and so the formula for the European call option under the discrete-time binomial model, Equation (3.3.7), can also be used to value the American counterpart.

At the end of Section 6, the paper gives an incomplete proof that the formula in Corollary 4 can be used to value the American exotic call option with an intrinsic value of

$$I_t = \max\{0, \max_{1 \le j \le n}(S_{j,t}) - K\}.$$

Let me give a detailed proof that the best early exercise rule is to exercise at the last exercise date, τ. Consider the discounted price process $\{e^{-\delta t}\mathbf{S}_t, \mathscr{H}_t\}_{t \ge 0}$ adapted to the history $\mathscr{H}_t = \sigma\{\mathbf{S}_s: s \le t\}$, where $\mathbf{S}_t = (S_{1,t}, \ldots, S_{n,t})^T$. Let us assume that there is a measure P^* and an expectation operator E^*, so that the discounted prices are martingales. Using Jensen's inequality twice, we find that the process $\{e^{-\delta t}I_t, \mathscr{H}_t\}_{t \ge 0}$ is a submartingale, which is a sufficient condition to invoke Doob's optional sampling theorem and find that

$$E^*(e^{-\delta\xi}I_\xi|\mathscr{H}_0) \le E^*(e^{-\delta\tau}I_\tau|\mathscr{H}_0)$$

for all $\xi \in \mathscr{C}$.

SAMUEL H. COX:

This paper is the beginning of a series of papers ([2], [3], [4]) on option pricing, in which the authors use the Esscher transform, optional sampling, and other martingale methods to dramatically improve known results and establish new ones. It is good to see actuaries making such valuable contributions to modern financial mathematics. I congratulate the authors of this fine work and thank them for choosing the *Transactions* to publish the first part of the story.

The Gerber and Shiu approach to finding a convenient equivalent martingale measure is closely related to recent work by Geman, El Karoui, and Rochet [1]. They work in a more general setting and obtain some formally similar results. For example, they obtain a general formula for the option to exchange two assets and a general formula for compound options. However, they do not obtain the explicit formulas that flow so

easily from the Gerber and Shiu method. For those who are prepared for the more general setting, I recommend reading Geman et al.

Another application of the Gerber and Shiu method is the valuing of compound European options. The valuation of these options under the usual assumption of lognormal prices is due to Geske [5]. I begin with the same framework as the authors and show ultimately that Geske's results are a special case, in the same way the authors generalized the Black-Scholes formula and other results.

Consider first a European call option on a European call option. This option, and the put option on a call option, is of interest because of the interpretation of equity of a levered firm as a call option. This is discussed by Merton [6, p. 394] and others. The gist of it is this: When a firm issues bonds, the shareholders are in a sense selling the assets of the firm to the bondholders for cash. If the firm is successful, the stockholders will pay the principal and interest and reclaim the assets. If the firm fails, the stockholders will default on the debt, leaving the bondholders with the assets. When the value of a firm is assumed to be the value of a call option in this way, then publicly traded options on the firm's stock are compound options. In this way, calls on calls and puts on calls become interesting contracts to price.

Consider a European call option written on the stock with exercise price K_1 at time T_1. The value of the call option at time $t \leq T_1$ can be found by applying (2.11) at time t. In applying the formula, use

$$S(T_1) = S(t) \exp(X(T_1) - X(t))$$

and the independent and stationary increments properties of X. This allows us to replace $S(0)$ by $S(t)$ and τ by $T_1 - t$ and calculate the expectation at time t. The parameter h^* does not change. Hence, according to (2.11), the price of the call is

$$C_1(t, S(t)) = E_t[e^{-\delta(T_1 - t)}(S(t)e^{X(T_1) - X(t)} - K_1)_+; h^*]$$
$$= E_t[e^{-\delta(T_1 - t)}(S(T_1) - K_1)_+; h^*],$$

where the t in the symbol E_t means that the expectation is calculated at time t conditionally on the value of S at time t. The authors' main result, the theorem of Section 6, includes many special cases given earlier in the paper, as the authors are aware. For example, if we let $g(s)=I(s>K_1)$ be the indicator function, 0 if $s \leq K_1$ and 1 if $s > K_1$, then the theorem yields Formula (2.15) as follows:

$$C_1(t, S(t)) = E_t[e^{-\delta(T_1-t)}(S(T_1) - K_1)_+; h^*]$$

$$= E_t[e^{-\delta(T_1-t)}(S(T_1) - K_1)g(S(T_1)); h^*]$$

$$= E_t[e^{-\delta(T_1-t)}S(T_1)g(S(T_1)); h^*] - E_t[e^{-\delta(T_1-t)}K_1g(S(T_1)); h^*]$$

$$= S(t)E_t[g(S(T_1)); h^* + 1] - e^{-\delta(T_1-t)}K_1E_t[g(S(T_1)); h^*].$$

The compound option is a call option written on the call C_1 with exercise price K_2 at time $T_2 < T_1$. The discounted option price process $e^{-\delta t}C_1(t, S(t))$ is a martingale under the distribution determined by h^*. This is clear because of the first formula immediately above and the law of iterated expectations. This means that we do not need to change h^* when evaluating options on the call option.

The critical stock price, $S^*(T_2)$, for exercising the call on the call at time T_2 can be determined by solving $C_1(T_2, S) = K_2$ for the stock price. The events $S(T_2) > S^*(T_2)$ and $C_1(T_2, S(T_2)) > K_2$ are equivalent. Now, applying (2.11) again but with C_1 in place of S, we find that price of the compound call on a call option is

$$C_2(0, S(0)) = E[e^{-\delta T_2}C_1(T_2, S(T_2))I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_2}K_2E[I(S(T_2) - S^*(T_2)); h^*],$$

where I is the indicator function. Now we can substitute for the call on the stock and simplify to obtain

$$C_2(0, S(0)) = E[e^{-\delta T_2}S(T_2)E_{T_2}[I(S(T_1) - K_1); h^* + 1]I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_1}K_1E[E_{T_2}[I(S(T_1) - K_1); h^*]I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_2}K_2E[I(S(T_2) - S^*(T_2)); h^*].$$

Now apply the theorem again but use

$$g(s) = E_{T_2}[I(S(T_1) - K_1)|S(T_2) = s; h^* + 1]I(s - S^*(T_2))$$

in the first term to obtain

$$C_2(0, S(0)) = E[e^{-\delta T_2}S(T_2)g(S(T_2)); h^*]$$

$$- e^{-\delta T_1}K_1E[I(S(T_1) - K_1)I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_2}K_2E[I(S(T_2) - S^*(T_2)); h^*]$$

$$= S(0)E[g(S(T_2)); h^* + 1]$$

$$- e^{-\delta T_1}K_1E[I(S(T_1) - K_1)I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_2}K_2E[I(S(T_2) - S^*(T_2)); h^*]$$

$$= S(0)E[E_{T_2}[I(S(T_1) - K_1); h^* + 1]I(S(T_2) - S^*(T_2)); h^* + 1]$$

$$- e^{-\delta T_1}K_1E[I(S(T_1) - K_1)I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_2}K_2E[I(S(T_2) - S^*(T_2)); h^*]$$

$$= S(0)E[I(S(T_1) - K_1)I(S(T_2) - S^*(T_2)); h^* + 1]$$

$$- e^{-\delta T_1}K_1E[I(S(T_1) - K_1)I(S(T_2) - S^*(T_2)); h^*]$$

$$- e^{-\delta T_2}K_2E[I(S(T_2) - S^*(T_2)); h^*].$$

Now let $\kappa_1 = \ln(K_1/S(0))$ and $\kappa_2 = \ln(S^*(T_2)/S(0))$. These substitutions yield a formula that generalizes Geske's formula. It involves the joint distribution of $X(T_1)$ and $X(T_2)$ relative to the Esscher parameters h^* and $h^* + 1$.

$$C_2(0, S(0)) = S(0)\Pr[X(T_1) > \kappa_1, X(T_2) > \kappa_2; h^* + 1]$$

$$- e^{-\delta T_1}K_1 \Pr[X(T_1) > \kappa_1, X(T_2) > \kappa_2; h^*]$$

$$- e^{-\delta T_2}K_2 \Pr[X(T_2) > \kappa_2; h^*]$$

In order to obtain formulas under various distribution assumptions, one has to be able to determine the joint probability of the event

$$X(T_1) > \kappa_1, X(T_2) > \kappa_2.$$

Under the authors' specifications for $\{X(t)\}$, one can calculate this joint probability.

Due to the stationary independent increments of X, it is easy to show that

$$E[X(t); h] = t\mu(h)$$

$$\mathrm{Var}[X(t); h] = t\sigma^2(h),$$

where $\mu(h)$ and $\sigma(h)$ are constants. For $i = 1$ and 2, define new standardized variables as

$$Z_i(h) = \frac{\mu(h) - X(T_i)}{\sigma(h)\sqrt{T_i}}.$$

The new variables have zero means and unit variances. Their covariance is

$$\text{Cov}(Z_1(h), Z_2(h); h) = \frac{1}{\sigma^2(h)\sqrt{T_1 T_2}} \text{Cov}(X(T_1), X(T_2); h)$$

$$= \frac{1}{\sigma^2(h)\sqrt{T_1 T_2}} \text{Cov}(X(T_2) + \{X(T_1) - X(T_2)\}, X(T_2); h)$$

$$= \frac{1}{\sigma^2(h)\sqrt{T_1 T_2}} \{\text{Var}(X(T_2); h) + \text{Cov}(X(T_1) - X(T_2), X(T_2); h)\}.$$

Because the Esscher transform of X has independent increments, the second term is zero, so we have

$$\text{Cov}(Z_1(h), Z_2(h); h) = \frac{1}{\sigma^2(h)\sqrt{T_1 T_2}} \sigma^2(h)T_2$$

$$= \sqrt{\frac{T_2}{T_1}}.$$

In the case in which $\{X(t)\}$ is a Wiener process with parameters μ and σ^2, the Esscher transform is a Wiener process with parameters $\mu + h\sigma^2$ and σ^2. The new variables $Z_1(h)$ and $Z_2(h)$ have a bivariate standard normal distribution with covariance parameter $\sqrt{T_2/T_1}$. The compound call-option-on-call-option formula can be written as follows:

$$C_2(0, S(0)) = S(0)\Phi\left(d_1, d_2; \sqrt{\frac{T_2}{T_1}}\right)$$

$$- e^{-\delta T_1}K_1\Phi\left(d_1 - \frac{\sigma}{2}\sqrt{T_1}, d_2 - \frac{\sigma}{2}\sqrt{T_1}; \sqrt{\frac{T_2}{T_1}}\right)$$

$$- e^{-\delta T_2}K_2\Phi\left(d_2 - \frac{\sigma}{2}\sqrt{T_1}\right),$$

where $\Phi(x, y; r)$ denotes the standard bivariate normal distribution with covariance parameter r; $\Phi(x)$ denotes the standard normal distribution; and the parameters are determined as follows:

$$d_1 = \frac{-\kappa_1 + \mu(h^* + 1)T_1}{\sigma\sqrt{T_1}} = \frac{\ln(S(0)/K_1) + (\delta + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}$$

$$d_2 = \frac{-\kappa_2 + \mu(h^* + 1)T_2}{\sigma\sqrt{T_2}} = \frac{\ln(S(0)/S^*(T_2)) + (\delta + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}.$$

$S^*(T_2)$ is obtained by solving this equation for S:

$$S\Phi\left(\frac{\ln(S/K_1) + (\delta + \sigma^2/2)(T_1 - T_2)}{\sigma\sqrt{T_1 - T_2}}\right)$$

$$- e^{-\delta(T_1 - T_2)}\Phi\left(\frac{\ln(S/K_1) + (\delta - \sigma^2/2)(T_1 - T_2)}{\sigma\sqrt{T_1 - T_2}}\right) = K_2.$$

In the authors' reference [47], Hull discusses Geske's formula for a stock that pays dividends at a constant known rate q. This is actually only a slight generalization of the authors' framework. The authors' later work applies to stocks with dividends of this type, and this deviation of Geske's formula is valid in that generality. The formula for the put option on a call option can be obtained easily by applying the put-call parity relation.

REFERENCES

1. GEMAN, HÉLYETTE, EL KAROUI, NICOLE, AND ROCHET, JEAN-CHARLES. "Change of Numéraire, Changes of Probability Measure and Option Pricing," AFFI Conference, Paris, 1992; *Journal of Applied Probability*, in press.

2. GERBER, HANS U., AND SHIU, E.S.W. "Martingale Approach to Pricing Perpetual American Options," *Proceedings of the 1994 4th AFIR International Colloquium*, Orlando, April 20–22, 1994, 659–89.
3. GERBER, HANS U., AND SHIU, E.S.W. "From Perpetual Strangles to Russian Options," *Insurance: Mathematics and Economics,* in press.
4. GERBER, HANS U., AND SHIU, E.S.W. "Pricing Financial Contracts with Indexed Homogeneous Payoff," *Bulletin of the Swiss Association of Actuaries* 94 (1994): 143–66.
5. GESKE, R. "The Valuation of Compound Options," *Journal of Financial Economics* 7 (1979): 63–81.
6. MERTON, ROBERT C. *Continuous-Time Finance.* Cambridge, Mass.: Basil Blackwell, 1990.

F. DELBAEN*, W. SCHACHERMAYER[†] and M. SCHWEIZER[‡]:

The authors use the well-known Esscher transform to give a partial solution to the problem of finding option prices. It is only a partial solution since, as this discussion will show, there are many other ways of obtaining a reasonable price. The technique used in mathematical finance to study a price process is related to martingale theory via a change of measure. The basic papers are Harrison and Kreps [7], Harrison and Pliska [8], and Kreps [9]. A general version of the theorem, which clarifies how far this change of measure technique goes, can be found in a recent paper by Delbaen and Schachermayer [3]. The martingale technique is illustrated in this paper by Gerber and Shiu.

The paper considers exponentials of processes with stationary and independent increments. The use of the moment-generating function and the Levy formula is restricted to these processes. The authors then use these exponentials to give an interpretation of the Esscher transform as a change of measure. The new measure is equivalent to the original measure, as stated but not proved. The parameter is chosen in such a way that the price process becomes a martingale under the new measure. The authors explicitly mention the existence of infinitely many other measures such that the original process becomes a martingale. In such a

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situation, the Esscher transform is one of many ways of assigning a value to a European option.

In the case in which there is only one risk-neutral measure, the case of the examples given in Section 3, the choice is obvious. Pricing of contingent claims is reduced to taking expectations with respect to this unique risk-neutral measure. Even more is true (and this is particularly important in risk management); any contingent claim can be hedged! This means that an economic agent can construct a portfolio of the risk-less asset and of the risky asset in such a way that at the end of the horizon, the value of the portfolio gives exactly the same payoff as the contingent claim.

If the set of equivalent risk-neutral measures is not reduced to one point, then finding such hedging strategies is no longer possible. The initial investment needed to reproduce the contingent claim is not de-fined, and in this sense there is no natural price for the claim under consideration. This is, for example, the case when dealing with the ex-ponential of a shifted compound Poisson process. If the price process is of such a form, then a European option has no uniquely defined price, and statements as in the second section, Formula (2.11), can be ambig-uous. There is no price of the option that is uniquely determined by arbitrage arguments. The best one can say is that there is an interval of values with the property that as soon as an agent is offered a price outside this interval, then she can construct a position consisting of the option, the riskless asset and the risky asset in such a way that she can make arbitrage profits. The hedging problem calls for new concepts and tech-niques that can, for example, be found in work of El Karoui, Föllmer-Schweizer, Karatzas, Kramkov, Shäl, and Schweizer.

The minimax theorem proved by Delbaen and Schachermayer [4] shows the following result: The minimal investment needed to construct a port-folio that gives a terminal outcome dominating the contingent claim f is given by

$$\alpha = \sup\{\mathbb{E}_Q[f] | Q \in M^e(\mathbb{P})\},$$

where $M^e(\mathbb{P})$ is the set of all equivalent martingale mesaures for the given price process. It follows that an agent who has sold the claim f, for example, an option, and wants to hedge the risk taken will need at least the amount α. A smaller investment will not allow her to construct a portfolio that yields a final outcome that is at least as big as f, that is,

is on the safe side. Under certain boundedness assumptions on the claim
f, a similar interpretation can be given for the quantity

$$\beta = \inf\{\mathbb{E}_Q[f] | Q \in M^e(\mathbb{P})\}.$$

All numbers in the interval $[\beta, \alpha]$ represent prices that are feasible and
do not allow arbitrage profits. A price bigger than α allows the agent

(1)   To sell the option
(2)   To use the amount α to construct a hedging portfolio that gives an
      outcome of at least f
(3)   To cash the difference between the price of the option and α.
(4)   At the end of the horizon, the constructed portfolio has a value of
      at least f, and selling this portfolio covers the position taken and
      possibly gives an extra income.

If the price is below β, a profit can in the same way be made from the
claim $-f$ by buying the option and selling a hedge portfolio.

In an economic setting in which utility functions are present and prices
will be fixed to obtain an equilibrium, the market price will agree with
a price in the interval $[\beta, \alpha]$. A justification for a particular price does
not come from arbitrage considerations alone, but from utility consid-
erations, that is, from the risk averseness of all the agents.

The three examples in Section 3 have the property that there is only
one risk-neutral measure. Every contingent claim can be hedged, and the
price of, for example, a European option is uniquely defined. It is both
the expected value of the discounted contingent claim with respect to the
risk-neutral measure and the initial investment needed to finance the
hedging portfolio.

In the examples of Section 4, the situation is different. Both models,
the shifted gamma as well as the shifted inverse normal, have infinitely
many risk-neutral measures. Even when the property of stationary and
independent increments has to be preserved, there is still an infinite num-
ber of risk-neutral measures. Only under the very restrictive assumption
that the process under the new measure should remain a shifted gamma
and a shifted inverse normal, respectively, do we obtain uniqueness. The
paper gives no economic or actuarial argument why this has to be the
case. The authors mention also that all but one parameter could be re-
covered from looking at a trajectory. This is indeed the case, but a hint
as to how it could be done would have been welcome.

BIBLIOGRAPHY

1. ANSEL, J.P., AND STRICKER, C. "Lois de Martingale, Densitée et Décomposition de Föllmer Schweizer," *Annales de L'Institut Henri Poincaré* 28 (1992): 375–92.
2. DUFFIE, D., AND RICHARDSON, H.R. "Mean Variance Hedging in Continuous Time," *Annals of Applied Probability* 1 (1991): 1–15.
3. DELBAEN, F., AND SCHACHERMAYER, W. "A General Version of the Fundamental Theorem of Asset Pricing," *Mathematische Annalen* 300 (1994): 463–520.
4. DELBAEN, F., AND SCHACHERMAYER, W. "The No-Arbitrage Property under a Change of Numeraire," working paper.
5. DELBAEN, F., AND SCHACHERMAYER, W. "The Variance Optimal Martingale Measure for Continuous Processes," working paper.
6. FÖLLMER, H., AND SCHWEIZER, M. "Hedging of Contingent Claims under Incomplete Information." In *Applied Stochastic Analysis, Stochastic Monographs*, edited by M.H.A. Davis and R.J. Elliott. London/New York: Gordon and Breach, 1991, 389–414.
7. HARRISON, M.J., AND KREPS, D.M. "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* 20 (1979): 381–408.
8. HARRISON, M.J., AND PLISKA, S.R. "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications* 11 (1981): 215–60.
9. KREPS, D.M. "Arbitrage and Equilibrium in Economics with Infinitely Many Commodities," *Journal of Mathematical Economics* 8 (1981): 15–35.
10. SCHÄL, M. "On Quadratic Cost Criteria for Option Hedging," *Mathematics of Operations Research* 19 (1994): 121–31.
11. SCHWEIZER, M. "Mean-Variance Hedging for General Claims," *Annals of Applied Probability* 2 (1992): 171–79.
12. SCHWEIZER, M. "Martingale Densities for General Asset Prices," *Journal of Mathematical Economics* 21 (1992): 363–78.
13. SCHWEIZER, M. "Approximation Pricing and the Variance Optimal Martingale Measure," working paper.
14. SCHWEIZER, M. "Variance-Optimal Hedging in Discrete Time," *Mathematics of Operations Research* 20, no. 1 (February 1995): 1–32.

HÉLYETTE GEMAN*:

The paper by Drs. Gerber and Shiu proposes a new approach to the pricing of contingent claims by introducing the Esscher transforms of price processes. These Esscher transforms are shown to transform Wiener processes into Wiener processes, that is, preserve the "canonical" assumption in finance on the dynamics of the stock price; the same property holds for a shifted Poisson process. But the main, and important, result in the paper is the fact that in the class of Esscher transforms, there is a unique one for which the corresponding probability measure, the so-called risk-neutral Esscher measure, makes martingales out of the discounted prices of the basic securities.

This property is then extended to contingent claims, just by stating that any contingent claim has a unique price equal to the expectation under this risk-neutral Esscher measure of the discounted terminal payoff.

In my comments, I focus more on the economics than on the mathematics.

The paper, not to be misleading, should state clearly that the probability measures associated with these Esscher transforms represent a very particular subset of the set of all equivalent measures, and that the uniqueness of the equivalent martingale measure holds within this particular subset. In the same spirit, a discussion on this new approach would have been welcome, as would a discussion of the important work on arbitrage pricing started by Harrison and Kreps and by Harrison and Pliska and developed more recently by Dalang, Morton and Willinger; Stricker; Artzner and Delbaen; and Schachermayer, where the "fundamental theorem of asset pricing" has precisely been extended to a more and more general setting.

The paper does not offer an alternative to the notion of self-financing replicating portfolios or attainable contingent claims that are fundamental for pricing and hedging contingent claims, and this latter problem is in fact the crucial one in finance. Consequently, one may conclude from the paper that every derivative security has a unique price and that markets are complete, while the issue at stake today is the incompleteness of *real* financial markets (stochastic volatility is already an obvious source of incompleteness).

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The property of preserving the nature of some classes of stochastic processes such as geometric Brownian motions and Poisson processes, which could argue in favor of these Esscher transforms, does not appear to be fundamental.

The authors need to assume a constant interest rate, because the discounted price process

$$\left\{ S_t \exp\left(-\int_0^t r(s)\mathrm{d}s \right) \right\}_{t \geq 0}$$

is not of same nature as $\{S_t\}_{t \geq 0}$ when interest rates are stochastic (their dynamics being driven by any of the models of interest rates that have been developed over the last 17 years).

In 1989, Geman (Doctoral Dissertation, University Paris I), for general models of interest rates, and Jamshidian [7], for Gaussian interest rates, independently offered the forward neutral probability (or forward risk-adjusted, depending on the author) as a general method for pricing a random cash flow under stochastic interest rates.

This methodology has been used in a great number (see the Bibliography) of problems in finance and insurance, the first one being the extension of the Black-Scholes formula to stochastic interest rates. This change of probability measure consists in taking as a new numéraire the zero coupon bond maturing at time T. The methodology of numéraire changes can be successfully extended to the pricing of different types of contingent claims (exchange options, quanto options, and so on). Certainly, the nature of the price process is changed, but the gain in pricing and hedging a given contingent claim by using the appropriate numéraire is remarkable, as far as both the mathematics and the economic intuition are concerned. For instance, in the option of exchanging an asset S_2 for an asset S_1, the important quantity is the price of S_2 expressed in the numéraire S_1. This explains why the interest rate does not appear in the option price formula (see the remark by Gerber and Shiu, p. 129), and consequently why Margrabe's formula also holds under stochastic interest rates, the relevant volatility being the volatility of S_2/S_1 (involving in particular the correlation between S_1 and S_2). In the same manner, floating-strike Asian options are "easily" priced using the stock price itself as the numéraire.

In summary, my point is that:

1. These numéraire changes do not preserve the nature of the initial price process.
2. They induce, however, a remarkable simplification in the valuation of nonstandard contingent claims (a few lines to prove Margrabe's formula in a general setting or to price a cross currencies option or a quanto option).
3. They have a clear economic interpretation, which extends to exhibiting the hedging portfolio.

In conclusion, I think that in order to make their interesting and well-written paper even more convincing, Drs. Gerber and Shiu should put more emphasis on the fact that the explicit formulation of the risk-neutral Esscher transform may allow one to compute option prices in a framework less well-known than the one addressed in their paper. This would give more power to this elegant manner of obtaining results fairly classical by now.

BIBLIOGRAPHY

1. ALBIZZATI, M.O., AND GEMAN, H. "Interest Rate Risk Management and Valuation of the Surrender Option in Life Insurance Policies," *Journal of Risk and Insurance* 61, no. 4 (December 1994): 616–37.
2. EL KAROUI, N., AND GEMAN, H. "Noteworthy: A Stochastic Approach to Pricing FRNs," *RISK* 4, no. 3 (1991): 58–60.
3. EL KAROUI, N., AND GEMAN, H. "A Probabilistic Approach to the Valuation of Floating Rate Notes with an Application to Interest Rate Swaps," *Advances in Options and Futures Research* 1 (1994): 41–64.
4. GEMAN, H. "The Importance of the Forward Neutral Probability in a Stochastic Approach to Interest Rates," ESSEC Working Paper 1989; copies available upon request from ESSEC (Ecole Superieure des Sciences Economiques at Commerciales), Avenue Bernard Hirsch, B.P. 105/95021, Cergy-Pontoise Cedex, France.
5. GEMAN, H., EL KAROUI, N., AND ROCHET, J.C. "Changes of Numéraire; Changes of Probability Measures and Option Pricing," AFFI Conference, Paris, 1992; *Journal of Applied Probability*, in press.
6. GEMAN, H., AND YOR, M. "Bessel Processes Asian Options and Perpetuities," *Mathematical Finance* 3, no. 4 (October 1993): 349–75.
7. JAMSHIDIAN, F. "An Exact Bond Option Formula," *Journal of Finance* 44, no. 1 (March 1989): 205–209.

JAMES C. HICKMAN AND VIRGINIA R. YOUNG:

We congratulate Drs. Gerber and Shiu for ingeniously deriving results that usually come from a great deal of messy calculating. Even though

the second discussant knew next to nothing about martingales when she first read their paper, she was able to follow their reasoning because of the elegance of their development. We thank the authors for bringing an otherwise inaccessible topic within the grasp of the average actuary.

The reason we write is to comment on statements of Drs. Shiu and Gerber that appear toward the end of their introduction and in Section 2.1, Remarks. They point out that the absence of arbitrage is "essentially equivalent" to the existence of an equivalent risk-neutral probability measure, or martingale measure. We discuss this topic only to expand on what the authors have written, not to contradict it. Much of what we have learned about this equivalence comes from Back and Pliska (ref. [5] in the paper), a very illuminating article. In addition to describing the results of Back and Pliska, we comment on how the absence of arbitrage is connected with the axioms of subjective probability.

An arbitrage opportunity is an investment strategy that guarantees a positive payoff in some contingency with no chance of a negative payoff and with no net investment. One may also think of arbitrage as the simultaneous purchase and sale of the same or equivalent security to profit from price discrepancies. The no-arbitrage assumption has also been called a requirement that the market be internally consistent.

As Drs. Gerber and Shiu note, in a finite, discrete-time model, the absence of arbitrage is completely equivalent to the existence of an equivalent martingale measure. Simply stated, an equivalent risk-neutral probability measure, or martingale measure, is a probability measure that has the same null events as the original measure and for which the value of a security is the expectation of its present value, as Drs. Gerber and Shiu mention in their introduction. Also, examine their equation at the bottom of page 103.

Back and Pliska consider the care for which the state space is infinite. They remark that it is known that the existence of a martingale measure implies the absence of arbitrage. They next show that the converse is not generally true by giving an example of a market in which there is no arbitrage and for which no risk-neutral probability measure exists. In their example, the market has a countably infinite number of possible trading dates in a finite interval $[0, T]$, $T < \infty$.

Back and Pliska consider other conditions related to the absence of arbitrage and the existence of a martingale measure. For example, "in great generality," one can show that the existence of an optimal demand for some agent who prefers more to less implies no arbitrage. Such an

investor has an optimal demand if there exists a contingent claim such that the price of the claim is less than or equal to the initial wealth of the investor and if the investor prefers that claim to any other affordable claim. Buried in this definition is a utility function that is increasing with respect to increasing wealth.

Also, they show that the existence of a linear pricing rule implies no arbitrage. A linear pricing rule is a positive linear function on the space of contingent claims such that the value of a contingent claim is its price. A linear function is a function that is commutative with respect to sums of claims and scalar products of claims.

Back and Pliska finish by stating conditions under which the absence of arbitrage implies the existence of a martingale measure. One set of circumstances is that there are only finitely many trading dates and that trading strategies are bounded stochastic processes. Another set of conditions is given in a rather long proposition, so we will not repeat it here. Please refer to their paper for more details.

When an equivalent martingale measure exists, the price for a contingent claim that pays 1 unit is simply the probability of the contingent event occurring, ignoring the value of time (Harrison and Kreps, ref. [42] in the paper). This property and the internal consistency required by no-arbitrage call to mind the axioms for personalistic, or subjective, probability. Jones [2] introduced the personalistic interpretation of probability to the Society of Actuaries. His development is based on work by de Finetti, an Italian actuary, mathematician, and philosopher. De Finetti proposed that one could elicit a given person's subjective probability as follows: The probability, $p(E)$, of an event, E, occurring is the amount of money that the person is willing to pay in exchange for 1 unit if the event were to occur. Conversely, one requires that the person also be willing to accept the reverse bet, that is, pay $1-p(E)$ in exchange for 1 unit if the event were not to occur.

In addition, the probabilities are to be consistent by not allowing a person to set up a series of gambles that guarantees a gain for at least one outcome and no loss for any outcome; that is, arbitrage is not permitted. This absence of arbitrage is a key consistency requirement for individuals assigning probabilities and is a characteristic of markets to which the developments of this paper are applicable.

In his survey article, Ellerman [1] points out this close relationship between the no-arbitrage assumption of a market and the requirement for consistency in subjective probability. On the other hand, one can argue

that this resemblance is only superficial because if a martingale measure determines the prices for the market, then how can it be a *subjective* probability measure? Does the market somehow "crunch" the subjective probability measures of all the market players? Is this "crunching" a weighted average according to the money each invests? Or can invest?

As one can see, our knowledge is limited, but our curiosity is not.

REFERENCES

1. ELLERMAN, D.P. "Arbitrage Theory: A Mathematical Introduction," *SIAM Review* 26, no. 2 (1984): 241–261.
2. JONES, D.A. "Bayesian Statistics," *TSA* XVII (1965): 33–57.

A.W. KOLKIEWICZ* AND K. RAVINDRAN[†]:

Compliments and congratulations to Drs. Hans Gerber and Elias Shu for providing yet another alternative and mathematically interesting approach to option pricing. Before discussing four major points, we would first like to bring to the authors' attention the article by Ravindran [14], which discusses a recreational approach to option pricing.

As both practitioners and academicians, we would like to add further insights to both the valuation and characteristics of a spread option. A spread option can be more generally defined as a derivative security that pays off at the option maturity time, τ, an amount that is max[0, $S_1(\tau) - S_2(\tau) - c$], where c, which is called an offset, can be any real number. The uses of this option in practice are given in Ravindran [16], [17]. When the offset, c, in a spread option payoff is set to 0, this option simplifies to a Magrabe or an exchange option. References [16] and [17] also illustrate the use of an exchange option in practice. As first shown by Magrabe [12], it is easy to obtain an analytical expression to value a spread option with zero offset. It is also important to note that when this offset is non-zero, it is not possible to obtain a clean analytical expression, and as such, one has to resort to numerical methods. See, for example, Barrett, Moore and Wilmott [2]. Despite this, we find that the valuation and the trading of a spread option can be made intuitive if

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we employ the concept of conditional expectations, which has been discussed by Ravindran [15].

Furthermore, the authors make an interesting comment that δ, the risk-free force of interest, is not present in the analytical expression at all. To intuit this comment, it is easier to think of a bird flying from east to west at a speed of x m/sec in a train that is also traveling in the same direction at the same speed. Thus, as long as both the bird and train are traveling in the same direction at the same speed, the relative speed of one to the other will be independent of x. In the same regard, if both the assets are growing at a rate of δ, the relative growth of one to the other will be independent of δ. This remark is true even if we assume that both asset 1 and 2 pay a continuously compounded dividend yield of q_1 and q_2, respectively. To intuit this, we should first observe that asset 1 grows at a rate of $\delta - q_1$ and asset 2 grows at a rate of $\delta - q_2$. As such, the relative growth of one asset to the other is simply the difference $q_1 - q_2$, which is again independent of δ. The analytical expression given in Hull [9] attests to this intuition. For a non-zero offset, c, however, it would be reasonable to expect δ to contribute to the option premium.

The Black-Scholes equation, which was discovered 20 years ago, is still being used after two decades. Although during this time the derivatives community has been exposed to more sophisticated and accurate models, the Black-Scholes model has withstood its competition and the test of time. The reason for this is simply that the Black-Scholes model is intuitively very appealing. Although five input variables (that is, current stock price, risk-free rate of interest, strike price of option, maturity date of option, and volatility of the stock) are required to value an option on a non-dividend-paying stock, in liquid markets, the only source of uncertainty is volatility. Furthermore, in such markets, instead of inputting the volatility values to obtain option premiums, traders in practice input the market option values into the Black-Scholes equation to extract the volatility values. Volatility numbers obtained in this fashion are called implied volatilities. The traders then trade volatility by determining whether this implied volatility, which is the market participants' view of volatility, is in reality correct. The use of a lognormal distributional assumption therefore allows for the existence of only one source of uncertainty. This sort of convenience is sacrificed when one uses the Poisson, gamma or the inverse Gaussian distribution. As such, we think that in considering the latter distributions, the authors have forsaken practical convenience for mathematical finesse.

Finally, the methodology presented in the paper can be extended to encompass a broader class of stock return distributions. Although stock return distributions have been widely studied, no single distribution has emerged as a clear winner from these studies, despite the common agreement that the returns' distributions should have tails that are fatter than the traditional lognormal distribution. See, for example, Becker [3], Kon [11], and Badrinath and Chatterjee [1].

As mentioned earlier, in addition to the lognormal process, the paper also discusses option valuation using both the gamma and the inverse Gaussian processes. Although the latter process allows for fatter tails, both the distributions have tails that decay exponentially. Given the fact that this rate of tail convergence is necessary for the existence of an Esscher transform, this should not come as any surprise. Such tail behavior constraints, however, can be avoided by considering shifted processes and distributions that are supported, for example, on the domain $(0, \infty)$. Since this is the approach adopted by the authors for the two above-mentioned processes, it still remains to be answered whether Equation (2.10), which defines h^*, can always be solved for a general distribution. Some insight to this question could have possibly been addressed by the authors' earlier result (Gerber and Shiu [7]). Because of our inability to access this work, we briefly discuss this issue here.

When discussing heavy-tailed distributions in the context of this paper, it is not unreasonable to think about stable Pareto distributions. Stable Pareto distributions are prominent members of the class of infinitely divisible distributions that, subsequent to the original works of Mandelbrot [13] and Fama [5], have often been used to explain the stochastic behavior of stock prices. The interest in stable distributions is largely due to the facts that only stable laws have domains of attraction (generalized central limit theorem) and that stable distributions belong to their own domain of attraction (stability). From a practical viewpoint, stable laws are flexible, empirical models that are capable of explaining the observed leptokurtosis and skewness in return distributions. Moreover, they allow us to capture the essentials of probability structures when sample moments exhibit a nonstationary behavior over time.

A stable Pareto distribution can fundamentally be described by the shape (denoted by α, where $0<\alpha<2$), skewness (denoted by β, where $|\beta|\leq1$), location, and scale parameters. Amongst these, the most influential is the shape parameter, which when decreased increases the tail probabilities. The interested reader is referred to Feller [6], Zolotarev

[18], and Klein [10] for further details on stable distributions. Two obvious drawbacks of these distributions are the lack of second moments (also the first if $\alpha < 1$) and the absence of explicit expressions for the density functions. These disadvantages, however, are not major obstacles when one considers asset pricing using the notion of risk-neutral valuation. This is due to the fact that all that is needed is the knowledge of the measure under which the discounted process is a martingale.

Suppose, for example, that X is α-stable, where $0 < \alpha < 2$. Then the random variable e^X has no finite moments except when X is totally skewed to the left (that is, $\beta = -1$). It is important to note that in this instance when $\alpha > 1$, the support of this distribution is the interval $(-\infty, \infty)$. Thus, when $\beta = -1$ and $\alpha > 1$, all moments of e^X are finite, and setting the location parameter to 0 results in zero expectation. Hence, one can consider the modeling of the stock price movement using the process $S(t) = S(0)e^{X(t)}$, where $t \geq 0$.

The value of β that was used in deriving the above process also forces the right tails of the distribution of $X(t)$ to decay rapidly, and as a consequence, the moment-generating function, $E(e^{\gamma X})$, $\gamma \geq 0$, exists for all $0 < \alpha \leq 2$ and was shown by Gupta and Waymire [8] to be equal to

$$\exp\left(-\frac{\sigma^\alpha \gamma^\alpha}{A}\right) \quad \text{if } \alpha \neq 1$$

$$\exp\left(\frac{2\sigma\gamma \ln \gamma}{\pi}\right) \quad \text{if } \alpha = 1, \text{ where } A = \cos\frac{\alpha\pi}{2}.$$

Hence, one can now consider the approach proposed by the authors for a shifted α-stable process $X(t) = Y(t) + \mu t$, where $Y(t)$ is a process with independent increments and an α-stable distribution with $\beta = -1$, shift parameter of 0 and scale parameter of $\sqrt[\alpha]{0.5\sigma^2 t}$ (see Breiman [4] for definition of such processes). Using notations from the paper and letting A be as above, we have that

$$M(z, t) = E(e^{zX(t)}) = \exp\left[\left(\mu z - \frac{\sigma^2 z^\alpha}{2A}\right)t\right], \text{ where } z \geq 0$$

In particular, when $\alpha = 2$, we get the classic lognormally distributed stock price process. However, this transformed process does not have many of the nice properties of a Wiener process because

$$M(z, t; h) = \frac{M(z + h, t)}{M(h, t)}$$

$$= \exp\left\{\mu zt - \frac{\sigma^2 t[(z + h)^\alpha - h^\alpha]}{2A}\right\}, \text{ where } h \geq 0, z + h \geq 0$$

implies that when $h \neq 0$, the Esscher transform of a shifted α-stable process is no longer an α-stable process. Despite this drawback, we still have a process with stationary and independent increments whose expected values exist for all $h \geq 0$. Hence, Equation (2.10) now takes the form

$$\mu - \frac{\sigma^2}{2A}[(1 + h^*)^\alpha - h^{*\alpha}] = \delta.$$

When $\alpha > 1$ and $h^* \geq 0$, the above equation has an unique solution only if $\delta \geq \mu - (\sigma^2/2A)$. Since it is easy to show that $\mu - (\sigma^2/2A)$ is the expected rate of return on the stock $S(t)$, we can readily conclude from the last inequality that the risk-free rate of interest, δ, should be greater than or equal to the expected rate of return on the security. Intuitively, this would mean that it would be reasonable for us to demand compensation when we are forced to choose between investing in a risky asset over a risk-free asset. Mathematically, this implies that a martingale measure for the discounted stock process exists only if the expected rate of return on the process is equal to the risk-free rate of interest. Then equality (2.10) can only hold when $h^* = 0$, and as such, the process itself is martingale under the original measure and μ is equal to $\delta + (\sigma^2/2A)$, where the parameters σ and α need to be either estimated from the observed stock process or implied from the market. It is interesting to note that when $\alpha = 2$, the process under the martingale measure is the same as the one derived by the authors for the Wiener process, although in this case there are no restrictions on the parameters.

Obviously, the requirement that the parameters of the process satisfy the condition $\delta = \mu - (\sigma^2/2A)$ may in some cases be too restrictive. However, it seems that for stable distributions this is the only possible result one can get using the Esscher transforms. Thus, it would be interesting to apply the method proposed by the authors to other distributions that could explain the observed leptokurtosis.

REFERENCES

1. BADRINATH, S.G., AND CHATTERJEE, S. "On Measuring Skewness and Elongation in Common Stock Return Distributions: The Case of the Market Index," *Journal of Business* 61 (1988): 451–72.

2. BARRETT, J., MOORE, G., AND WILMOTT, P. "Inelegant Efficiency," *RISK* 5, no. 9 (October 1992): 82–84.

3. BECKER, D.N. "Statistical Tests of the Lognormal Distribution as a Basis for Interest Rate Changes," *TSA* XLIII (1991): 7–72.

4. BREIMAN, L. *Probability*. Reading, MA.: Addison-Wesley, 1968.

5. FAMA, E. "The Behaviour of Stock Market Prices," *Journal of Business* 38 (1965): 34–105.

6. FELLER, W. *An Introduction to Probability Theory and Its Applications*, Volume 2. New York: John Wiley, 1971.

7. GERBER, H.U. AND SHIU, E.S.W. "Martingale Approach to Pricing Perpetual American Options," *Proceedings of the 4th AFIR International Colloquium*, Orlando, April 20-22, 1994, 659–89.

8. GUPTA, V.K. AND WAYMIRE, E. "Multiscaling Properties of Spatial Rainfall and River Flow Distributions," *Journal of Geophysical Research* 95, no. D3 (1990): 1999–2009.

9. HULL, J.C. *Options, Futures, and Other Derivative Securities*. 2nd ed. Englewood Cliffs, N.J.: Prentice Hall, 1993.

10. KLEIN, G.E. "The Sensitivity of Cash-Flow Analysis to the Choice of Statistical Model for Interest Rate Changes," *TSA* XLV (1992): 79–186.

11. KON, S.J. "Models of Stock Returns—A Comparison," *Journal of Finance* 36 (1984): 149–65.

12. MAGRABE, W. "The Value of an Option to Exchange One Asset for Another," *Journal of Finance* XXXIII (1978): 177–86.

13. MANDELBROT, B. "The Variation of Certain Speculative Prices," *Journal of Business* 36 (1963): 394–419.

14. RAVINDRAN, K. "Option Pricing: An Offspring of the Secretary Problem?" *Mathematicae Japonica* 38 (1993): 905–912.

15. RAVINDRAN, K. "Low Fat Spreads," *RISK* 6, no. 10 (October 1993): 66–67.

16. RAVINDRAN, K. "Exotic Options." Chapter 5 in *Advanced Interest Rate and Currency Swaps*. Chicago, Ill.: Probus Publishing, 1994.

17. RAVINDRAN, K. "Exotic Options: The Basic Building Blocks and Their Applications." In *Handbook of Derivative Instruments*, 2nd ed., in press.

18. ZOLOTAREV, V.M. *One-Dimensional Stable Distributions*. Providence, R.I.: American Mathematical Society, 1986.

FRÉDÉRIC MICHAUD:

First, I congratulate the authors for a very-well written paper. It shows how much actuaries can contribute to finance, and I am certainly looking forward to reading other such papers. The intention of this discussion is to stress the fact that it is natural to use Esscher transforms for pricing options.

The key idea in option pricing is to replace the *objective* or *physical* probability measure by another probability measure. This new measure has to be an equivalent probability measure that is consistent with current market observations. It is the so-called *risk-neutral* probability measure. When searching for such a measure, one has to distinguish two cases.

The first case is models for which only two outcomes can happen in a unit or infinitesimal time interval. This is the case of the models of Sections 3.2 and 3.3. In such cases, there exists only one risk-neutral probability measure. The model of Section 3.1 also has a unique risk-neutral measure, because it can be interpreted as a limiting case of the models of Section 3.2 or 3.3. For these models, the price of an option is unique.

The second case is the one of "richer" models. For them, there are many risk-neutral probability measures. As a consequence, taking expected values of the discounted payoff with respect to such measures can lead to different results, and therefore there does not exist a unique "price" for the option. Then the following question comes up: Why should one choose Esscher transforms? The reason is clearly given in this paper: for many important cases, the Esscher transform remains in the same family of models as the objective probability measure. Thus it is very natural to use the risk-neutral Esscher transform to price an option, even though one can argue that a unique price does not exist.

To illustrate this, we can generalize the model of Section 3.2 by replacing the constant jumps of size k by random jumps. (Esscher transforms of compound Poisson distribution are discussed in references [6], [35] and [38] of the paper.) Let us assume (2.1) and (4.1.1), where the process $\{Y(t); t \geq 0\}$ is a compound Poisson process with Poisson parameter λ and jump size distribution $P(\cdot)$. The moment-generating function of $X(t)$ is given by

$$M(z, t) = \exp[-ctz + \lambda t(m(z) - 1)]$$

with

$$m(z) = \int_0^x e^{zx} dP(x).$$

According to (2.6), we find

$$M(z, t; h) = \exp\left[-ctz + \lambda m(h)t\left(\frac{m(z + h)}{m(h)} - 1\right)\right].$$

This shows that the *risk-neutral* Esscher transform remains in the family of compound Poisson processes, with unchanged value of c, new Poisson parameter $\lambda m(h)$, and jump amount distribution that is the Esscher transform of the original jump amount distribution. As with processes, the Esscher transform of a distribution remains in the family of the original distributions for most important cases. In particular, the gamma, exponential, normal, inverse Gaussian, negative binomial, geometric, and Poisson distributions are examples of such distributions. So not only does the process remain compound Poisson under the risk-neutral measure, but also the jumps often are in the same family of distributions as the original jumps.

As an example, let us suppose that the jump sizes have an exponential distribution with moment-generating function

$$m(z) = \frac{\beta}{\beta - z}.$$

From (2.10), we see that the risk-neutral Esscher parameter is implicitly defined by the equation

$$\delta = -c + \lambda\left(\frac{\beta}{\beta - h^* - 1} - \frac{\beta}{\beta - h^*}\right), h^* < \beta.$$

This yields

$$\beta^* = \beta - h^* = 0.5 + \sqrt{0.25 + \frac{\lambda\beta}{\delta + c}}.$$

Since

$$\lambda m(h^*) = \frac{\lambda\beta}{\beta^*}$$

and

$$\frac{m(z + h^*)}{m(h^*)} = \frac{\beta^*}{\beta^* - z},$$

the risk-neutral Esscher transform is a compound Poisson process with parameter $\lambda^* = \lambda\beta/\beta^*$ and exponential jump distribution with mean $1/\beta^*$. Using (2.15) and $G(x; \alpha, \beta)$ as defined as in Section 4.1.1, we get the following expression for a price of the option:

$$S(0)\left[1 - \sum_{n=0}^{\infty} \frac{e^{-\lambda^*}(\lambda^*)^n}{n!} G(\kappa + c\tau; n, \beta^* - 1)\right]$$

$$- Ke^{-\delta\tau}\left[1 - \sum_{n=0}^{\infty} \frac{e^{-\lambda^*}(\lambda^*)^n}{n!} G(\kappa + c\tau; n, \beta^*)\right].$$

HAL W. PEDERSEN:

Drs. Gerber and Shiu are to be congratulated for their insightful and instructive paper. In addition to presenting elegant original results, the paper offers new insights into several classical option-pricing formulas. My study of this paper has impressed me with the computational power of the Esscher transform framework. Formulas (2.15) and (6.6) and their applications illustrate this point. As the paper shows, the power of these formulas comes from the fact that we explicitly know the distribution of $X(t)$ under the Esscher-transformed probability measure.

It is by now a standard result that for complete models of securities markets, the price of a European call option on a stock with strike price K and expiration date τ may be obtained by the expectation $E^Q[e^{-\delta\tau}(S(\tau)-K)_+]$, where Q is the unique probability measure under which $\{e^{-\delta\tau}S(t)\}$ is a martingale. This equation is merely Equation (2.11) of the paper in a different notation. The determination of Q by the method of Esscher transforms is very elegant and is a nice result in the paper. However, I think that one of the more important insights of the paper concerns the calculation of expressions such as $E^Q[e^{-\delta\tau}(S(\tau)-K)_+]$ within the Esscher transform framework. Indeed, when one wishes to calculate this expression for the Black-Scholes model, one normally uses a formula

for the truncated mean of a lognormal distribution. More complicated options require even more laborious calculations. Why is it that such cumbersome calculations do not arise when one works in the Esscher transform setting? The answer lies in Equation (2.15) and the closure property of the Esscher transform for the types of processes considered in the paper. Equation (2.15) tells us that the price of the option may be expressed in terms of the Esscher-transformed distribution functions $F(x, t; h)$. Therefore, if we know the form of these transformed distribution functions, then (2.15) allows us to immediately write down the price of the option. On the other hand, if we do not know $F(x, t; h)$, then Formula (2.15) is of little practical importance.

Although we will know the explicit form of $F(x, t)$ for any given model, there is no reason to expect that we will also know $F(x, t; h)$. Direct calculations of $M(z, t; h)$, which are carried out in the paper, show that the form of $F(x, t)$ is preserved under the Esscher transform for the models considered in the paper. For the one-dimensional Wiener, Poisson, and gamma models, the following table illustrates this point.

Model	$F(x, t)$	$M(z, t)$
Wiener	$N(x; \mu t, \sigma^2 t)$	$\exp(t[\mu z + \tfrac{1}{2}\sigma^2 z^2])$
Poisson	$\Lambda((x+ct)/k; \lambda t)$	$\exp(t[\lambda(e^{zk}-1)-cz])$
Gamma	$G(x+ct; \alpha t, \beta)$	$\exp\left(t\left[\alpha \log\left(\dfrac{\beta}{\beta-z}\right)-cz\right]\right)$

Model	$M(z, t; h)$	$F(x, t; h)$
Wiener	$\exp(t[(\mu+\sigma^2 h)z + \tfrac{1}{2}\sigma^2 z^2])$	$N(x; (\mu+\sigma^2 h)t, \sigma^2 t)$
Poisson	$\exp(t[\lambda e^{hk}(e^{zk}-1)-cz])$	$\Lambda((x+ct)/k; \lambda e^{hk}t)$
Gamma	$\exp\left(t\left[\alpha \log\left(\dfrac{\beta-h}{\beta-h-z}\right)-cz\right]\right)$	$G(x+ct; \alpha t, \beta-h)$

The table emphasises that, for each of these three examples, the effect of the Esscher transform is to alter one parameter of the process, X. Consequently, $F(x, t; h)$ will be known for any value of the Esscher parameter and Formula (2.15) will yield an immediate valuation formula for these models.

It would be nice to have a proof that for a certain class of distributions, the Esscher transform has the necessary closure property. Is it possible to use some general theory to establish the closure property of the Esscher transform? Is there some common property of the types of processes used in the paper that would allow us to conclude ex ante that the Esscher transforms of these processes are of the same type?

We note that Formula (6.6) is another example of the insight that certain option prices can be given explicit expressions in terms of Esscher-transformed distributions. This is perhaps the most important original insight of the paper. Here again, however, the utility of this result depends on the closure property of the Esscher transform. Although Formulas (2.15) and (6.6) are valid only for the risk-neutral Esscher measure parameterized by h^*, the factorization formula that is implicit in the proof of (2.15) and (6.6) is valid for arbitrary h.

This paper offers a welcome new perspective on the computation of option values. The Esscher transform framework allows the authors to write down explicit formulas for option prices that have hitherto been computed in a cumbersome fashion. Section 6 of the paper concisely illustrates this point.

WOJCIECH SZATZSCHNEIDER*:

This paper is a user-friendly option-pricing theory. It deals specifically with European call options but can be extended to other derivative securities.

The method unifies the theory of option pricing for known models but, moreover, introduces the study of new ones. In the case of geometric Brownian motion, the method is equivalent to the one obtained by Doléans exponential.

The method applied by the authors to Esscher similar processes [where, if process $X(t)$ belongs to some well-defined class of processes, then $\tilde{X}$ (~ being the Esscher transform) belongs to the same class] is applicable to more general cases.

To conclude, there are two questions that spring to mind:

(1) Is it true that the Esscher transform is the only one that leads to the unified theory?

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(2) What may be done with respect to α-stable processes, where unfortunately, the method is ruled out? By this, I mean through a method similar to the one presented and not one analogous to the binomial model approach and the weak convergence argument.

YONG YAO*:

Drs. Gerber and Shiu have written an interesting paper, giving an elegant and practical method for valuing derivative securities.

In general, there are two techniques for valuing derivative securities. One is to use a no-arbitrage argument to get a partial differential equation or system ([1] and [3, Chapters 5 and 10]). The other is by means of martingales. The purpose of this discussion is to derive a stochastic exponential formula relating the method used in the paper with the customary martingale approach ([3, Chapter 9], [4, Chapter 6], [5]).

We fix a complete probability space (Ω, F, P) equipped with a filtration $\{F_t\}_{0 \le t \le \infty}$. By a *filtration* we mean an increasing family of σ-algebras $\{F_t\}_{0 \le t \le \infty}$, $F_s \subset F_t$ if $s \le t$. A probability measure Q is said to be *equivalent* to P provided that, for each event A, $Q(A) > 0$ if and only if $P(A) > 0$. For a stochastic process $X = \{X(t)\}_{t \ge 0}$ on (Ω, F, P), a probability measure Q is called an *equivalent martingale measure* if Q is equivalent to P and X is a martingale with respect to Q. A recent paper on equivalent martingale measures is the one by Christopeit and Musiela [2].

A stochastic process is called a *semimartingale* if it is the sum of a local martingale and an adapted process with paths of finite variation on compact time intervals. A *Lévy process* is a stochastic process with stationary and independent increments. The Lévy processes, which include the Brownian motion and the Poisson process as special cases, are prototypic examples of semimartingales. The *stochastic exponential* of a semimartingale $X = \{X(t)\}_{t \ge 0}$ with $X(0) = 0$, written $\epsilon(X)$, is the unique semimartingale $\{Z(t)\}_{t \ge 0}$ that is the solution of the equation:

$$Z(t) = 1 + \int_0^t Z(s-)dX(s), \, t > 0,$$

or

$$dZ(t) = Z(t-)dX(t), \, t > 0.$$

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For an introduction to semimartingales and stochastic exponentials, see Chapter 10 and Section 11.4 of Dothan's book [3] or Chapters 1 and 3 of Protter's book [6].

The martingale approach is based on the "Fundamental Theorem of Asset Pricing," which states that, for certain frictionless securities market models, the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure. There are two important elements in this approach: a stochastic process $\{X(t)\}_{t\geq 0}$ on a complete probability space (Ω, F, P) with filtration $\{F_t\}_{0\leq t\leq \infty}$ modeling the movements of the prices of the primitive securities, and an equivalent martingale measure, Q, with respect to which the price of any derivative security is calculated as the expectation of its discounted payoffs. The probability measure, Q, is usually constructed via the Radon-Nikodym derivative, dQ/dP. In the paper, the authors use the Radon-Nikodym derivative:

$$\frac{dQ}{dP} = \frac{e^{hX}}{E[e^{hX}]}.$$

The customary martingale approach chooses a stochastic exponential as the Radon-Nikodym derivative.

For a Brownian motion $B=\{B(t)\}_{t\geq 0}$ with mean per unit time 0 and variance per unit time σ^2, we have ([3, p. 284], [6, pp. 77–79])

$$\epsilon(hB)(t) = \exp(hB(t) - \frac{1}{2}[hB, hB](t))$$

$$= \exp(hB(t) - \frac{1}{2}\sigma^2 h^2 t).$$

Because

$$E[e^{hB}(t)] = \exp\left(\frac{1}{2}\sigma^2 h^2 t\right),$$

we obtain

$$\frac{e^{hB}}{E[e^{hB}]} = \epsilon(hB).$$

In general, for a Lévy process $X=\{X(t)\}_{t\geq 0}$, there exists a local martingale $V=\{V(t)\}_{t\geq 0}$ such that

$$\frac{e^{hX}}{E[e^{hX}]} = \epsilon(V).$$

To determine V, we state three lemmas without proof.

Lemma 1. (Itô's Formula) ([3, Theorem 11.21], [6, p. 74, Theorem 33]).

Let $X=(X^1, \ldots, X^n)$ be an n-tuple of semimartingale, and let $f: R^n \to R$ have continuous second-order partial derivatives. Then $f(X)$ is a semimartingale and the following formula holds:

$$f(X(t)) - f(X(0)) = \sum_{i=1}^{n} \int_{0+}^{t} \frac{\partial}{\partial x_i} f(X(s-))dX^i(s)$$

$$+ \frac{1}{2} \sum_{1\leq i,j\leq n} \int_{0+}^{t} \frac{\partial^2}{\partial x_i \partial x_j} f(X(s-))d[X^i, X^j]^c(s)$$

$$+ \sum_{0<s\leq t} \left\{ f(X(s)) - f(X(s-)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(X(s-))\Delta X^i(s) \right\}$$

Lemma 2. (Lévy Decomposition) [6, p. 32, Theorem 42].

Let X be a Lévy process. Then X has a decomposition:

$$X(t) = B(t) + \mu t + K(t),$$

where $B=\{B(t)\}_{t\geq 0}$ is a Brownian motion with zero drift, and $K=\{K(t)\}_{t\geq 0}$ is a pure jump semimartingale.

Lemma 3.

Let $K=\{K(t)\}_{t\geq 0}$ be a pure jump semimartingale, and let

$$J(t) = \sum_{0<s\leq t} \{e^{h\Delta K(s)} - 1\}.$$

Then $J=\{J(t)\}_{t\geq 0}$ is also a pure jump semimartingale, and

$$\Delta J(t) = e^{h\Delta K(t)} - 1.$$

We are now ready to determine V. Let

$$L(h, t) = \frac{e^{hX(t)}}{E[e^{hX(t)}]}.$$

From Section 2.1 of the paper (because X is a Lévy process),

$$E[e^{hX(t)}] = M(h, t) = [M(h, 1)]^t = \exp[t \ln M(h, 1)].$$

Applying Itô's Formula (with $n=2$) to

$$L(h, t) = \exp[hX(t) - t \ln M(h, 1)],$$

we have

$$L(h, t) - 1 = \int_0^t L(h, s-)[hdX(s) - \ln M(h, 1)ds]$$

$$+ \frac{1}{2}\sigma^2 h^2 \int_0^t L(h, s-)ds$$

$$+ \sum_{0<s\le t} [L(h, s) - L(h, s-) - hL(h, s-)\Delta X(s)].$$

From Lemma 2,

$$\int_0^t L(h, s-)dX(s) = \int_0^t L(h, s-)d[B(s) + \mu s + K(s)]$$

$$= \int_0^t L(h, s-)d[B(s) + \mu s] + \int_0^t L(h, s-)dK(s).$$

Because of Lemma 2 and Lemma 3,

$$L(h, s) - L(h, s-) = L(h, s-)(e^{h\Delta K(s)} - 1) = L(h, s-)\Delta J(s).$$

For the pure jump semimartingale J, we can write

$$\sum_{0<s\le t} L(h, s-)\Delta J(s) = \int_0^t L(h, s-)dJ(s),$$

so we get

$$\sum_{0<s\le t} [L(h, s) - L(h, s-)] = \int_0^t L(h, s-)dJ(s).$$

Similarly, we have

$$\sum_{0<s\leq t} L(h, s-)\Delta X(s) = \sum_{0<s\leq t} L(h, s-)\Delta K(s) = \int_0^t L(h, s-)dK(s).$$

Thus

$$L(h, t) - 1 = \int_0^t L(h, s-)d\{hB(s) + [\mu h - \ln M(h, 1) + \frac{1}{2}\sigma^2 h^2]s + J(s)\}.$$

Let

$$V(t) = hB(t) + [\mu h - \ln M(h, 1) + \frac{1}{2}\sigma^2 h^2]t + J(t),$$

then $V=\{V(t)\}_{t\geq 0}$ is a semimartingale, with $V(0)=0$, satisfying

$$L(h, t) - 1 = \int_0^t L(h, s-)dV(s),$$

or

$$\frac{e^{hX}}{E[e^{hX}]} = \epsilon(V).$$

Because

$$V(t) = \int_0^t \frac{dL(h, s)}{L(h, s-)},$$

$V=\{V(t)\}_{t\geq 0}$ is a local martingale.

To illustrate the above, we present two examples. In Section 3.1 of the paper,

$$X(t) = B(t) + \mu t, t \geq 0;$$

then

$$V(t) = hB(t), t \geq 0.$$

In Section 3.2 of the paper,

$$X(t) = kN(t) - ct, t \geq 0;$$

then

$$V(t) = (e^{hk} - 1)[N(t) - \lambda t], \ t \geq 0.$$

REFERENCES

1. BLACK, F., AND SCHOLES, M. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (1973): 637–59.
2. CHRISTOPEIT, N., AND MUSIELA, M. "On the Existence and Characterization of Arbitrage-Free Measures in Contingent Claim Valuation," *Stochastic Analysis and Applications* 12 (1994): 41–63.
3. DOTHAN, M. *Prices in Financial Markets.* New York and Oxford: Oxford University Press, 1990.
4. DUFFIE, D. *Dynamic Asset Pricing Theory.* Princeton: Princeton University Press, 1992.
5. ELLIOT, R. AND KOPP, P. "Option Pricing and Hedge Portfolio for Poisson Process," *Stochastic Analysis and Applications* 8 (1990): 157–67.
6. PROTTER, P. *Stochastic Integration and Differential Equations: A New Approach.* New York: Springer-Verlag, 1990.

(AUTHORS' REVIEW OF DISCUSSIONS)

HANS U. GERBER AND ELIAS S.W. SHIU:

We are grateful for receiving ten discussions that add much breadth and depth to the paper. We thank the discussants for their thoughtful contributions. Before responding to their comments individually, we clarify and review some ideas in the paper, present a model extension to include dividend-paying stocks, and discuss an application of Esscher transforms to the classical actuarial problem of ruin probability.

Consider a one-period model, in which there are only one riskless asset (a one-period bond with force of interest δ) and one risky asset (a non-dividend-paying stock). With the notation in Section 3.3, the stock price at time 1 is

$$S(1) = S(0)e^{X_1}.$$

Let Ω denote the set of points on which X_1 has positive probability. Assume that Ω is finite and consists of more than one point; let a be its smallest element and b its largest, with

$$a < \delta < b.$$

If Ω consists of two points only, $\Omega=\{a, b\}$, then there is exactly one equivalent martingale measure; see (3.3.6) in the paper. [In this simple model, a martingale measure is a probability measure (indicated by ^) such that

$$\hat{E}[S(1)] = S(0)e^{\delta},$$

or

$$\hat{E}(e^{-\delta+X_1}) = 1.]$$

If Ω consists of more than two points, $|\Omega|>2$, then there are infinitely many equivalent martingale measures, and we have an *incomplete market* model. In terms of linear algebra, the payoffs of the stock and bond can be viewed as two linearly independent vectors in the linear space $R^{|\Omega|}$. If there are more than two states of nature at time 1 ($|\Omega|>2$), one bond and one stock cannot *span* all possible outcomes, as the vectors $(e^a, \ldots, e^b)$ and $(e^{\delta}, \ldots, e^{\delta})$ can only span a two-dimensional subspace in $R^{|\Omega|}$. In order to obtain a complete market model, or a unique equivalent martingale measure, the number of independent assets must be the same as the number of states of nature at time 1.

A main theme of the paper is that, for a certain class of stock price processes, there is a "natural" equivalent martingale measure. In the first part of the paper we consider a non-dividend-paying stock whose price is given by

$$S(t) = S(0)e^{X(t)}, \tag{R.1}$$

where the process $\{X(t)\}$ has independent and stationary increments. For a theoretical "justification" that stock prices should be governed by such stochastic processes, see Samuelson [27] or Parkinson [23]. We assume that the moment-generating function of $X(t)$,

$$M(h, t) = E[e^{hX(t)}],$$

exists and that

$$M(h, t) = M(h, 1)^t. \tag{R.2}$$

The process

$$\{e^{hX(t)}M(h, 1)^{-t}\} \tag{R.3}$$

is a positive martingale and can be used to define a change of probability measure. That is, it can be used to define the Radon-Nikodym derivative

dQ/dP, where P is the original probability measure and Q is the Esscher measure of parameter h. The *risk-neutral Esscher measure* is the Esscher measure of parameter $h=h^*$ such that the process

$$\{e^{-\delta t}S(t)\} \tag{R.4}$$

is a martingale. The parameter h^* is unique; for a proof see Section 2 of our paper [12]. We reiterate that there may be many other equivalent martingale measures.

In some statistical literature, the Esscher transform is known under the name *exponential tilting*.

Representative Investor with Power Utility Function

Some discussants have raised an important question. When there is more than one equivalent martingale measure (incomplete market), why should the option price be the expectation, with respect to the risk-neutral Esscher measure, of the discounted payoff? This particular choice may be justified within a utility function framework. Consider a simple economy with only a stock and a risk-free bond and their derivative securities. There is a representative investor who owns m shares of the stock and bases his decisions on a risk-averse utility function $u(x)$. Consider a derivative security that provides a payment of $\pi(\tau)$ at time τ, $\tau>0$; $\pi(\tau)$ is a function of the stock price process until time τ. What is the representative investor's *price* for the derivative security, such that it is optimal for him not to buy or sell any fraction or multiple of it? Let $V(0)$ denote this price. Then, mathematically, this is the condition that the function

$$\phi(\eta) = E[u(mS(\tau) + \eta[\pi(\tau) - e^{\delta\tau}V(0)])] \tag{R.5}$$

is maximal for $\eta=0$. From

$$\phi'(0) = 0$$

we obtain

$$V(0) = e^{-\delta\tau}\frac{E[\pi(\tau)u'(mS(\tau))]}{E[u'(mS(\tau))]} \tag{R.6}$$

(as a necessary and sufficient condition, since $\phi''(\eta)<0$ if $u''(x)<0$). In the particular case of a power utility function with parameter $c>0$,

$$u(x) = \begin{cases} \dfrac{x^{1-c}}{1-c} & \text{if } c \neq 1 \\[2ex] \ln x & \text{if } c = 1 \end{cases}, \tag{R.7}$$

we have $u'(x) = x^{-c}$, and

$$V(0) = e^{-\delta\tau} \frac{E[\pi(\tau)[mS(\tau)]^{-c}]}{E[[mS(\tau)]^{-c}]}$$

$$= e^{-\delta\tau} \frac{E[\pi(\tau)S(\tau)^{-c}]}{E[S(\tau)^{-c}]}. \tag{R.8}$$

Formula (R.8) must hold for all derivative securities. For $\pi(\tau) = S(\tau)$ and therefore $V(0) = S(0)$, (R.8) becomes

$$S(0) = e^{-\delta\tau} \frac{E[S(\tau)^{1-c}]}{E[S(\tau)^{-c}]}$$

$$= e^{-\delta\tau} S(0) \frac{M(1-c, \tau)}{M(-c, \tau)},$$

or

$$e^{\delta} = \frac{M(1-c, 1)}{M(-c, 1)}$$

$$= M(1, 1; -c). \tag{R.9}$$

On comparing (R.9) with (2.9) in the paper, we see that the value of the parameter c is $-h^*$. Hence $V(0)$ is indeed the discounted expectation of the payoff $\pi(\tau)$, calculated with respect to the Esscher measure of parameter $h^* = -c$.

By considering different points in time τ, we get a consistency requirement. This is satisfied if the representative investor has a power utility function. We conjecture that it is violated for any other risk-averse

utility function, which implies that the pricing of an option by the risk-neutral Esscher measure is a consequence of the consistency requirement. Some related papers are Rubinstein [26], Bick ([5], [6]), Constantinides [9], Naik and Lee [22], Stapleton and Subrahmanyam [29], He and Leland [15], Heston [17], and Wang [31].

Extension to Dividend-Paying Stocks

The model can be extended to the case in which the stock pays dividends continuously, at a rate proportional to its price. In other words, we assume that there is a nonnegative number φ such that the dividend paid between time t and $t+dt$ is

$$\varphi S(t)dt. \tag{R.10}$$

Thus, if all dividends are reinvested in the stock, each share of the stock at time 0 grows to $e^{\varphi t}$ shares at time t. The risk-neutral Esscher measure is the Esscher measure of parameter $h=h^*$ such that the process

$$\{e^{-(\delta-\varphi)t}S(t)\} \tag{R.11}$$

is a martingale. (The number φ may be called the dividend-yield rate.) Because, for $t \geq 0$,

$$e^{hX(t)}M(h, 1)^{-t} = \frac{e^{hX(t)}}{E[e^{hX(t)}]} = \frac{S(t)^h}{E[S(t)^h]}, \tag{R.12}$$

we have the following: let g be a measurable function and h, k and t be real numbers, $t \geq 0$; then

$$\begin{aligned}
E[S(t)^k g(S(t)); h] &= E[S(t)^k\, g(S(t))\, e^{hX(t)}M(h, 1)^{-t}] \\
&= \frac{E[S(t)^{h+k}g(S(t))]}{E[S(t)^h]} \\
&= \frac{E[S(t)^{h+k}]}{E[S(t)^h]} \frac{E[g(S(t))\, S(t)^{h+k}]}{E[S(t)^{h+k}]} \\
&= E[S(t)^k; h]\, E[g(S(t)); h + k]. \tag{R.13}
\end{aligned}$$

This factorization formula simplifies many calculations and is a main reason why the method of Esscher transforms is an efficient device for valuing certain derivative securities. For example, applying (R.13) with $k=1$, $g(x)=I(x>K)$, $t=\tau$ and $h=h^*$, we obtain

$$E[S(\tau)I(S(\tau) > K); h^*] = E[S(\tau); h^*] E[I(S(\tau) > K); h^* + 1]$$

$$= E[S(\tau); h^*] [1 - F(\kappa, \tau; h^* + 1)].$$

Since (R.11) is a martingale with respect to the risk-neutral Esscher measure, we have

$$E[S(\tau); h^*] = S(0)e^{(\delta - \varphi)\tau}. \tag{R.14}$$

Thus we obtain a pricing formula for a European call option on a dividend-paying stock,

$$E[e^{-\delta\tau}(S(\tau) - K)_+; h^*] = E[e^{-\delta\tau}(S(\tau) - K)I(S(\tau) > K); h^*]$$

$$= e^{-\delta\tau}\{E[S(\tau)I(S(\tau) > K); h^*]$$

$$- KE[I(S(\tau) > K); h^*]\}$$

$$= S(0)e^{-\varphi\tau} [1 - F(\kappa, \tau; h^* + 1)]$$

$$- Ke^{-\delta\tau}[1 - F(\kappa, \tau; h^*)], \tag{R.15}$$

which is a generalization of (2.15) in the paper. Formula (R.15) may also be used to price currency exchange options, with $S(\tau)$ denoting the spot exchange rate at time τ, δ the domestic force of interest, and φ the foreign force of interest. For $\{S(t)\}$ being a geometric Brownian motion, (R.15) is known as the *Garman-Kohlhagen formula.*

The paper also considers the case of n stocks. Again, we can extend the model to dividend-paying stocks. For each j, $j = 1, 2, \ldots, n$, we assume that there exists a nonnegative constant φ_j such that stock j pays dividends of amount

$$\varphi_j S_j(t) dt$$

between time t and $t + dt$. Now, the risk-neutral Esscher measure is the Esscher measure of parameter vector $h = h^*$ such that, for each j, $j = 1, 2, \ldots, n$,

$$\{e^{-(\delta - \varphi_j)t} S_j(t)\} \tag{R.16}$$

is a martingale. As pointed out in the second proof of the Theorem in Section 6, we have the factorization formula:

$$E[S(t)^k g(S(t)); h] = E[S(t)^k; h]E[g(S(t)); h + k]. \tag{R.17}$$

Formula (6.6) in the Theorem becomes

$$E[e^{-\delta t}S_j(t)g(S(t)); \mathbf{h}^*] = S_j(0)e^{-\varphi_j t}E[g(S(t)); \mathbf{h}^* + \mathbf{1}_j]. \quad (R.18)$$

Similarly, Corollary 1 (Margrabe option formula) is generalized as

$$E[e^{-\delta \tau}(S_1(\tau) - S_2(\tau))_+; \mathbf{h}^*]$$

$$= S_1(0)e^{-\varphi_1 \tau}Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1]$$

$$- S_2(0)\ e^{-\varphi_2 \tau}Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2], \quad (R.19)$$

a special case of which is (R.15). For the n-dimensional Wiener process model, Formulas (7.1) and (7.2) are generalized as

$$\boldsymbol{\mu} + \mathbf{V}\mathbf{h}^* = \delta \mathbf{1} - (\varphi_1 + \tfrac{1}{2}\sigma_{11}, \varphi_2 + \tfrac{1}{2}\sigma_{22}, \ldots, \varphi_n + \tfrac{1}{2}\sigma_{nn})' \quad (R.20)$$

and

$$\boldsymbol{\mu} + \mathbf{V}(\mathbf{h}^* + \mathbf{1}_j)$$

$$= \delta \mathbf{1} - (\varphi_1 - \sigma_{1j} + \tfrac{1}{2}\sigma_{11}, \varphi_2$$

$$- \sigma_{2j} + \tfrac{1}{2}\sigma_{22}, \ldots, \varphi_n - \sigma_{nj} + \tfrac{1}{2}\sigma_{nn})', \quad (R.21)$$

respectively. Here,

$$\mathbf{1} = (1, 1, 1, \ldots, 1)'$$

and

$$\mathbf{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0)',$$

where the 1 in the column vector $\mathbf{1}_j$ is in the j-th position.

Semicontinuous Stock Prices

The sample paths of the stock prices considered in Sections 3.1, 3.2, 4.1, and 4.2 of the paper are skip-free downward (jump-free downward). A stochastic process $\{X(t)\}$ with stationary and independent increments and skip-free downward sample paths has the decomposition:

$$X(t) = Y(t) + v^2 B(t) - ct, t \geq 0, \quad (R.22)$$

where $\{Y(t)\}$ is either a compound Poisson process with positive increments or the limit of such processes, $\{B(t)\}$ is an independent standardized Wiener process (with zero drift and unit variance per unit time), and the last term, ct, represents a deterministic drift. The cumulant generating function of the random variable $X(t)$ is of the form

$$\ln [M(z, t)] = t \left\{ \int_0^\infty (e^{zx} - 1)[-dQ(x)] + v^2z^2/2 - cz \right\}, \quad \text{(R.23)}$$

where $Q(x)$ is some non-negative and nonincreasing function with $Q(\infty)=0$. Because

$$\ln[M(z, t; h)] = \ln[M(z + h, t)] - \ln[M(h, t)]$$

$$= t \left\{ \int_0^\infty (e^{zx} - 1)e^{hx}[-dQ(x)] \right.$$

$$\left. + v^2z^2/2 - (c - v^2h)z \right\}, \quad \text{(R.24)}$$

under the Esscher measure of parameter h $\{X(t)\}$ is a similar process with the following modifications:

$$dQ(x) \rightarrow e^{hx}dQ(x), \quad \text{(R.25)}$$

$$v^2 \rightarrow v^2 \qquad \text{(unchanged)}, \quad \text{(R.26)}$$

$$c \rightarrow c - v^2h. \quad \text{(R.27)}$$

To obtain the models proposed in Sections 4.1 and 4.2, let $v=0$ and

$$Q'(x) = -ax^{\alpha-1}e^{-bx}, \quad x > 0, \quad \text{(R.28)}$$

where $a>0$, $b>0$ and $\alpha>-1$ are three parameters. [In the context of risk theory, we [10] have considered such a $Q(x)$ function.] Then $\{Y(t)\}$ is a compound Poisson process for $\alpha>0$, a gamma process for $\alpha=0$, and an inverse Gaussian process for $\alpha=-\frac{1}{2}$. For a proof of these results, see the Appendix of our paper [12].

Hedging

Some discussants have stated that derivative securities should be priced by constructing hedging portfolios. Since we never addressed this issue in the paper and some members of the Society may be unfamiliar with the idea, let us give an illustration using the model in Section 3.2, where the price of the non-dividend-paying stock is

$$S(t) = S(0)e^{kN(t)-ct}, \quad \text{(R.29)}$$

with k and c being constants and $\{N(t)\}$ a Poisson process. We are to determine the price of a derivative security, $V(S(t), t)$, from a portfolio of stock and risk-free bond replicating its payoff. Consider a portfolio with the amount

$$\eta = \eta(S(t), t) \qquad (R.30)$$

invested in the stock at time t and the amount $V(S(t), t) - \eta$ in the risk-free bond. The amount η is such that the derivative security price and the portfolio value have equal instantaneous change. By considering whether there will be an instantaneous jump in the stock price, we have the following two conditions:

$$V(Se^k, t) - V(S, t) = \eta e^k - \eta, \qquad (R.31)$$

and

$$V_t(S, t) - cSV_S(S, t) = -c\eta + \delta[V(S, t) - \eta]$$

$$= \delta V(S, t) - (\delta + c)\eta. \qquad (R.32)$$

Formula (R.31) yields

$$\eta = \frac{V(Se^k, t) - V(S, t)}{e^k - 1}. \qquad (R.33)$$

Thus (R.32) becomes

$$V_t(S, t) - cSV_S(S, t) = \delta V(S, t) - \lambda^*[V(Se^k, t) - V(S, t)], \quad (R.34)$$

where

$$\lambda^* = \frac{\delta + c}{e^k - 1}, \qquad (R.35)$$

the same as (3.2.6) in the paper.

Now, let $W(S(t), t)$ denote the value at time t of the expected discounted payoffs of the derivative security; the expectation is taken with respect to the probability measure corresponding to the Poisson parameter λ^*. Let s be a very small positive number. By the Poisson process assumption, the probability that a jump in the stock price will occur in the time interval $(t, t+s)$ is $\lambda^* s + o(s)$. Thus, conditioning on whether there are stock-price jumps in the interval $(t, t+s)$, we have

$$W(S, t) = e^{-\delta s}[(1 - \lambda^* s)W(Se^{-cs}, t + s)$$
$$+ \lambda^* s W(Se^{k-cs}, t + s)] + o(s), \qquad (R.36)$$

or

$$(1 + \delta s)W(S, t) - W(Se^{-cs}, t + s)$$
$$= \lambda^* s[W(Se^{k-cs}, t + s) - W(Se^{-cs}, t + s)] + o(s).$$

Dividing the last equation by s and letting s tend to 0 yields

$$\delta W(S, t) + cSW_S(S, t) - W_t(S, t) = \lambda^*[W(Se^k, t) - W(S, t)], \quad (R.37)$$

which is identical to (R.34). Consequently, the price of the derivative security, $V(S, t)$, is calculated as the expected discounted payoffs according to the provisions of the contract; the expectation is taken with respect to the measure corresponding to the Poisson process with parameter λ^*.

We note that, in constructing the replicating portfolio, we did not use the assumption that $\{N(t)\}$ is a Poisson process. Thus $N(t)$ in (R.29) may be assumed to come from a counting process; the equivalent martingale measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter λ^* given by (R.35). A replicating portfolio can be constructed because at each point of time the stock price has only two possible movements, both with known magnitude.

It is interesting to consider the limiting case where $k \rightarrow 0$ and $c \rightarrow \infty$ such that the variance per unit time of the exponent in (R.29) is constant:

$$\lambda^* k^2 = \frac{\delta + c}{e^k - 1} k^2 = \sigma^2. \qquad (R.38)$$

This is the classical lognormal model (see Section 3.1). In the limit (R.33) becomes

$$\eta = SV_S(S, t),$$

showing that the ratio, $\eta(S(t), t)/S(t)$, is given by $V_S(S(t), t)$, which is usually called *delta*, Δ, in the option literature. Also, by the Taylor expansion,

$$\lambda^*[V(Se^k, t) - V(S, t)] = \lambda^*\{(e^k - 1)SV_S(S, t)$$
$$+ [(e^k - 1)S]^2V_{SS}(S, t)/2 + O(k^3)\}$$
$$= (\delta + c)SV_S(S, t) + \sigma^2S^2V_{SS}(S, t)/2 + O(k).$$

Thus in the limit (R.34) becomes

$$V_t(S, t) = \delta V(S, t) - \delta SV_S(S, t) - \frac{\sigma^2}{2} S^2V_{SS}(S, t). \tag{R.39}$$

This partial differential equation was first derived by Black and Scholes
[7] with a replicating portfolio argument.

Some discussants have referred to research papers on hedging. Perhaps
we can add one more reference. The paper [30] by Stricker is a brief
survey of some recent results on pricing contingent claims in an incom-
plete market, where there are multiple equivalent martingale measures
and therefore multiple "prices" for a general contingent claim; it studies
the relationship between the maximum price and the existence of hedging
strategy and gives necessary and sufficient conditions for a contingent
claim to be representable with respect to the discounted price process.

Change of Numéraire and Homogeneous Payoff Function

The remark that there is no interest rate in the European Margrabe
option formula (7.6) has prompted explanations from two discussants.
The result may be viewed in a more general setting. Consider the div-
idend-paying extension of the n-stock model in Section 6. Let the payoff
of a European option or derivative security with exercise date τ be

$$\Pi(S_1(\tau), \ldots, S_n(\tau)). \tag{R.40}$$

For example, the Margrabe option has the payoff function

$$\Pi(s_1, s_2) = (s_1 - s_2)_+. \tag{R.41}$$

Let $E_t[\cdot]$ denote the expectation conditional on all information up to time
t. For $0 \leq t \leq \tau$, let $V(t)$ denote the price of the security at time t, calculated
with respect to the risk-neutral Esscher measure,

$$V(t) = E_t[e^{-\delta(\tau - t)}\Pi(S_1(\tau), \ldots, S_n(\tau)); \mathbf{h}^*]$$

$$= E_t[e^{-\delta(\tau - t)}S_j(\tau)\Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^*]$$

$$= E_t[e^{-\delta(\tau - t)}S_j(\tau); \mathbf{h}^*]\, E_t[\Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j]$$

$$= e^{-\varphi_j(\tau - t)}S_j(t)E_t[\Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j].$$

Thus

$$\frac{V(t)}{e^{\varphi_j't}S_j(t)} = E_t\left[\frac{1}{e^{\varphi_j'\tau}S_j(\tau)}\Pi(S_1(\tau), \ldots, S_n(\tau)); \mathbf{h}^* + \mathbf{1}_j\right], \quad (\text{R}.42)$$

from which it follows that, with respect to the Esscher measure of parameter vector $\mathbf{h}^* + \mathbf{1}_j$, the process

$$\left\{\frac{V(t)}{e^{\varphi_j't}S_j(t)}; 0 \le t \le \tau\right\} \qquad (\text{R}.43)$$

is a martingale. In particular, with respect to the Esscher measure of parameter vector $\mathbf{h}^* + \mathbf{1}_j$, the processes

$$\left\{\frac{e^{\delta t}}{e^{\varphi_j't}S_j(t)}\right\} \qquad (\text{R}.44)$$

and

$$\left\{\frac{e^{\varphi_k't}S_k(t)}{e^{\varphi_j't}S_j(t)}\right\} \qquad (\text{R}.45)$$

are martingales. To explain the denominator $e^{\varphi_j't}S_j(t)$, we consider stock j as a standard of value or a *numéraire*. In other words, we consider a mutual fund consisting of stock j only and all dividends are reinvested. All other securities are measured in terms of the value of this mutual fund.

Now, we assume that the payoff function Π is positively homogeneous of degree one with respect to the j-th variable,

$$\Pi(s_1, \ldots, s_n) = s_j\,\Pi(s_1/s_j, \ldots, s_{j-1}/s_j, 1, s_{j+1}/s_j, \ldots, s_n/s_j), \ s_j > 0, \tag{R.46}$$

which is a condition satisfied by (R.41) with both $j=1$ and $j=2$. Then (R.42) becomes

$$\frac{V(t)}{e^{\varphi_j'}S_j(t)} = E_t\left[\Pi\left(\frac{S_1(\tau)}{e^{\varphi_j^\tau}S_j(\tau)}, \ldots, \frac{S_n(\tau)}{e^{\varphi_j^\tau}S_j(\tau)}\right); \mathbf{h}^* + \mathbf{1}_j\right]. \qquad (R.47)$$

The right-hand side is an expectation, with respect to the Esscher measure of parameter vector $\mathbf{h}^* + \mathbf{1}_j$, of a function of the $(n-1)$-dimensional random vector

$$(X_1(\tau) - X_j(\tau), \ldots, X_{j-1}(\tau) - X_j(\tau), X_{j+1}(\tau) - X_j(\tau), \ldots, X_n(\tau) - X_j(\tau))'. \qquad (R.48)$$

In the case that $\{\mathbf{X}(t)\}$ is an n-dimensional Wiener process, (R.48) is a normal random vector, and it follows from (R.21) that its mean does not involve the force of interest δ, and of course its $(n-1)$-dimensional covariance matrix, which is the same for all $\mathbf{h}$, does not depend on δ. Thus the price of the derivative security, $V(t)$, does not depend on δ.

Probability of Ruin

The idea of replacing the original probability measure by an Esscher measure with an appropriately chosen parameter has an elegant application in classical actuarial risk theory. Let $\{U(t)\}$ be the surplus process,

$$U(t) = u + X(t), \qquad (R.49)$$

where $u \geq 0$ is the initial surplus and $X(t)$ the aggregate gains (premiums minus claims) up to time t. We suppose that the process $\{X(t)\}$ has independent and stationary increments, satisfies (R.2), and has a positive drift,

$$E[X(1)] > 0. \qquad (R.50)$$

Let

$$T = \inf\{t|U(t) < 0\} \qquad (R.51)$$

be the time of ruin. The probability of ruin before time m, $m > 0$, is

$$\psi(u, m) = \text{Prob}(T < m) = E[I(T < m)]. \qquad (R.52)$$

Let $a \wedge b$ denote the minimum of a and b. By a change of measure,

$$\psi(u, m) = E[I(T < m)e^{-hX(T \wedge m)}M(h, 1)^{T \wedge m}; h]$$
$$= E[I(T < m)e^{-hX(T)}M(h, 1)^T; h], \qquad (R.53)$$

which can be simplified if h is chosen as the nontrivial solution of the equation

$$M(h, 1) = 1. \qquad (R.54)$$

For simplicity we write

$$M(h) = M(h, 1).$$

It follows from

$$M''(h) = E[X(1)^2 e^{hX(1)}] > 0$$

that $M(h)$ is a convex function. Thus Equation (R.54) has at most one other solution besides $h=0$. Because

$$M'(0) = E[X(1)] > 0,$$

the nontrivial solution for (R.54) is a negative h. Following the usual notation in risk theory, we write this solution of (R.54) as $-R$. (R is called the adjustment coefficient.) With $h=-R$, (R.53) becomes

$$\psi(u, m) = E[I(T < m)e^{RX(T)}; -R]. \qquad (R.55)$$

The probability of ruin over an infinite horizon is

$$\begin{aligned} \psi(u) &= \psi(u, \infty) \\ &= E[I(T < \infty)e^{RX(T)}; -R]. \end{aligned} \qquad (R.56)$$

Now,

$$\begin{aligned} E[X(1); -R] &= E[X(1)e^{-RX(1)}] \\ &= M'(-R) \\ &< 0, \end{aligned}$$

because M is a convex function. An aggregate gains process with a negative drift means that ruin is certain. Thus, under the Esscher measure of parameter $-R$,

$$I(T < \infty) = 1$$

almost surely, and (R.56) simplifies as

$$\begin{aligned} \psi(u) &= E[e^{RX(T)}; -R] \\ &= E[e^{RU(T)}; -R]e^{-Ru} \end{aligned} \qquad (R.57)$$

This approach to the ruin problem can be found in Chapter XII of Asmussen's book [1], and he has attributed the idea to von Bahr [3] and Siegmund [28]. Formula (R.57) should be compared with (12.3.4) in *Actuarial Mathematics* [8, p. 352],

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} \mid T < \infty]},$$

where the conditional expectation in the denominator is taken with respect to the original probability measure.

Responses

We now respond to the discussions alphabetically. Some of our replies already appear in the above.

Dr. Carriere has provided a rigorous proof that an American call option on a non-dividend-paying stock is never optimally exercised before its maturity date. In other words, the option is always worth more alive than dead. Indeed, his proof also applies to the Margrabe option, whose payoff function is given by (R.41). As long as stock 1 does not pay dividends, an American Margrabe option is never optimally exercised before its maturity date. In other words, the price of an American Margrabe option is the same as the corresponding European Margrabe option, if stock 1 pays no dividends. Similarly, Dr. Carriere's second proof can be generalized to the case in which the exercise price, K, is replaced by a stock price, $S_{n+1}(t)$. The condition remains that none of the first n stocks pays dividends.

Dr. Cox has presented an elegant derivation of the Geske compound option formula. As he points out at the end of his discussion, the formula can be further generalized to the case of a dividend-paying stock. This provides an excellent exercise for the interested reader, who can derive the more general formula by applying the property that (R.11), not (R.4), is a martingale under the risk-neutral Esscher measure.

Drs. Delbaen, Schachermayer and Schweizer point out that there are many ways of obtaining a "reasonable" price for an option. They state a minimax theorem defining an interval of option prices that are feasible and do not allow arbitrage profit. It would be interesting to know what these intervals are for the two models in Section 4 of the paper. Their list of references contains cutting-edge papers on the mathematical theory of option pricing. Their last comment concerns how one may recover all

but one parameter from a trajectory. Section 4 of our paper [10] shows how one recovers such parameters for the gamma process.

We have already responded to some of Dr. Geman's comments in the above. One approach to extending the method of Esscher transforms to a stochastic interest rate setting is to start with the assumption that the discounted stock price process,

$$\left\{ S(t) \exp\left(- \int_0^t \delta_s ds \right) \right\},$$

has stationary and independent increments. The concept of forward neutral probability or forward risk-adjusted measure is indeed an important one for pricing European options. The paper by Pedersen and Shiu [24] has used it to price the GIC rollover option. We certainly agree that the ease in pricing "a given contingent claim by using the appropriate numéraire is remarkable." Indeed, Margrabe [20] applied this change-of-numéraire technique to obtain his option-pricing formula. We ([13] and [14]) have used the method to derive closed-form formulas for pricing perpetual American options whose payoff functions are positively homogeneous with respect to two stock prices; the perpetual American Margrabe option is a special case.

Drs. Hickman and Young have given a precise summary of the results in the paper by Back and Pliska [2]. They also point out that the principle of no arbitrage is related to the work of de Finetti on subjective probability. In a 1937 paper, de Finetti showed that, if odds are posted on each set in a finite partition of a probability space, then either the odds are consistent with a finite additive measure or a sure win is possible. Members of the Society were introduced to the theory of subjective probability by Jones [18]. In a complete market, there is a unique equivalent martingale measure; if an investor's subjective probability measure is equivalent to this measure (that is, there is an agreement on what future states of nature are possible), the investor's price of a security is the value calculated with the equivalent martingale measure. However, in an incomplete market, the investor would use his utility function and subjective probabilities to determine acceptable prices. An illustration is (R.6), where the expectations are calculated with the investor's subjective probability measure.

Mr. Kolkiewicz and Dr. Ravindran have provided many practical references and argued, by "general reasoning," why the Margrabe option

formula involves no interest rate. They also consider modeling the logarithm of the stock price as a shifted α-stable process. It is perhaps important to stress that the formulas are for $\beta = -1$; otherwise, $E[S(t)] = \infty$.

Mr. Michaud demonstrates that the Esscher transform of a compound Poisson process is again a compound Poisson process with a changed Poisson parameter and a jump distribution that is the Esscher transform of the original jump distribution. He has presented an elegant option formula for the exponential-Poisson model. In the finance literature, there is another approach to pricing options on stocks with a compound Poisson component; see Merton [21] and Kim, Oh and Brooks [19].

Mr. Pedersen points out that, to effectively use the method of Esscher transforms, one should pick a process $\{X(t)\}$ with known $F(x, t; h)$. His table is a nice summary of results. We do not have explicit answers to his questions. However, (R.25) and Mr. Michaud's discussion give a partial answer.

We are not sure what is meant by "the unified theory" in Dr. Szatzschneider's first question. With respect to his second question, the last part of the discussion by Mr. Kolkiewicz and Dr. Ravindran provides a partial answer; Chapter 15 of Peters' book [25] describes McCulloch's work on pricing "European options with log-stable uncertainty."

Mr. Yao has given several interesting calculations in his discussion. Stochastic calculus is now in the syllabus of a Fellowship examination. When candidates find such formulas in a future examination, they will know who is responsible for giving the examiners the idea.

We thank all 14 discussants for their thought-provoking comments. It is appropriate to repeat Dr. Hickman and Dr. Young's closing remark, which we share: "Our knowledge is limited, but our curiosity is not."

REFERENCES

1. ASMUSSEN, S. *Applied Probability and Queues*. New York: Wiley, 1987.
2. BACK, K., AND PLISKA, S.R. "On the Fundamental Theorem of Asset Pricing with an Infinite State Space," *Journal of Mathematical Economics* 20 (1991): 1–18.
3. VON BAHR, B. "Ruin Probabilities Expressed in Terms of Ladder Height Distributions," *Scandinavian Actuarial Journal* (1974): 190–204.
4. BARNDORFF-NIELSEN, O.E. "Gaussian\Inverse Gaussian Processes and the Modelling of Stock Returns," working paper, Aarhus University, 1994.

5. BICK, A. "On the Consistency of the Black-Scholes Model with a General Equilibrium Framework," *Journal of Financial and Quantitative Analysis* 22 (1987): 259–75.
6. BICK, A. "On Viable Diffusion Price Processes of the Market Portfolio," *Journal of Finance* 45 (1990): 673–89.
7. BLACK, F., AND SCHOLES, M. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (1973): 637–59.
8. BOWERS, N.L., JR., GERBER, H.U., HICKMAN, J.C., JONES, D.A., AND NESBITT, C.J. *Actuarial Mathematics.* Itasca, Ill.: Society of Actuaries, 1986.
9. CONSTANTINIDES, G.M. "Theory of Valuation: Overview and Recent Developments." In *Theory of Valuation: Frontier of Modern Financial Theory*, Volume 1, edited by S. Bhattacharya and G.M. Constantinides, 1–23. Totowa, N.J.: Rowman & Littlefield, 1989.
10. DUFRESNE, F., GERBER H.U., AND SHIU, E.S.W. "Risk Theory with the Gamma Process," *ASTIN Bulletin* 21 (1991): 177–92.
11. GERBER, H.U., AND SHIU, E.S.W. Discussion of J.A. Tilley's "Valuing American Options in a Path Simulation Model," *TSA* XLV (1993): 524–35.
12. GERBER, H.U., AND SHIU, E.S.W. "Martingale Approach to Pricing Perpetual American Options," *ASTIN Bulletin* 24 (1994), 195–220. This is a revision of [40] of the paper.
13. GERBER, H.U., AND SHIU, E.S.W. "Pricing Financial Contracts with Indexed Homogeneous Payoff," *Bulletin of the Swiss Association of Actuaries* 94 (1994): 143–66.
14. GERBER, H.U., AND SHIU, E.S.W. "Martingale Approach to Pricing Perpetual American Options on Two Stocks," Submitted for publication.
15. HE, H., AND LELAND, H. "On Equilibrium Asset Price Processes," *Review of Financial Studies* 6 (1993): 593–617.
16. HEATH, D.C., AND SUDDERTH, W.D. "On a Theorem of De Finetti, Oddsmaking, and Game Theory," *Annals of Mathematical Statistics* 43 (1972): 2072–77.
17. HESTON, S.L. "Invisible Parameters in Option Prices," *Journal of Finance* 48 (1993): 933–47.
18. JONES, D.A. "Bayesian Statistics," *TSA* XVII (1965): 33–57; Discussion 181–200.
19. KIM, M.-J., OH, Y.-H., AND BROOKS, R. "Are Jumps in Stock Returns Diversifiable? Evidence and Implications for Option Pricing," *Journal of Financial and Quantitative Analysis* 29 (1994): 609–31.
20. MARGRABE, W. "The Value of an Option to Exchange One Asset for Another," *Journal of Finance* 33 (1978): 177–86.
21. MERTON, R.C. "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Financial Economics* 3 (1976): 125–44.

22. NAIK, V., AND LEE, M. "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns," *Review of Financial Studies* 3 (1990): 493–521.

23. PARKINSON, M. "Option Pricing: The American Put," *Journal of Business* 50 (1977): 21–36.

24. PEDERSEN, H.W., AND SHIU, E.S.W. "Evaluation of the GIC Rollover Option," *Insurance: Mathematics and Economics* 14 (1994): 117–27.

25. PETERS, E.E. *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics.* New York: Wiley, 1994.

26. RUBINSTEIN, M. "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics* 7 (1976): 407–25.

27. SAMUELSON, P.A. "Rational Theory of Warrant Pricing," *Industrial Management Review* 6, issue 2 (1965): 13–32.

28. SIEGMUND, D. "The Time Until Ruin in Collective Risk Theory," *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* 75 (1975): 157–66.

29. STAPLETON, R.C., AND SUBRAHMANYAM, M.G. "Risk Aversion and the Intertemporal Behavior of Asset Prices," *Review of Financial Studies* 3 (1990): 677–93.

30. STRICKER, C. "Some Remarks on Hedging of Contingent Claims." In *Stochastic Process and Optimal Control*, edited by Engelbert and I. Karatzas, 177–81. New York: Gordon & Breach, 1993.

31. WANG, S. "The Integrability Problem of Asset Prices," *Journal of Economic Theory* 59 (1993): 199–213.